Covering functors without groups

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Abstract

Coverings in the representation theory of algebras were introduced for the Auslander-Reiten quiver of a representation finite algebra in [15] and later for finite dimensional algebras in [2, 7, 11]. The best understood class of covering functors is that of Galois covering functors $F : A \to B$ determined by the action of a group of automorphisms of A. In this work we introduce the balanced covering functors which include the Galois class and for which classical Galois covering-type results still hold. For instance, if $F : A \to B$ is a balanced covering functor, where A and B are linear categories over an algebraically closed field, and B is tame, then A is tame.

Introduction and notation

Let k be a field and A be a finite dimensional (associative with 1) k-algebra. One of the main goals of the representation theory of algebras is the description of the category of finite dimensional left modules $_A$ mod. For that purpose it is important to determine the representation type of A. The finite representation type (that is, when A accepts only finitely many indecomposable objects in $_A$ mod, up to isomorphism) is well understood. In that context, an important tool is the construction of Galois coverings $F: \tilde{A} \to A$ of A since \tilde{A} is a locally representation-finite category if and only if A is representation-finite [7, 12]. For a tame algebra A and a Galois covering $F: \tilde{A} \to A$, the category \tilde{A} is also tame, but the converse does not hold [9, 14].

Coverings were introduced in [15] for the Auslander-Reiten quiver of a representationfinite algebra. For algebras of the form A = kQ/I, where Q is a quiver and I an admissible ideal of the path algebra kQ, the notion of covering was introduced in [2, 7, 11]. Following [2], a functor $F: A \to B$, between two locally bounded k-categories A and B, is a covering functor if the following conditions are satisfied:

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- (a) F is a k-linear functor which is onto on objects;
- (b) the induced morphisms

$$\bigoplus_{Fb'=j} A(a,b') \to B(Fa,j) \text{ and } \bigoplus_{Fa'=i} A(a',b) \to B(i,Fb)$$

are bijective for all i, j in B and a, b in A.

We denote by $({}_{b'}f_{a}^{\bullet})_{b'} \mapsto f$ and $({}_{b}^{\bullet}f_{a'})_{a'} \mapsto f$ the corresponding bijections. We shall consider $F_{\lambda}: {}_{A} \mod \to {}_{B} \mod$ the left adjoint to the pull-up functor $F_{\bullet}: {}_{B} \mod \to {}_{A} \mod$, $M \mapsto MF$, where ${}_{C} \mod$ denotes the category of left modules over the k-category C, consisting of covariant k-linear functors.

The best understood examples of covering functors are the Galois covering functors $A \to B$ given by the action of a group of automorphisms G of A acting freely on objects and where $F: A \to B = A/G$ is the quotient defined by the action. See [2, 5, 7, 11, 12] for results on Galois coverings. Examples of coverings which are not of Galois type will be exhibited in Section 1.

In this work we introduce balanced coverings as those coverings $F: A \to B$ where ${}_{b}f_{a}^{\bullet} = {}_{b}^{\bullet}f_{a}$ for every $f \in B(Fa, Fb)$. Among many other examples, Galois coverings are balanced, see Section 2. We shall prove the following:

THEOREM 0.1 Let $F: A \to B$ be a balanced covering. Then every finitely generated A-module X is a direct summand of $F_{\bullet}F_{\lambda}X$.

In fact, according to the notation in [1], we show that a balanced covering functor is a *cleaving* functor, see Section 3. This is essential for extending Galois covering-type results to more general situations. For instance we show the following result.

THEOREM 0.2 Assume that k is an algebraically closed field and let $F: A \rightarrow B$ be a covering functor. Then the following hold:

- (a) If F is induced from a map $f : (Q, I) \to (Q', I')$ of quivers with relations, where A = kQ/I and B = kQ'/I', then B is locally representation-finite if and only if so is A;
- (b) If F is balanced and B is tame, then A is tame.

More precise statements are shown in Section 4. For a discussion on the representation type of algebras we refer to [1, 13, 9, 6, 14].

1 Coverings: examples and basic properties

1.1 The pull-up and push-down functors

Following [2, 7], consider a locally bounded k-category A, that is, A has a (possibly infinite) set of non-isomorphic objects A_0 such that

- (a) A(a, b) is a k-vector space and the composition corresponds to linear maps $A(a, b) \otimes_k A(b, c) \to A(a, c)$ for every a, b, c objects in A_0 ;
- (b) A(a, a) is a local ring for every a in A_0 ;
- (c) $\sum_{b} A(a, b)$ and $\sum_{b} A(b, a)$ are finite dimensional for every a in A_0 .

