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Hochschild Cohomology of Triangular Matrix Algebras[1]

Sandra Michelena and María Inés Platzeck[2]

Instituto de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina

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We study the Hochschild cohomology of triangular matrix rings $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$, where A and R are finite dimensional algebras over an algebraically closed field K and M is an A-R-bimodule. We prove the existence of two long exact sequences of K-vector spaces relating the Hochschild cohomology of A, R, and B.

Let A be a finite dimensional algebra (associative, with identity) over an algebraically closed field K.

The Hochschild cohomology groups $\mathrm{H}^i(A, X)$ of A with coefficients in a finitely generated A-A-bimodule X were originally defined by Hochschild in 1945 [Ho]. When $X = A$ we write $\mathrm{H}^i(A)$ instead of $\mathrm{H}^i(A, A)$ and $\mathrm{H}^i(A)$ is called the ith-Hochschild cohomology group of A.

Computations of the Hochschild cohomology groups for semicommutative schurian algebras and algebras arising from narrow quivers have been provided in [H, C1], respectively. The case of monomial and truncated algebras has been considered in [B, BLM, BM, L]. Recently, M. J. Redondo and A. Gatica have computed these groups for some incidence algebras [GR]. However, the actual calculations of Hochschild cohomology groups have been fairly limited. The reader can refer to [H] for a summary of the work in this area.

In this paper we will study the Hochschild cohomology of a triangular matrix algebra $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$ where A, R are arbitrary finite dimensional K-algebras and M is a finitely generated A-R-bimodule. The main result of this paper is the following.

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[2] The second author is a researcher from CONICET, Argentina.

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THEOREM. *Let* $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. *Then there exist long exact sequences of K-vector spaces of the form*

$$0 \to \mathrm{Ann}(M_R) \cap Z(R) \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \xrightarrow{\Delta_0} \mathrm{Ext}^1_{B \otimes_K R^{op}}(R, P_e) \to$$

$$\to \cdots \to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \xrightarrow{\Delta_i} \mathrm{Ext}^{i+1}_{B \otimes_K R^{op}}(R, P_e) \to \cdots,$$

where e is the idempotent $\begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$ *of B and* $P_e = Be$, *and*

$$0 \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \oplus \mathrm{H}^0(R) \to \mathrm{Ext}^1_{B \otimes_K R^{op}}(R, M) \to \ldots \to$$

$$\to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \oplus \mathrm{H}^i(R) \to \mathrm{Ext}^{i+1}_{B \otimes_K R^{op}}(R, M) \to \cdots.$$

The existence of the first long exact sequence is proved in Section 1.

In Section 2 we show that $\mathrm{Ext}^{i+1}_{B\otimes R^{op}}(R, M) \simeq \mathrm{Ext}^i_{B \otimes R^{op}}(M, M) \simeq \mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ for all $i \geq 0$, and so the second sequence takes the form

$$0 \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \oplus \mathrm{H}^0(R) \to \mathrm{Hom}_{A \otimes_K R^{op}}(M, M) \to \cdots \to$$

$$\to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \oplus \mathrm{H}^i(R) \to \mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M) \to \cdots.$$

This sequence has been obtained also by C. Cibils in [C] and generalizes a result obtained by D. Happel for one point extensions of artin algebras [H].

We also study the relationship between the groups $\mathrm{Ext}^i_{B \otimes_K R^{op}}(R, P_e)$ and $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ which occur in the above sequences. We prove, in addition, that there exists another long exact sequence connecting these groups with the Hochschild cohomology groups of R.

It is thus of interest to know the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$. When ${}_AM$ is projective we know a simple way to construct a projective resolution of M over $A \otimes_K R^{op}$ in terms of a projective resolution of R over R^e. We apply this construction to the particular case $R = K|x|/\langle x^i \rangle$, by considering an appropriate well known projective resolution of R over R^e. Then the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ can be obtained with a straightforward calculation from the resulting resolution of M over $A \otimes_K R^{op}$.

We recall that a triangular matrix algebra $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$ is said to be a local extension of A by M if R is a local algebra. The above methods are appropriate to study the Hochschild cohomology of a particular class of standardly stratified algebras Λ, those having the property that all their idempotent ideals are projective left Λ-modules. The reason is that these algebras can be obtained from a local algebra A_0 by successive local extensions $A_{i+1} = \begin{pmatrix} R_{i+1} & 0 \\ M_i & A_i \end{pmatrix}$, $i = 0, \ldots, n$ by bimodules M_i which are projective left A_i-modules.

Section 3 ends the paper providing examples illustrating the use of the results obtained.

PRELIMINARIES

Throughout this paper K will denote an algebraically closed field. By an algebra we mean a finite dimensional K-algebra which we shall also assume to be basic and indecomposable. So an algebra Λ can be written as a bound quiver algebra $\Lambda \cong KQ/I$ where Q is a finite connected quiver and I is an admissible ideal of the path algebra KQ.

Given an algebra Λ, all the modules considered here are finitely generated left Λ-modules, and we denote by mod Λ the category of finitely generated left Λ-modules and by ind Λ the full subcategory of mod Λ consisting of one chosen representative of each isoclass of indecomposable Λ-modules. We will denote by r_Λ the Jacobson radical of Λ and the Jacobson radical of a Λ-module M will be indicated by rad M.

For a given quiver Q, we will denote by Q_0 the set of vertices of Q and by Q_1 the set of arrows between vertices. For each arrow α, $s(\alpha)$ and $e(\alpha)$ will be the start and end vertices of α, respectively.

For each i in Q_0, we denote S_i the simple Λ-module associated to i and P_i, I_i will denote the projective cover and injective envelope of S_i, respectively. Clearly, if $\Lambda = KQ/I$ and e_i is the idempotent element of Λ corresponding to the vertex i of Q then $P_i = \Lambda e_i$. In order to be more clear, sometimes we will write P_{e_i} instead of P_i.

For a pair of Λ-modules X, Y, we denote by $\tau_X Y$ the trace of X in Y, that is, the submodule of Y generated by all homomorphic images of X. The Λ-module $\tau_X Y$ is an $\mathrm{End}_\Lambda Y$-submodule of Y. Furthermore, $\tau_X Y$ is a Λ-$(\mathrm{End}_\Lambda Y)^{op}$-subbimodule of Y. For a given K-algebra Λ we will denote by Λ^e the enveloping algebra of Λ, that is $\Lambda^e = \Lambda \otimes_K \Lambda^{op}$, and for $\lambda \in \Lambda$, λ° will be λ considered as an element in Λ^{op}.

If Q is a quiver with vertices $1, \ldots, n$ then for each indecomposable projective Λ-module P_i we will call $\hat{P}_i$ the sum $\coprod_{j \neq i} P_j$.

1. A GENERALIZATION OF THE HAPPEL SEQUENCE FOR TRIANGULAR MATRIX ALGEBRAS

This section is devoted to proving the existence of the first exact sequence of the theorem stated in the introduction.

The Hochschild cohomology groups of a given algebra are generally hard to compute by using the definition. For this reason, one often tries to find alternative methods for computing these groups. An example of this fact is the following result due to D. Happel [H] related to one point extensions, which shows the existence of a long exact sequence of K-vector spaces connecting the Hochschild cohomology groups of a one point extension B with the Hochschild cohomology groups of a particular quotient A of B. Sometimes this sequence allows one to compute the Hochschild cohomology groups of B in terms of the Hochschild cohomology groups of A, and the advantage now is that the number of nonisomorphic simple A-modules is smaller than the number of nonisomorphic simple B-modules.

THEOREM (Happel). *Let B be a one point extension of A by an A-module M. Then there exists an exact sequence of K-vector spaces connecting the Hochschild cohomology of A and B of the form*

$$0 \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \to \mathrm{Hom}_A(M, M)/K \to \mathrm{H}^1(B) \to \mathrm{H}^1(A) \to \cdots$$
$$\to \mathrm{Ext}_A^{i-1}(M, M) \to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \to \mathrm{Ext}_A^i(M, M) \to \cdots.$$

Let now B be the path algebra of a quiver Q with relations, that means $B \simeq KQ/I$ for some admissible ideal I. If i is a source in Q and e_i is the corresponding idempotent element in B then B can be written as the one point extension of $A = (1 - e_i)B(1 - e_i)$ by the A-module $(1 - e_i)Ae_i$, that is, $B \simeq \left(\begin{smallmatrix} K & 0 \\ (1-e_i)Ae_i & A \end{smallmatrix}\right)$. So, in this case the above theorem holds. However, there exist many algebras which cannot be written as a one point extension of other algebras.

For two K-algebras A, R and an A-R-bimodule ${}_AM_R$, we will consider the triangular matrix K-algebra $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$. In this section we prove the following result which generalizes the preceding result to triangular matrix K-algebras.

THEOREM. *Let $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$. Then there exists an exact sequence of K-vector spaces connecting the Hochschild cohomology of A and B of the form*

$$0 \to \mathrm{Ann}\, M_R \cap Z(R) \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \to \mathrm{Ext}^1_{B \otimes_K R^{op}}(R, P_e) \to$$
$$\to \mathrm{H}^1(B) \to \mathrm{H}^1(A) \to \mathrm{Ext}^2_{B \otimes_K R^{op}}(R, P_e) \to \cdots \to \mathrm{H}^i(B) \to$$
$$\to \mathrm{H}^i(A) \to \mathrm{Ext}^{i+1}_{B \otimes_K R^{op}}(R, P_e) \to \cdots,$$

where e is the idempotent $\left(\begin{smallmatrix} 1_R & 0 \\ 0 & 0 \end{smallmatrix}\right)$ of B and $P_e = Be$.

We start by recalling the construction of the Hochschild cohomology groups of a K-algebra A by a finite dimensional A-bimodule X.

