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Global dimensions for endomorphism algebras of tilting modules

By

S. Gastaminza, D. Happel, M. I. Platzeck, M. J. Redondo and L. Unger

Abstract. Let Λ be an Artin algebra of finite global dimension and let T be a tilting module over Λ. We develop bounds for the global dimension of the endomorphism algebra Γ of T in terms of homological data of T.

Let Λ be an Artin algebra over a commutative Artin ring R and let $\operatorname{mod}\Lambda$ be the category of finitely generated left Λ-modules. For $X \in \operatorname{mod}\Lambda$ we denote by $\operatorname{pd}_\Lambda X$ (resp. $\operatorname{id}_\Lambda X$) the projective dimension (resp. injective dimension) of X. We will say that a module X is exceptional if $\operatorname{pd}_\Lambda X < \infty$ and $\operatorname{Ext}^i_\Lambda(X, X) = 0$ for $i > 0$. An exceptional module T is called a tilting module if there exists an exact sequence

$$0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to \ldots \to T^m \to 0$$

with $T^j \in \operatorname{add} T$ for all j, where $\operatorname{add} T$ is the full subcategory of $\operatorname{mod}\Lambda$ whose objects are direct sums of direct summands of T.

The purpose of tilting theory is to compare $\operatorname{mod}\Lambda$ with $\operatorname{mod}\Gamma$ where Γ is the endomorphism algebra $\operatorname{End}_\Lambda T$ of a tilting module ${}_\Lambda T$. In this article we are mainly interested in the comparison of the global dimensions $\operatorname{gl.dim}\Lambda$ of Λ and $\operatorname{gl.dim}\Gamma$ of Γ. We will assume that Λ is of finite global dimension. It is well-known (compare for example [4], III, 3.4) that in this case also $\operatorname{gl.dim}\Gamma < \infty$ and even

$$\operatorname{gl.dim}\Lambda - \operatorname{pd}_\Lambda T \leqq \operatorname{gl.dim}\Gamma \leqq \operatorname{gl.dim}\Lambda + \operatorname{pd}_\Lambda T.$$

Since Λ is of finite global dimension we have that ${}_\Lambda T$ is of finite injective dimension. In section two we improve the bound for the global dimension of Γ, namely

$$\operatorname{id}_\Lambda T \leqq \operatorname{gl.dim}\Gamma \leqq \operatorname{pd}_\Lambda T + \operatorname{id}_\Lambda T.$$

This will follow from the general fact that a tilting module T induces an equivalence $\operatorname{Hom}_\Lambda(T, -)$ between certain subcategories of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$.

In sections three and four we continue our investigations of global dimensions for endomorphism algebras of tilting modules.

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In the third section we consider classical tilting modules, i.e. $\mathrm{pd}_\Lambda T \leqq 1$. In this case the result in section two states that $s = \mathrm{id}_\Lambda T \leqq \mathrm{gl.dim}\, \Gamma \leqq \mathrm{id}_\Lambda T + 1$. We prove that $\mathrm{gl.dim}\, \Gamma = s$ if and only if $\mathrm{Ext}^s_\Lambda(\tau T, T) = 0$, where τ denotes the Auslander-Reiten translation. This can be thought of as a generalized splitting condition. Namely if $s = 1$, then it is straightforward to see that the tilting torsion pair on $\mathrm{mod}\, \Lambda$ induced by T splits if and only if $\mathrm{Ext}^1_\Lambda(\tau T, T) = 0$, or equivalently that T is a complete slice, or further equivalently that $\Gamma = \mathrm{End}_\Lambda T$ is a hereditary Artin algebra.

In section four we consider certain Ext-injective tilting modules. For $0 \leqq i \leqq d = \mathrm{gl.dim}\, \Lambda$ we denote by $\mathscr{P}^{\leqq i}(\Lambda)$ the full subcategory of $\mathrm{mod}\, \Lambda$ consisting of those Λ–modules ${}_\Lambda X$ with $\mathrm{pd}_\Lambda X \leqq i$. In case $\mathscr{P}^{\leqq i}(\Lambda)$ is contravariantly finite in $\mathrm{mod}\, \Lambda$ it was shown in [7] that there exists a tilting module $T_i \in \mathscr{P}^{\leqq i}(\Lambda)$ which is Ext-injective in $\mathscr{P}^{\leqq i}(\Lambda)$ and has $\mathrm{pd}_\Lambda T_i = i$. If $\Gamma_i = \mathrm{End}_\Lambda T_i$ and $d_i = \mathrm{gl.dim}\, \Gamma_i$ we will show that $d - i \leqq d_i \leqq d$. Hence $\mathrm{gl.dim}\, \Gamma_i$ will never exceed $\mathrm{gl.dim}\, \Lambda$. A special case of this result has been shown by different methods in [10].