For a locally bounded k-category A, we denote by ${}_{A}Mod$ (resp. Mod_{A}) the category of covariant (resp. contravariant) functors $A \to Mod_k$; by ${}_{A}mod$ (resp. mod_A) we denote the full subcategory of locally finite-dimensional functors $A \to mod_k$ of the category ${}_{A}Mod$ (resp. Mod_A). In case A_0 is finite, A can be identified with the finitedimensional k-algebra $\bigoplus_{a,b\in A_0}A(a,b)$; in this case the category ${}_{A}Mod$ (resp. ${}_{A}mod$) is equivalent to the category of left A-modules (resp. finitely generated left A-modules).

According to [6], in case k is algebraically closed, there exist a quiver Q and an ideal I of the path category kQ, such that A is equivalent to the quotient kQ/I. Then any module $M \in {}_A$ Mod can be identified with a *representation* of the *quiver* with relations (Q, I). Usually our examples will be presented by means of quivers with relations.

Let $F: A \to B$ be a k-linear functor between two locally bounded k-categories. The *pull-up* functor $F_{\bullet}: {}_{B}Mod \to {}_{A}Mod, M \mapsto MF$ admits a left adjoint $F_{\lambda}: {}_{A}Mod \to {}_{B}Mod$, called the *push-down* functor, which is uniquely defined (up to isomorphism) by the following requirements:

- (i) $F_{\lambda}A(a, -) = B(Fa, -);$
- (ii) F_{λ} commutes with direct limits.

In particular, F_{λ} preserves projective modules. Denote by $F_{\rho}: {}_{A}Mod \to {}_{B}Mod$ the right adjoint to F_{\bullet} .

For covering functors $F: A \to B$ we get an explicit description of F_{λ} and F_{ρ} as follows:

Lemma 1.1 [2]. Let $F: A \to B$ be a covering functor. Then

(a) For any $X \in A \mod and f \in B(i, j)$,

$$F_{\lambda}X(f) = (X({}_{b}f_{a}^{\bullet})) : \bigoplus_{Fa=i} X(a) \to \bigoplus_{Fb=j} X(b), \text{ with } \sum_{Fb=j} F({}_{b}f_{a}^{\bullet}) = f$$

In particular, $F_{\bullet}(a, -): F_{\lambda}A(a, -) \to B(Fa, -)$ is the natural isomorphism given by $({}_{b}f_{a}^{\bullet})_{b} \mapsto f$.

(b) For any $X \in A \mod and f \in B(i, j)$

$$F_{\rho}X(f) = (X({}^{\bullet}_{b}f_{a})) \colon \prod_{Fa=i} X(a) \to \prod_{Fb=j} X(b), \text{ with } \sum_{Fa=i} F({}^{\bullet}_{b}f_{a}) = f.$$

In particular, $F_{\bullet}D(-,b)$: $F_{\rho}DA(-,b) \to DB(-,Fb)$ is the natural isomorphism induced by $({}_{b}^{\bullet}f_{a})_{a} \mapsto f$.

1.2 The order of a covering

The following lemma allows us to introduce the notion of *order* of a covering.

Lemma 1.2 Let $F: A \to B$ be a covering functor. Assume that B is connected and a fiber $F^{-1}(i)$ is finite, for some $i \in B_0$. Then the fibers have constant cardinality.

Proof. Let $i \in B_0$ and $0 \neq f \in B(i, j)$. For $a \in F^{-1}(i)$, $\sum_{Fb=j} \dim_k A(a, b) = \dim_k B(i, j)$. Hence $|F^{-1}(i)|\dim_k B(i, j) = \sum_{Fa=i} \sum_{Fb=j} \dim_k A(a, b) = \sum_{Fb=j} \sum_{Fa=i} \dim_k A(a, b) = |F^{-1}(j)|\dim_k B(i, j)$ and $|F^{-1}(i)| = |F^{-1}(j)|$. Since B is connected, the claim follows.

In case $F: A \to B$ is a covering functor with B connected and A_0 is finite, we define the *order* of F as $\operatorname{ord}(F) = |F^{-1}(i)|$ for any $i \in B_0$. Thus $\operatorname{ord}(F)|B_0| = |A_0|$.

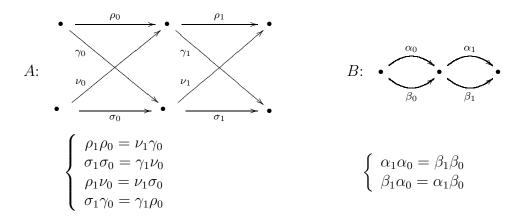
We recall from the Introduction that a covering functor $F: A \to B$ is balanced if ${}_{b}f_{a}^{\bullet} = {}_{b}^{\bullet}f_{a}$ for every couple of objects a, b in A.

Lemma 1.3 Let $F: A \to B$ be a balanced covering functor, then $F_{\lambda} = F_{\rho}$ as functors $_{A} \mod \to _{B} \mod$.