Let A be an algebra and let $A^{\otimes i}$ be the i-fold tensor product over K of A with itself, that is, $A^{\otimes i} = A \otimes_K \cdots \otimes_K A$. For an A-bimodule ${}_AX_A$ of finite dimension over K, the Hochschild complex $C^{\cdot} = (C^i, d^i)_{i \in \mathbb{Z}}$ associated to A and X is defined as $C^i = 0$, $d^i = 0$, for all $i < 0$, $C^0 = {}_AX_A$, $C^i = \mathrm{Hom}_K(A^{\otimes i}, X)$, for all $i > 0$, $d^0 : X \to \mathrm{Hom}_K(A, X)$ is given by $(d^0x)(a) = ax - xa$, for all $x \in X$ and $a \in A$, and $d^i : C^i \to C^{i+1}$ with

$$(d^if)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}$$

for $f \in C^i$ and $a_1, \dots, a_{i+1} \in A$.

Then the ith Hochschild cohomology group $\mathrm{H}^i(A, X)$ of A with coefficients in X is by definition $\mathrm{H}^i(C^{\cdot}) = \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$. When $X = A$ we write $\mathrm{H}^i(A)$ instead of $\mathrm{H}^i(A, A)$ and $\mathrm{H}^i(A)$ is called the ith Hochschild cohomology group of A.

The following are well known interpretations of $\mathrm{H}^0(A)$ and $\mathrm{H}^1(A, X)$, respectively. By definition $\mathrm{H}^0(A)$ coincides with the center $Z(A)$ of A and $\mathrm{H}^1(A, X) = \mathrm{Der}(A, X)/\mathrm{Der}^0(A, X)$, where $\mathrm{Der}(A, X) = \{\delta \in \mathrm{Hom}_K(A, X) : \delta(ab) = a\delta(b) + \delta(a)b\}$ is the K-vector space of derivations of A in X, and $\mathrm{Der}^0(A, X) = \{\delta_x \in \mathrm{Hom}_K(A, X) : \delta_x(a) = ax - xa \ \forall x \in X\}$ is the subspace of inner derivations from A to X.

A different way of defining the Hochschild cohomology groups of A is to consider the enveloping algebra A^e of A. Any A-bimodule X may be regarded as a left A^e-module by setting $(a \otimes b^{\circ})x = axb$ for all $a, b \in A$, and $x \in X$.

In particular, A is a left A^e-module and $\mathrm{H}^i(A, X) = \mathrm{Ext}^i_{A^e}(A, X)$ for all $i \geq 0$. This is proven by constructing an A^e-projective resolution $S.(A) = (S_i(A), \delta_i^A)_{i \geq 0}$ of A, which is called the standard resolution of A [Ho, CE].

Our aim now is to prove the main result of this section. The following considerations will be useful throughout the paper.

Remark 1.1. Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. From now on we denote by e the idempotent element $\begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$ of B and, as we said before, we denote by P_e the indecomposable projective B-module Be. Consider the ring morphisms $\psi : B \to A$, $\psi(\begin{pmatrix} r & 0 \\ m & a \end{pmatrix}) = a$, and $\pi : B \to R$, $\pi(\begin{pmatrix} r & 0 \\ m & a \end{pmatrix}) = r$.

The A-R-bimodule ${}_AM_R$ is a B-B-bimodule via ψ and π, respectively.

On the other hand, the two-sided ideal $\begin{pmatrix} 0 & 0 \\ {}_AM_R & 0 \end{pmatrix}$ of B may be considered as an A-R-bimodule since $(\mathrm{Ker}\,\psi).\begin{pmatrix} 0 & 0 \\ {}_AM_R & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ {}_AM_R & 0 \end{pmatrix}.(\mathrm{Ker}\,\pi) = 0$.

Now, it is easy to verify that the mapping $M \to \left(\begin{smallmatrix} 0 & 0 \\ {}_AM_R & 0 \end{smallmatrix}\right)$ which maps m into $\left(\begin{smallmatrix} 0 & 0 \\ m & 0 \end{smallmatrix}\right)$ is a B, B^e, $B \otimes_K R^{op}$, and $A \otimes_K R^{op}$-isomorphism.

Similarly the B^e-module P_e may be regarded as a $B \otimes_K R^{op}$-module via π, since $P_e(\operatorname{Ker} \pi) = 0$, and finally, the morphism π makes R a B^e-module.

According to the above remark, the canonical sequence

$$0 \to M \to P_e \to R \to 0$$

is B, B^e, and $B \otimes_K R^{op}$-exact.

Let $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$. Any B-R-bimodule ${}_BX_R$ may be considered as a B^e-module via the morphism π of Remark 1.1, and the B-module structures of R and X make $\operatorname{Hom}_{R^{op}}(R, X)$ a B^e-module by setting $[(a \otimes b^\circ)f](r) = af(br)$ for all $(a \otimes b^\circ) \in B^e$ and $f \in \operatorname{Hom}_{R^{op}}(R, P_e)$.

In addition, the mapping $\operatorname{Hom}_{R^{op}}(R, X) \to X$ which sends any R^{op}-morphism $f: R \to X$ into the element $f(1_R)$ of X is a B^e-isomorphism. Using now the fact that the canonical sequence

$$0 \to P_e \to B \to A \to 0 \tag{1}$$

is B^e-exact we obtain the following result.

THEOREM 1.2. *Let* $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$. *Then*

$$\operatorname{Hom}_{B^e}(B, A) \simeq \operatorname{Hom}_{B^e}(A, A) \simeq \operatorname{Hom}_{A^e}(A, A) = \operatorname{H}^0(A)$$
$$\operatorname{Ext}^i_{B^e}(B, A) \simeq \operatorname{Ext}^i_{B^e}(A, A), \qquad \textit{for all } i \geq 1.$$

Proof. The above sequence (1) gives rise to the long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{B^e}(A, A) \to \operatorname{Hom}_{B^e}(B, A) \to \operatorname{Hom}_{B^e}(P_e, A) \to \\ &\to \operatorname{Ext}^1_{B^e}(A, A) \to \operatorname{Ext}^1_{B^e}(B, A) \to \operatorname{Ext}^1_{B^e}(P_e, A) \to \cdots \to \\ &\to \operatorname{Ext}^i_{B^e}(A, A) \to \operatorname{Ext}^i_{B^e}(B, A) \to \operatorname{Ext}^i_{B^e}(P_e, A) \to \cdots. \end{aligned}$$

It follows from the structure of B^e-module of A that $\operatorname{Hom}_{B^e}(A, A) \simeq \operatorname{Hom}_{A^e}(A, A) = \operatorname{H}^0(A)$. Thus, in order to prove the first part of the theorem it suffices to show that $\operatorname{Hom}_{B^e}(A, A) \simeq \operatorname{Hom}_{B^e}(B, A)$. This result follows from the equality $\operatorname{Hom}_{B^e}(P_e, A) = 0$, which is a direct consequence of $\operatorname{Hom}_B(P_e, A) = 0$. The rest of the theorem follows from the fact that $\operatorname{Ext}^i_{B^e}(P_e, A) = 0$ for all $i \geq 1$, which will be proved next in Corollary 1.4. ∎

To prove that $\operatorname{Ext}^i_{B^e}(P_e, A) = 0$ for all $i \geq 1$ we will construct a projective resolution of P_e over B^e.

First we recall some basic properties of the enveloping algebra Λ^e of an algebra Λ over an algebraically closed field K.

Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents in Λ. Since K is an algebraically closed field, $\{e_i \otimes e_j^\circ\}_{1 \le i, j \le n}$ is a complete set of primitive orthogonal idempotents in $\Lambda^e = \Lambda \otimes_K \Lambda^{op}$. As we pointed out under Preliminaries, we denote by $P_{e_i \otimes e_j^\circ}$ the indecomposable projective Λ^e-module $\Lambda^e.(e_i \otimes e_j^\circ)$. Let $S_{e_i \otimes e_j^\circ}$ be the simple Λ^e-module top $(P_{e_i \otimes e_j^\circ})$. It is known that $S_{e_i \otimes e_j^\circ} \simeq \mathrm{Hom}_K(S_j, S_i)$ where S_i is the simple Λ-module $\Lambda e_i / r_\Lambda e_i = P_i / r_\Lambda P_i$.

We are ready now to describe a B^e-projective resolution of P_e.

PROPOSITION 1.3. *Let* $B = \begin{pmatrix} R & 0 \\ {}_A M_R & A \end{pmatrix}$ *and let* $\{e_1, e_2, \dots, e_n\}$ *be a complete set of primitive orthogonal idempotents in* B. *Suppose that* $e = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} = e_1 + \cdots + e_t$. *Then there exists a projective resolution of* P_e *over* B^e *of the form*

$$\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to P_e \to 0$$

with $Q_n \in \mathrm{add}(\coprod_j P_{e \otimes e_j^\circ})$, *for all* $n \ge 0$.

Proof. We use the following general fact true for any module X over a K-algebra Λ. Let

$$\cdots \to R_n \to \cdots \to R_1 \to R_0 \to X \to 0$$

be a minimal projective resolution of X and S a simple Λ-module. Then the multiplicity of $P_0(S)$ is a summand of R_0 is $\dim_K \mathrm{Hom}_\Lambda(X, S)$. Using this fact it follows that the multiplicity of $P_0(S)$ as a summand of R_i is $\dim_K \mathrm{Ext}^i_\Lambda(X, S)$ for all $i \ge 0$.

