Lagrangian Reduction by Stages

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Fall, 1996; this version, February 18, 2000

Abstract

This paper studies the geometry of the reduction of Lagrangian systems with symmetry in a way that allows the reduction process to be repeated; that is, it develops a context for Lagrangian reduction by stages. The Lagrangian reduction procedure focusses on the geometry of variational structures and how to reduce them. This philosophy is well known for the classical cases, such as those of Routh (where the symmetry group is Abelian) and the Euler–Poincaré equations (for the case in which the configuration space is a Lie group).

The context established for this theory is a Lagrangian analogue of the bundle picture on the Hamiltonian side. In this picture, a category is developed that includes, as a special case, the realization of the quotient of a tangent bundle as the Whitney sum of the tangent of the quotient bundle with the associated adjoint bundle. The elements of

this new category, called the Lagrange-Poincaré category, have enough geometric structure so that the category is stable under the procedure of Lagrangian reduction. Thus, reduction may be repeated, giving the desired context for reduction by stages.

We also give an intrinsic and geometric way of writing the reduced equations, called the Lagrange-Poincaré equations, using covariant derivatives and connections. In addition, the context includes the interpretation of cocycles as curvatures of connections and is general enough to include interesting situations involving both semidirect products and central extensions. Examples are given to illustrate the general theory.

Contents

1 Introduction

Reduction theory for mechanical systems with symmetry has its origins in the classical works of Euler, Lagrange, Hamilton, Routh, Jacobi, Liouville and Poincaré, who studied the extent to which one can reduce the dimension of the phase space of the system by making use of any available symmetries and associated conservation laws. Corresponding to the main two views of mechanics, namely Hamiltonian and Lagrangian mechanics, one can also adopt two views of reduction theory.

In symplectic and Poisson reduction, which are now well developed and much studied subjects, one focusses on how to pass the symplectic two form and the Poisson bracket structure as well as any associated Hamiltonian dynamics to a quotient space for the action of a symmetry group (see, for example, Meyer [1973], Marsden and Weinstein [1974], Marsden and Ratiu [1986] and the expositions in Abraham and Marsden [1978], Arnold [1989], Lieberman and Marle [1987] and Marsden [1992]).

In Lagrangian reduction theory, which proceeds in a logically independent way, one emphasizes how the variational structure passes to a quotient space (see, for example, Cendra and Marsden [1987], Cendra, Ibort and Marsden [1987], Marsden and Scheurle [1993a,b], Bloch, Krishnaprasad, Marsden, and Ratiu [1996], Cendra, Holm, Marsden and Ratiu [1998], Holm, Marsden and Ratiu [1998], and Jalnapurkar and Marsden [1999]). Of course, the two methodologies are related by the Legendre transform, although not always in a straightforward way.

The main purpose of this work is to further the development of Lagrangian reduction theory. There are several aspects to this program. First, we provide a context that allows for repeated Lagrangian reduction by the action of a symmetry group. Second, we provide the geometry that is useful for the

expression of the reduced equations, called the *Lagrange-Poincaré equations.* Further details concerning the main results of this work are given shortly.

1.1 Background

As we have mentioned, in the last few years there has been considerable activity in the area of Lagrangian reduction in which one focusses on the reduction of variational principles. We shall review the background for this theory briefly, starting with the best known classical results.

Classical Cases. Several classical instances of Lagrangian reduction are well known, such as Routh reduction which was developed by Routh [1877] in connection with his studies of the stability of relative equilibria. Routh began the development of what we would call today Lagrangian reduction for Abelian groups. One thinks of this case as covering that for systems with cyclic variables.

Another fundamental case is that of Euler–Poincaré reduction, which occurs for the case in which the configuration is a Lie group. One thinks of this case as primarily intended for systems governed by Euler equations, such as those of a rigid body and a fluid. This case has its origins in the work of Lagrange [1788] and Poincaré [1901]. Both of them clearly had some idea of the reduction process.

In these classical works, many important ideas were developed. However, for both of these cases, some of the clarifications and generalizations are remarkably recent. For example, only in Bretherton [1970] was the reduced variational principle established for the fluid equations, but this was done by ad hoc rather than general methods. Both the intrinsic (coordinate free) formulation of Routh reduction as well as the general formulation of Euler–Poincaré reduction in terms of variational principles were given in Marsden and Scheurle [1993a,b]). The Euler–Poincaré case was further developed in Bloch, Krishnaprasad, Marsden and Ratiu [1996]. An exposition of Lagrangian reduction for both the Routh and Euler–Poincaré cases can be found in Marsden and Ratiu [1994].