1.3 Examples

(a) Let A be a locally bounded k-category and let G be a group of k-linear automorphisms acting freely on A (that is, for $a \in A_0$ and $g \in G$ if ga = a, then g = 1). The quotient category A/G has as objects the G-orbits in the objects of A; a morphism $f: i \to j$ in A/G is a family $f: ({}_bf_a) \in \prod_{a,b} A(a,b)$, where a (resp. b) ranges in i (resp. j) and $g \cdot {}_bf_a = {}_{gb}f_{ga}$ for all $g \in G$. The canonical projection $F: A \to A/G$ is called a Galois covering defined by the action of G.

A particular situation is illustrated by the following algebras (given as quivers with relations):



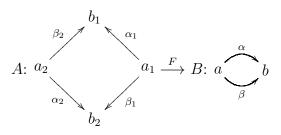
The algebra A is tame, but B is wild when char k = 2 [9]. The cyclic group C_2 acts freely on A and A/C_2 is isomorphic to B.

(b) Consider the algebras given by quivers with relations and the functor F as follows:

$$a_{2}\underbrace{\bigcap_{\beta_{2}}}^{\alpha_{2}}b_{2}\underbrace{\bigcap_{\rho_{2}}}^{\rho_{1}}b_{1}\underbrace{\bigcap_{\beta_{1}}}^{\alpha_{1}}a_{1}\xrightarrow{F}a\underbrace{\bigcap_{\beta}}^{\alpha}b\bigcirc\rho$$

both algebras with rad $^2 = 0$ and $F\alpha_1 = \alpha$, $F\alpha_2 = \alpha + \beta$, $F\beta_i = \beta$, $F\rho_i = \rho$, i = 1, 2. It is a simple exercise to check that F is a balanced covering, but obviously it is not of Galois type.

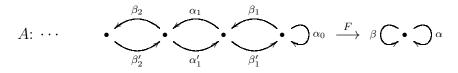
(c) Consider the functor



where $F\alpha_i = \alpha$, i = 1, 2, $F\beta_1 = \beta$, $F\beta_2 = \alpha + \beta$. Since $F(\beta_2 - \alpha_2) = \beta$ and $F(\beta_1) = \beta$, then ${}_{b_2}\beta_{a_2}^{\bullet} = -\alpha_2$ and ${}_{b_2}^{\bullet}\beta_{a_2} = 0$. Hence F is a non-balanced covering functor.

For the two dimensional indecomposable A-module X given by $X(a_2) = k, X(b_2) = k, X(\alpha_2) = id$ and zero otherwise, it follows that $F_{\bullet}F_{\lambda}X$ is indecomposable and hence X is not a direct summand of $F_{\bullet}F_{\lambda}X$.

(d) As a further example, consider the infinite category A and the balanced covering functor defined in the obvious way:



where both categories A and B have $rad^2 = 0$.

1.4 Coverings of schurian categories

We say that a locally bounded k-category B is schurian if for every $i, j \in B_0$, $\dim_k B(i, j) \leq 1$.

Lemma 1.4 Let $F: A \to B$ be a covering functor and assume that B is schurian, then F is balanced.

Proof. Let $0 \neq f \in B(i, j)$ and Fa = i, Fb = j. Since B is schurian, there is a unique $0 \neq {}_{b'}f_a^{\bullet} \in A(a, b')$ with Fb' = j and a unique $0 \neq {}_{b}^{\bullet}f_{a'} \in A(a', b)$ with Fa' = i satisfying $F_{b'}f_a^{\bullet} = f = F_b^{\bullet}f_{a'}$. In case b = b', then a = a' and ${}_{b}f_a^{\bullet} = {}_{b}^{\bullet}f_a$. Else $b \neq b'$ and ${}_{b}f_a^{\bullet} = 0$. In this situation $a \neq a'$ and ${}_{b}f_a = 0$.

Proposition 1.5 Let $F: A \to B$ be a covering functor with finite order and B schurian. Then for every $M \in {}_B \mod, F_{\lambda}F_{\bullet}M \cong M^{\operatorname{ord}(F)}$.

Proof. For any $0 \neq f \in B(i, j)$ we get

Since for each *a* there is a unique *b* with ${}_{b}f_{a}^{\bullet} \neq 0$ such that $F_{b}f_{a}^{\bullet} = f$, then the square commutes.

Remark: If B is not schurian the result may not hold as shown in [9, (3.1)] for a Galois covering $F: B \to C$ with B as in Example (1.3.a).

