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Comments on the $U(2)$ noncommutative instanton

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Abstract

We discuss the 't Hoof ansatz for instanton solutions in noncommutative $U(2)$ Yang–Mills theory. We show that the extension of the ansatz leading to singular solutions in the commutative case, yields to non self-dual (or self-antidual) configurations in noncommutative space–time. A proposal leading to selfdual solutions with $Q = 1$ topological charge (the equivalent of the regular BPST ansatz) can be engineered, but in that case the gauge field and the curvature are not Hermitian (although the resulting Lagrangian is real). © 2001 Published by Elsevier Science B.V.

1. Introduction

After the connection between noncommutative quantum field theory and string theory was discovered [1,2], instantons in gauge theories, originally introduced in [3] for noncommutative R^4 space, were seen to play a central role in the quantization of strings ending in D-branes in the presence of a B-field [4]. Later on, they were the object of many investigations [5–13].

Although one can envisage to construct instantons for $U(N)$ gauge group for arbitrary N, most of the results reported correspond to the case of $U(1)$, for which, in contrast with what happens in ordinary space, there also exist non-trivial multi-instantons. The explicit solutions were constructed mostly by applying the ADHM recipe.

Concerning solutions for $N \geqslant 2$, appart from discussions on the ADHM construction [7,8,11], the possibility of extending the so called 't Hooft ansatz for multi-instanton solutions to the noncommutative case was already suggested in [3]. However, we will show that the ansatz proposed in that article for the case of $U(2)$ leads to a configuration which is not self-dual (or self-antidual) and hence does not correspond to a bound of the action.

It is the purpose of this note to carefully test the 't Hooft ansatz for $U(2)$ noncommutative gauge theory, showing that the naive extension of the ordinary ansatz leads to a non self-dual (or non self-antidual) configuration which does not extremize the action. The problem cannot be solved by projecting out an appropriate state from the Fock space, as it can be done in the ADHM approach (see [8] and references therein). Interestingly enough, although the resulting topological charge is $Q = 0$, the configuration does coincide with the ordinary $Q = 1$ instanton solution in the singular gauge when the noncommutative parameters $\theta_{\mu\nu}$ are put to zero. This shows that the $\theta_{\mu\nu} \to 0$ limit is not smooth. We shall then analyse an alterna-

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tive — Belavin–Polyakov–Schwarz–Tyupkin (BPST) like — ansatz for $U(2)$ self-dual (self-antidual) solutions which requires a different internal $U(2)$ structure for the gauge field. However, hermiticity of the gauge fields and curvature is lost, although the resulting instanton Lagrangian is real. Finally, we discuss possible issues which, starting from BPST like ansatz, may lead to hermitian gauge field configurations corresponding to noncommutative instanton.

2. 't Hooft ansatz in commutative space

The first order instanton equations for (ordinary) $SU(2)$ Yang–Mills theory in 4-dimensional Euclidean space are

$$F_{\mu\nu} = \pm \widetilde{F}_{\mu\nu} \tag{1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu + i[A_\mu, A_\nu] \tag{2}$$

and

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{3}$$

Here A_μ are Hermitian gauge fields taking values in the Lie algebra of $SU(2)$, $A_\mu = A^a_\mu \sigma^a/2$, with σ^a the Pauli matrices.

The self-dual ('$+$' sign) and self-antidual ('$-$' sign) solutions to equations (1) correspond to positive and negative topological charge Q,

$$Q = \frac{1}{16\pi^2}\int d^4x \ \mathrm{tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}. \tag{4}$$

The well-honored 't Hooft ansatz [14–16] for a *self-dual* multi-instanton solution is

$$A_\mu(x) = \overline{\Sigma}_{\mu\nu} j_\nu, \tag{5}$$

$$j_\nu = \phi^{-1}(x)\partial_\nu\phi(x). \tag{6}$$

Here

$$\overline{\Sigma}_{\alpha\beta} = \frac{1}{2}\sigma^a \bar{\eta}_{a\alpha\beta}, \tag{7}$$

where $\bar{\eta}_{a\mu\nu}$ is the 't Hooft tensor,

$$\bar{\eta}_{a\mu\nu} = \varepsilon_{a\mu\nu}, \quad \text{if } \mu,\nu = 1,2,3,$$
$$\bar{\eta}_{a\mu 4} = -\bar{\eta}_{a4\mu} = -\delta_{a\mu}, \quad \bar{\eta}_{a44} = 0. \tag{8}$$

