HOCHSCHILD COHOMOLOGY AND FUNDAMENTAL GROUPS OF INCIDENCE ALGEBRAS

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Abstract

The aim of this paper is to report on some computations of Hochschild cohomology and fundamental groups of incidence algebras.

Let A be a finite dimensional algebra (associative with unit) over an algebraically closed field. We assume that the algebra A is basic, that is, $A = kQ/I$ for some finite quiver Q and an admissible ideal I of the path algebra kQ [ARS].

The Hochschild cohomology groups $\mathrm{H}^i(A, X)$ of A with coefficients in an A-bimodule X were defined by Hochschild [Ho]. When $X = A$ we shall denote $\mathrm{H}^i(A) = \mathrm{H}^i(A, A)$ the ith- Hochschild cohomology group of A.

In general it is not easy to compute the Hochschild cohomology groups of a given algebra. The purpose of this paper is to compute them when A is an incidence algebra, that is, A is a subalgebra of the algebra $M_n(k)$ of square matrices over k with elements $(x_{ij}) \in M_n(k)$ satisfying $x_{ij} = 0$ if $i \not\leq j$, for some partial order $\leq$ defined in the *poset* (partially order set) $\{1, \ldots, n\}$. It is well known that to a finite poset P we may associate a simplicial complex $\sum_P$ whose i-simplices are the chains of length i, such that the cohomology of $\sum_P$ with coefficients in k is isomorphic to the Hochschild cohomology of the incidence algebra associated to the poset P [C, GS, GS1].

We prove first that if A is an incidence algebra such that its poset has a unique maximal (minimal) element then $\mathrm{H}^i(A) = 0$ for all $i \geq 1$.

It is known that the Hochschild cohomology groups of an incidence algebra vanish if the associated poset does not contain crowns (see [D, IZ]). Assume that A is an incidence algebra such that its poset P contains crowns. There is an algorithm given by Igusa and Zacharia [IZ] that allows us to find the so-called *reduced subposet* $\overline{P}$ of P which has the property that all elements $x \in \overline{P}$, neither minimal nor maximal elements, are such that $\{y \in \overline{P} : y \geq x\}$ has at least two minimal elements and $\{z \in \overline{P} : x \geq z\}$ has at least two maximal elements. The Hochschild cohomology groups are invariant under this construction.

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We compute the Hochschild cohomology groups of the incidence algebras associated to posets with reduced subposet given by $([qn+s] \times [n-1], <)$ where $[m] = \{0, \ldots, m\}$, $(l,j) < (l+1,j)$, $(l,j) < (l+1,j+1)$ and $(l,n) = (l,0)$.

For each pair (Q,I) such that $A \simeq kQ/I$, called a *presentation* of A, one can define the fundamental group $\pi_1(Q,I)$ (see Section 1.5). Assume that Q has no oriented cycles. Then A is called *simply connected* if, for every presentation (Q,I) of A, the fundamental group $\pi_1(Q,I)$ is trivial [AS]. The importance of simply connected algebras in representation theory follows from the fact that often we may reduce, with the help of coverings, the study of indecomposable modules over an algebra to that of the corresponding simply connected algebras. This is the case for representation-finite algebras (see [BG]).

In [H] Happel shows that a representation-finite algebra A is simply connected if and only if A is representation directed and $\mathrm{H}^1(A) = 0$. This suggests the existence of a relation between $\mathrm{H}^1(A)$ and the fundamental group of A. Moreover, the existence of an injective morphism of abelian groups $s : \mathrm{Hom}(\pi_1(Q,I), k^+) \to \mathrm{H}^1(A)$ is known, for any presentation (Q,I) of A, where k^+ denotes the underlying additive group of the field k [AP]. For the algebras considered in this paper $\mathrm{Hom}(\pi_1(Q,I), k^+) \simeq \mathrm{H}^1(A)$, and this allows us to describe their fundamental groups.

The article is organized as follows: in Section 1 we fix notations and briefly recall the definitions and results that will be needed throughout this paper. In Section 2 we compute the Hochschild cohomology groups $\mathrm{H}^i(A)$ for incidence algebras given by: i) posets with a unique maximal or minimal element, ii) posets with reduced subposet of the given type above. Finally in Section 3 we compute the fundamental group of any presentation of the incidence algebras considered in ii) using their relation with the first Hochschild cohomology group.

1 Preliminaries

1.1 Path algebras

A *quiver* $Q = (Q_0, Q_1)$ is a finite oriented graph, Q_0 the set of vertices and Q_1 the set of arrows. We denote by $s, e : Q_1 \to Q_0$ the maps associating to each arrow its starting and ending point respectively. The *path algebra* kQ is the k-vector space with basis all the paths in Q, including trivial paths e_x of length zero, one for each vertex $x \in Q_0$. The multiplication on two basis elements is the composition of paths if they are composable, and zero otherwise.