In case two consecutive categories $\mathscr{P}^{\leqq i}(\Lambda)$ and $\mathscr{P}^{\leqq i+1}(\Lambda)$ are both contravariantly finite in $\mathrm{mod}\, \Lambda$ we will show that $d_i - 1 \leqq d_{i+1} \leqq d_i + 1$.

If $\mathscr{P}^{\leqq i}(\Lambda)$ is contravariantly finite in $\mathrm{mod}\, \Lambda$ for all $0 \leqq i \leqq d$ (this is for example satisfied for representation-finite algebras or Auslander algebras) we may associate a sequence $(d_0, \ldots, d_d)$ of global dimensions with Λ. Note that $d_0 = d_d = d$ and the absolute difference of two consecutive numbers is at most one. We finish with examples showing what kind of sequences may occur.

1. Preliminaries. In this section we will briefly recall the basic definitions and results we will use in the main part of this article. For unexplained representation-theoretic terminology we refer to [9] or [2]. We denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$ in a given category $\mathscr{K}$ by fg.

Let $\mathscr{C}$ be a full subcategory of $\mathrm{mod}\, \Lambda$. It is always assumed to be closed under direct sums, direct summands and isomorphisms. The subcategory $\mathscr{C}$ is called contravariantly finite in $\mathrm{mod}\, \Lambda$, if every $X \in \mathrm{mod}\, \Lambda$ has a right $\mathscr{C}$-approximation, i.e. there is a morphism $F_X \to X$ with $F_X \in \mathscr{C}$ such that the induced morphism $\mathrm{Hom}_\Lambda(C, F_X) \to \mathrm{Hom}_\Lambda(C, X)$ is surjective for all $C \in \mathscr{C}$. If X admits a right $\mathscr{C}$-approximation then it admits also a minimal right $\mathscr{C}$-approximation. This is a right $\mathscr{C}$-approximation which in addition is also right minimal in the sense that its restriction to any nonzero summand is nonzero.

All subcategories in this paragraph are subcategories of $\mathrm{mod}\, \Lambda$. The subcategory $\mathscr{C}$ is called resolving if $\mathscr{C}$ is closed under extensions, kernels of surjective maps and contains ${}_\Lambda\Lambda$. Note that for a contravariantly finite subcategory which is resolving every right approximation is surjective [1]. Indeed, the projective cover has to factor over the right approximation.

If $\mathscr{C}$ is a resolving subcategory and $C \in \mathscr{C}$ is $\mathscr{C}$-injective, so satisfies $\mathrm{Ext}^1_\Lambda(X, C) = 0$ for all $X \in \mathscr{C}$, then $\mathrm{Ext}^i_\Lambda(X, C) = 0$ for all $X \in \mathscr{C}$ and all $i > 0$.

The notions of covariantly finite and coresolving subcategories are defined dually.

Following Auslander and Reiten [1] we associate with a subcategory $\mathscr{C}$ of $\mathrm{mod}\, \Lambda$ the following full subcategories of $\mathrm{mod}\, \Lambda$:

$$\mathscr{C}^\perp = \{X \mid \mathrm{Ext}^i_\Lambda(C, X) = 0 \text{ for all } i > 0 \text{ and all } C \in \mathscr{C}\}$$

$$^\perp\mathscr{C} = \{X \mid \mathrm{Ext}^i_\Lambda(X, C) = 0 \text{ for all } i > 0 \text{ and all } C \in \mathscr{C}\}$$

We will be mainly interested in these categories when $\mathscr{C} = \mathrm{add}_\Lambda T$ for a tilting module ${}_\Lambda T$. In this case they will be denoted by $T^\perp, {}^\perp T$ respectively. For a tilting module ${}_\Lambda T$ we denote by $\Gamma = \mathrm{End}_\Lambda T$ the endomorphism algebra of T. We will also consider the full subcategory $\mathscr{T}_0$ of $\mathrm{mod}\,\Gamma$ whose objects Y satisfy $\mathrm{Tor}_j^\Gamma(T,Y) = 0$ for all $j > 0$.