It is important to stress that $\overline{\Sigma}_{\alpha\beta}$ is anti-selfdual,

$$\overline{\Sigma}_{\alpha\beta} = -\widetilde{\overline{\Sigma}}_{\alpha\beta}. \tag{9}$$

Inserting ansatz (5) in Eq. (2) one finds that the curvature takes the form

$$F^a_{\mu\nu} = \bar{\eta}_{a\mu\beta} v_{\beta\nu}[\phi] - \bar{\eta}_{a\nu\beta} v_{\beta\mu}[\phi] - \frac{1}{2}\bar{\eta}_{a\mu\nu} v_{\beta\beta}[\phi] - \frac{1}{2}\bar{\eta}_{a\mu\nu} a[\phi], \tag{10}$$

where

$$v_{\mu\sigma} = -\frac{2}{\phi^2}\partial_\mu\phi\partial_\sigma\phi + \frac{1}{\phi}\partial_\mu\partial_\sigma\phi, \tag{11}$$

$$a[\phi] = \frac{1}{\phi}\nabla^2\phi. \tag{12}$$

It is easy to prove that $F^a_{\mu\nu}$ as given in (10) can be written in the form

$$F^a_{\mu\nu} = \widetilde{F}^a_{\mu\nu} - \bar{\eta}_{a\mu\nu} a[\phi], \tag{13}$$

so that, in order to have selfduality, one should impose $a[\phi] = 0$ or

$$\frac{1}{\phi}\nabla^2\phi = 0. \tag{14}$$

A general solution to this equation is

$$\phi(x) = 1 + \sum_{i=1}^{N} \frac{\lambda_i^2}{(x^\mu - a_i^\mu)^2}. \tag{15}$$

Although this solution has singularities at $x = a_i$ ($\nabla^2\phi \propto \sum \delta(x - a_i)$), Eq. (14) is satisfied everywhere.

This ansatz leads to a singular self-dual gauge field. It is instructive to write it in the simplest $Q = 1$ case (with $a_1 = 0$)

$$A^{\mathrm{sing}}_\alpha(x) = -2\lambda^2 \overline{\Sigma}_{\alpha\beta}\frac{x_\beta}{x^2(x^2+\lambda^2)}. \tag{16}$$

Now, the singularity can be removed by an appropriate (singular) gauge transformation leading to

$$A^{\mathrm{reg}}_\alpha(x) = -2\Sigma_{\alpha\beta}\frac{x_\beta}{x^2+\lambda^2}, \tag{17}$$

which coincides with the BPST instanton solution. It is important to note that the self-antidual $\overline{\Sigma}_{\alpha\beta}$ tensor appearing in the singular ansatz has been traded for a self-dual tensor $\Sigma_{\alpha\beta}$,

$$\Sigma_{\alpha\beta} = \frac{1}{2}\sigma^a \eta_{a\alpha\beta}, \tag{18}$$

$$\eta_{a\mu\nu} = \varepsilon_{a\mu\nu}, \quad \text{if } \mu, \nu = 1, 2, 3,$$
$$\eta_{a\mu 4} = -\eta_{a4\mu} = \delta_{a\mu}, \quad \eta_{a44} = 0. \tag{19}$$

The same procedure can be applied to get regular multi-instanton solutions with arbitrary topological charge Q [17].