The proof of the theorem follows now the arguments of Happel [H, 1.5]. First we will prove that each indecomposable projective P_{e_k}, $1 \le k \le t$, has a B^e-projective resolution of the form

$$\cdots \to R_n^{(k)} \to \cdots \to R_1^{(k)} \to R_0^{(k)} \to P_{e_k} \to 0$$

with $R_n^{(k)} \in \mathrm{add}(\coprod_j P_{e_k \otimes e_j^\circ})$ for all $n \ge 0$. Assume that

$$\cdots \to R_n^{(k)} \to \cdots \to R_1^{(k)} \to R_0^{(k)} \to P_{e_k} \to 0$$

is a minimal projective resolution of P_{e_k} over B^e. Then

$$R_n^{(k)} \simeq \coprod_{i,j} \left[P_{e_i \otimes e_j^\circ} \right]^{r_{ij}}$$

where $r_{ij} = \dim_K \mathrm{Ext}^n_{B^e}(P_{e_k}, S_{e_i \otimes e_j^\circ}) = \dim_K \mathrm{Ext}^n_{B^e}(P_{e_k}, \mathrm{Hom}_K(S_j, S_i))$.

Now, [CE, Theorem 2.8a, Chap. IX] in this case states that

$$\operatorname{Ext}^n_{B^e}\big(P_{e_k}, \operatorname{Hom}_K(S_j, S_i)\big) \simeq \operatorname{Ext}^n_{B^{op}\otimes_K B}\big(P_{e_k}, \operatorname{Hom}_K(DS_i, DS_j)\big)$$
$$\simeq \operatorname{Ext}^n_B\big(P_{e_k} \otimes_{B^{op}} DS_i, DS_j\big)$$

and so, $r_{ij} = \dim_K \operatorname{Ext}^n_B(P_{e_k} \otimes_{B^{op}} DS_i, DS_j)$, for all i, j. Since for each i, $P_{e_k} \otimes_{B^{op}} DS_i \simeq DS_i \otimes_B Be_k$ and $DS_i \otimes_B Be_k$ is DS_{e_k} if $i = e_k$ and 0 otherwise, it follows that $\operatorname{Ext}^n_B(P_{e_k} \otimes_{B^{op}} DS_i, DS_j) \simeq \operatorname{Ext}^n_B(DS_i, DS_j)$ is different from zero just for $i = e_k$. Observe that, here $\operatorname{Ext}^n_B(DS_i, DS_j)$ denotes the extensions group between the right B-modules DS_i and DS_j. Finally the desired result follows from the fact that $\operatorname{Ext}^n_B(DS_i, DS_j) \simeq \operatorname{Ext}^n_B(S_j, S_i)$, where $\operatorname{Ext}^n_B(S_j, S_i)$ denotes the extension group between the left B-modules S_j and S_i.

Hence, P_e has a projective resolution of the form

$$\cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to P_e \to 0$$

with $Q_n \in \operatorname{add}(\coprod_j P_{e\otimes e_j^\circ})$ for all $n \geq 0$. ▮

This proposition has the following corollary which completes the proof of Theorem 1.2.

COROLLARY 1.4. *With the hypothesis of Theorem* 1.2, $\operatorname{Ext}^i_{B^e}(P_e, A) = 0$, *for all* $i \geq 1$.

Proof. Applying the functor $\operatorname{Hom}_{B^e}(\ , A)$ to the B^e-projective resolution of P_e given in the above theorem we obtain the long exact sequence

$$0 \to \operatorname{Hom}_{B^e}(P_e, A) \to \operatorname{Hom}_{B^e}(Q_0, A) \to \cdots \to \operatorname{Hom}_{B^e}(Q_n, A) \to \cdots,$$

where $Q_n \in \operatorname{add}[\coprod_j (P_{e\otimes e_j^\circ})]$ for all $n \geq 0$.

So it is enough to prove that $\operatorname{Hom}_{B^e}(P_{e\otimes e_j^\circ}, A) = 0$, for all $j = 1, \dots, n$. This is an immediate consequence of the equality $\operatorname{Hom}_{B^e}(P_{e\otimes e_j^\circ}, A) = eAe_j$ since eAe_j clearly vanishes and ends the proof of the corollary. ▮

Keeping our aim in mind, we apply now the functor $\operatorname{Hom}_{B^e}(B,\ )$ to the B^e-exact sequence (1) above considered:

$$0 \to P_e \to B \to A \to 0.$$

Then we obtain a long exact sequence of the form

$$0 \to \operatorname{Hom}_{B^e}(B, P_e) \to \operatorname{Hom}_{B^e}(B, B) \to \operatorname{Hom}_{B^e}(B, A) \to$$
$$\to \operatorname{Ext}^1_{B^e}(B, P_e) \to \operatorname{Ext}^1_{B^e}(B, B) \to \operatorname{Ext}^1_{B^e}(B, A) \to \cdots. \qquad (2)$$

Remark 1.5. For any B-R-bimodule ${}_BX_R$ we have $\mathrm{Ext}^i_{B^e}(B,X) \simeq \mathrm{Ext}^i_{B^e}(B, \mathrm{Hom}_{R^{op}}(R,X)) \simeq \mathrm{Ext}^i_{B^e}(B, \mathrm{Hom}_R(DX, DR))$ and the formula given in [CE, Chap. IX, Theorem 2.8a] yields in this case

$$\begin{aligned} \mathrm{Ext}^i_{B^e}(B, \mathrm{Hom}_R(DX, DR)) &\simeq \mathrm{Ext}^i_{R\otimes_K B^{op}}(DX \otimes_B B, DR) \\ &\simeq \mathrm{Ext}^i_{B\otimes_K R^{op}}(R, X) \end{aligned}$$

for all $i \geq 0$.

Now, using our last remark for $X = P_e$, we have that $\mathrm{Ext}^i_{B^e}(B, P_e) \simeq \mathrm{Ext}^i_{B\otimes_K R^{op}}(R, P_e)$, and so the sequence (2) induces a long exact sequence of the form

$$\begin{aligned} 0 &\to \mathrm{Hom}_{B\otimes_K R^{op}}(R, P_e) \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \xrightarrow{\Delta_0} \mathrm{Ext}^1_{B\otimes_K R^{op}}(R, P_e) \to \\ &\to \mathrm{H}^1(B) \to \mathrm{Ext}^1_{B^e}(A,A) \to \cdots \to \mathrm{Ext}^i_{B\otimes_K R^{op}}(R, P_e) \to \mathrm{H}^i(B) \to \\ &\to \mathrm{Ext}^i_{B^e}(A,A) \to \cdots \qquad (3) \end{aligned}$$

via the above isomorphisms and the isomorphisms given in Theorem 1.2.

Let now $B = KQ/I$ for some finite quiver Q and an admissible ideal I of the path algebra KQ. As is usual in representation theory, we consider Λ also as the K category whose objects are the vertices of Q and the set of morphisms from x to y is the vector space $KQ(x,y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x,y) = I \cap KQ(x,y)$.

We say that an algebra A is a convex subcategory of B if there is a path-closed full subquiver Q' of Q such that $A = KQ'/(I \cap KQ')$.

For example, if B is the triangular matrix algebra $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$ then A and R are convex subcategories of B.

The following known fact will be necessary in the sequel.

LEMMA 1.6. *Let A be a convex subcategory of $B = KQ/I$. Then* $\mathrm{Ext}^i_B(X,Y) \simeq \mathrm{Ext}^i_A(X,Y)$, *for all $i \geq 1$ and X, Y in* mod A.

Let Λ and Λ' be two K-algebras (not necessarily finite dimensional) with fixed presentations $\Lambda = KQ/I$ and $\Lambda' = KQ'/I'$. Assume $Q = (Q_0, Q_1)$ and $Q' = (Q'_0, Q'_1)$ are finite quivers; that means Q_0, Q_1, Q'_0, Q'_1 are finite sets.

Then $\Lambda \otimes_K \Lambda'$ is also a K-algebra, and so $\Lambda \otimes_K \Lambda'$ is the path algebra of a quiver $Q_{\Lambda\otimes_K\Lambda'}$ with relations. To know how to describe $\Lambda \otimes_K \Lambda'$ as the path algebra of a quiver with relations motivated some of the proofs and examples of this paper. So, even though such description is known [GM, 3] we describe briefly the construction of the quiver $Q_{\Lambda\otimes_K\Lambda'}$ and the ideal $I_{\Lambda\otimes_K\Lambda'}$ of relations of $\Lambda \otimes_K \Lambda'$.

Let $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_m\}$ be complete sets of primitive orthogonal idempotents in Λ and Λ', respectively. Since K is an algebraically closed field, $\{e_i \otimes e'_j\}_{1 \le i \le n, 1 \le j \le m}$ is a complete set of primitive orthogonal idempotents in $\Lambda \otimes_K \Lambda'$. So, for each primitive orthogonal idempotent $e_i \otimes e'_j$ of $\Lambda \otimes_K \Lambda'$ there exists a vertex $v_{e_i \otimes e'_j}$ in $Q_{\Lambda \otimes_K \Lambda'}$.

We shall investigate now the set $(Q_{\Lambda \otimes_K \Lambda'})_1$ of arrows of $Q_{\Lambda \otimes_K \Lambda'}$. For any pair of vertices $v_{e_i \otimes e'_j}$ and $v_{e_s \otimes e'_t}$ the number of arrows from $v_{e_i \otimes e'_j}$ to $v_{e_s \otimes e'_t}$ is $\dim_K[(e_s \otimes e'_t) r_{\Lambda \otimes_K \Lambda'}/r^2_{\Lambda \otimes_K \Lambda'}(e_i \otimes e'_j)]$.

Since $r_{\Lambda \otimes_K \Lambda'} = r_\Lambda \otimes_K \Lambda' + \Lambda \otimes_K r_{\Lambda'}$ we get that $r^2_{\Lambda \otimes_K \Lambda'} = r^2_{\Lambda \otimes_K \Lambda'} + r_\Lambda \otimes_K r_{\Lambda'} + \Lambda \otimes_K r^2_{\Lambda'}$.