1.5 Coverings induced from a map of quivers

Let $q: Q' \to Q$ be a covering map of quivers, that is, q is an onto morphism of oriented graphs inducing bijections $i^+ \to q(i)^+$ and $i^- \to q(i)^-$ for every vertex i in Q', where x^+ (resp. x^-) denotes those arrows $x \to y$ (resp. $y \to x$). For the concept of covering and equitable partitions in graphs, see [10].

Assume that Q is a finite quiver. Let I be an *admissible ideal* of the path algebra kQ, that is, $J^n \subset I \subset J^2$ for J the ideal of kQ generated by the arrows of Q. We say that I is *admissible with respect* to q if there is an ideal I' of the path category kQ' such that the induced map $kq: kQ \to kQ'$ restricts to isomorphisms $\bigoplus_{q(a)=i} I'(a,b) \to I(i,j)$ for q(b) = j and $\bigoplus_{q(b)=j} I'(a,b) \to I(i,j)$ for q(a) = i. Observe that most examples in (1.3) (not Example (c)) are built according to the following proposition:

Proposition 1.6 Let $q: Q' \to Q$ be a covering map of quivers, I an admissible ideal of kQ and I' an admissible ideal of kQ' making I admissible with respect to q as in the above definition. Then the induced functor $F: kQ'/I' \to kQ/I$ is a balanced covering functor.

Proof. Since q is a covering of quivers, it has the unique lifting property of paths. Hence for any pair of vertices i in Q and a in Q' with q(a) = i, we have that

is a commutative diagram with F an isomorphism. This shows that F is a covering functor.

For any arrow $i \xrightarrow{\alpha} j$ in Q and q(a) = i, there is a unique b in Q' and an arrow $a \xrightarrow{\alpha'} b$ with $q(\alpha') = \alpha$. Hence the class ${}_{b}f_{a}^{\bullet}$ of α' in kQ'/I'(a,b) satisfies that $F({}_{b}f_{a}^{\bullet})$ is the class $f = \bar{\alpha}$ of α in kQ/I(i,j). By symmetry, ${}_{b}f_{a}^{\bullet} = {}_{b}^{\bullet}f_{a}$. For arbitrary $f \in kQ/I(i,j)$, f is the linear combination $\sum_{\alpha} \lambda_{i}f_{i}$, where f_{i} is the product of classes of arrows in Q. Observe that for arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} m$ we have ${}_{c}(\bar{\beta}\bar{\alpha})_{a}^{\bullet} = ({}_{c}\bar{\beta}_{b}^{\bullet})({}_{b}^{\bullet}\bar{\alpha}_{a}) = ({}_{c}\bar{\beta}_{b}^{\bullet})({}_{b}\bar{\alpha}_{a}^{\bullet})$. It follows that F is balanced. \Box

In the above situation we shall say that the functor F is *induced* from a map $q: (Q', I') \to (Q, I)$ of quivers with relations.

2 On Galois coverings

2.1 Galois coverings are balanced

Proposition 2.1 Let $F: A \to B$ be a Galois covering, then F is balanced.

Proof. Assume F is determined by the action of a group G of automorphisms of A, acting freely on the objects A_0 . Let i, j be objects of B and $f \in B(i, j)$. Consider a, b in A with Fa = i, Fb = j and $\binom{b'f_a}{b'} \in \bigoplus_{Fb'=j} A(a, b')$ with $\sum_{Fb'=j} F({}_{b'}f_a^{\bullet}) = f$. For each object b' with Fb' = j, there is a unique $g_{b'} \in G$ with $g_{b'}(b') = b$. Then $(g_{b'}({}_{b'}f_a^{\bullet}))_{b'} \in \bigoplus_{b'} A(g_{b'}(a), b) = \bigoplus_{Fa'=i} A(a', b)$ with $\sum_{b'} F(g_{b'}({}_{b'}f_a^{\bullet})) = \sum_{b'} F({}_{b'}f_a^{\bullet}) = f$. Hence $g_{b'}({}_{b'}f_a^{\bullet}) = {}_{b}^{\bullet}f_{g_{b'}(a)}$ for every Fb' = j. In particular, for $g_b = 1$ we get ${}_{b}f_a^{\bullet} = {}_{b}^{\bullet}f_a$.

2.2 The smash-product

We say that a k-category B is G-graded with respect to the group G if for each pair of objects i, j there is a vector space decomposition $B(i, j) = \bigoplus_{g \in G} B^g(i, j)$ such that the composition induces linear maps

$$B^g(i,j) \otimes B^h(j,m) \to B^{gh}(i,m).$$

Then the smash product B # G is the k-category with objects $B_0 \times G$, and for pairs $(i, g), (j, h) \in B_0 \times G$, the set of morphisms is

$$(B \# G)((i,g),(j,h)) = B^{g^{-1}h}(i,j)$$

with compositions induced in natural way.