3. 't Hooft ansatz in noncommutative space

Let us start by defining the Moyal $*$ product of two functions $f(x)$ and $g(x)$,

$$(f * g)(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\right) f(x)g(y)\bigg|_{y=x} \tag{20}$$

with $\theta_{\mu\nu}$ a constant antisymmetric matrix. Then, the Moyal bracket is defined as

$$\{f, g\} = f * g - g * f. \tag{21}$$

The Moyal bracket for (Euclidean) space–time coordinates then reads

$$\{x_\mu, x_\nu\} = i\theta_{\mu\nu}. \tag{22}$$

In 4-dimensional space one can always make $\theta_{12} = \theta_1$, $\theta_{34} = \theta_2$ (with θ_1 and θ_2 real numbers) while all other components vanish.

An alternative approach to noncommutative theories which has shown to be very useful in finding soliton solutions [18] is to directly work with operators in the phase space $\{x_\mu\}$ with commutator (22). Then, $*$ product is just the product of operators and integration over R^4 becomes a trace,

$$\int d^4x\, f(x) = 4\pi^2\theta_1\theta_2 \operatorname{Tr} f(x). \tag{23}$$

In this framework, one considers operators a_b and $a_b^\dagger$ with $b = 1, 2$ in the form

$$a_1 = \frac{1}{\sqrt{2\theta_1}}(x_1 + ix_2) = \frac{z_1}{\sqrt{\theta_1}},$$
$$a_1^\dagger = \frac{1}{\sqrt{2\theta_1}}(x_1 - ix_2) = \frac{\bar{z}_1}{\sqrt{\theta_1}},$$
$$a_2 = \frac{1}{\sqrt{2\theta_2}}(x_3 + ix_4) = \frac{z_2}{\sqrt{\theta_2}},$$
$$a_2^\dagger = \frac{1}{\sqrt{2\theta_2}}(x_3 - ix_4) = \frac{\bar{z}_2}{\sqrt{\theta_2}}, \tag{24}$$

satisfying the algebra (consistent with (22))

$$[a_b, a_c^\dagger] = \delta_{bc}, \qquad [a_b, a_c] = 0,$$
$$[a_b^\dagger, a_c^\dagger] = 0. \tag{25}$$

With this conventions, derivatives should be written as

$$\partial_{z_b} = \frac{-1}{\sqrt{\theta_b}}[a_b^\dagger,\ ], \qquad \partial_{\bar{z}_b} = \frac{1}{\sqrt{\theta_b}}[a_b,\ ]. \tag{26}$$

From the eigenstates of the number operators $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$ one constructs the Fock space $|n_1 n_2\rangle$ so that $|00\rangle$ is the vacuum. It will be important in what follows the following correspondence between projectors and functions

$$|n_1 n_2\rangle\langle n_1 n_2| \longrightarrow 4(-1)^{n_1+n_2} \exp\left(-\frac{r_1^2}{\theta_1} - \frac{r_2^2}{\theta_2}\right) \times L_{n_1}\left(\frac{2r_1^2}{\theta_1}\right) L_{n_2}\left(\frac{2r_2^2}{\theta_2}\right), \tag{27}$$

where $\vec{r}_1 = (x_1, x_2)$ and $\vec{r}_2 = (x_3, x_4)$ and L_n are the Laguerre polynomials.

Coming back to the instanton solution, let us first see that a naive extension of the (commutative) $SU(2)$ 't Hooft ansatz does not work in noncommutative space [21,22]. To begin with, it is well known that consistency requires that the gauge group for noncommutative gauge theories has to be $U(N)$ (or certain subgroups of $U(N)$ [23,24]). We then consider the $U(2)$ case and write

$$A_\mu(x) = A_\mu^a \frac{\sigma^a}{2} + A_\mu^4 \frac{I}{2}. \tag{28}$$

The natural extension of the commutative 't Hooft ansatz (5), (6), to be supplemented with an appropriate ansatz for A_μ^4, is then

$$A_\mu^a \frac{\sigma^a}{2} = \overline{\Sigma}_{\mu\nu} J_\nu[\Phi], \tag{29}$$

where

$$J_\nu = \Phi^{-1} * \partial_\nu \Phi + \partial_\nu \Phi * \Phi^{-1}. \tag{30}$$

Here we have taken a real Φ ($\Phi = \Phi^\dagger$) and hence the combination in (30) leads to an Hermitian gauge field (in fact, (29), (30) is the Hermitian version of the proposal in [3]). Note that in the θ_1, $\theta_2 \to 0$ limit, J_μ coincides with j_μ in (6) if one identifies Φ with $\phi^{1/2}$.