Semidirect Product Theory. Another well developed subject in Hamiltonian reduction theory is that of semidirect product theory. This theory has its origins in the work of Guillemin and Sternberg [1980] (see also Guillemin and Sternberg [1984]) and Marsden, Ratiu and Weinstein [1984a,b]. This theory has many interesting applications, such as to the heavy top, MHD, and the dynamics of underwater vehicles (Leonard and Marsden [1997]). This semidirect product theory was a direct precursor to the development of symplectic reduction by stages (Marsden, Misiolek, Perlmutter, and Ratiu [1998a,b]).

Lagrangian analogues of the semidirect product theory were developed in Holm, Ratiu and Marsden [1998] with applications to many fluid mechanical problems of interest. The point of view was to extend the Euler–Poincaré theory to the case of systems such as the heavy top and compressible flows in which there are advected parameters. This methodology was applied to the case of the Maxwell-Vlasov equations by Cendra, Holm, Hoyle and Marsden [1998] and Cendra, Holm, Marsden and Ratiu [1998] showed how it fits into the general framework of Lagrangian reduction.

Symplectic versus Poisson Reduction. We should emphasize that in the framework of the theory of Poisson manifolds, Poisson reduction by stages is quite simple, while in the framework of symplectic manifolds, symplectic reduction by stages is more sophisticated. On the other hand, if one wants to study the reduction of cotangent bundles and to retain as much of this structure as possible, then even the Poisson point of view is quite nontrivial.

The Lagrangian analogue of symplectic reduction is *nonabelian Routh reduction* (Marsden and Scheurle [1993a,b], Jalnapurkar and Marsden [1999] and Marsden, Ratiu and Scheurle [2000]) and its full development in the context of reduction by stages is the subject of a future publication. The present work represents a Lagrangian analogue of the Poisson version of reduction by stages but keeping the structure of the tangent bundle as much as possible. One of the things that makes the Lagrangian side interesting has been the lack of a general category that is the Lagrangian analogue of Poisson manifolds. Such a category, that of Lagrange-Poincaré bundles is proposed here, with the tangent bundle of a configuration manifold and a Lie algebra as its most basic examples. We also develop the Lagrangian analogue of reduction for central extensions and, as in the case of symplectic reduction by stages, cocycles and curvatures enter in this context in a natural way.

Nonholonomic Mechanics. The ideas of Lagrangian reduction have had a significant impact on the theory of nonholonomic systems, as in Bloch, Krishnaprasad, Marsden, and Murray [1996] and Koon and Marsden [1997b,c, 1998]. These references also develop Lagrangian reduction methods in the context of nonholonomic mechanics with symmetry (such as systems with rolling constraints). These methods have been quite useful in many control problems and in robotics. The techniques of the present paper can be used to give an intrinsic geometric meaning to the reduction of the Lagrange-d'Alembert equations of nonholonomic mechanics. This is the subject of another work, Cendra, Marsden and Ratiu [2000].

Optimal Control. Lagrangian reduction theory has also had a significant impact on optimal control theory. See Vershik and Gershkovich [1987, 1994],

Bloch and Crouch [1994], Montgomery [1990, 1993], Koon and Marsden [1997a] and references therein.

1.2 The Main Results of This Paper.

We now give a few more details concerning the main results. The first of these, given in §5, develops the theory of *Lagrange-Poincaré bundles*, which enable one to perform Lagrangian reduction in stages. Lagrange-Poincaré bundles may be regarded as the Lagrangian analogue of a Poisson manifold in symplectic geometry. Lagrange-Poincaré bundles include, of course, the case of reduced tangent bundles TQ. This in turn, includes important examples such as Euler-Poincaré reduction for the special case $Q = G$, a Lie group. Euler-Poincaré reduction is now a textbook topic that can be found in Marsden and Ratiu [1994]. We show that when a general tangent bundle is reduced by a group action, one ends up in the category of Lagrange-Poincaré bundles.

We mention that Lagrange-Poincaré bundles are in particular, Lie algebroids but carry additional structure. We will not use any theory of groupoids or algebroids in this work, but we will comment on the literature in the body of the work.

The *Lagrange-Poincaré equations* are expressed using connections and curvature. These equations are obtained using the idea of reducing variational principles. The Lagrange-Poincaré category is stable under reduction and the structure carried by Lagrange-Poincaré bundles is exactly what is needed to write the Lagrange-Poincaré equations, in a covariant form.

In §5.3 and §5.4 we show that if the symmetry group has a normal subgroup (*i.e.,* one has a *group extension*), then reducing by the whole group is shown to be isomorphic to what one gets by reducing in stages, first by the normal subgroup followed by reduction by the quotient group. This result is a Lagrangian analogue of doing Poisson reduction by stages, but keeping track of the local structure of Poisson manifolds, as in the Lie-Weinstein theorem (see Weinstein [1983]).