A *relation* from x to y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ such that, for each $1 \leq i \leq m$, λ_i is a non-zero scalar and w_i a path of length at least two from x to y. A set of relations on Q generates an ideal I, said to be *admissible*, in the path algebra kQ of Q. The pair (Q,I) is then called a *bound quiver*. It is well-known that for every basic algebra A there exists a surjective k-algebra morphism $v : kQ \to A$ whose kernel I_v is admissible, where Q is the ordinary quiver of A. Thus we have $A \simeq kQ/I_v$. The bound quiver (Q, I_v) is called a *presentation* of A.

It is well-known that if $A = kQ/I$ then the category $\mathrm{mod}\, A$ of finitely generated left A-modules is equivalent to the category of all bound (finite-dimensional) representations of (Q, I). Therefore we may identify a module M with the corresponding representation $(M(x), M(\alpha))_{x\in Q_0,\alpha\in Q_1}$. For each $x \in Q_0$ we denote by S_x the corresponding simple A-module, and by P_x the projective cover of S_x. It can be seen that $\mathrm{Hom}_A(P_x, M) \simeq M(x)$.

We refer to [ARS, BG] for more details.

1.2 Incidence algebras

An incidence algebra A is a subalgebra of the algebra $M_n(k)$ of square matrices over k with elements $(x_{ij}) \in M_n(k)$ satisfying $x_{ij} = 0$ if $i \not\leq j$, for some partial order $\leq$ defined in the *poset* (partially order set) $\{1, \ldots, n\}$.

Incidence algebras can equivalently be viewed as path algebras of quivers with relations in the following way. Let Q be a finite quiver without oriented cycles and such that for each arrow $x \overset{\alpha}{\to} y \in Q_1$ there is no oriented path other than α joining x to y. These quivers are called *ordered*. The set Q_0 of vertices of Q is then a finite poset as follows: $x \geq y$ if and only if there exists an oriented path from x to y. Conversely, if Q_0 is a finite poset, we construct a quiver Q with set of vertices Q_0, and with an arrow from x to y if and only if $x > y$ and there is no $u \in Q_0$ such that $x > u > y$. Clearly we obtain in this way an ordered quiver and a bijection between finite posets and ordered quivers.

Let us consider kQ the path algebra of Q and I be the *parallel ideal* of kQ, that is, I is the two-sided ideal of kQ generated by all the differences $\gamma - \delta$ where γ and δ are *parallel paths* (i.e. γ and δ have the same starting and ending points). The algebra $A = kQ/I$ is the *incidence algebra* of the poset associated to the ordered quiver Q.

Remark 1.1 *If $A = kQ/I$ is an incidence algebra it is easy to describe the representations of the indecomposable projective modules P_x for each $x \in Q_0$. In fact, $P_x(y) = k$ if $y \leq x$ and it is zero otherwise, and $P_x(\alpha) = id_k$ if $s(\alpha) \leq x$ and it is zero otherwise.*

1.3 Hochschild cohomology groups

We recall the construction of the Hochschild cohomology groups $\mathrm{H}^i(A)$ of an algebra A. Consider the A-bimodule A and the complex $C = (C^i, d^i)$ defined by: $C^i = 0$, $d_i = 0$ for $i < 0$, $C^0 = A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_k \cdots \otimes_k A$, $d_0 : A \to \mathrm{Hom}_k(A, A)$ the map $d^0(x)(a) = ax - xa$ and $d^i : C^i \to C^{i+1}$ defined by

$$\begin{aligned}(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ &+ \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ &+ (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}\end{aligned}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $\mathrm{H}^i(A) = \mathrm{H}^i(C^\bullet) = \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$ is the i-th Hochschild cohomology group of A with coefficients in A, see [Ho].

Recall the interpretation of the low-dimensional groups: $\mathrm{H}^0(A)$ and $\mathrm{H}^1(A)$. By definition $\mathrm{H}^0(A)$ coincides with the center $Z(A)$ of A. So if A is a basic connected finite dimensional k-algebra whose quiver has no oriented cycles then $\mathrm{H}^0(A) \simeq k$.

To compute $\mathrm{H}^1(A)$ we may use an alternative complex given by Cibils [C1] for path algebras of quivers with relations. In this case $A = E \oplus r$, where E is the subalgebra of A generated by the trivial paths $\{e_i, i \in Q_0\}$ and r is the E-E-bimodule $\operatorname{rad} A$. So, $\mathrm{H}^1(A) = \operatorname{Ker} d^1 / \operatorname{Im} d^0$ with

$$d^0 : A^E \to \operatorname{Hom}_{E-E}(r, A) \qquad d^0(a)(x) = ax - xa$$

$$d^1 : \operatorname{Hom}_{E-E}(r, A) \to \operatorname{Hom}_{E-E}(r^{\otimes 2}, A) \qquad d^1(f)(x \otimes y) = xf(y) - f(xy) + f(x)y$$

where $A^E = \{a \in A : ae = ea \quad \text{for all } e \in E\}$

Remark 1.2 *If $A = kQ/I$ is an incidence algebra, $[\delta] \in H^1(A)$ and $\alpha : i \to j \in Q_1$ then $\delta(\alpha) = \delta(e_j \alpha e_i) = e_j \delta(\alpha) e_i = \lambda_\alpha \alpha$ for some $\lambda_\alpha \in k$. Moreover, if $[\delta] = 0$ then $\delta(x) = ax - xa$, for some $a \in A^E$. Now, $ae_i = e_i a$ implies that $a = \sum_{i=1}^n ae_i = \sum_{i=1}^n e_i a e_i = \sum_{i=1}^n \mu_i e_i$, $\mu_i \in k$, since Q has no oriented cycles.*