The next lemma enables us to describe the set of arrows $(Q_{\Lambda \otimes_K \Lambda'})_1$.

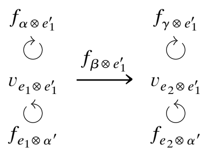
LEMMA 1.7. *There exists an isomorphism of K-vector spaces* $\Phi : r_{\Lambda \otimes_K \Lambda'}/r^2_{\Lambda \otimes_K \Lambda'} \to (r_\Lambda / r^2_\Lambda \otimes_K \coprod_{j=1}^m K e'_j) \times (\coprod_{i=1}^n K e_i \otimes_K r_{\Lambda'}/r^2_{\Lambda'})$ *where* $\{e_1, \ldots, e_n\} \subseteq \Lambda$ *and* $\{e'_1, \ldots, e'_m\} \subseteq \Lambda'$ *are complete sets of primitive orthogonal idempotents of* Λ *and* Λ', *respectively.*

According to the preceding result the arrows of $Q_{\Lambda \otimes_K \Lambda'}$ are given as follows. For each arrow $\beta : i \to j$ in Q_1 we have m arrows $f_{\beta \otimes e'_s} : v_{e_i \otimes e'_s} \to v_{e_j \otimes e'_s}$, $s = 1, \ldots, m$ and for each arrow $\beta' : k \to l$ in Q'_1 we have n arrows $f_{e_t \otimes \beta'} : v_{e_t \otimes e'_k} \to v_{e_t \otimes e'_l}$, $t = 1, \ldots, n$ in $(Q_{\Lambda \otimes_K \Lambda'})_1$.

Note that if $\Lambda = KQ/I$ and $\Lambda' = KQ'/I'$ then $Q_{\Lambda \otimes_K \Lambda'} = Q_{KQ \otimes_K KQ'}$.

We illustrate the construction of $Q_{\Lambda \otimes_K \Lambda'}$ in the following example.

EXAMPLE 1.8. Let Q be the quiver $\underset{\alpha}{\circlearrowleft} 1 \xrightarrow{\beta} 2 \underset{\gamma}{\circlearrowleft}$ and Q' the quiver $\underset{\alpha'}{\circlearrowleft}^{1}$. Then $Q_{KQ \otimes_K KQ'}$ is



We will describe now the ideal $I_{\Lambda \otimes_K \Lambda'}$ of relations of $\Lambda \otimes_K \Lambda'$.

First we assume that $\Lambda = KQ$ and $\Lambda' = KQ'$, that is, $I = I' = 0$. Consider the morphism of K-algebras $\theta : KQ_{\Lambda \otimes \Lambda'} \to \Lambda \otimes \Lambda' = KQ \otimes_K KQ'$ defined over $(Q_{\Lambda \otimes \Lambda'})_0$ and $(Q_{\Lambda \otimes \Lambda'})_1$ as

$$\theta(v_{e_i \otimes e'_j}) = e_i \otimes e'_j$$

$$\theta(f_{\beta \otimes e'_j}) = \beta \otimes e'_j; \qquad \theta(f_{e_i \otimes \beta'}) = e_i \otimes \beta'$$

for all $v_{e_i \otimes e'_j} \in (Q_{\Lambda \otimes \Lambda'})_0$ and $f_{\beta \otimes e'_j}, f_{e_i \otimes \beta'} \in (Q_{\Lambda \otimes \Lambda'})_1$.

Then θ is an epimorphism and we can now state the first result relative to the ideal $I_{\Lambda \otimes_K \Lambda'}$ of relations of $\Lambda \otimes_K \Lambda'$.

LEMMA 1.9. *The set*

$$\mathscr{S} = \{(e(\alpha) \otimes \beta)(\alpha \otimes s(\beta)) - (\alpha \otimes e(\beta))(s(\alpha) \otimes \beta) \quad \text{with } \alpha \in Q_1, \beta \in Q_1'\}$$

generates the ideal $I_{\Lambda \otimes_K \Lambda'}$ *of relations of* $\Lambda \otimes_K \Lambda'$.

Let J be the ideal generated by the set $\mathscr{S}$ of the preceding result. Then we have the following description of $I_{\Lambda \otimes_K \Lambda'}$.

LEMMA 1.10. *Let* $\Lambda = KQ/I$ *and* $\Lambda' = KQ'/I'$. *Then*

$$I_{\Lambda \otimes_K \Lambda'} \simeq [KQ \otimes_K I' + I \otimes_K KQ'] \coprod J$$

as K-vector spaces.

Finally, the next result gives a set of generators of the ideal of relations of $\Lambda \otimes_K \Lambda' = KQ/I \otimes_K KQ'/I'$.

COROLLARY 1.11. *The following set generates the ideal* $I_{\Lambda \otimes_K \Lambda'}$ *of relations of* $\Lambda \otimes_K \Lambda'$:

$$\mathscr{S} \cup \left\{ \sum k_{\delta_1, \dots, \delta_t} f_{\delta_t \otimes e_j'} \dots f_{\delta_1 \otimes e_j'} : \sum k_{\delta_1, \dots, \delta_t} \delta_1 \dots \delta_t \text{ is a relation of } \Lambda \right\}$$
$$\cup \left\{ \sum k_{\delta_1', \dots, \delta_t'} f_{e_i \otimes \delta_t'} \dots f_{e_i \otimes \delta_1'} : \sum k_{\delta_1', \dots, \delta_t'} \delta_1' \dots \delta_t' \text{ is a relation of } \Lambda' \right\}.$$

Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. As we had already mentioned, A and R are convex subcategories of B. Hence, it is not difficult to see, using the above description, that A^e, R^e, and $A \otimes R^{op}$ are convex subcategories of B^e.

Then, we have the following straightforward consequence of Lemma 1.6.

PROPOSITION 1.12. *Let* $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. *Then for all* $i \geq 1$,

$$\operatorname{Ext}^i_{B^e}(X,Y) \simeq \operatorname{Ext}^i_{A^e}(X,Y), \qquad X,Y \text{ in } \operatorname{mod} A^e$$
$$\operatorname{Ext}^i_{B^e}(X,Y) \simeq \operatorname{Ext}^i_{R^e}(X,Y), \qquad X,Y \text{ in } \operatorname{mod} R^e$$
$$\operatorname{Ext}^i_{B \otimes R^{op}}(X,Y) \simeq \operatorname{Ext}^i_{A \otimes R^{op}}(X,Y), \qquad X,Y \text{ in } \operatorname{mod}(A \otimes R^{op}).$$

In particular,

$$\operatorname{Ext}^i_{B^e}(A,A) \simeq \operatorname{Ext}^i_{A^e}(A,A) = \operatorname{H}^i(A)$$
$$\operatorname{Ext}^i_{B^e}(R,R) \simeq \operatorname{H}^i(R)$$
$$\operatorname{Ext}^i_{B^e}(M,M) \simeq \operatorname{Ext}^i_{A \otimes R^{op}}(M,M).$$

The next lemma will be needed in the proof of the main result of this section.

LEMMA 1.13. *Let* $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. *Then*

$$\operatorname{Hom}_{B\otimes_K R^{op}}(R, M) = 0$$
$$\operatorname{Hom}_{B\otimes_K R^{op}}(R, P_e) \simeq \operatorname{Ann}(M_R) \cap Z(R).$$

Proof. The equality $\operatorname{Hom}_{B\otimes_K R^{op}}(R, M) = 0$ follows from the fact that $\operatorname{Hom}_B(R, M) = 0$.

Let now $f: R \to P_e$ be a B-R-morphism. Then f is uniquely determined by the element $f(1_R) = \begin{pmatrix} r & 0 \\ m & 0 \end{pmatrix}$. In addition, for all $b = \begin{pmatrix} r' & 0 \\ m' & a \end{pmatrix} \in B$, we have that $f(b) = f(b\,.\,1_R) = b\,.\,f(1_R)$ and $f(b\,.\,1_R) = f(r') = f(1)r'$. Thus, it easily follows that $f(1_R) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ with $r \in \operatorname{Ann}(M_R) \cap Z(R)$. Hence the map $\theta: \operatorname{Ann}(M_R) \cap Z(R) \to \operatorname{Hom}_{B\otimes_K R^{op}}(R, P_e)$ given by $\theta(r)(1) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$ is a group isomorphism. ∎

We are now in a position to prove the desired result.

THEOREM 1.14. *Let B be the triangular matrix algebra* $\begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. *Then there exists a long exact sequence of K-vector spaces connecting the Hochschild cohomology of A and B of the form*

$$0 \to \operatorname{Ann}(M_R) \cap Z(R) \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \xrightarrow{\Delta_0} \operatorname{Ext}^1_{B\otimes_K R^{op}}(R, P_e) \to$$
$$\to \cdots \to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \xrightarrow{\Delta_i} \operatorname{Ext}^{i+1}_{B\otimes_K R^{op}}(R, P_e) \to \cdots. \tag{4}$$

Proof. Combining Theorem 1.2 with Proposition 1.12 and Lemma 1.13 we obtain

$$\operatorname{Ext}^i_{B^e}(B, A) \simeq \operatorname{Ext}^i_{B^e}(A, A) \simeq \operatorname{Ext}^i_{A^e}(A, A) = \mathrm{H}^i(A)$$

and

$$\operatorname{Hom}_{B\otimes_K R^{op}}(R, P_e) \simeq \operatorname{Ann}(M_R) \cap Z(R)$$

Therefore, the exact sequence (3) induces an exact sequence of the desired form. ∎

The following easy lemma gives necessary and sufficient conditions for $\operatorname{Ann}(M_R)$ being zero when R is a local ring.