2. General properties. We keep the notation from the previous sections.

We now turn to the proof of a bound for the global dimension of the endomorphism algebra of a tilting module. We denote by ΩX the kernel of a projective cover of X, and define inductively $\Omega^i X = \Omega(\Omega^{i-1}X)$ for $i > 0$. For convenience we sometimes set $\Omega^0 X = X$.

Proposition 2.1. *Let Λ be an Artin algebra with finite global dimension d. Let ${}_\Lambda T$ be a tilting module with $\mathrm{pd}_\Lambda T = r$ and $\mathrm{id}_\Lambda T = s$. Let $\Gamma = \mathrm{End}_\Lambda T$, then for each $Y \in \mathscr{T}_0$ we have $\mathrm{pd}_\Gamma Y \leqq s$. Moreover we have that $s \leqq \mathrm{gl.dim}\,\Gamma \leqq r + s$.*

Proof. We first observe that T is also a cotilting module since Λ is of finite global dimension, and so there exists a long exact sequence

$$0 \to T^s \to \ldots \to T^0 \to D(\Lambda_\Lambda) \to 0$$

with $T^i \in \mathrm{add}\,T$ for all i (compare for example [4], III, 2.2). We will assume that we have chosen such a sequence with T^i minimal for all i.

Next we apply the functor $\mathrm{Hom}_\Lambda(T,-)$ to the sequence above and conclude that $\mathrm{pd}_\Gamma \mathrm{Hom}_\Lambda(T, D(\Lambda_\Lambda)) = s$, which shows the first inequality. In a similar way we may show that for each $Y \in T^\perp$ we have that $\mathrm{pd}_\Gamma \mathrm{Hom}_\Lambda(T,Y) \leqq s$. In fact, it is well-known that for each $Y \in T^\perp$ we have an exact sequence

$$0 \to T^t \to \ldots \to T^0 \to Y \to 0$$

with $T^i \in \mathrm{add}\,T$ for all i and some $t \leqq s$. Note that here we use the assumption that Λ is of finite global dimension.

By tilting theory we infer that $\mathrm{Hom}_\Lambda(T,-)$ induces an equivalence between $T^\perp$ and the full subcategory $\mathscr{T}_0$ of $\mathrm{mod}\,\Gamma$ whose objects Y satisfy $\mathrm{Tor}_j^\Gamma(T,Y) = 0$ for all $j > 0$ (compare for example [4], III, 3.2). So we see that each $Y \in \mathscr{T}_0$ satisfies $\mathrm{pd}_\Gamma Y \leqq s$. Now $\mathrm{pd}\,T_\Gamma = r$, hence $\mathrm{Tor}_j^\Gamma(T,Y) = 0$ for all $j > r$. Thus for all $Y \in \mathrm{mod}\,\Gamma$ we have that $\Omega^r Y \in \mathscr{T}_0$. Now $s \geqq \mathrm{pd}_\Gamma \Omega^r Y = -r + \mathrm{pd}_\Gamma Y$, which shows the second inequality.

3. Classical tilting modules. We now consider global dimensions for endomorphism algebras of tilting modules.

We consider the special case of a tilting module T with $\mathrm{pd}_\Lambda T = 1$. Let $s = \mathrm{id}_\Lambda T$ and let $\Gamma = \mathrm{End}_\Lambda T$. By Proposition 2.1 we know that $\mathrm{gl.dim}\,\Gamma = s$ or $\mathrm{gl.dim}\,\Gamma = s + 1$. We will develop a criterion which decides which case will actually occur.

We will first recall some of the special features of this case (for details we refer to [5] or [9], 4.1). A tilting module ${}_\Lambda T$ with $\mathrm{pd}_\Lambda T = 1$ induces a torsion pair $(\mathscr{T}, \mathscr{F})$ on $\mathrm{mod}\,\Lambda$ as follows:

$$\mathscr{T} = \mathrm{Fac}\,T = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(T,X) = 0\}$$

and

$$\mathscr{F} = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}_\Lambda(T,X) = 0\},$$

where $\operatorname{Fac} T$ is the subcategory generated by T. It may be shown that $\mathscr{F}$ is the subcategory $\operatorname{Sub} \tau T$ cogenerated by τT, where τ is the Auslander-Reiten translation.