1.4 One-point extensions

Let $A = kQ_A/I$ be an algebra and let x be a *source* in Q_A, that is, there is no arrow α in Q_A with $e(\alpha) = x$. Let $B = A/ < e_x >$, where $< e_x >$ denotes the two-sided ideal in A generated by e_x. The A-module $M = \operatorname{rad} P_x$ has a canonical B-module structure, and A is isomorphic to the one point extension algebra

$$B[M] = \begin{pmatrix} k & 0 \\ M & B \end{pmatrix}$$

where the operations are the usual addition of matrices and the multiplication is induced by the B-module structure of M.

The next theorem due to Happel [H] is useful for the computation of the Hochschild cohomology groups of the algebras considered in this paper.

Theorem 1.3 *[H, page 124] Let $A = B[M]$ be a one point extension of B by a B-module M. Then there exists a long exact sequence of k-vector spaces connecting the Hochschild cohomology of A and B:*

$$0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to \operatorname{End}_B(M)/k \to \mathrm{H}^1(A) \to \mathrm{H}^1(B) \to \operatorname{Ext}^1_B(M, M) \to \dots$$

$$\dots \to \mathrm{H}^i(A) \to \mathrm{H}^i(B) \to \operatorname{Ext}^i_B(M, M) \to \mathrm{H}^{i+1}(A) \to \dots$$

1.5 Fundamental groups

Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^m \lambda_i w_i$ in $I(x, y)$ is called *minimal* if $m \geq 2$ and for every non-empty proper subset $J \subset \{1, 2, \dots, m\}$ we have that $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse. A walk from x to y in Q is a formal composition $\alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \dots \alpha_t^{\epsilon_t}$ (where $\alpha_i \in Q_1$, $\epsilon_i = ^+_- 1$ for $1 \leq i \leq t$) starting at x and ending at y. We denote by e_x the trivial path at x.

Let $\sim$ be the smallest equivalence relation on the set of all walks in Q such that:

a) If $\alpha : x \to y$ is an arrow then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.

b) If $\rho = \sum_{i=1}^m \lambda_i w_i$ is a minimal relation then $w_i \sim w_j$ for all $1 \leq i, j \leq m$.

c) If $u \sim v$ then $wuw' \sim wvw'$ whenever these compositions make sense.

We denote by $[u]$ the equivalence class of a walk u. Let $x_0 \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x_0)$ of equivalence classes of all the closed walks starting and ending at x_0 has a group structure defined by the operation $[u][v] = [uv]$. Clearly the group $\pi_1(Q, I, x_0)$ does not depend on the choice of the base point x_0. We denote it simply by $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I) (see [G, MP]).

Recall that an algebra $A = kQ/I_v$ is called *triangular* if Q has no oriented cycles. There is a close relation between the first Hochschild cohomology group $\mathrm{H}^1(A)$ and the fundamental group $\pi_1(Q, I_v)$ of a triangular algebra A. The following result makes this relation explicit:

Theorem 1.4 *[AP, page 200] Let A be a triangular algebra and (Q, I_v) be a presentation of A. Then there exists an injective morphism of abelian groups*

$$s : \mathrm{Hom}(\pi_1(Q, I_v), k^+) \to \mathrm{H}^1(A)$$

(where k^+ denotes the underlying group of the field k).

De la Peña and Saorín give necessary and sufficient conditions for $\mathrm{H}^1(A)$ to be isomorphic to $\mathrm{Hom}(\pi_1(Q, I_v), k^+)$ (see [PS]). In the particular case of incidence algebras this result can be proved using a construction given by De la Peña in [P].

Let (Q, I_v) be a presentation of a k-algebra A. Let $C^0(A, I_v, k^+)$ be the set of all k^+-valued functions on Q_0. Let $Z^1(A, I_v, k^+)$ be the set of all k^+-valued functions g on Q_1 such that $\sum_{i=1}^s g(\alpha_i) = \sum_{j=1}^t g(\beta_j)$ whenever there exists a minimal relation $\rho = \sum_{i=1}^m \lambda_i w_i$ such that $w_1 = \alpha_1\alpha_2 \ldots \alpha_s$ and $w_2 = \beta_1\beta_2 \ldots \beta_t$.