LEMMA 1.15. *Let* $B = KQ_B/I$ *and R be a local ring. Assume that* Q_B *has no oriented cycles which are not loops and let a be a vertex of* $(Q_B)_0$. *Suppose* *that there are no arrows except loops ending at a. Then the following conditions are equivalent*:

(a) $\mathrm{Hom}_B(S_a, P_a) = 0$

(b) $\mathrm{Ann}((\tau_{\hat{P}_a} P_a)_{\mathrm{End}\, P_a^{op}}) = 0$

(c) *If* α *is a loop at a and i is minimum such that* $\alpha^i = 0$ *in B, then for all* $j < i$ *there exists an arrow* β *which is not a loop such that* $\beta\alpha^j \neq 0$.

We finish this section with the following remark.

Remark 1.16. If B is the one point extension of A by M, that is, the ring R is the field K, D. Happel proved in [H, 5.3] that $\mathrm{Ext}^i_{B \otimes_K R^{op}}(R, P_e) \simeq \mathrm{Ext}^{i-1}_A(M, M)$, for all $i \geq 2$ and $\mathrm{Ext}^1_{B \otimes_K R^{op}}(R, P_e) \simeq \mathrm{Hom}_A(M, M)/K$. Clearly, in this case we have that $\mathrm{Ann}\, M_R = 0$ and therefore our main theorem coincides with the result given by D. Happel [H, 5.3] for one point extensions.

2. OTHER LONG EXACT SEQUENCES

Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$ with R and A arbitrary K-algebras. In Section 1 we gave a long exact sequence of K-vector spaces connecting the Hochschild cohomology of B with the Hochschild cohomology of A and the groups $\mathrm{Ext}^i_{B \otimes_K R^{op}}(R, P_e)$. A natural question to ask is if it is possible to get information about the Hochschild cohomology of B in terms of the Hochschild cohomology of R and A. In this section we prove the following result which provides a long exact sequence of K-vector spaces connecting directly the Hochschild cohomology of B, A, and R with the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ (see also [C]).

PROPOSITION. *Let* $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. *Then there exists a long exact sequence of K-vector spaces connecting the Hochschild cohomology groups of A, B, and R of the form*

$$0 \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \oplus \mathrm{H}^0(R) \to \mathrm{Hom}_{A \otimes_K R^{op}}(M, M) \to \cdots \to$$

$$\to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \oplus \mathrm{H}^i(R) \to \mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M) \to \cdots.$$

Since the groups $\mathrm{Ext}^i_{B \otimes_K R^{op}}(R, P_e)$ appear in the main result of Section 1 and the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ in the just stated result, it is of interestto relate them. We end the section by proving the existence of another long exact sequence which connect these groups with the Hochschild cohomology groups of R.

Consider first the canonical B^e-exact sequence

$$0 \to M \xrightarrow{j} B \to A \oplus R \to 0.$$

Applying the functor $\operatorname{Hom}_{B^e}(B, \ )$ to this sequence we have a long exact sequence of K-vector spaces of the form

$$\begin{aligned} 0 &\to \operatorname{Hom}_{B^e}(B, M) \to \operatorname{Hom}_{B^e}(B, B) \to \operatorname{Hom}_{B^e}(B, A \oplus R) \to \cdots \\ &\to \operatorname{Ext}^i_{B^e}(B, M) \xrightarrow{\operatorname{Ext}^i_{B^e}(B, j)} \operatorname{Ext}^i_{B^e}(B, B) \to \operatorname{Ext}^i_{B^e}(B, A \oplus R) \to \cdots. \end{aligned} \tag{$*$}$$

It is easy to see that $\operatorname{Hom}_{B^e}(B, M) = 0$. We know that $\operatorname{Ext}^i_{B^e}(B, X) \simeq \operatorname{Ext}^i_{B \otimes_K R^{op}}(R, X)$ for any B-R-bimodule X (see Remark 1.5), and $\operatorname{Ext}^i_{B^e}(B, A) \simeq \operatorname{Ext}^i_{B^e}(A, A)$, as we proved in Theorem 1.2. Combining this with the fact that A^e, R^e, and $B \otimes_K R^{op}$ are convex subcategories of B^e we get (see Proposition 1.12)

$$\operatorname{Ext}^i_{B^e}(B, M) \simeq \operatorname{Ext}^i_{B \otimes_K R^{op}}(R, M)$$

$$\operatorname{Ext}^i_{B^e}(B, A \oplus R) \simeq \mathrm{H}^i(A) \oplus \mathrm{H}^i(R).$$

So the above sequence induces a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \oplus \mathrm{H}^0(R) \to \operatorname{Ext}^1_{B \otimes_K R^{op}}(R, M) \to \cdots \to \\ &\to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \oplus \mathrm{H}^i(R) \to \operatorname{Ext}^{i+1}_{B \otimes_K R^{op}}(R, M) \to \cdots. \end{aligned} \tag{5}$$

We shall prove now that

$$\operatorname{Ext}^{i+1}_{B^e}(R, M) \simeq \operatorname{Ext}^i_{B^e}(M, M) \simeq \operatorname{Ext}^i_{A \otimes_K R^{op}}(M, M)$$

for all $i \geq 0$. This will easily follow from the fact that $\operatorname{Ext}^i_{B^e}(P_e, M) = 0$. It is worthwhile to mention that this fact can be proven by using the B^e-projective resolution of P_e constructed in Proposition 1.3. However, we think that the next construction of a new projective resolution of P_e over $B \otimes_K R^{op}$ is useful for computing examples. With this purpose in mind, the next general lemmas will be needed.

Let Λ and Λ' be two K-algebras and let $\mathscr{C} = (C_i, \delta_i)_{i \geq 0}$ be a projective resolution of Λ over Λ^e.

LEMMA 2.1. *Let X_Λ be a right Λ-module. Then $X \otimes_\Lambda \mathscr{C} = (X \otimes_\Lambda C_i, 1 \otimes \delta_i)_{i \geq 0}$ is an exact sequence.*

Proof. Let $K_i = \operatorname{Ker} \delta_i$. Since Λ is a projective left Λ-module, it follows that the sequence

$$0 \to K_0 \to C_0 \xrightarrow{\delta_0} \Lambda \to 0$$

splits in $\operatorname{mod} \Lambda$. Thus K_0 is a projective left Λ-module. Iterating this argument we have that each short exact sequence

$$0 \to K_i \to C_i \xrightarrow{\delta_i} K_{i-1} \to 0$$

splits in $\operatorname{mod} \Lambda$, which proves the statement. ∎

LEMMA 2.2. *Let ${}_{\Lambda'}X_{\Lambda}$ be a Λ'-Λ-bimodule. Assume that ${}_{\Lambda'}X$ is projective. Then $X \otimes_{\Lambda} \mathscr{C} = (X \otimes_{\Lambda} C_i, 1 \otimes \delta_i)_{i \geq 0}$ is a projective resolution of X over $\Lambda' \otimes_K \Lambda^{op}$. In particular, if C_i is the i-fold $\Lambda^{\otimes i} = \Lambda \otimes_K \cdots \otimes_K \Lambda$ then $(X \otimes_{\Lambda} \mathscr{C})_i = X \otimes_{\Lambda} C_i \simeq X \otimes_K \Lambda^{\otimes i-1}$.*

EXAMPLE 2.3. Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$ and let P_e be the B-R-module $B . e = B . \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$. Then P_e has a $B \otimes_K R^{op}$-projective resolution of the form

$$\mathscr{C}: \quad \cdots \to Q_n \to \cdots \to Q_1 \to Q_0 \to P_e \to 0$$

with $Q_i \in \operatorname{add}(P_e \otimes_K R)$.

In fact, let $\mathscr{C}$ be the standard resolution of R over R^e. That is,

$$\mathscr{C}: \quad \cdots \xrightarrow{\delta_4} R^{\otimes 3} \xrightarrow{\delta_3} R^{\otimes 2} \to R \to 0.$$

According to Lemma 2.2, $P_e \otimes_K \mathscr{C} = (P_e \otimes_R R^{\otimes i}, 1 \otimes \delta_i)$ is a projective resolution of P_e over $B \otimes_K R^{op}$.

Now, $P_e \otimes_R R^{\otimes i} \simeq P_e \otimes_K R^{\otimes i-1} \simeq (P_e \otimes_K R)^{t^{(i-2)}}$ where $t = \dim_K R$, and therefore $P_e \otimes_R R^{\otimes i} \in \operatorname{add}(P_e \otimes_K R)$.

We are now ready to prove that $\operatorname{Ext}^{i+1}_{B^e}(R, M) \simeq \operatorname{Ext}^i_{B^e}(M, M) \simeq \operatorname{Ext}^i_{A \otimes_K R^{op}}(M, M)$ for all $i \geq 0$.

PROPOSITION 2.4. *Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. Then $\operatorname{Ext}^{i+1}_{B^e}(R, M) \simeq \operatorname{Ext}^i_{B^e}(M, M) \simeq \operatorname{Ext}^i_{A \otimes_K R^{op}}(M, M)$ for all $i \geq 0$.*

Proof. Applying the functor $\operatorname{Hom}_{B^e}(\ , M)$ to the canonical short exact sequence

$$0 \to M \to P_e \to R \to 0$$

and using the fact that $\operatorname{Hom}_{B^e}(R, M) = \operatorname{Hom}_{B^e}(P_e, M) = 0$, we obtain a long exact sequence of the form

$$0 \to \operatorname{Hom}_{B^e}(M, M) \to \operatorname{Ext}^1_{B \otimes_K R^{op}}(R, M) \to \operatorname{Ext}^2_{B \otimes_K R^{op}}(P_e, M) \to$$

$$\to \operatorname{Ext}^1_{B \otimes_K R^{op}}(M, M) \to \operatorname{Ext}^1_{B \otimes_K R^{op}}(R, M) \to \cdots.$$

We apply now the same functor to the $B \otimes_K R^{op}$-projective resolution of P_e given in Example 2.3. Since $\mathrm{Hom}_{B\otimes_K R^{op}}((P_e \otimes_K R)^t, M) \simeq \mathrm{Hom}_{B^e}((P_e \otimes_K R)^t, M) \simeq (eMe)^t = 0$ for all $t \geq 1$, it follows that $\mathrm{Ext}^i_{B\otimes_K R^{op}}(P_e, M) = 0$ for all $i \geq 1$. ▌

Finally we have the following result which shows the existence of the desired long exact sequence.

