Stochastic Relaxation Oscillator Model for the Solar Cycle

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We perform a detailed analysis of the sunspot number time series to reconstruct the phase space of the underlying dynamical system. The features of this phase space allow us to describe the behavior of the solar cycle in terms of a simple relaxation oscillator in two dimensions. The absence of systematic self-crossings suggests that the complexity of the sunspot time series does not arise as a consequence of chaos. Instead, we show that it can be adequately modeled through the introduction of a stochastic fluctuation in one of the parameters of the dynamic equations.

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The solar cycle was discovered by Schwabe in 1843. Using observations of the sunspot number as a function of time, Schwabe determined a period of approximately 10 years, and also described an irregular behavior, with fluctuations in the cycle duration as well as in the individual shape and maximum intensity [1]. In 1848 Wolf introduced the criteria currently used to measure the sunspot number, and in 1852 he reported the well-known 11-year cycle.

Even though the global aspects of the solar cycle are well explained by dynamo theory [2], the nature of the irregularities displayed by the sunspot time series is still being debated. Numerous attempts to explain these irregularities have been invoked in the literature, which rely on either of two rather different mechanisms: chaos or stochasticity.

The chaotic approach consists of modeling the dynamics of the system by a set of equations of reduced dimensionality, including nonlinearities able to display chaotic behavior for reasonable values of the relevant parameters (see, for instance [3,4], and references therein). Although the chaotic approach seems appealing, there are no firm indicators of chaos in the sunspot number time series [5]. Recent studies have shown that the standard algorithms used to search for signatures of a low-dimensional chaotic attractor in time series (like Lyapunov exponent estimators, correlation dimensions, and increase of a prediction error with a prediction horizon) can lead to spurious convergence when applied to a limited time series (see [6,7] for a review). Also, it was shown that the increase in the prediction error with an increase in the prediction horizon can in fact be observed in systems with a deterministic skeleton and stochastic components [8–10].

Considering the spatial and temporal complexity associated with the turbulent dynamics of the dynamo region (the solar convective region), we show that it is reasonable to look for alternative mechanisms, other than a chaotic attractor. For instance, the scenario of a stochastically driven solar dynamo has been studied in a number of recent papers [11–13]. Within this framework, the spatial and temporal irregularities observed in the time series can be described as the end result of a stochastic process underlying the deterministic equations governing the dynamics of the magnetic fields.

Recently, Paluš and Novotná [14] have shown evidence for a randomly driven nonlinear oscillator underlying the dynamics of the sunspot cycle. They found a correlation between the instantaneous amplitude and frequency in the yearly sunspot number time series, and the statistical significance of this correlation was tested against other models like the one proposed by Barnes *et al.* [15] as well as surrogate sets of the original data.

In this Letter we perform an embedding of the daily sunspot number time series (Fig. 1a), to reconstruct the phase space of the underlying dynamical system and generate a simple model of the solar cycle. As an important result, we find that the features of this phase space allow us to describe the global behavior of the solar cycle in terms of a deterministic skeleton given by a rather simple relaxation oscillator in two dimensions. Also, we propose an intrinsic stochastic component to model the irregularities, associated with the turbulent processes in the rising and sinking of convective cells at the solar interior.

The physical quantity of interest in solar cycle studies is the magnetic field. Many authors choose the sunspot number as a quadratic quantity in the magnetic field components (for instance, [16] chose the toroidal component

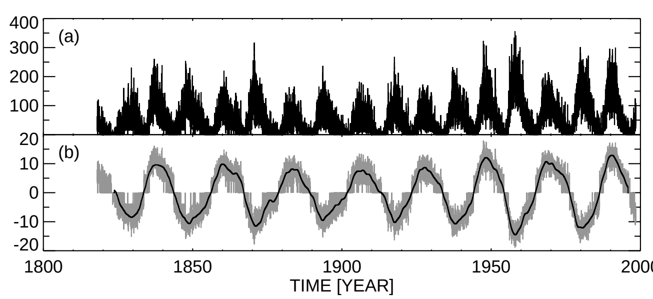

FIG. 1. (a) Daily sunspot number from 1816 to the present (Royal Observatory of Belgium), and (b) time series proportional to the magnetic field obtained from the daily sunspot number (thin line), and smoothed version of the time series (thick line).