We have an exact sequence of abelian groups

$$0 \to k^+ \xrightarrow{d^0} C^0(A, I_v, k^+) \xrightarrow{d^1} Z^1(A, I_v, k^+) \xrightarrow{p} \mathrm{Hom}(\pi_1(Q, I_v), k^+) \to 0$$

where d^0 associates to the element $m \in k^+$ the constant function $f : Q_0 \to k^+$ with value m; d^1 associates to a function $f : Q_0 \to k^+$ the function $g : Q_1 \to k^+$ which maps $\alpha : y \to z$ to $g(\alpha) = f(y) - f(z)$, and finally p maps a function g to the morphism of groups $h : \pi_1(Q, I_v) \to k^+$ defined by $h([\alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2} \ldots \alpha_t^{\epsilon_t}]) = \sum_{i=1}^t \epsilon_i g(\alpha_i)$.

Recall the description of $\mathrm{H}^1(A)$ given in Remark 1.2. Let $\xi : Z^1(A, I_v, k^+) \to \mathrm{H}^1(A)$ be given by $\xi(g) = [\delta]$ where

$$\delta(\overline{v}) = (\sum_{k=1}^p g(\alpha_k))\, \overline{v}$$

for $v = \alpha_1 \ldots \alpha_p$ a non trivial path in kQ with residual class $\overline{v} \in r = \mathrm{rad}\, A$. A direct computation shows that ξ is well defined and is a group morphism.

Proposition 1.5 *Let A be an incidence algebra and (Q, I_v) be a presentation of A. Then there exists an exact sequence of abelian groups*

$$0 \to k^+ \xrightarrow{d^0} C^0(A, I_v, k^+) \xrightarrow{d^1} Z^1(A, I_v, k^+) \xrightarrow{\xi} \mathrm{H}^1(A) \to 0$$

Proof: The exactness can be checked by considering the description of $\mathrm{H}^1(A)$ given in Remark 1.2, using the following trivial fact: if $[\delta] \in \mathrm{H}^1(A)$ we have that $\delta(\alpha) = \lambda_\alpha \alpha$ for any arrow $\alpha \in Q_1$. Then $\xi(g) = [\delta]$ for $g \in Z^1(A, I_v, k^+)$ given by $g(\alpha) = \lambda_\alpha$. $\square$

Corollary 1.6 *Let A be an incidence algebra and (Q, I_v) be a presentation of A. Then*

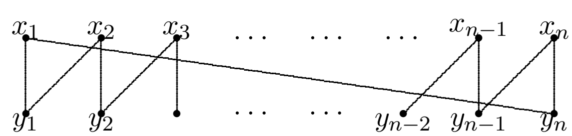
$$\mathrm{Hom}(\pi_1(Q, I_v), k^+) \simeq \mathrm{H}^1(A)$$

It follows from the definition that the fundamental group $\pi_1(Q, I)$ depends essentially on I, thus it is not an invariant of A. Bardzell and Marcos [BM] proved that if the algebra is *constricted* then the fundamental group does not depend on the presentation. Constricted algebras include incidence algebras.

Theorem 1.7 *[BM] If $A = kQ/I$ is an incidence algebra then $\pi_1(Q, I) \simeq \pi_1(Q, I_v)$ for any presentation (Q, I_v) of A.*

2 Hochschild cohomology computations

It is known [D, IZ] that the Hochschild cohomology groups of an incidence algebra vanish if the associated poset does not contain crowns, that is, subposets of the form



where $x_i > y_i$ and $x_i > y_{i-1}$, for $1 \le i \le n$, with $y_0 = y_n$, $n \ge 2$.

Now we are going to prove that the Hochschild cohomology groups of incidence algebras also vanish if the associated posets have a unique maximal or minimal element. In order to prove this, given an incidence algebra $A = kQ/I$, we construct a new poset $\widetilde{Q_0}$ by adding to Q_0 two new elements a and b. The partial order between the elements of $\widetilde{Q_0}$ coincides with the partial order on Q_0 and $a > u$, $u > b$, $\forall u \in Q_0$.

A *source* of Q is a vertex $x \in Q_0$ such that there is no arrow in Q_1 ending on x and a *sink* of Q is a vertex $y \in Q_0$ such that there is no arrow in Q_1 starting on y.

The corresponding quiver $\widetilde{Q}$ is the quiver Q with two new vertices a and b, a new arrow from a to each source of Q and a new arrow from each sink of Q to b. Then $\widetilde{Q_0}$ is a finite poset having a unique maximal element a and a unique minimal element b. Let $\widetilde{I}$ be the parallel ideal of $k\widetilde{Q}$ and $\widetilde{A} = k\widetilde{Q}/\widetilde{I}$. We denote by S_a and S_b the simple left $\widetilde{A}$-modules corresponding to a and b.