It also induces a torsion pair $(\mathscr{X}, \mathscr{Y})$ on $\operatorname{mod} \Gamma$:

$$\mathscr{X} = \{Y \in \operatorname{mod} \Gamma \mid T \otimes_\Gamma Y = 0\}$$

and

$$\mathscr{Y} = \{Y \in \operatorname{mod} \Gamma \mid \operatorname{Tor}_1^\Gamma(T, Y) = 0\}.$$

Lemma 3.1 *Assume that Λ is of finite global dimension. Let ${}_\Lambda T$ be a tilting module with $\operatorname{pd}_\Lambda T = 1$ and $\operatorname{id}_\Lambda T = s$. Then $\operatorname{Ext}^s_\Lambda(\tau T, T) = 0$ if and only if $\operatorname{Ext}^s_\Lambda(\mathscr{F}, \mathscr{T}) = 0$.*

Proof. Since $\tau T \in \mathscr{F}$ and $T \in \mathscr{T}$ one direction is clear.

For the converse we first recall that $\mathscr{F} = \operatorname{Sub} \tau T$. Since $\operatorname{id}_\Lambda T = s$, we infer that $\operatorname{Ext}^s(-, T)$ is right exact, so $\operatorname{Ext}^s_\Lambda(\tau T, T) = 0$ implies that $\operatorname{Ext}^s(\mathscr{F}, T) = 0$. Next, for each $X \in \mathscr{T}$ there is an exact sequence

$$0 \to T^m \to \ldots \to T^0 \to X \to 0,$$

with $T^i \in \operatorname{add} T$ and with $m \leqq s$. Hence for $Y \in \mathscr{F}$ we get that $\operatorname{Ext}^s_\Lambda(Y, X) = \operatorname{Ext}^{s+m}(Y, T^m) = 0$ completing the proof of the lemma.

Theorem 3.2. *Assume that Λ is of finite global dimension. Let ${}_\Lambda T$ be a tilting module with $\operatorname{pd}_\Lambda T = 1$ and $\operatorname{id}_\Lambda T = s \geqq 1$. Then $s = \operatorname{gl.dim} \Gamma$ if and only if $\operatorname{Ext}^s_\Lambda(\tau T, T) = 0$. Moreover $\operatorname{Ext}^s_\Lambda(\tau T, T) = 0$ implies that $\operatorname{pd}_\Lambda \tau T \leqq s$.*

Proof. By the previous lemma it is enough to show that $s = \operatorname{gl.dim} \Gamma$ if and only if $\operatorname{Ext}^s_\Lambda(\mathscr{F}, \mathscr{T}) = 0$.

By 2.1 we know that each $Y \in \mathscr{Y}$ satisfies $\operatorname{pd}_\Gamma Y \leqq s$, for $\mathscr{T}_0 = \mathscr{Y}$ in this context. Next let $X \in \mathscr{X}$. Then $X = \operatorname{Ext}^1_\Lambda(T, Z)$ for some $Z \in \mathscr{F}$. We consider the universal extension of Z by T

$$0 \to Z \to E \to T^t \to 0.$$

It is easy to see that $E \in \mathscr{T}$. Apply $\operatorname{Hom}_\Lambda(T, -)$ to this sequence. This yields the following exact sequence of Γ-modules:

$$0 \to \operatorname{Hom}_\Lambda(T, E) \to \operatorname{Hom}_\Lambda(T, T^t) \to \operatorname{Ext}^1_\Lambda(T, Z) \to 0.$$

Then $\operatorname{Hom}_\Lambda(T, T^t)$ is a projective Γ-module and $\operatorname{Ext}^i(Z, T) = \operatorname{Ext}^i(E, T)$ for all $i \geqq 1$. Let

$$0 \to T^m \to \ldots \to T^0 \to E \to 0.$$

with $T^i \in \operatorname{add} T$ for all i and m minimal. Then $\operatorname{pd}_\Gamma \operatorname{Hom}_\Lambda(T, E) = m < s$ if and only if $\operatorname{Ext}^s(E, T) = 0$. So $\operatorname{pd}_\Gamma \operatorname{Ext}^1(T, Z) \leqq s$ for all $Z \in \mathscr{F}$ if and only if $\operatorname{Ext}^s(Z, T) = 0$ for all $Z \in \mathscr{F}$.