PROPOSITION 2.5 [C]. *Let $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. Then there exists a long exact sequence of K-vector spaces connecting the Hochschild cohomology groups of A, B, and R of the form*

$$0 \to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \oplus \mathrm{H}^0(R) \to \mathrm{Hom}_{A\otimes_K R^{op}}(M, M) \to \cdots \to$$
$$\to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \oplus \mathrm{H}^i(R) \to \mathrm{Ext}^i_{A\otimes_K R^{op}}(M, M) \xrightarrow{\Delta_i} \mathrm{H}^{i+1}(B) \cdots. \tag{6}$$

Proof. The result is an immediate consequence of the sequence (5) and Proposition 2.4. ▌

In particular when M is a projective $A \otimes_K R^{op}$-module we obtain the following nice consequence of Proposition 2.5.

COROLLARY 2.6 [C]. *Let B be the triangular matrix algebra $\begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. Suppose that M is a projective $A \otimes_K R^{op}$-module. Then $H^i(B) \simeq \mathrm{H}^i(A) \oplus \mathrm{H}^i(R)$ for all $i \geq 2$.*

We observe here that Proposition 2.5 follows from the work of Cibils [C]. However, his approach is quite different.

The main point of our proof is to consider the cohomology long exact sequence associated to the sequence $0 \to M \xrightarrow{j} B \to A \oplus R \to 0$. The rest of the proof relies on certain Cartan–Eilenberg identifications, the fact that A^e, R^e, and $B \otimes_K R^{op}$ are convex subcategories of the enveloping algebra of $B = \begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$, and on the construction of some projective resolutions of B over B^e, given in Lemma 2.2 and Example 2.3. This approach applies to finite dimensional (basic) algebras and the projective resolutions constructed turn out to be useful also in calculations, as we show in the next section.

In a more general context, Cibils considers in [C] the tensor algebra T corresponding to a ring A and an A-bimodule M, and he studies the Hochschild homology and cohomology of factors T/I of T by positive ideals I. In the particular case when the quiver of M is an arrow the corresponding tensor algebra T is a triangular matrix algebra and he obtains a long exact sequence similar to the one given in Proposition 2.5. However, his result is more general so that it involves the Hochschild cohomology groups of T with coefficients in an arbitrary bimodule X, and he also describes the connecting homomorphism in terms of a cup product associated to the arrow of the quiver of M. The main tool used by Cibils is a projective resolution of T obtained through a certain separable sub-algebra of T.

We end this section by showing the existence of another long exact sequence connecting the groups $\operatorname{Ext}^i_{B\otimes_K R^{op}}(R, P_e)$, of the main result of Section 1, and the groups $\operatorname{Ext}^i_{A\otimes_K R^{op}}(M, M)$, which occur in Proposition 2.5, with the Hochschild cohomology groups of R.

Applying the functor $\operatorname{Hom}_{B\otimes_K R^{op}}(R, \ )$ to the $B \otimes_K R^{op}$-exact sequence

$$0 \to M \to P_e \to R \to 0$$

we obtain a long exact sequence of K-vector spaces of the form

$$\begin{aligned} 0 &\to \operatorname{Hom}_{B\otimes_K R^{op}}(R, P_e) \to \operatorname{Hom}_{B\otimes_K R^{op}}(R, R) \to \operatorname{Ext}^1_{B\otimes_K R^{op}}(R, M) \to \\ &\to \operatorname{Ext}^1_{B\otimes_K R^{op}}(R, P_e) \to \operatorname{Ext}^1_{B\otimes_K R^{op}}(R, R) \to \cdots . \end{aligned} \tag{7}$$

Then, arguments similar to those used throughout the section prove the following result.

THEOREM 2.7. *Let B be the triangular matrix algebra $\begin{pmatrix} R & 0 \\ {}_AM_R & A \end{pmatrix}$. Then there exists a long exact sequence of K-vector spaces of the form*

$$\begin{aligned} 0 &\to \operatorname{Ann}(M_R) \cap Z(R) \to \mathrm{H}^0(R) \to \operatorname{Ext}^1_{A\otimes_K R^{op}}(M, M) \to \\ &\to \operatorname{Ext}^1_{B\otimes_K R^{op}}(R, P_e) \to \mathrm{H}^1(R) \to \cdots \to \operatorname{Ext}^i_{A\otimes_K R^{op}}(M, M) \to \\ &\to \operatorname{Ext}^i_{B\otimes_K R^{op}}(R, P_e) \to \mathrm{H}^i(R) \to \cdots . \end{aligned}$$

3. EXAMPLES

In this section we develop some examples illustrating how to use the results of Sections 1 and 2.

We start by recalling that the Hochschild cohomology groups of the polynomial ring $R = K[x]/\langle x^n \rangle$ are known [W, p. 304].

Remark 3.1. Let $R = K[x]/\langle x^n \rangle$. Then

$$\mathrm{H}^i(R) = \begin{cases} R & \text{if } i = 0 \\ R/\langle x^{n-1} \rangle & \text{if } i > 0. \end{cases}$$

Our first example is the following.

3.2. Let Q be the quiver $\circlearrowleft_{\alpha} 1 \xrightarrow{\beta} 2 \circlearrowleft_{\gamma}$ and B be the K-algebra $KQ/\langle \alpha^2, \gamma^2, \beta\alpha - \gamma\beta \rangle$. Then

$$B \simeq \begin{pmatrix} \operatorname{End} P_1^{op} & 0 \\ \tau_{P_2} P_1 & A \end{pmatrix} \simeq \begin{pmatrix} K[x]/\langle x^2 \rangle & 0 \\ \tau_{P_2} P_1 & K[x]/\langle x^2 \rangle \end{pmatrix}.$$

It is not difficult to see that $\mathrm{H}^0(B) \simeq \mathrm{H}^0(A) \simeq K^2$. We will prove that $\mathrm{H}^i(B) \simeq \mathrm{H}^i(A) = \mathrm{H}^i(K[x]/\langle x^2 \rangle) = K$, for all $i > 0$.

Since $\operatorname{Ann}(M_R) = 0$ we have, using Theorem 1.14, the following long exact sequence of K-vector spaces:

$$\begin{aligned} 0 &\to \mathrm{H}^0(B) \to \mathrm{H}^0(A) \to \operatorname{Ext}^1_{B \otimes_K R^{op}}(R, P_1) \to \mathrm{H}^1(B) \to \mathrm{H}^1(A) \to \\ &\to \cdots \to \operatorname{Ext}^i_{B \otimes_K R^{op}}(R, P_1) \to \mathrm{H}^i(B) \to \mathrm{H}^i(A) \to \\ &\to \operatorname{Ext}^{i+1}_{B \otimes_K R^{op}}(R, P_1) \to \cdots. \end{aligned}$$

We will prove that for all $j \geq 1$, $\operatorname{Ext}^j_{B \otimes_K R^{op}}(R, P_1) = 0$ by constructing a $B \otimes_K R^{op}$-injective resolution of P_1.

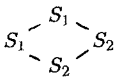
We begin by showing that the map $\phi : P_1 = B . e_1 \to P_{e_1 \otimes 1} . (e_1 \otimes \alpha + \alpha \otimes 1)$ given by $\phi(b . e_1) = (b \otimes 1) . (e_1 \otimes \alpha + \alpha \otimes 1)$ for all $b \in B$, is a $B \otimes_K R^{op}$-isomorphism.

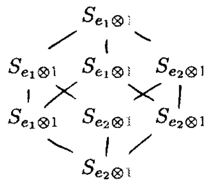
It is easy to verify that ϕ is a $B \otimes_K R^{op}$-morphism. Now, since $\dim_K P_1 = \dim_K P_{e_1 \otimes 1}(e_1 \otimes \alpha + \alpha \otimes 1) = 4$, it is enough to prove that ϕ is a monomorphism. Let $b = k_0 e_1 + k_1 e_2 + k_2 \alpha + k_3 \beta + k_4 \gamma + k_5 \gamma\beta \in B$. Then

$$\begin{aligned} \phi(be_1) = 0 \Leftrightarrow 0 &= (b \otimes 1)(e_1 \otimes \alpha + \alpha \otimes 1) = be_1 \otimes \alpha + b\alpha \otimes 1 \\ &= k_0(e_1 \otimes \alpha) + k_2(\alpha \otimes \alpha) + k_3(\beta \otimes \alpha) \\ &\quad + k_5(\beta\alpha \otimes \alpha) + k_0(\alpha \otimes 1) + k_3(\beta\alpha \otimes 1). \end{aligned}$$

Since $\{e_1 \otimes \alpha, \alpha \otimes \alpha, \beta \otimes \alpha, \beta\alpha \otimes \alpha, \alpha \otimes 1, \beta\alpha \otimes 1\}$ is linearly independent, it follows that $k_0 = k_2 = k_3 = k_5 = 0$. Then $b = k_1 e_2 + k_4 \gamma$ and so $b . e_1 = 0$, which proves that ϕ is a monomorphism.