Theorem 2.1 *[C, page 225] Let $A = kQ/I$ be an incidence algebra. Then* $\mathrm{H}^i(A) = \mathrm{Ext}^{i+2}_{\widetilde{A}}(S_a, S_b)$*, $\forall i \geq 1$.*

Theorem 2.2 *Let $A = kQ/I$ be an incidence algebra. If Q_0 has a unique maximal (minimal) element then $\mathrm{H}^i(A) = 0$ for all $i \geq 1$.*

Proof: Suppose Q_0 has a unique maximal element x. Then x is a source of Q. Since $\mathrm{H}^i(A) = \mathrm{Ext}^{i+2}_{\widetilde{A}}(S_a, S_b)$, $\forall i \geq 1$, it is enough to show that $\mathrm{Ext}^{i+2}_{\widetilde{A}}(S_a, S_b) = 0$, $\forall i \geq 1$. The result follows immediately from the short exact sequence in mod $\widetilde{A}$

$$0 \to P_x \to P_a \to S_a \to 0$$

$\square$

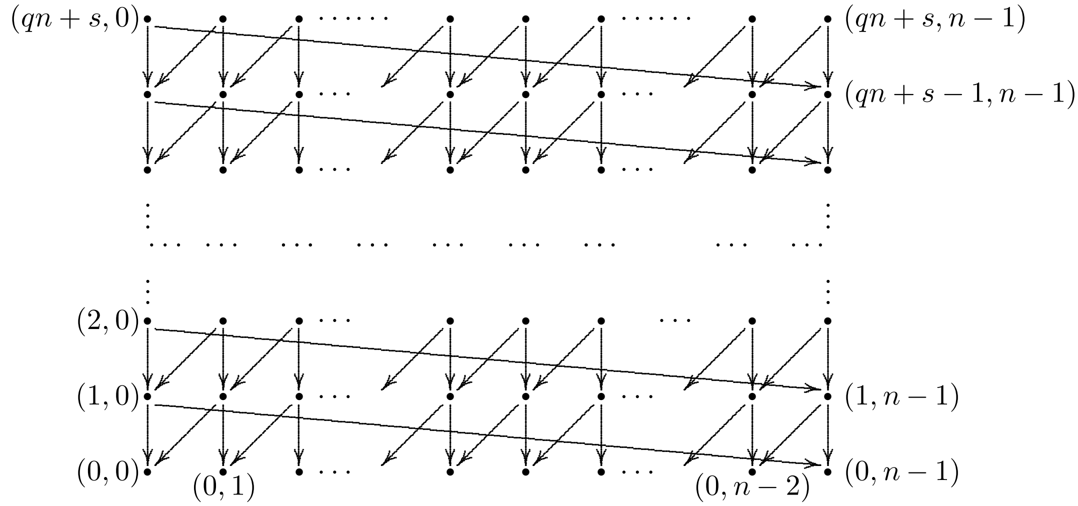
In [IZ] Igusa and Zacharia give a combinatorial algorithm to find an upper bound for the cohomological dimension of $\mathrm{Ext}^{i}_{\widetilde{A}}(S_a, S_b)$. They show how to construct the so-called *reduced subposet* $\overline{Q_0}$ of Q_0 which has the property that all elements $x \in \overline{Q_0}$, neither minimal nor maximal elements, are such that $\{y \in \overline{Q_0} : y \geq x\}$ has at least two minimal elements and $\{z \in \overline{Q_0} : x \geq z\}$ has at least two maximal elements. The Hochschild cohomology groups are invariant under this construction. Hence it is enough to compute them for incidence algebras associated to reduced posets that contain crowns.

Now we are going to compute the Hochschild cohomology groups of the incidence algebras A_{qn+s} associated to reduced posets P where all elements $x \in P$, neither minimal nor maximal elements, are such that $\{y \in P : y \geq x\}$ has exactly two minimal elements and $\{z \in P : x \geq z\}$ has exactly two maximal elements.

We will denote by A_{qn+s} $(n \geq 2,\ q \geq 0,\ 0 \leq s < n)$ the incidence algebra associated to the poset $(Q_{qn+s})_0$ given by

$$(Q_{qn+s})_0 = [qn + s] \times [n - 1]$$

where $[m] = \{0, \ldots, m\}$, $(l, j) < (l + 1, j)$, $(l, j) < (l + 1, j + 1)$ and $(l, n) = (l, 0)$. This means that A_{qn+s} is the incidence algebra with ordered quiver:



We will prove two lemmas that will be useful for the computations of $\mathrm{H}^i(A_{qn+s})$. Given an A-module M, we denote $\operatorname{supp} M = \{x \in Q_0 : M(x) \neq 0\}$. Consider the following conditions:

1) for any $x \in \operatorname{supp} M$, $M(x) = k$;

2) if $\alpha \in Q_1$ and $s(\alpha), e(\alpha) \in \operatorname{supp} M$ then $M(\alpha) = id_k$.