We now show the last assertion. By definition of a tilting module we have a short exact sequence

$$0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to 0$$

with $T^0, T^1 \in \operatorname{add} T$. Applying $\operatorname{Hom}_\Lambda(\tau T, -)$ to this sequence yields the following exact sequence for all $t \geqq s$:

$$\mathrm{Ext}^t_\Lambda(\tau T, T^1) \to \mathrm{Ext}^{t+1}_\Lambda(\tau T, {}_\Lambda\Lambda) \to \mathrm{Ext}^{t+1}(\tau T, T^0).$$

The outer terms vanish by assumption, so does the middle term, hence $\mathrm{pd}_\Lambda \tau T \leqq s$.
This finishes the proof of the theorem.

We point out that in general the converse of the last implication will not hold. For example let H be a hereditary Artin algebra and ${}_H T$ a non-injective, non-projective tilting module. So $\mathrm{pd}_H T = \mathrm{id}_H T = 1$. Then we always have that $\mathrm{pd}_H \tau T \leqq 1$, but it is easily seen that there are plenty of examples with $\mathrm{Ext}^1_H(\tau T, T) \neq 0$.

In Section 4 we will give a reformulation of this result for special tilting modules.

From the theorem we deduce the following well-known characterisation of tilted algebras. Recall that an Artin algebra Λ is called a tilted algebra if there exists a hereditary Artin algebra H and a tilting module ${}_H T$ with $\mathrm{End}_H T = \Lambda$.

Corollary 3.3. *Let Λ be an Artin algebra of finite global dimension. Then Λ is a tilted algebra if and only if there exists a tilting module ${}_\Lambda T$ with $\mathrm{pd}_\Lambda T = \mathrm{id}_\Lambda T = 1$ and $\mathrm{Ext}^1_\Lambda(\tau T, T) = 0$.*

Proof. If such a tilting module exists the theorem implies that $\Gamma = \mathrm{End}_\Lambda T$ is hereditary, and therefore Λ is a tilted algebra. For the converse use the usual proof from tilting theory (compare for example [9], 4.2).

4. Ext-injective tilting modules. As before let Λ be an Artin algebra of finite global dimension d. For each $0 \leqq i \leqq d$ we denote by $\mathscr{P}^{\leqq i}(\Lambda)$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of those $\Lambda-$modules ${}_\Lambda X$ with $\mathrm{pd}_\Lambda X \leqq i$. In case $\mathscr{P}^{\leqq i}(\Lambda)$ is contravariantly finite in $\mathrm{mod}\,\Lambda$ it was shown in [7] that there exists a tilting module $T_i \in \mathscr{P}^{\leqq i}(\Lambda)$ which is Ext-injective in $\mathscr{P}^{\leqq i}(\Lambda)$ and has $\mathrm{pd}_\Lambda T_i = i$. If $\Gamma_i = \mathrm{End}_\Lambda T_i$ we denote by $d_i = \mathrm{gl.dim}\,\Gamma_i$.

Let us pause for a moment to comment on our assumptions. In general the subcategories $\mathscr{P}^{\leqq i}(\Lambda)$ will not be contravariantly finite in $\mathrm{mod}\,\Lambda$ except for the extreme values $i = 0$ or $i = d$. An example for this can be found for algebras of finite global dimension in [6], which is a modification of an example in [8]. However if Λ is a representation-finite algebra of finite global dimension all these subcategories will be contravariantly finite in $\mathrm{mod}\,\Lambda$. Another general class of algebras where these assumptions are satisfied is the class of Auslander algebras. Recall that all Auslander algebras are obtained as follows: Let Λ be an arbitrary representation-finite algebra and let $M_1, \ldots, M_n$ be a complete list of representatives from the isomorphism classes of the indecomposable Λ-modules. Let $M = \bigoplus_{i=1}^{n} M_i$. Then Γ is an Auslander algebra if Γ is of the form $\mathrm{End}_\Lambda M$. It is well-known that the global dimension of an Auslander algebra is two. It follows from a result in [8] that $\mathscr{P}^{\leqq 1}(\Gamma)$ is contravariantly finite in $\mathrm{mod}\,\Gamma$ for an Auslander algebra Γ.

We start with an easy general observation.