Concerning the A_μ^4 choice, we shall use as a guide that the appropriate ansatz should lead, together with

A^a_μ, to a self-dual $F_{\mu\nu}$ and in particular a selfdual $F^4_{\mu\nu}$. Now, using ansatz (29), (30) one has

$$F^4_{\mu\nu} = \frac{i}{2}\{J_\mu, J_\nu\} + \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\{J_\alpha, J_\beta\} + f_{\mu\nu}, \quad (31)$$

where

$$f_{\mu\nu} = \partial_\mu A^4_\nu - \partial_\nu A^4_\mu + \frac{i}{2}\{A^4_\mu, A^4_\nu\}. \quad (32)$$

Now, one can easily see that the simple ansatz

$$A^4_\mu = -i\left(\Phi^{-1} * \partial_\mu\Phi - \partial_\mu\Phi * \Phi^{-1}\right) \quad (33)$$

leads to a self dual $F^4_{\mu\nu}$ field,

$$F^4_{\mu\nu} = \widetilde{F}^4_{\mu\nu} = i\{J_\mu, J_\nu\} + \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\{J_\alpha, J_\beta\}. \quad (34)$$

Concerning the components of $F_{\mu\nu}$ on the Lie algebra of $SU(2)$, ansatz (29)–(33) gives, in terms of Φ,

$$F^a_{\mu\nu} = \bar{\eta}_{a\mu\beta}V_{\beta\nu} - \bar{\eta}_{a\nu\beta}V_{\beta\mu} - \frac{1}{2}\bar{\eta}_{a\mu\nu}V_{\beta\beta} - \frac{1}{2}\bar{\eta}_{a\mu\nu}A, \quad (35)$$

where

$$V_{\mu\nu} = \left(\partial_\mu\partial_\nu\Phi^{-1}\right) * \Phi + \Phi * \left(\partial_\mu\partial_\nu\Phi^{-1}\right) + \partial_\mu\Phi * \Phi^{-2} * \partial_\nu\Phi + \partial_\nu\Phi * \Phi^{-2} * \partial_\mu\Phi, \quad (36)$$

and

$$A = \Phi^{-1} * \nabla^2\Phi^2 * \Phi^{-1}. \quad (37)$$

Now, after some work, one can see that (35) can be written in the form

$$F^a_{\mu\nu} = \widetilde{F}^a_{\mu\nu} - \bar{\eta}_{a\mu\nu}A, \quad (38)$$

so that, in order to satisfy selfduality, one just has to impose

$$\Phi^{-1} * \nabla^2\Phi^2 * \Phi^{-1} = 0. \quad (39)$$

This is the noncommutative version of the derivation summarized by Eqs. (10)–(14). As in that case, we conclude now that if one finds a field Φ satisfying

$$\nabla^2\Phi^2 = 0 \quad (40)$$

then one has obtained an explicit solution for the $U(2)$ noncommutative instanton. Now, as explained for the ordinary case, Eq. (40) has no nontrivial solution so that one has to look for singular solutions (with eventual sources in the r.h.s. of (40)) but still satisfying (39).

Paralleling the treatment in the ordinary case, one should then introduce, for finite θ_1 and θ_2, a regular, Gaussian-like source for the Laplacian, producing a delta-function when $\theta_1, \theta_2 \to 0$. That is, we propose to solve, instead of a sourceless Laplace equation, the following one

$$\nabla^2\Phi^2(x;\theta_1,\theta_2) = -\frac{4\lambda^2}{\theta_1\theta_2}\exp\left(-\frac{r_1^2}{\theta_1} - \frac{r_2^2}{\theta_2}\right), \quad (41)$$

where λ defines the instanton size.