Lemma 2.3 *Let M_1, M_2 be two A-modules satisfying conditions (1) and (2) above. If $\emptyset \neq \operatorname{supp} M_1 \subseteq \operatorname{supp} M_2$ and $\operatorname{supp} M_1$ is connected then $\mathrm{Hom}_A(M_1, M_2) \simeq k$.*

Proof: Let $A = kQ/I$ be an algebra and let f be an A-homomorphism from M_1 to M_2. Hence, f is given by a family of linear maps $(f_x : M_1(x) \to M_2(x))_{x \in Q_0}$ such that for any arrow $\alpha : x \to y$ in Q_1 the corresponding diagram is commutative. From our assumptions we deduce that $f_x = f_y$ whenever there exists an arrow $\alpha : x \to y$, $x, y \in \operatorname{supp} M_1$. Now, $\operatorname{supp} M_1$ is connected so $f_x = f_y$ for all $x, y \in \operatorname{supp} M_1$ and $f_x = 0$ for all $x \in Q_0 \setminus \operatorname{supp} M_1$. Therefore our claim follows. $\square$

Lemma 2.4 *Let $A = B[M]$ be a one point extension of B by a B-module M, $\mathrm{H}^i(A) = 0$ for all $i > 0$ and $\mathrm{End}_B(M) = k$. Then $\mathrm{H}^0(A) = \mathrm{H}^0(B)$ and $\mathrm{H}^i(B) = \mathrm{Ext}^i_B(M, M), \forall i > 0$.*

Proof: We get the desired result using the long exact sequence given in Theorem 1.3. $\square$

We are now in a position to compute the Hochschild cohomology groups of the incidence algebras A_{qn+s}.

Theorem 2.5 *Let A_{qn+s} ($n \geq 2$, $q \geq 0$, $0 \leq s < n$) be the incidence algebra associated to the poset $(Q_{qn+s})_0$. Then:*

$$\mathrm{H}^i(A_{qn+s}) = \begin{cases} k & \textit{if } i = 0 \\ k & \textit{if } i = 2q+1 \\ 0 & \textit{otherwise} \end{cases} \quad , \quad s \neq 0$$

$$\mathrm{H}^i(A_{qn}) = \begin{cases} k & \textit{if } i = 0 \\ k^{n-1} & \textit{if } i = 2q \\ 0 & \textit{otherwise} \end{cases}$$

Proof: The result is known for $n = 2$ (see [GPPRT]). Assume $n \geq 3$.

Let $A = A_{qn+s}$ be the incidence algebra whose ordered quiver is $Q = Q_{qn+s}$. This quiver is connected and has no oriented cycles, then $\mathrm{H}^0(A) = Z(A) = k$.

In order to compute $\mathrm{H}^i(A)$ for $i > 0$ we construct the quiver $\widehat{Q}$ by adding to Q one new element a and a new arrow from a to each source of Q. Let $\widehat{A} = k\widehat{Q}/\widehat{I}$ be the incidence algebra associated to the ordered quiver $\widehat{Q}$. Then $\widehat{A} = A[M]$, with $M = \operatorname{rad} P_a$, and the associated poset has a unique maximal element a. Hence Theorem 2.2 implies that $\mathrm{H}^i(\widehat{A}) = 0$ for all $i > 0$. By the previous lemmas we get that $\mathrm{H}^i(A) = \mathrm{Ext}^i_A(M, M)$, $\forall i > 0$. So, it is enough to compute $\mathrm{Ext}^i_A(M, M)$, $\forall i > 0$.

Let us denote $M = M_{qn+s}$. Then for each p such that $0 \leq p \leq q$ we consider the following short exact sequences in $\operatorname{mod} A$

$$\begin{aligned} 0 \to K_{pn+s} \xrightarrow{inc} \amalg_{i=0}^{n-1} P_{(pn+s,i)} \xrightarrow{f_{pn+s}} M_{pn+s} \to 0 \\ 0 \to M_{(p-1)n+s} \xrightarrow{\Delta} \amalg_{i=0}^{n-1} P_{(pn+s-1,i)} \xrightarrow{g_{pn+s}} K_{pn+s} \to 0 \end{aligned} \qquad (1)$$

where $M_{-n+s} = 0$. The morphisms f_{pn+s}, Δ and g_{pn+s} are induced respectively by the linear maps

$$f_{pn+s}(l, j)(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} x_i$$

$$\Delta(l, j)(x) = (x, \ldots, x)$$

$$g_{pn+s}(l, j)(x_0, \ldots, x_{n-1}) = (-x_{n-1} + x_0, -x_0 + x_1, \ldots, -x_{n-2} + x_{n-1})$$