In [4] it was shown that B # G accepts a free action of G such that

$$(B \# G)/G \xrightarrow{\sim} B.$$

Moreover, if B = A/G is a quotient, then B is a G-graded k-category and

$$(A/G) \# G \xrightarrow{\sim} A.$$

Proposition 2.2 Let $F: A \to B$ be a covering functor and assume that B is a G-graded k-category. Then

(i) Assume A accepts a G-grading compatible with F, that is, F(A^g(a, b)) ⊆ B^g(Fa, Fb), for every pair a, b ∈ A₀ and g ∈ G. Then there is a covering functor F # G: A # G → B # G completing a commutative square

$$\begin{array}{c} A \ \# \ G \xrightarrow{F \ \# \ G} B \ \# \ G \\ \downarrow \qquad \qquad \downarrow \\ A \xrightarrow{F \ B} B \end{array}$$

where the vertical functors are the natural quotients. Moreover F is balanced if and only if F # G is balanced.

(ii) In case B is a schurian algebra, then A accepts a G-grading compatible with F.

Proof. (i): For each $a, b \in A_0$, consider the decomposition $A(a, b) = \bigoplus_{g \in G} A^g(a, b)$ and $B(Fa, Fb) = \bigoplus_{g \in G} B^g(Fa, Fb)$. Since these decompositions are compatible with F, then $A^g(a, b) = F^{-1}(B^g(Fa, Fb))$, for every $g \in G$. For $\alpha \in (A \# G)((a, g), (b, h)) = A^{g^{-1}h}(a, b) = F^{-1}(B^{g^{-1}h}(Fa, Fb))$, we have

$$(F \# G)(\alpha) = F\alpha \in B^{g^{-1}h}(Fa, Fb) = (B \# G)((Fa, g), (Fb, h)).$$

(ii): Assume B is schurian and take $a, b \in A_0$ and $g \in G$. Either $B^g(Fa, Fb) = B(Fa, Fb) \neq 0$, if $A(a, b) \neq 0$ or $B^g(Fa, Fb) = 0$, correspondingly we set $A^g(a, b) = A(a, b)$ or $A^g(a, b) = 0$. Observe that the composition induces linear maps $A^g(a, b) \otimes A^h(b, c) \to A^{gh}(a, c)$, hence A accepts a G-grading compatible with F. \Box

Remark: In the situation above, the fact that A and B # G are connected categories does not guaranty that A # G is connected. For instance, if B = A/G, then $A \# G = A \times G$.

The following result is a generalization of Proposition 2.2(ii).

Proposition 2.3 Let $F: A \to B$ be a (balanced) covering functor induced from a map of quivers with relations. Let $F': B' \to B$ be a Galois covering functor induced from a map of quivers with relations defined by the action of a group G. Assume moreover that B' is schurian. Then A accepts a G-grading compatible with F making the following diagram commutative

$$\begin{array}{c} A \ \# \ G \xrightarrow{F \ \# G} B' \\ \downarrow \qquad \qquad \downarrow_{F'} \\ A \xrightarrow{F \ B} \end{array}$$

Proof. Let $A = k\Delta/J$, B = kQ/I and B' = kQ'/I' be the corresponding presentations as quivers with relations, F induced from the map $\delta: \Delta \to Q$, while F' induced from the map $q: Q' \to Q$. For each vertex a in Δ fix a vertex a' in Q' such that F'a' = Fa.

Consider an arrow $a \xrightarrow{\alpha} b$ in Δ and $\overline{\alpha}$ the corresponding element of A. We claim that there exists an element $g_{\alpha} \in G$ such that $F(\overline{\alpha}) \in B^{g_{\alpha}}(Fa, Fb)$. Indeed, we get $F(\overline{\alpha}) = \overline{\beta} = F'(\overline{\beta'})$ for arrows $Fa \xrightarrow{\beta} Fb$ and $a' \xrightarrow{\beta'} g_{\alpha}b'$ for a unique $g_{\alpha} \in G$. Therefore $F(\overline{\alpha}) \in B^{g_{\alpha}}(Fa, Fb)$. We shall define $A^{g_{\alpha}}(a, b)$ as containing the space $k\overline{\alpha}$. For this purpose, consider $g \in G$ and any vertices a, b in Δ , then $A^g(a, b)$ is the space generated by the classes \overline{u} of the paths $u : a \to b$ such that $F(\overline{u}) \in B^g(Fa, Fb)$. Since the classes of the arrows in Δ generate A, then $A(a, b) = \bigoplus_{g \in G} A^g(a, b)$. We shall prove that there are linear maps

$$A^g(a,b) \otimes A^h(b,c) \to A^{gh}(a,c).$$

Indeed, if $\overline{u} \in A^g(a, b)$ and $\overline{v} \in A^h(b, c)$ for paths $u : a \to b$ and $v : b \to c$ in Δ , let $F(\overline{u}) = F'(\overline{u'})$ and $F(\overline{v}) = F'(\overline{v'})$ for paths $u' : a' \to gb'$ and $v' : b' \to hc'$ in Q'. Since B' is schurian then the class of the lifting of $F(\overline{vu})$ to B' is (gv')u'. Therefore

$$F(\overline{v})F(\overline{u}) = F'(\overline{(gv')u'}) \in B^{gh}(Fa, Fb).$$

By definition, the G-grading of A is compatible with F. We get the commutativity of the diagram from Proposition 2.2. \Box