When $\theta_1 = \theta_2 = \theta$ one easily finds a solution to (41) in the form

$$\Phi^2(x;\theta,\theta) = 1 + \frac{\lambda^2}{r_1^2 + r_2^2}\left(1 - \exp\left(-\frac{r_1^2 + r_2^2}{\theta}\right)\right). \quad (42)$$

Using Eqs. (26) and (27) one can see that Eq. (41) takes, in operator language, the form

$$\frac{2}{\theta_1}\left[a_1^\dagger, \left[a_1, \Phi^2\right]\right] + \frac{2}{\theta_2}\left[a_2^\dagger, \left[a_2, \Phi^2\right]\right] = \frac{\lambda^2}{\theta_1\theta_2}|00\rangle\langle 00|. \quad (43)$$

The right hand side of Eq. (43) corresponds, in the Fock space framework, to the Gaussian source introduced in Eq. (41).

For $\theta_1 = \theta_2 = \theta$ the solution to (43) can be written in a simple form

$$\Phi^2(\theta,\theta) = 1 + \frac{\lambda^2}{2\theta}\sum_{n_1,n_2}\frac{1}{n_1+n_2+1}|n_1n_2\rangle\langle n_1n_2|. \quad (44)$$

Using sum rules for the Laguerre polynomials associated to projectors, one can easily see that (44) coincides with (42). Let us signal at this point that the $\theta_1 \neq \theta_2$ case does not present new difficulties: one starts by solving Eq. (41) and ends with the generalization of (44). We do not detail this derivation since the results are conceptually equivalent to those corresponding to $\theta_1 = \theta_2$.

Expression (44) for Φ^2 was originally presented in [3] as providing an instanton solution once the gauge field is written in terms of Φ. Now, for this to be true, one should verify that, although Φ does not satisfy the sourceless equation (40) but Eq. (41), it still verifies Eq. (39), which provides a necessary condition for selfduality. Now, using expression (44) one finds

$$\Phi^{-1}\nabla^2\Phi^2\Phi^{-1} = -\frac{2\lambda^2}{\theta(2\theta+\lambda^2)}|00\rangle\langle 00|. \tag{45}$$

This implies that $F_{\mu\nu}$ is not selfdual but satisfies

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu} + \overline{\Sigma}_{\mu\nu}\frac{2\lambda^2}{\theta(2\theta+\lambda^2)}|00\rangle\langle 00|. \tag{46}$$

A similar problem was found in [25] in the investigation of 2-dimensional instantons in noncommutative $CP(n)$ model. Indeed, when looking for a solution leading to a singular instanton in the $\theta \to 0$ limit, these authors find that selfduality was satisfied up to a vacuum projector exactly as what happens with the r.h.s. of Eq. (46). In the ADHM approach to 4-dimensional instantons, one also faces such a problem but in that case, it manifests through a non-normalizable zero-mode. However, in that case it is possible to find a "shift" transformation which makes the zero mode normalizable [7,8].

Let us compute the topological charge associated with the configuration (35), (44), using the formula

$$Q = \frac{1}{4}\theta^2 \,\mathrm{Tr}\,\mathrm{tr}\, F_{\mu\nu}\widetilde{F}_{\mu\nu}. \tag{47}$$

One finds, after a lengthy but straightforward calculation,

$$Q = 0. \tag{48}$$

It is interesting to note that $\lim_{\theta\to 0}\Phi^2 = \phi^{(1)}$ with $\phi^{(1)}$ the solution leading to the singular instanton solution (16), which corresponds to $Q = 1$. This shows that it is not safe to interchange the $\theta \to 0$ limit with the Tr operation.