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while [17] chose the poloidal one), since according to dynamo models the number of sunspots is proportional to the magnetic energy erupting to the photosphere. Dealing with the intensity time series might affect the topology of the attractor obtained after the embedding [18]. Therefore, it is necessary to obtain a field amplitude time series from the observational data before performing the embedding.

Since the solar magnetic field displays a period of 22 years with a polarity inversion every 11 years, it is customary to define the Bracewell number [19] to explicitly show these features. This number is defined as the sunspot number with a sign change at the beginning of each period, and therefore this index displays a period of 22 years with a sign change every 11 years. Considering the sunspot number as quadratic in the magnetic field, we take the square root of the sunspot number series and change sign at each minimum following [19], to obtain a time series proportional to the spatially averaged magnetic field. The location of solar activity minima was determined by filtering the time series with a low-pass filter to eliminate the high frequency components, and comparing the location of the minimum values thus obtained with those tabulated by the Zurich observatory. Figure 1b shows the time series obtained after applying this method.

The daily sunspot number has a considerable level of high frequency fluctuations. We need to smooth out the series and calculate its derivatives in order to build the underlying phase space. In Fig. 1b we also show a filtered time series in which we eliminated all frequencies whose amplitudes were smaller than 2% of the amplitude of the fundamental mode, and the spurious daily oscillation was removed using a low-pass filter. Without loss of generality and following [16], we will consider this time series to be proportional to the toroidal magnetic field B.

The daily time series of B has more than 60 000 data. To obtain a simple dynamical system of the solar cycle we reconstructed the phase space using a differential phase space embedding (see, for instance, [20,21] for a review). The temporal derivatives of $B(t)$ were calculated using centered finite difference formulas.

Even though the dimension of the embedding is not known, and would justify an analysis of the degrees of freedom of the data, the number of periods is insufficient for such analysis. As an example, Fig. 2 shows an embedding of the time series in two dimensions using a step of 200 days in the finite differences formulas.

At first glance, the presence of self-crossings of the trajectory in phase space seems to justify an embedding in a phase space of higher dimensionality. However, the crossings observed in Fig. 2 are not systematic and can be regarded as caused by the addition of noise to a simple limit cycle, rather than to projection effects from a larger phase space. The lower density of points in the horizontal sides is indicative of a slow dynamics along these sides and a much faster evolution along the vertical sides. To check the convergence, filter out noise, and clearly identify the attractor

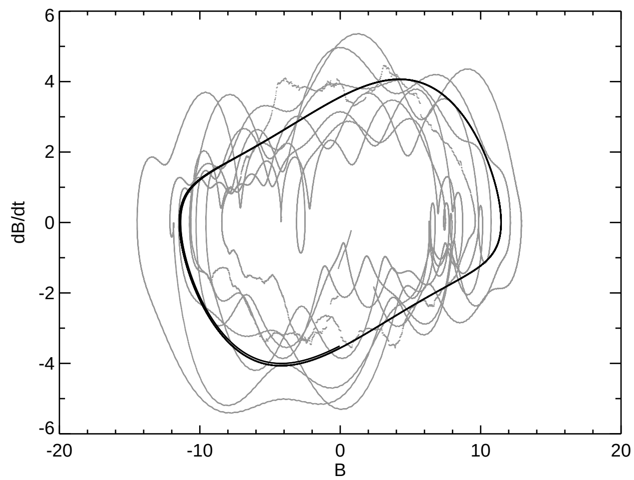

FIG. 2. Trajectories of the observed time series (thin line) and of the Van der Pol oscillator in phase space (thick line).

of the underlying dynamical system, we performed the following analysis.