In order to describe the corresponding representations of these modules we define the following subspaces of k^n:

$$C^{j,t} = \{(x_0, \ldots, x_{n-1}) \in k^n : x_i = 0 \;\; \forall i \not\equiv j, j+1, \ldots, j+t (\operatorname{mod} n)\}$$

and

$$C_0^{j,t} = \{(x_0, \ldots, x_{n-1}) \in C^{j,t} : \sum_{i=0}^{n-1} x_i = 0\}$$

with $0 \leq j < n$, $t \geq 0$. Now, for any $(l, j) \in (Q_{qn+s})_0$, we have that:

$$M_{pn+s}(l, j) = \begin{cases} 0 & \text{if } l \geq pn+s+1 \\ k & \text{if } l \leq pn+s \end{cases}$$

$$\amalg_{i=0}^{n-1} P_{(pn+s,i)}(l,j) = \begin{cases} 0 & \text{if } l \geq pn+s+1 \\ C^{j,pn+s-l} & \text{if } (p-1)n+s+2 \leq l \leq pn+s \\ C^{j,n-1} = k^n & \text{if } l \leq (p-1)n+s+1 \end{cases}$$

$$K_{pn+s}(l,j) = \begin{cases} 0 & \text{if } l \geq pn+s \\ C_0^{j,pn+s-l} & \text{if } (p-1)n+s+2 \leq l \leq pn+s-1 \\ C_0^{j,n-1} & \text{if } l \leq (p-1)n+s+1 \end{cases}$$

The corresponding linear maps are induced by the identity. Recall that $M = M_{qn+s}$. Hence, applying the functor $\mathrm{Hom}_A(-, M)$ to the short exact sequences (1) we get that, for $i \geq 0$,

$$\mathrm{Ext}_A^{2i+1}(M,M) = \begin{cases} \mathrm{Ext}_A^1(M_{(q-i)n+s}, M) & \text{if } i \leq q \\ \mathrm{Ext}_A^{2(i-q)+1}(M_s, M) & \text{if } i > q \end{cases}$$

and for $i > 0$,

$$\mathrm{Ext}_A^{2i}(M,M) = \begin{cases} \mathrm{Ext}_A^1(K_{(q-i+1)n+s}, M) & \text{if } i \leq q \\ \mathrm{Ext}_A^{2(i-q)+1}(K_{n+s}, M) & \text{if } i > q \end{cases}$$

Note that the short exact sequences (1) allows us to construct a projective resolution of M_s. A direct computation shows that

$$\mathrm{Ext}_A^{2(i-q)+1}(M_s, M) = 0 \quad \text{for } i > q$$
$$\mathrm{Ext}_A^1(M_0, M) = 0 \quad \text{and} \quad \mathrm{Ext}_A^1(M_s, M) = k \quad \text{if } s > 0$$

Analogously a projective resolution of K_{n+s} constructed using (1) shows that

$$\mathrm{Ext}_A^{2(i-q)+1}(K_{n+s}, M) = 0 \quad \text{for } i > q$$

To finish the proof we have to compute $\mathrm{Ext}_A^1(M_{pn+s}, M)$ and $\mathrm{Ext}_A^1(K_{pn+s}, M)$ for $p > 0$. This can be done by applying the functor $\mathrm{Hom}_A(-, M)$ to the short exact sequences (1). We compute now the dimensions of the k-vector spaces $\mathrm{Hom}_A(M_{pn+s}, M)$ and $\mathrm{Hom}_A(K_{pn+s}, M)$. Applying Lemma 2.3 we get that

$$\mathrm{Hom}_A(M_{pn+s}, M) \simeq k \quad \text{for } pn+s > 0$$

Now we are going to compute $\mathrm{Hom}_A(K_{pn+s}, M)$ for $p > 0$. If $r \in \mathrm{Hom}_A(K_{pn+s}, M)$ then $r(l-1,j)$ is uniquely determined by $r(l,j)$ and $r(l,j+1)$, for any l such that $1 \leq l \leq pn+s-1$. So, the map r is uniquely determined by the linear maps

$$r(pn+s-1, j) : C_0^{j,1} \to k$$

for all j such that $0 \leq j < n$. Denote by $v_i = (y_0, \ldots, y_{n-1}) \in k^n$ with $y_i = -y_{i+1} = 1$ and $y_r = 0$ otherwise, where $y_n = y_0$. All the subspaces $C_0^{j,pn+s-l}$ has as basis a subset of $\{v_0, \dots v_{n-1}\}$. Let $r(pn+s-1, j)(v_j) = \lambda_j$. This implies that $r(l,j)(v_i) = \lambda_i$, for all v_i in the given basis of $C_0^{j,pn+s-l}$ and the map r is uniquely determined by $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$. In particular, take $l = 0$. Since $p > 0$, $v_0, \dots, v_{n-1} \in C_0^{j,pn+s}$ and $\sum_{i=0}^{n-1} v_i = 0$. So

$$0 = r(0,j)(\sum_{i=0}^{n-1} v_i) = \sum_{i=0}^{n-1} \lambda_i$$

Hence, $\dim_{\mathrm{k}} \mathrm{Hom}_A(K_{pn+s}, M) \leq n-1$. On the other hand the k- linear maps

$$r_t(pn+s-1, j) : C_0^{j,1} \to k, \quad t = 0, \dots, n-2$$

given by

$$r_t(pn+s-1, j)(v_j) = \begin{cases} 1 & \text{if } j = t \\ 0 & \text{otherwise} \end{cases}$$