Now, for constructing the required injective resolution of P_1 over $B \otimes_K R^{op}$ we will use the fact that the indecomposable projective $B \otimes_K R^{op}$ module $P_{e_1 \otimes 1}$ coincides with the indecomposable injective $I_{e_2 \otimes 1}$. Although it is not necessary in the proof, it is useful to keep in mind the structures of the indecomposable projective B-module P_1 and the indecomposable





projective $B \otimes_K R^{op}$-module $P_{e_1 \otimes 1}$ which we indicate in the picture

To prove that $P_{e_1 \otimes 1} = I_{e_2 \otimes 1}$ it is enough to show that $\mathrm{Soc}(P_{e_1 \otimes 1}) = S_{e_2 \otimes 1}$ and $\mathrm{top}(I_{e_2 \otimes 1}) = S_{e_1 \otimes 1}$.

Let δ be the path $(\beta \otimes 1)(\alpha \otimes 1)(e_1 \otimes \alpha)$ of $KQ_{B \otimes_K R^{op}}$ and let $\bar{\delta}$ be the corresponding path in $B \otimes_K R^{op}$. Since $\bar{\delta}$ is the unique nonzero path of maximal length starting in the vertex $e_1 \otimes 1$, it follows that $\mathrm{soc}(P_{e_1 \otimes 1})$ is simple and generated by $\bar{\delta}$.

On the other hand, the path $\delta : (e_1 \otimes 1) \to (e_2 \otimes 1)$ induces by right multiplication a nonzero $B \otimes_K R^{op}$-morphism $(\,.\,\bar{\delta}) : P_{e_2 \otimes 1} \to P_{e_1 \otimes 1}$. It follows from the maximality of the length of δ that $(\,.\,\bar{\delta})(\mathrm{rad}\, P_{e_2 \otimes 1})) = 0$, and so there exists a monomorphism $S_{e_2 \otimes 1} \to P_{e_1 \otimes 1}$. Hence, $S_{e_2 \otimes 1} = \mathrm{soc}(P_{e_1 \otimes 1})$.

Similarly, we can prove that

$$\mathrm{soc}\big(P_0\big(DS_{e_2 \otimes 1}\big)\big) \simeq DS_{e_1 \otimes 1},$$

where $P_0(DS_{e_2 \otimes 1})$ denotes the projective cover of the simple $(B \otimes_K R^{op})^{op}$-module $DS_{e_2 \otimes 1}$.

Then we have that $S_{e_1 \otimes 1} \simeq D\,\mathrm{soc}(P_0(DS_{e_2 \otimes 1})) = \mathrm{top}(DP_0(DS_{e_2 \otimes 1})) = \mathrm{top}\, I_{e_2 \otimes 1}$, which completes the proof of $P_{e_1 \otimes 1} = I_{e_2 \otimes 1}$.

Now, it is not difficult to check that the following sequence is a $B \otimes_K R^{op}$-injective resolution of P_1,

$$0 \to P_1 \xrightarrow{\phi} P_{e_1 \otimes 1} \xrightarrow{d_0} P_{e_1 \otimes 1} \xrightarrow{d_1} \cdots \xrightarrow{d_i} P_{e_1 \otimes 1} \to \cdots,$$

where $d_{2k} = e_1 \otimes \alpha - \alpha \otimes 1$ and $d_{2k+1} = e_1 \otimes \alpha + \alpha \otimes 1$ for all $k \geq 0$.

Since $S_{e_1 \otimes 1} = \mathrm{top}({}_{(B \otimes_K R^{op})} R)$ and $S_{e_1 \otimes 1}$ is not a composition factor of the socle of $P_{e_1 \otimes 1}$, we have that $\mathrm{Hom}_{(B \otimes_K R^{op})}(R, P_{e_1 \otimes 1}) = 0$ and so $\mathrm{Ext}^i_{B \otimes R^{op}}(R, P_1) = 0$ for all $i \geq 1$.

3.3. Let $B = \begin{pmatrix} R & 0 \\ {}_A M_R & A \end{pmatrix}$ with $R = K[x]/\langle x^i \rangle$ and suppose that ${}_A M$ is projective.

In order to compute the groups $\mathrm{H}^i(B)$ using Proposition 2.5, we need to know the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ and the kernel of the morphisms Δ_i.

We indicate a way to compute easily the groups $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ using the projectivity of M as a left A-module.

It is known that the following is a R^e-projective resolution of R (cf. [W, 9.1.4]),

$$\cdots \xrightarrow{d_i} R \otimes_K R^{op} \to \cdots \xrightarrow{d_1} R \otimes_K R^{op} \xrightarrow{d_0} R \otimes_K R^{op} \xrightarrow{m} R \to 0,$$

where $m(r' \otimes r^\circ) = r' . r$, for all $r' \otimes r^\circ \in R \otimes_K R^{op}$ and the morphisms d_{2j}, d_{2j+1} are given by right multiplication by the elements $(1 \otimes x - x \otimes 1)$ and $\sum_{k=0}^{i-1}(x^{i-k-1} \otimes x^k)$, respectively, for all $j \geq 0$. Then we have that

$$\cdots \xrightarrow{d_i} M \otimes_K R^{op} \to \cdots \xrightarrow{d_1} M \otimes_K R^{op} \xrightarrow{d_0} M \otimes_K R^{op} \xrightarrow{m} M \to 0$$

is a projective resolution of M over $B \otimes_K R^{op}$, by Lemma 2.2. Here the morphism m is also the multiplication $(m' \otimes r^\circ) \to m'r$, and for all $j \geq 0$ the morphisms d_{2j} and d_{2j+1} are given by right multiplication by the elements $(1 \otimes x - x \otimes 1)$ and $\sum_{k=0}^{i-1}(x^{i-k-1} \otimes x^k)$, respectively, as above.

Since ${}_A M$ is projective, it is easy to compute $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ using this resolution.

We illustrate this considering the case ${}_A M = A = B.(1-e)$ and $M.x = 0$, which will be needed in the next example.

In this particular case, applying the functor $\mathrm{Hom}_{B \otimes_K R^{op}}(\ , M)$ to the latter long exact sequence and identifying $\mathrm{Hom}_{B \otimes_K R^{op}}(M \otimes_K R^{op}, M)$ with M we obtain that the morphisms $\mathrm{Hom}_{B \otimes_K R^{op}}(d_i, M)$ are given by right multiplication by x and so, $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M) \simeq M$ for all $i \geq 1$.

We can apply the same idea to the general case when ${}_A M$ is projective. As we mentioned at the beginning, to finish the calculation of the groups $\mathrm{H}^i(B)$ using Proposition 2.5, we should know the kernel of the map $\Delta_i : \mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M) \to \mathrm{H}^{i+1}(B)$. Through our identification of $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M)$ with $\mathrm{Ext}^{i+1}_{B \otimes_K R^{op}}(B, M)$ the kernel of Δ_i coincides with the kernel of the morphism $\mathrm{Ext}^{i+1}_{B \otimes_K R^{op}}(B, j)$ of the sequence $(*)$ given at the beginning of Section 2.

3.4. Let Q be the quiver $\underset{\alpha}{\circlearrowleft} 1 \xrightarrow{\beta} 2 \underset{\gamma}{\circlearrowleft}$ and let $B = KQ/\langle \alpha^2, \gamma^2, \beta\alpha \rangle$.

We will prove that $\mathrm{H}^0(B) \simeq K^2$, $\mathrm{H}^1(B) \simeq K^4$, $\mathrm{H}^{2k}(B) \simeq K$, and $\mathrm{H}^{2k+1}(B) \simeq K^3$ for all $k \geq 1$.

The algebra B is isomorphic to the triangular matrix algebra $\left(\begin{smallmatrix} R & 0 \\ M & A \end{smallmatrix}\right)$ where $R = K[\alpha]/\langle \alpha^2 \rangle$, $A = K[\gamma]/\langle \gamma^2 \rangle$, and $M = \tau_{Be_2} Be_1 = K\beta \oplus K\gamma\beta$. Then $\mathrm{Ann}\, M_R = K.\alpha$ and ${}_A M \simeq Be_2$. So, according to Example 3.3, we know that $\mathrm{Ext}^i_{A \otimes_K R^{op}}(M, M) \simeq M \simeq K^2$ for all $i \geq 1$.

On the other hand, both A and R are isomorphic to $K[x]/\langle x^2 \rangle$, so as we observed in Remark 3.1, $\mathrm{H}^0(A) \simeq \mathrm{H}^0(R) \simeq K^2$ and $\mathrm{H}^i(A) \simeq \mathrm{H}^i(R)$

$\simeq K$. Thus the exact sequence of Proposition 2.6 yields an exact sequence of the form

$$0 \to \mathrm{H}^0(B) \to K^4 \to K^2 \xrightarrow{\delta_1} \mathrm{H}^1(B) \to K^2 \to K^2 \xrightarrow{\delta_2} \mathrm{H}^2(B) \to \cdots. \quad (*)$$

So, to compute $\mathrm{H}^i(B)$ we need to know the kernel of δ_i. As we observed in Subsection 3.3, this amounts to knowing $\mathrm{Ker}\,\mathrm{Ext}^i_{B^e}(B, j) : \mathrm{Ext}^i_{B^e}(B, M) \to \mathrm{Ext}^i_{B^e}(B, B)$ where $j : M \to B$ denotes the inclusion map.

With this purpose in mind we construct a projective resolution $\mathscr{C} = (C_i, d_i)$ of B over $B \otimes_K B^{op}$.