2.3 Universal Galois covering

Let B = kQ/I be a finite dimensional k-algebra. According to [11] there is a kcategory $\tilde{B} = k\tilde{Q}/\tilde{I}$ and a Galois covering functor $\tilde{F}: \tilde{B} \to B$ defined by the action of the fundamental group $\pi_1(Q, I)$ which is *universal* among all the Galois coverings of B, that is, for any Galois covering $F: A \to B$ there is a covering functor $F': \tilde{B} \to A$ such that $\tilde{F} = FF'$. In fact, the following more general result is implicitely shown in [11]:

Proposition 2.4 [11]. The universal Galois covering $\tilde{F}: \tilde{B} \to B$ is universal among all (balanced) covering functors $F: A \to B$ induced from a map $q: (Q', I') \to (Q, I)$ of quivers with relations, where A = kQ'/I'.

3 Cleaving functors

3.1 Balanced coverings are cleaving functors

Consider the k-linear functor $F: A \to B$ and the natural transformation $F(a,b): A(a,b) \to B(Fa,Fb)$ in two variables. The following is the main observation of this work.

THEOREM 3.1 Assume $F: A \to B$ is a balanced covering, then the natural transformation $F(a, b): A(a, b) \to B(Fa, Fb)$ admits a retraction $E(a, b): B(Fa, Fb) \to A(a, b)$ of functors in two variables a, b such that $E(a, b)F(a, b) = \mathbf{1}_{A(a,b)}$ for all $a, b \in A_0$.

Proof. Set E(a, b): $B(Fa, Fb) \to A(a, b)$, $f \mapsto {}^{\bullet}_{b}f_{a}$ which is a well defined map. For any $\alpha \in A(a, a')$, $\beta \in A(b, b')$, we shall prove the commutativity of the diagrams:

$$\begin{array}{ccc} B(Fa,Fb) \xrightarrow{E(a,b)} A(a,b) & B(Fa',Fb) \xrightarrow{E(a',b)} A(a',b) \\ B(Fa,F\beta) & & & & \\ B(Fa,Fb') \xrightarrow{E(a,b')} A(a,b') & & & \\ B(Fa,Fb) & & & & \\ B(Fa,Fb) \xrightarrow{E(a,b)} A(a,b) & & \\ \end{array}$$

For the sake of clarity, let us denote by \circ the composition of maps. Indeed, let $f \in B(Fa, Fb)$ and calculate $\sum_{Fa'=Fa} F(\beta \circ {}^{\bullet}_{b}f_{a'}) = F\beta \circ f$, hence

$$A(a,\beta) \circ E(a,b)(f) = \beta \circ {}_{b}^{\bullet}f_{a} = {}_{b'}^{\bullet}(F\beta \circ f)_{a} = E(a,b') \circ B(Fa,F\beta)(f),$$

and the first square commutes. Moreover, let $h \in B(Fa', Fb)$ and calculate $\sum_{Fb'=Fb} F({}_{b'}h^{\bullet}_{a'} \circ \alpha) = h \circ F \alpha$ and therefore ${}_{b}h^{\bullet}_{a'} \circ \alpha = {}_{b}(h \circ F \alpha)^{\bullet}_{a}$. Using that F is balanced we get that $E(a, b) \circ B(Fa, Fb)(h) = {}_{b}^{\bullet}(h \circ F \alpha)_{a} = {}_{b}^{\bullet}h_{a'} \circ \alpha = A(\alpha, b) \circ E(a', b)(h)$. \Box

Given a k-linear functor $F: A \to B$ the composition $F_{\bullet}F_{\lambda}: {}_{A}\operatorname{Mod} \to {}_{A}\operatorname{Mod}$ is connected to the identity **1** of ${}_{A}\operatorname{Mod}$ by a canonical transformation $\varphi: F_{\bullet}F_{\lambda} \to \mathbf{1}$ determined by $F_{\bullet}F_{\lambda}A(a, -)(b) = \bigoplus_{Fb'=Fb}A(a, b') \to A(a, b), (f_{b'}) \mapsto f_{b}$, see [1, page 234]. Following [1], F is a cleaving functor if the canonical transformation φ admits a natural section $\varepsilon: \mathbf{1} \to F_{\bullet}F_{\lambda}$ such that $\varphi(X)\varepsilon(X) = \mathbf{1}_{X}$ for each $X \in {}_{A}\operatorname{Mod}$. The following statement, essentially from [1], yields Theorem 0.1 in the Introduction.