Differentiation increases the ratio between noise and the original signal. As a result, when the step used in the finite difference formulas is reduced, noise and self-crossings increase noticeably, but for large enough step sizes, the system shows a quite rectangular limit cycle with the dynamics just described. We calculated the slope of each side of the rectangle (two horizontal sides with slow dynamics and two vertical sides with fast evolution) using a least squares fit and finite differentiation with step sizes decreasing from 1600 to 200 days. In each of these regions the slope was found to converge gently for small step sizes, justifying the interpretation of the phase space as an almost rectangular cyclic attractor with two different time scales.

To simulate this behavior, we seek a dynamical system with an attracting solution sharing the features found in the data. The two key elements that we want to reproduce are (a) the difference of time scales in different portions of the trajectory (namely, the slow dynamics in its lateral sides and the much faster evolution in its horizontal sides), and (b) the rectangular shape of the average cycle. These two features together can be modeled by a dynamical system displaying relaxation oscillations.

Let us analyze the paradigmatic (and probably simplest) relaxation oscillator described in the literature: the Van der Pol oscillator. This dynamical system describes an autonomous oscillator where the friction coefficient depends on the amplitude of the oscillations,

$$\begin{cases} \dot{x} = -y - \mu x(\xi x^2 - 1), \\ \dot{y} = \varpi^2 x. \end{cases} \tag{1}$$

For positive values of μ, the origin is a repeller, and the phase space displays an attracting limit cycle which shows two time scales, like the ones observed in the solar cycle time series. By performing a best fit of the parameters, the shape of the limit cycle obtained from the observed series (see Fig. 2) can be correctly described by a Van der Pol

oscillator. Notice that it is possible to interpret these equations in physical terms: if x is proportional to the toroidal component of the magnetic field, and y is a linear combination of the toroidal and poloidal components, Eqs. (1) give rise to a truncated expression of the dynamo equations. The linear terms represent the role of a poloidal velocity field, and the α and ω effects, and the cubic nonlinearities correspond to α quenching and magnetic buoyancy (see, e.g., [2,3] for a review on the dynamo equations).

Since this set of equations is written in terms of two variables and only one of them is proportional to the observed time series, we used the so-called standard form [22,23] to reconstruct the vector field. Changing coordinates in Eqs. (1),

$$u = -y - \mu(\xi x^3 - x), \tag{2}$$

we obtain

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -\varpi^2 x - \mu u(3\xi x^2 - 1), \end{cases} \tag{3}$$

where the standard function is $F(x,u) = -\varpi^2 x - \mu u(3\xi x^2 - 1)$. Now we can build the original phase space by appropriately setting the free parameters in the standard function F. Assuming that x in Eqs. (3) corresponds to the observed time series B, we can compute u and $\dot{u}$ following a finite difference scheme,

$$u = \frac{x(t_{i+h}) - x(t_{i-h})}{t_{i+h} - t_{i-h}}, \tag{4}$$

$$\dot{u} = 4\,\frac{x(t_{i+h}) - 2x(t_i) + x(t_{i-h})}{(t_{i+h} - t_{i-h})^2}. \tag{5}$$

To set the free parameters that best fit the observational data, we randomly took 30 000 data from the time series $B(t)$ and minimized the mean square error between F and $\dot{u}$ using a simplex method. As a result, we obtained an adjustment with $\chi^2 = 0.4$ per point, and

$$\begin{cases} \varpi = 0.2993, \\ \mu = 0.2044, \\ \xi = 0.0102. \end{cases} \tag{6}$$

The number of degrees of freedom of the χ^2 distribution is given by the difference between the number of data points and the adjusted parameters. To give a quantitative measure of the goodness of fit we used the probability Q [24], from which we obtain that a probability of 99% corresponds to $\chi^2 = 0.98$ per point. Therefore, the quality of our fit ($\chi^2 = 0.4$) is remarkably good.

In Fig. 2 we overlay the trajectory of the Van der Pol oscillator in phase space, with the set of parameters listed in Eqs. (6), to the observed time series B. We confirm that the Van der Pol oscillator that best fits the observed series is indeed a reasonable dynamical system to describe the main features of the evolution of the variable B.