Lemma 4.1. *Let Λ be an Artin algebra with $\mathrm{gl.dim}\,\Lambda = d < \infty$ and let ${}_\Lambda T$ be a tilting module with $\mathrm{pd}_\Lambda T = r$ and $\mathrm{id}_\Lambda T = s$. Then $d - r \leqq s \leqq d$.*

Proof. Trivially we have that $\mathrm{id}_\Lambda T \leqq d$. By definition of a tilting module we have an exact sequence

$$0 \to {}_\Lambda\Lambda \to T^0 \to \ldots \to T^m \to 0$$

where $m = r = \mathrm{pd}_\Lambda T$ (compare [4], III, 2.2). From the above sequence we conclude that $\mathrm{id}_\Lambda \Lambda \leqq \mathrm{id}_\Lambda T + r$. Thus $d \leqq s + r$, or equivalently $d - r \leqq s$.

Proposition 4.2. *Let Λ be an Artin algebra with* $\mathrm{gl.dim}\,\Lambda = d < \infty$ *and let* $0 \leqq i \leqq d$. *If* $\mathscr{P}^{\leqq i}(\Lambda)$ *is contravariantly finite in* $\mathrm{mod}\,\Lambda$, *then* $\mathrm{pd}_\Lambda T_i = i$ *and* $\mathrm{id}_\Lambda T_i = d - i$.

Proof. The assertion on the projective dimension of T_i is contained in [7]. The previous lemma shows that $\mathrm{id}_\Lambda T_i \geqq d - i$. For the other inequality let $X \in \mathrm{mod}\,\Lambda$. By assumption we have that $\mathrm{pd}_\Lambda X \leqq d$. Thus $\mathrm{pd}_\Lambda \Omega^{d-i} X \leqq i$. Thus $\mathrm{Ext}^1_\Lambda(\Omega^{d-i}X, T_i) = 0$, for T_i is Ext-injective in $\mathscr{P}^{\leqq i}(\Lambda)$. But

$$\mathrm{Ext}^1_\Lambda(\Omega^{d-i}X, T_i) = \mathrm{Ext}^{d-i+1}_\Lambda(X, T_i),$$

hence $\mathrm{id}_\Lambda T_i \leqq d - i$.

Corollary 4.3. *With the notation and assumptions above we have that* $d \geqq d_i \geqq \max(d - i, i)$.

Proof. The first inequality follows immediately from 2.1 and the previous proposition. Again by 2.1 and the previous proposition we have that $d_i \geqq d - i$. Since $\mathrm{pd}_\Lambda T_i = i$ we infer that $\mathrm{id}_{\Gamma_i} D(T_i) = i$, hence $d_i \geqq i$. This shows the assertion.

We now come back to the situation of 3.2. We point out that this result was previously obtained in [10] by different methods.

Corollary 4.4. *Let Λ be an Artin algebra of finite global dimension $d > 1$. Assume that* $\mathscr{P}^{\leqq 1}(\Lambda)$ *is contravariantly finite in* $\mathrm{mod}\,\Lambda$ *and let T_1 be the Ext-injective tilting module in* $\mathscr{P}^{\leqq 1}(\Lambda)$. *Then* $d_1 = \mathrm{gl.dim}\,\mathrm{End}_\Lambda T_1 = d - 1$ *if and only if* $\mathrm{pd}_\Lambda \tau T \leqq d - 1$.

Proof. By the previous corollary and Theorem 3.2 it is enough to show that $\mathrm{Ext}^{d-1}_\Lambda(\tau T_1, T_1) = 0$ whenever $\mathrm{pd}_\Lambda \tau T_1 \leqq d - 1$. If this is the case we infer that $\mathrm{pd}_\Lambda \Omega^{d-2}(\tau T_1) \leqq 1$. So $\mathrm{Ext}^1_\Lambda(\Omega^{d-2}(\tau T_1), T_1) = 0$. But

$$\mathrm{Ext}^1_\Lambda(\Omega^{d-2}(\tau T_1), T_1) = \mathrm{Ext}^{d-1}_\Lambda(\tau T_1, T_1),$$

which shows the assertion.

We will now consider the case that two consecutive such categories are contravariantly finite in $\mathrm{mod}\,\Lambda$ and will compare the two global dimensions.

Theorem 4.5. *Let Λ be an Artin algebra of finite global dimension d. Assume that* $\mathscr{P}^{\leqq j}(\Lambda)$ *is contravariantly finite in* $\mathrm{mod}\,\Lambda$ *for* $j \in \{i, i+1\}$ *and* $i + 1 \leqq d$. *Let ${}_\Lambda T_j$ be the Ext-injective tilting module in* $\mathscr{P}^{\leqq j}(\Lambda)$ *and* $\Gamma_j = \mathrm{End}_\Lambda T_j$. *If* $d_j = \mathrm{gl.dim}\,\Gamma_j$, *then* $d_i - 1 \leqq d_{i+1} \leqq d_i + 1$.