Corollary 3.2 Let $F: A \to B$ be a balanced covering, then F is a cleaving functor.

Proof. Observe that $F_{\bullet}F_{\lambda}$ is exact, preserves direct sums and projectives (the last property holds since $F_{\bullet}B(i, -) = \bigoplus_{Fa=i}A(a, -)$). Hence to define $\varepsilon: \mathbf{1} \to F_{\bullet}F_{\lambda}$ it is enough to define $\varepsilon(A(a, -)): A(a, -) \to F_{\bullet}F_{\lambda}A(a, -)$ with the desired properties. For $b \in A_0$, consider $\varepsilon_b: A(a, b) \to \bigoplus_{Fb'=Fb} A(a, b') = F_{\bullet}F_{\lambda}A(a, -)(b)$ the canonical inclusion. For $h \in A(b, c)$ we shall prove the commutativity of the following diagram:

Let $f \in A(a,b)$, since F is balanced $A(a, {}_{c'}Fh_{b'}) \circ \varepsilon_b(f) = {}_{c'}Fh_b \circ f = {}_{c'}Fh_$

4 On the representation type of categories

4.1 Representation-finite case

Recall that a k-category A is said to be *locally representation-finite* if for each object a of A there are only finitely many indecomposable A-modules X, up to isomorphism, such that $X(a) \neq 0$. For a cleaving functor $F: A \rightarrow B$ is was observed in [1] that in case B is of locally representation-finite then so is A. In particular this holds when F is a Galois covering by [7]. We shall generalize this result for covering functors.

Part (a) of Theorem 0.2 in the Introduction is the following:

THEOREM 4.1 Assume that k is algebraically closed and let $F: A \to B$ be a covering induced from a map of quivers with relations. Then B is locally representationfinite if and only if so is A. Moreover in this case the functor $F_{\lambda}: {}_{A} \mod \to {}_{B} \mod$ preserves indecomposable modules and Auslander-Reiten sequences.

Proof. Let $F: A \to B$ be induced from $q: (Q', I') \to (Q, I)$ where A = kQ'/I' and B = kQ/I. Let $\tilde{B} = k\tilde{Q}/\tilde{I}$ be the universal cover of B and $\tilde{F}: \tilde{B} \to B$ the universal covering functor. By Proposition 2.4 there is a covering functor $F': \tilde{B} \to A$ such that $\tilde{F} = FF'$.

(1) Assume that B is a connected locally representation-finite category. Since F is induced by a map of quivers with relations, then Proposition 1.6 implies that F is balanced. Hence Corollary 3.2 implies that F is a cleaving functor. By [1, (3.1)], A is locally representation-finite; for the sake of completness, recall the simple argument: each indecomposable A-module $X \in A$ mod is a direct summand of

 $F_{\bullet}F_{\lambda}X = \bigoplus_{i=1}^{n} F_{\bullet}N_{i}^{n_{i}}$ for a finite family N_{1}, \ldots, N_{n} of representatives of the isoclasses N of the indecomposable B-modules with $N(i) \neq 0$ for some i = F(a) with $X(a) \neq 0$.

(2) Assume that A is a locally representation-finite category. First we show that B is representation-finite. Indeed, by case (1), since $F': \tilde{B} \to A$ is a covering induced by a map of a quiver with relations, then \tilde{B} is locally representation-finite. By [12], B is representation-finite. In particular, [7] implies that \tilde{F}_{λ} preserves indecomposable modules, hence F_{λ} and F'_{λ} also preserve indecomposable modules.

Let X be an indecomposable A-module. We shall prove that X is isomorphic to $F'_{\lambda}N$ for some indecomposable \tilde{B} -module N. Since indecomposable projective A-modules are of the form $A(a, -) = F'_{\lambda}\tilde{B}(x, -)$ for some x in \tilde{B} , using the connectedness of Γ_A , we may assume that there is an irreducible morphism $Y \xrightarrow{f} X$ such that $Y = F'_{\lambda}N$ for some indecomposable \tilde{B} -module N. If N is injective, say $N = D\tilde{B}(-, j)$, there is a surjective irreducible map $(h_i) : N \to \bigoplus_i N_i$ such that all N_i are indecomposable modules and $0 \longrightarrow S_j \longrightarrow N \xrightarrow{(h_i)} \bigoplus_i N_i \longrightarrow 0$ is an exact sequence. Then Y = DA(-, F'j) and the exact sequence