At this point, one could think that projecting out from $F_{\mu\nu}$ its $|00\rangle\langle 00|$ one should obtain a selfdual curvature with nontrivial ($n = 1$) topological charge. However, if one just eliminates the terms containing $|00\rangle\langle 00|$ from $F_{\mu\nu}$ one obtains a selfdual expression but the corresponding Q is not integer (and depends on λ and θ). This is due to the fact that such projected $F_{\mu\nu}$ cannot be written as the curvature of an adequate gauge connection. We have also tried to find a kind of shift S transformation, as defined in [5,19,20], acting on A_μ so that the resulting curvature has no $|00\rangle\langle 00|$ but we did not succeed in it. In summary, in the form proposed in [3] (Eq. (4.9) of [3]) or as modified in (29)–(33), the extension of 't Hooft ansatz does not lead to a regular noncommutative instanton solution with $Q = 1$.

In ordinary space, appart from 't Hooft ansatz, there is an alternative approach, at least for $Q = 1$, which corresponds to search for a regular solution from the beginning, as done in the pioneering work of Belavin, Polyakov, Schwarz and Tyupkin [26]. In the present case, this amounts to propose, instead of ansatz (29) one in the form

$$A_\mu^a\frac{\sigma^a}{2} = \Sigma_{\mu\nu}J_\nu[\Phi] \tag{49}$$

with J_μ defined as in (30). Note that, in contrast with ansatz (29) here we use $\Sigma_{\mu\nu}$, the tensor arising in the regular selfdual solution in commutative space, Eq. (17). This ansatz leads to an $F_{\mu\nu}^4$ in the form

$$F_{\mu\nu}^4 = \frac{i}{2}\left\{J_\mu[\Phi], J_\nu[\Phi]\right\} - \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\left\{J_\alpha[\Phi], J_\beta[\Phi]\right\} + f_{\mu\nu}[\Phi] \tag{50}$$

with $f_{\mu\nu}$ defined as in (32). Now, the choice for A_μ^4 leading to a selfdual $F_{\mu\nu}^4$ is

$$A_\mu^4[\Phi] = i\left(\Phi^{-1} * \partial_\mu\Phi + 3\partial_\mu\Phi * \Phi^{-1}\right), \tag{51}$$

which is manifestly non-Hermitian. From ansatz (49)–(51) one can construct $F_{\mu\nu}$ and determine the conditions under which $F_{\mu\nu}^a$ is also selfdual. One has

$$F_{\mu\nu}^a[\Phi] = -\eta_{a\mu\nu}J_\alpha[\Phi] * J_\alpha[\Phi] + \eta_{a\nu\alpha}D_{\mu\alpha}[\Phi] - \eta_{a\mu\alpha}D_{\nu\alpha}[\Phi], \tag{52}$$

where

$$D_{\mu\alpha} = -\Phi * \partial_\mu\partial_\alpha\Phi^{-2} * \Phi + \left\{J_\mu[\Phi], J_\alpha[\Phi]\right\}. \tag{53}$$

Now, it is easy to see from the expression for $F_{\mu\nu}^a$ as given by (52) that selfduality is ensured whenever the symmetric part of $D_{\mu\alpha}$ satisfies

$$-\Phi * \partial_\mu\partial_\alpha\Phi^{-2} * \Phi = D[\Phi]\delta_{\mu\alpha}, \tag{54}$$

where $D[\Phi]$ is an arbitrary function. Equation (54) has the simple solution

$$\Phi^{-2} = 1 + \frac{1}{\lambda^2}\left(r_1^2 + r_2^2\right), \tag{55}$$

which in the operator language reads

$$\Phi^{-2} = 1 + \frac{2\theta}{\lambda^2}\sum_{n_1 n_2}(n_1 + n_2 + 1)|n_1 n_2\rangle\langle n_1 n_2|. \tag{56}$$

With this expression one can compute A_μ from Eq. (49) and then the field strength which reads

$$F^a_{\mu\nu} = \eta_{a\mu\alpha}\{J_\nu, J_\alpha\} - \eta_{a\nu\alpha}\{J_\mu, J_\alpha\} - \eta_{a\mu\nu}\big(J_\alpha * J_\alpha + 2D[\Phi]\big), \qquad (57)$$

where D, as defined by (54) takes for the solution (55) the form

$$D = -\frac{2}{\lambda^2}\Phi^2. \qquad (58)$$

Now, as one should expect from (51), $F_{\mu\nu}$ is not Hermitian. However the Lagrangian and, a fortiori, the action S and the topological charge Q, are real. In fact, one can check by explicit computation that $S = Q = 1$, a result consistent with the fact that $F_{\mu\nu}$ as given by (57) is a selfdual curvature, which necessarily satisfies the (noncommutative) Yang–Mills equations of motion.