induce $n-1$ morphisms in $\mathrm{Hom}_A(K_{pn+s}, M)$ which are linearly independent. Therefore

$$\mathrm{Hom}_A(K_{pn+s}, M) \simeq k^{n-1} \quad \text{for } p > 0$$

So, applying $\mathrm{Hom}_A(-, M)$ to the short exact sequences (1), we get that:

$$\mathrm{Ext}_A^1(M_{pn+s}, M) = 0 \quad \text{for all } p > 0$$
$$\mathrm{Ext}_A^1(K_{pn+s}, M) = 0 \quad \text{for all } p > 0,\ pn+s \neq n$$
$$\mathrm{Ext}_A^1(K_n, M) = k^{n-1}$$

□

Remark 2.6 *Since the Hochschild cohomology of incidence algebras is isomorphic to the cohomology of simplicial complexes, it would be nice to describe the simplicial complexes associated to the given posets* $(Q_{qn+s})_0$.

This is not difficult in particular cases: if $n = 2$*, the underlying space of the simplicial complex is homeomorphic to the* $(2q+s)$*-sphere* S^{2q+s} *(see [GPPRT]); if* $q = 0$ *and* $s = 1$*, it is homeomorphic to* S^1*; if* $q = 0$ *and* $s = 2$*, it is homeomorphic to a cylinder if* n *is even, and to a Möbius band if* n *is odd.*

3 The fundamental group of any presentation of A_{qn+s}

In Section 2 we compute the first Hochschild cohomology group $\mathrm{H}^1(A_{qn+s})$. Using the close relation between the first Hochschild cohomology group and the fundamental group, we can prove the following theorem.

Theorem 3.1 *For any presentation* (Q_{qn+s}, I_v) *of the incidence algebra* A_{qn+s}*,* $qn+s > 0$*,*

$$\pi_1(Q_{qn+s}, I_v) \simeq \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

Proof: Since the fundamental group does not depend on the presentation of the incidence algebra A_{qn+s}, we may consider $I_v = I$ the parallel ideal. We denote the arrows in $(Q_{qn+s})_1$ in the following way:

$$\alpha_{(l,j)} : (l,j) \to (l-1,j) \qquad \text{and} \qquad \beta_{(l,j)} : (l,j) \to (l-1,j-1)$$

First we are going to prove that the fundamental group of $\pi_1(Q_{qn+s}, I)$ is cyclic. Let $\sim$ be the equivalence relation defined in Section 1.5. Considering the minimal relations in I we have that $\alpha_{(l,j)}\beta_{(l,j)}{}^{-1} \sim \beta^{-1}_{(l-1,j)}\alpha_{(l-1,j-1)}$. Then any closed walk u starting and ending at $(0,0)$ is equivalent to a walk which is a composition of some arrows $\alpha_{(1,j)}$, $\beta_{(1,j)}$ and their formal inverses. Hence

$$[u] = \left[\alpha_{(1,0)}\beta_{(1,0)}{}^{-1}\alpha_{(1,n-1)}\beta_{(1,n-1)}{}^{-1}\dots\alpha_{(1,2)}\beta_{(1,2)}{}^{-1}\alpha_{(1,1)}\beta_{(1,1)}{}^{-1}\right]^r$$

for some $r \in \mathbb{Z}$.

By Corollary 1.6 we know that $\mathrm{Hom}(\pi_1(Q_{qn+s}, I), k^+) \simeq \mathrm{H}^1(A_{qn+s})$ and from Theorem 2.5 we have that $\mathrm{H}^1(A_{qn+s}) = k$ if $q = 0$, and it is zero if $q > 0$.

If $q > 0$, then $\mathrm{Hom}(\pi_1(Q_{qn+s}, I), k^+) = 0$ for any field k. Since $\pi_1(Q_{qn+s}, I)$ is a cyclic group then $\mathrm{Hom}(\pi_1(Q_{qn+s}, I), k^+) = 0$ implies that $\pi_1(Q_{qn+s}, I)$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for some m. If $m > 1$ we get a contradiction if we take a field k with car $k = p$ and p/m. Therefore the fundamental group of any presentation of A_{qn+s} is the trivial group when $q > 0$.

If $q = 0$, we have that $\mathrm{Hom}(\pi_1(Q_s, I), k^+) \simeq \mathrm{H}^1(A) = k$, for any field k. Therefore, $\pi_1(Q_s, I) \neq 0$. If $\pi_1(Q_s, I) \simeq \mathbb{Z}/m\mathbb{Z}$ for some m, we get a contradiction since $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, k^+) = 0$ for any field k such that car k does not divide m. Therefore our claim follows. □

Recall that a triangular algebra is simply connected if, for every presentation (Q, I) of A, the fundamental group $\pi_1(Q, I)$ is trivial.

Corollary 3.2 *The incidence algebra A_{qn+s} is simply connected if and only if $q \geq 1$.*

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