However, the sunspot number time series (and consequently the observed time series B) also has an irregular behavior with fluctuations in its frequency and in the intensity of its maxima. Considering the spatial and temporal complexity associated with the turbulent convective motions of the dynamo region, these irregularities can be understood as the result of a stochastic process.

The parameter ξ in Eqs. (3) is responsible for the nonlinear saturation of the system. In dynamo theories [25], the source of poloidal magnetic field is the eruption of magnetic structures caused by instabilities driven by turbulent photospheric motions. The saturation is provided by the Lorentz force, acting to locally limit turbulent motions. Therefore, it seems perfectly natural to assume that ξ is a stochastic variable, i.e.,

$$\xi = \xi_0 + r\xi_s, \tag{7}$$

where ξ_s is assumed to be a zero mean Gaussian stochastic process, describing white noise with dispersion equal to unity, and the dimensionless parameter r is the rms value of the stochastic part of ξ.

The Van der Pol equations (3) can now be written in the following form:

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -\varpi^2 x - \mu u[3(\xi_0 + r\xi_s)x^2 - 1]. \end{cases} \tag{8}$$

In Fig. 3 we show the result of integrating the Van der Pol equations (8) for a period of 150 years, with $r = 0.02$, and the observed time series B. The value for ξ_s was randomly changed every 30 days, which is a typical lifetime for giant cells [11]. However, changing the correlation time from 1 day to 30 days did not change our results appreciably.

This set of equations models the shape of the peaks and the relaxation oscillations of the system quite adequately. Taking the square of the time series we obtain the sunspot number. The theoretical time series displays fast increase and slow decrease in the sunspot number, reaching each maximum in approximately 4 years and decaying in 6 years, which agrees fairly well with the average times observed in the observed sunspot number series. Six historically important observable quantities are generated out of our synthesized data and displayed in Table I, namely, mean values and deviations of the cycle period, rise time and maximum sunspot number. Note that the theoretical dispersions of all the parameters listed in Table I (i.e., period, rise time, and maximum) have been adjusted with only one parameter (r).

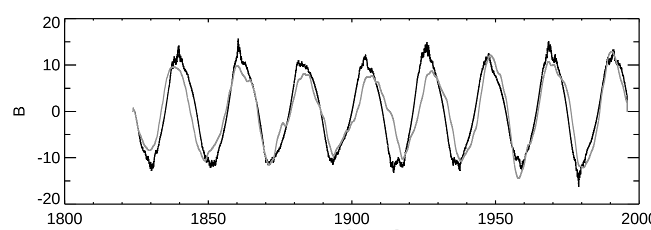

FIG. 3. Observed times series (thin line) and the stochastic Van der Pol oscillator (thick line).

TABLE I. Mean values and deviations for the observational data and for the theoretical dynamical system driven by noise.

	Period (years)	Rise time[a] (years)	Maximum value[b]
Van der Pol	21.4 ± 0.8	4.3 ± 0.4	152 ± 38
Time series *B*	21.4 ± 1.6	4.2 ± 0.4	113 ± 40

[a]From zero to maximum.
[b]For the square of the time series.

In summary, we used the daily sunspot number to reconstruct the phase space of the underlying dynamical system and build a simple model of the solar cycle. The lack of systematic self-crossings in the reconstructed phase space allows us to propose a two-dimensional phase space to describe the deterministic part of the dynamics of the solar cycle. The features of the phase space thus obtained supports a description of the global behavior of the solar cycle in terms of a Van der Pol oscillator. We adjusted the free parameters in the equations using the so-called standard form. We find that the Van der Pol oscillator that best fits the observed series is able to describe the main features of the observations quite adequately.

The irregularities of the solar cycle were reasonably modeled through the introduction of a stochastic parameter in the equations. The mean values and deviations obtained from the theoretical model for the rising times, periods, and peak values, are in good agreement with the corresponding values obtained from the observations.

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