We have discussed the case $\theta_1 = \theta_2$ for the sake of simplicity, but the general $\theta_1 \neq \theta_2$ case can be equally treated just by noting that the solution (56) becomes

$$\Phi^{-2} = 1 + \frac{1}{\lambda^2}\sum_{n_1 n_2}(2\theta_1 n_1 + 2\theta_2 n_2 + \theta_1 + \theta_2) \times |n_1 n_2\rangle\langle n_1 n_2|. \qquad (59)$$

We have seen that noncommutative versions of 't Hooft ansatz yields to $U(2)$ configurations which are either non selfdual or non-Hermitian. One might then expect that a less restrictive ansatz could overcome these problems. One possibility is to consider, instead of (49)

$$A_\mu = \Sigma_{\mu\nu} a_\nu, \qquad A^4_\mu = b_\mu \qquad (60)$$

with a_μ and b_μ Hermitian. The curvature $F^4_{\mu\nu}$ takes then the form,

$$F^4_{\mu\nu} = \frac{i}{2}\{a_\mu, a_\nu\} - \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\{a_\alpha, a_\beta\} + \partial_\mu b_\nu - \partial_\nu b_\mu + \frac{i}{2}\{b_\mu, b_\nu\}. \qquad (61)$$

Now, the selfduality condition applied to $F^4_{\mu\nu}$ implies a relation between a_μ and b_μ as well as a condition over their curvatures. A simple choice satisfying all these restrictions is

$$b_\mu = a_\mu, \qquad (62)$$

$$\partial_\mu a_\nu - \partial_\nu a_\mu + 2i\{a_\mu, a_\nu\} = 0. \qquad (63)$$

Now, in order to also achieve selfduality for $F^a_{\mu\nu}$, the following identity should hold (compare with (53), (54))

$$\partial_\mu a_\nu + \partial_\nu a_\mu - \{a_\mu, a_\nu\}_+ = D[a]\delta_{\mu\nu}. \qquad (64)$$

Concerning condition (63), it is trivially satisfied by

$$a_\mu = -\frac{i}{2}U(x)^\dagger * \partial_\mu U(x), \qquad (65)$$

with $U(x)$ an element of noncommutative $U(1)$ group. In terms of $U(x)$, Eq. (64) becomes

$$-\frac{i}{2}U^{-1}\partial_\mu\partial_\nu U - \frac{1}{8}(1+2i)\big(\partial_\mu U^{-1}\partial_\nu U + \partial_\nu U^{-1}\partial_\mu U\big) = D[U]\delta_{\mu\nu}. \qquad (66)$$

Unlike Eq. (54), we were unable to find a solution of Eq. (66).

In summary, we have shown that the natural extension of the 't Hooft ansatz for $U(2)$ instanton solutions to noncommutative spacetime, as proposed in [3], does not work since it leads to a non-selfdual (or self-antidual) field strength. An alternative ansatz allows to find a self-dual instanton solution which however corresponds to a non-Hermitian gauge field. Nevertheless, this configuration leads to a real Lagrangian and corresponds to a bound of the action S: its topological charge is $Q = S = 1$.

The connection between commutative and noncommutative instantons can then be schematized as follows: the extension of 't Hooft ansatz leading to singular instantons yields, in the noncommutative case, to non-selfdual configurations. Concerning the ansatz leading to regular ordinary instantons, it does give, in the noncommutative case, self-dual solutions. The explicit one that we found was not Hermitian but one should expect that more general ansatz would lead to selfdual Hermitian instantons. We hope to come back to this problem in the future.

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