We know by [H, 1.5] that the multiplicity of $P_{e_i \otimes e_j^\circ}$ as a summand of C_k is $\dim_K \mathrm{Ext}^k_B(S_j, S_i)$. From this fact we get that $C_0 = P_{e_1 \otimes e_1^\circ} \oplus P_{e_2 \otimes e_2^\circ}$ and $C_i = P_{e_1 \otimes e_1^\circ} \oplus P_{e_2 \otimes e_1^\circ} \oplus P_{e_2 \otimes e_2^\circ}$. We consider next the chain complex

$$\cdots \to C_{i+1} \xrightarrow{d_i} C_i \to \cdots C_1 \xrightarrow{d_0} C_0 \xrightarrow{m} B \to 0$$

where m is the multiplication map,

$$d_0 = \begin{pmatrix} [\alpha \otimes e_1^\circ - e_1 \otimes \alpha & \beta \otimes e_1^\circ & 0 \\ 0 & -e_2 \otimes \beta^\circ & \gamma \otimes e_2^\circ - e_2 \otimes \gamma^\circ \end{pmatrix}$$

and

$$d_i = \begin{pmatrix} \left[\alpha \otimes e_1^\circ + (-1)^{i+1} e_1 \otimes \alpha^\circ\right] & \beta \otimes e_1^\circ & 0 \\ 0 & (-1)^{i+1} e_2 \otimes \alpha^\circ & 0 \\ 0 & 0 & \left[\gamma \otimes e_2^\circ + (-1)^{i+1} e_2 \otimes \gamma^\circ\right] \end{pmatrix}$$

for all $i > 0$.

A straightforward computation shows us that $\dim_K \mathrm{Im}\, d_i = 10$ for all $i \geq 0$. Since $\dim_K C_i = 20$ for all $i > 0$, $\dim_K C_0 = 16$, and $\dim_K B = 6$ it follows that $\mathscr{C} = (C_i, d_i)$ is exact and so, it gives a minimal projective resolution of B over B^e.

We consider now the cochain complex $\mathrm{Hom}_{B^e}(\mathscr{C}, B)$. Since

$$\mathrm{Hom}_{B^e}\left(P_{e_i \otimes e_j^\circ}, B\right) = e_i B e_j \qquad \text{and} \qquad e_2 B e_1 = M$$

this complex is

$$0 \to e_1 B e_1 \oplus e_2 B e_2 \xrightarrow{(d_0, B)} \cdots \to e_1 B e_1 \oplus M \oplus e_2 B e_2$$
$$\xrightarrow{(d_i, B)} e_1 B e_1 \oplus M \oplus e_2 B e_2 \to \cdots,$$

where

$$(d_0, B) = \begin{pmatrix} \alpha \otimes e_1^\circ - e_1 \otimes \alpha^\circ & 0 \\ \beta \otimes e_1^\circ & -e_2 \otimes \beta^0 \\ 0 & \gamma \otimes e_2^\circ - e_2 \otimes \gamma^\circ \end{pmatrix}$$

and

$$(d_i, B) = \begin{pmatrix} \left[\alpha \otimes e_1^\circ + (-1)^{i+1} e_1 \otimes \alpha^\circ\right] & 0 & 0 \\ \beta \in e_1^\circ & (-1)^{i+1} e_2 \otimes \alpha^\circ & 0 \\ 0 & 0 & \left[\gamma \otimes e_2^\circ + (-1)^{i+1} e_2 \otimes \gamma^\circ\right] \end{pmatrix}$$

for all $i \geq 1$,

and

$$(d_i, B) = \begin{pmatrix} \left[\alpha \otimes e_1^\circ + (-1)^{i} e_2 \otimes \alpha^0\right] & 0 & 0 \\ \beta \otimes e_1^\circ & e_2 \otimes \alpha^\circ & 0 \\ 0 & 0 & \left[\gamma \otimes e_2^\circ + (-1)^{i} e_2 \otimes \gamma^\circ\right] \end{pmatrix}$$

for all $i \geq 1$.

On the other hand, $\operatorname{Ext}^i_{B^e}(B, M)$ is given by the homology of the complex

$$\operatorname{Hom}_{B^e}(\mathscr{C}, M): 0 \to M \to \cdots \to M \xrightarrow{0} M \to \cdots \xrightarrow{0} M \cdots.$$

So we have to see which elements of M are boundaries in $\operatorname{Hom}_{B^e}(\mathscr{C}, B)$.

Let $B_i = \operatorname{Im}(d_i, B)$. Since $(d_0, B)(e_1) = \beta$, $(d_0, B)(\gamma) = \gamma\beta$ we have that $M \cap B_0 = M$. It is not difficult to see that $M \cap B_i = K\beta \simeq K$ for all odd $i > 1$, and $M \cap B_i = 0$ if i is even. Thus, $\operatorname{Ker}\operatorname{Hom}_{B^e}(B, j) = M \simeq K^2$, $\operatorname{Ker}\delta_i = \operatorname{Ker}\operatorname{Ext}^i_{B^e}(B, j)$ is isomorphic to K^2 if $i = 1$, to K for all odd $i > 1$ and 0 if $i > 1$ is even.

Therefore the long exact sequence $(*)$ yields short exact sequences

$$0 \to \mathrm{H}^0(B) \to K^4 \to K^2 \to 0$$

$$0 \to K^2 \to \mathrm{H}^1(B) \to K^2 \to 0$$

$$0 \to \mathrm{H}^{2k}(B) \to K^2 \to K \to 0$$

$$0 \to K \to \mathrm{H}^{2k+1}(B) \to K^2 \to 0$$

for all $k \geq 1$.

Hence, $\mathrm{H}^0(B) \simeq K^2$, $\mathrm{H}^1(B) \simeq K^4$, $\mathrm{H}^{2k}(B) \simeq K$, and $\mathrm{H}^{2k+1}(B) \simeq K^3$ for all $k \geq 1$.

3.5. We recall that a triangular matrix algebra $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$ is said to be a local extension of A by the A-R-bimodule M if R is a local algebra.

On the other hand, an algebra Λ is called an IIP-algebra if all the idempotent ideals of Λ are projective left Λ-modules.

The results obtained in Section 1 are useful to study the Hochschild cohomology of IIP-algebras, since they can be obtained from a local algebra by successive local extensions by appropriate bimodules. And, moreover, these bimodules are projective left modules [CMaMP].

The algebras given in Subsections 3.2 and 3.4 are, in fact, IIP-algebras. Another example of an IIP-algebra is an algebra $B = KQ/I$ where Q is a quiver with loops, without other oriented cycles, and I is generated by relations involving only the loops of Q.

Let $B = KQ/I$ be such an IIP-algebra and assume that $B = \left(\begin{smallmatrix} R & 0 \\ {}_AM_R & A \end{smallmatrix}\right)$. Then M is a projective $A \otimes_K R^{op}$-module. In fact, let $\beta_1, \dots, \beta_r$ be all the arrows starting in the new vertex e of Q and $P = \coprod_{i=1}^r P_{e(\beta_i)} = \coprod_{i=1}^r A \,.\, e(\beta_i)$. We shall prove briefly that $M \simeq P \otimes_K R$ as A-R-bimodules.

Each arrow β_i induces an A-morphism $m_i : P_{e(\beta_i)} \to M$ by right multiplication. Let $m = [m_1 \dots m_r] : P \to M$. Clearly, m is an A-morphism. We define now $f : P \otimes_K R \to M$ by $f(p \otimes r) = m(p)r$.

Then f is an A-R-epimorphism by definition. Since the relations in B involve only loops it follows that f is also a monomorphism.

As we said before, B can be obtained from a local ring R_0 by successive local extensions $A_{i+1} = \left(\begin{smallmatrix} R_{i+1} & 0 \\ M_i & A_i \end{smallmatrix}\right)$, $i = 0, \dots, n-1$, $A_0 = R_0$, and $A_n = B$. Moreover, at each step we have that M_i is a projective left $A_i \otimes_K R_{i+1}^{op}$-module.

We see now that the groups $\mathrm{H}^j(B)$ can be computed in terms of the groups $\mathrm{H}^j(R_0), \mathrm{H}^j(R_1), \dots, \mathrm{H}^j(R_n)$ for all $j \geq 2$, and for $\mathrm{H}^1(B)$ we are able to give its dimension over K.

Suppose $M_i = (A_i \otimes_K R_{i+1}^{op}) \,.\, f_i$ for some idempotent element f_i of $A_i \otimes_K R_{i+1}^{op}$. Then it follows from Corollary 2.6 that

$$\mathrm{H}^j(A_{i+1}) \simeq \mathrm{H}^j(R_{i+1}) \oplus \mathrm{H}^j(A_i)$$

for all $j \geq 2$ and $i = 0, \dots, n-1$. Hence,

$$\mathrm{H}^j(B) \simeq \bigoplus_{i=0}^{n} \mathrm{H}^j(R_i)$$

for all $j \geq 2$.

Concerning $\mathrm{H}^1(B)$, for each $i = 0, \ldots, n-1$ we have the exact sequence

$$0 \to \mathrm{H}^0(A_{i+1}) \to \mathrm{H}^0(A_i) \oplus \mathrm{H}^0(R_{i+1}) \to \mathrm{Hom}_{A_i \otimes_K R_{i+1}^{op}}(M_i, M_i) \to$$
$$\to \mathrm{H}^1(A_{i+1}) \to \mathrm{H}^1(A_i) \oplus \mathrm{H}^1(R_{i+1}) \to 0.$$

Since $\mathrm{Hom}_{A_i \otimes_K R_{i+1}^{op}}(M_i, M_i) \simeq f_i M_i$ for all $i = 0, \ldots, n-1$ we have that

$$\dim_K \mathrm{H}^1(B) = \dim_K Z(B) - \sum_{i=0}^{n} \dim_K Z(R_i) + \sum_{i=0}^{n-1} \dim_K (f_i M_i) + \sum_{i=0}^{n} \dim_K \mathrm{H}^1(R_i).$$

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