Proof. Since $\mathscr{P}^{\leqq i}(\Lambda)$ is contravariantly finite in $\mathrm{mod}\,\Lambda$ we can consider the minimal $\mathscr{P}^{\leqq i}(\Lambda)$–approximation of T_{i+1}. This gives the following exact sequence

$$0 \to K_{i+1} \to F_{i+1} \to T_{i+1} \to 0$$

with $F_{i+1} \in \mathscr{P}^{\leqq i}(\Lambda)$ and $\mathrm{Ext}^1(\mathscr{P}^{\leqq i}(\Lambda), K_{i+1}) = 0$ by Wakamatsu's Lemma (compare for example [1]). Since $\mathscr{P}^{\leqq i}(\Lambda) \subset \mathscr{P}^{\leqq i+1}(\Lambda)$ and T_{i+1} is Ext-injective in $\mathscr{P}^{\leqq i+1}(\Lambda)$ we also have that $\mathrm{Ext}^1(\mathscr{P}^{\leqq i}(\Lambda), T_{i+1}) = 0$, hence $\mathrm{Ext}^1(\mathscr{P}^{\leqq i}(\Lambda), F_{i+1}) = 0$. Since $\mathrm{pd}_\Lambda T_{i+1} = i+1$ and $\mathrm{pd}_\Lambda F_{i+1} \leqq i$, we infer from the sequence above that $\mathrm{pd}_\Lambda K_{i+1} = i$, and so $K_{i+1} \in \mathscr{P}^{\leqq i}(\Lambda)$. Since $\mathscr{P}^{\leqq i}(\Lambda)$ is resolving we have $\mathrm{Ext}^t(\mathscr{P}^{\leqq i}(\Lambda), K_{i+1}) = \mathrm{Ext}^t(\mathscr{P}^{\leqq i}(\Lambda), F_{i+1}) = 0$ for all $t > 0$. But then K_{i+1} and F_{i+1} are both Ext-injective in $\mathscr{P}^{\leqq i}(\Lambda)$, hence $K_{i+1}, F_{i+1} \in \mathrm{add}\, T_i$. For convenience we rewrite the sequence to

$$0 \to T_i^1 \to T_i^0 \to T_{i+1} \to 0$$

with $T_i^1, T_i^0 \in \mathrm{add}\, T_i$. This also shows that $T_{i+1} \in T_i^{\perp}$. Next we apply the functor $\mathrm{Hom}_\Lambda(T_i, -)$ to this sequence and obtain an exact sequence of Γ_i-modules

$$0 \to \mathrm{Hom}_\Lambda(T_i, T_i^1) \to \mathrm{Hom}_\Lambda(T_i, T_i^0) \to \mathrm{Hom}_\Lambda(T_i, T_{i+1}) \to 0.$$

So we see that $\mathrm{pd}_{\Gamma_i} \mathrm{Hom}_\Lambda(T_i, T_{i+1}) \leqq 1$. Moreover it is easy to verify that the Γ_i-module $\mathrm{Hom}_\Lambda(T_i, T_{i+1})$ is a tilting module with $\Gamma_{i+1} = \mathrm{End}_{\Gamma_i}(\mathrm{Hom}_\Lambda(T_i, T_{i+1}))$. So the assertion follows from 2.1 in combination with 4.1.

We point out that the Γ_i-tilting module $\mathrm{Hom}_\Lambda(T_i, T_{i+1})$ will in general not be Ext-injective in $\mathscr{P}^{\leqq 1}(\Gamma_i)$. This can be easily verified in examples but also follows from the following general remark.

Let Λ be an Artin algebra of finite global dimension d and assume that $\mathscr{P}^{\leqq i}(\Lambda)$ is contravariantly finite in $\mathrm{mod}\,\Lambda$ for all $0 \leqq i \leqq d$. Let T_i be the Ext-injective tilting module in $\mathscr{P}^{\leqq i}(\Lambda)$. Let $\Gamma_i = \mathrm{End}_\Lambda T_i$ and $d_i = \mathrm{gl.dim}\,\Gamma_i$. Then we may form the global dimension sequence $(d_0, \ldots, d_d)$. Trivially we have that $d_0 = d_d = d$. From 4.3 we conclude that $d \geqq d_i \geqq \max(d-i, i)$ and the last assertion shows that $d_i - 1 \leqq d_{i+1} \leqq d_i + 1$. So in case there is j with $d_j < d$ there has to be some index $t+1 \geqq j$ where the Γ_t-tilting module $\mathrm{Hom}_\Lambda(T_t, T_{t+1})$ will not be Ext-injective in $\mathscr{P}^{\leqq 1}(\Gamma_t)$.