$$0 \longrightarrow S_{F'j} \longrightarrow Y \xrightarrow{(F'_{\lambda}(h_i))} \oplus_i F'_{\lambda}(N_i) \longrightarrow 0$$

yields the irreducible maps starting at Y (ending at the indecomposable modules $F'_{\lambda}(N_i)$). Therefore $X = F'_{\lambda}(N_r)$ for some r, as desired. Next, assume that N is not injective and consider the Auslander-Reiten sequence $\xi: 0 \longrightarrow N \xrightarrow{g} N' \xrightarrow{g'} N'' \longrightarrow 0$ in $_{\tilde{B}}$ mod. We shall prove that the push-down $F'_{\lambda}\xi: 0 \longrightarrow F'_{\lambda}N \xrightarrow{F'_{\lambda}g} F'_{\lambda}N' \xrightarrow{F'_{\lambda}g'} F'_{\lambda}N'' \longrightarrow 0$ is an Auslander-Reiten sequence in $_{A}$ mod. This implies that there exists a direct summand \bar{N} of N' such that $X \xrightarrow{\sim} F'_{\lambda}\bar{N}$ which completes the proof of the claim.

To verify that $F'_{\lambda}\xi$ is an Auslander-Reiten sequence, let $h: F'_{\lambda}N \to Z$ be nonsplit mono in $_A$ mod. Consider $\operatorname{Hom}_A(F'_{\lambda}N, Z) \xrightarrow{\sim} \operatorname{Hom}_{\tilde{B}}(N, F'_{\bullet}Z), h \mapsto h'$ which is not a split mono (otherwise, then $\operatorname{Hom}_{\tilde{B}}(F'_{\bullet}Z, N) \xrightarrow{\sim} \operatorname{Hom}_A(Z, F'_{\rho}N), \nu \mapsto \nu'$ with $\nu h' = 1_{F'_{\bullet}Z}$. By Lemma 1.3, $F'_{\lambda} = F'_{\rho}$ and $\nu' h = 1_Z$). Then there is a lifting $\bar{h}: N' \to F'_{\bullet}Z$ with $\bar{h}g = h'$. Hence $\operatorname{Hom}_{\tilde{B}}(N', F'_{\bullet}Z) \xrightarrow{\sim} \operatorname{Hom}_A(F'_{\lambda}N', Z), \bar{h} \mapsto \bar{h}'$ with $\bar{h}'F'_{\lambda}g = h$.

We show that F_{λ} preserves Auslander-Reiten sequences. Let X be an indecomposable A-module of the form $X = F'_{\lambda}N$ for an indecomposable \tilde{B} -module N. Then $F_{\lambda}X = F_{\lambda}F'_{\lambda}N = \tilde{F}_{\lambda}N$. Since by [12], \tilde{F}_{λ} preserves indecomposable modules, then $F_{\lambda}X$ is indecomposable. Finally, as above, we conclude that F_{λ} preserves Auslander-Reiten sequences.

4.2 Tame representation case

Let k be an algebraically closed field. We recall that A is said to be of tame representation type if for each dimension $d \in \mathbb{N}$ and each object $a \in A_0$, there are finitely many A - k[t]-bimodules M_1, \ldots, M_s which satisfy:

(a) M_i is finitely generated free as right k[t]-module $i = 1, \ldots, s$;

(b) each indecomposable $X \in {}_{A}$ mod with $X(a) \neq 0$ and $\dim_{k} X = d$ is isomorphic to some module of the form $M_{i} \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

In fact, it is shown in [13] that A is tame if (a) and (b) are substituted by the weaker conditions:

(a') M_i is finitely generated as right k[t]-module $i = 1, \ldots, s$;

(b') each indecomposable $X \in {}_{A}$ mod with $X(a) \neq 0$ and $\dim_{k} X = d$ is a direct summand of a module of the form $M_{i} \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

The following statement covers claim (b) of Theorem 0.2 in the Introduction.

THEOREM 4.2 Let $F: A \to B$ be a balanced covering functor. If B is tame, then A is tame.

Proof. Let $a \in A_0$ and $d \in \mathbb{N}$. Let M_1, \ldots, M_s be the B - k[t]-bimodules satisfying (a) and (b): each indecomposable $M \in B$ mod with $M(Fa) \neq 0$ and $\dim_k M \leq d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$. By Corollary 3.2 each indecomposable $X \in A$ mod with $X(a) \neq 0$ and $\dim_k X =$ d is a direct summand of some $F_{\bullet}(M_i \otimes_{k[t]} (k[t]/(t-\lambda)))$, which is isomorphic to $F_{\bullet}M_i \otimes_{k[t]} (k[t]/(t-\lambda))$, for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$. Hence A satisfies conditions

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(a') and (b').

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