We finish with some examples which illustrate what kind of sequences may occur. The calculations are straightforward and therefore will not be given. For the examples we choose a field k and the algebras will be path algebras of a finite quiver $\vec{\Delta}$ over k modulo an admissible twosided ideal of $k\vec{\Delta}$. To each vertex i of $\vec{\Delta}$ we denote by $S(i)$ the simple module concentrated in i and by $P(i)$ the projective cover of $S(i)$.

For the first example let $\vec{\Delta}$ be the following quiver and let I be the twosided ideal generated by all paths of length two in $\vec{\Delta}$. Let $\Lambda_n = k\vec{\Delta}/I$.

Note that $\mathrm{gl.dim}\,\Lambda_n = n-1$ and that the Ext-injective tilting module T_i in $\mathscr{P}^{\leqq i}(\Lambda_n)$ is of the form

$$T_{i-1} = \bigoplus_{j=2}^{n} P(j) \oplus S(i)$$

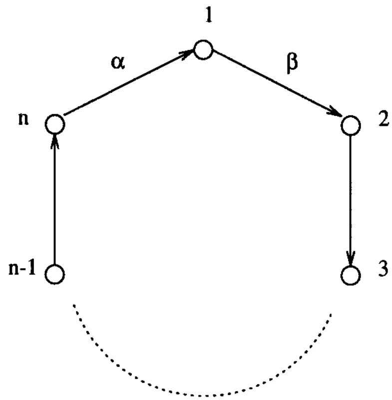
for $1 \leqq i \leqq n$. Let $\Gamma_i = \mathrm{End}_{\Lambda_n} T_i$ and $d_i = \mathrm{gl.dim}\,\Gamma_i$ for $0 \leqq i \leqq n-1$.

We distinguish two cases.

First we consider the case that $n = 2m + 1$ is odd. Then using the notation from above we have that $d_{m+i} = d_{m-i} = m + i$ for $0 \leqq i \leqq m$.

If $n = 2m$ is even. Then using the notation from above we have that $d_{m+i} = m + i$ for $0 \leqq i \leqq m - 1$ and $d_{m-i} = m + i - 1$ for $1 \leqq i \leqq m$.

For the next example let $\vec{\Delta}$ be the following quiver and let I be the twosided ideal generated by all paths of length two except $\alpha\beta$. Let $\Lambda_n = k\vec{\Delta}/I$. The indecomposable projectives Λ_n-modules $P(i)$ have length two for $1 \leqq i \leqq n - 1$ and $P(n)$ has length three.



Note that gl.dim $\Lambda_n = n$ and that the Ext-injective tilting module T_i in $\mathscr{P}^{\leqq i}(\Lambda_n)$ is of the form

$$T_i = \bigoplus_{j=2}^{n} P(j) \oplus S(n - i + 1)$$

for $1 \leqq i \leqq n - 1$. Clearly $T_0 = {}_{\Lambda_n}\Lambda_n$ and $T_n = D(\Lambda_{n\,\Lambda_n})$. Let $\Gamma_i = \mathrm{End}_{\Lambda_n} T_i$ and let $d_i = \text{gl.dim}\,\Gamma_i$ for $0 \leqq i \leqq n$. Then it is easy to see that $\Gamma_i \simeq \Lambda_n$ for all $0 \leqq i \leqq n$. In particular we see that $d_i = n$ for $0 \leqq i \leqq n$.

These two examples show that the upper and lower bounds established in 4.3 are best possible.

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Anschrift der Autoren:

S. Gastaminza, M. I. Platzeck
and M. J. Redondo
Departamento de Matemática
Universidad Nacional del Sur
Bahia Blanca
Argentina

D. Happel
Fakultät für Mathematik
Technische Universität Chemnitz
09107 Chemnitz
Germany

L. Unger
Fachbereich für Mathematik
Fernuniversität Hagen
58084 Hagen
Germany