The theory we establish may be viewed as the Lagrangian analogue of the bundle picture on the Hamiltonian side developed by Montgomery [1986] and Montgomery, Marsden, and Ratiu [1984]. This bundle picture was in turn influenced by work on Wong's equations for a particle in a Yang–Mills field, as studied by Sternberg [1977], Weinstein [1978], Montgomery [1984], and Koon and Marsden [1997a]. As we shall see in §3.3, this theory has a very beautiful Lagrangian analogue. In §6 we give a number of additional examples. In future works we plan to establish additional links with the Hamiltonian side.

1.3 Future Work and Related Issues.

Our theory naturally suggests a number of additional things that warrant further investigation.

Geometric Phases. The development of the theory of geometric phases in the Lagrangian context is natural to develop. The paper of Marsden, Ratiu and Scheurle [2000] gives results along these lines in the context of Routh reduction. The development of geometric phases by stages would be of interest. In fact, the Lagrangian setting provides natural connections and also a natural setting for averaging which is one of the basic ingredients in geometric phases.

Nonholonomic Mechanics. As mentioned above, the work of Cendra, Marsden and Ratiu [2000] extends the notion of Lagrange-Poincaré bundles to those of Lagrange-d'Alembert-Poincaré bundles that are appropriate for nonholonomic mechanics. This extension may be regarded as the Lagrangian analogue of the notion of an almost Poisson manifold (in which Jacobi's identity can fail), as in Koon and Marsden [1998] and Cannas da Silva and Weinstein [1999]. Furthering the links with almost Poisson manifolds and also developing a nonholonomic reduction by stages theory would of course be of interest.

Further Relations with the Hamiltonian Side. It would also be interesting to investigate the precise relationship of the work here with the Hamiltonian reduction by stages theory in more detail, in particular, the relation with symplectic reduction by stages applied to cotangent bundles. This requires the nonabelian Routh reduction analogue of the work here, namely that of Marsden, Ratiu and Scheurle [2000], which generalizes the work of Marsden and Scheurle [1993a] and Jalnapurkar and Marsden [1999].

Relations with Poisson Geometry. This paper develops a Lagrangian category that locally has the dual of the local structure for Poisson manifolds under the hypotheses of the Lie-Weinstein theorem, namely that a Poisson manifold is locally the product of a symplectic manifold and the dual of a Lie algebra. Our bundles actually have more structure than this, which is important for carrying out the reduction, namely we also carry along a connection and a two-form that helps keep track of curvature (or magnetic terms). This structure is also very useful for writing covariant versions of the reduced equations, that is, the Lagrange-Poincaré equations.

Variational Integrators and Discrete Reduction. As Weinstein [1996] points out, there is a more general context for Lagrangian mechanics that also includes discrete mechanics in the sense of Veselov [1988, 1990]. Our category

of Lagrange-Poincaré bundles does not include this literally, but still, there is a well defined discrete analogue of these bundles. This picture is useful in the understanding of the reduced Hamilton-Jacobi equation (see Ge and Marsden [1988]).

One of the interesting developments in symplectic integration algorithms has been the progress made in variational integrators. These are based on direct discretizations of Hamilton's principle following some of the ideas of Veselov [1988]. See, for example, Wendlandt and Marsden [1997], Marsden, Patrick and Shkoller [1998] and Kane, Marsden, Ortiz and West [1999]. There is a discrete reduction theory for these as well that is still under development. See, for example, Marsden, Pekarsky and Shkoller [1999].

Infinite Dimensional Examples. In this paper we will be dealing with Lagrangian reduction theory in the context of *finite dimensional manifolds*. Of course the theory formally applies to many interesting infinite dimensional examples. In the infinite dimensional context, many of the expressions that appear here as pure partial derivatives must be written in the notation of functional derivatives (see Marsden and Ratiu [1994] for some of the basic examples, an explanation of the function derivative notation and additional references to the literature).

Multisymplectic Context. Another area of much current interest is that of multisymplectic geometry. See, for example, Marsden and Shkoller [1999] and Marsden, Patrick and Shkoller [1998]. This theory has both a Lagrangian and a Hamiltonian view and it has allowed, for example, a development of the Moser-Veselov theory to the context of PDE's. Reduction theory in this context is in its infancy (see, for example, Montgomery, Morrison and Thompson [1986] and Castrillón, Ratiu and Shkoller [1999]). Obviously it would be of interest to develop such a theory from the Lagrangian reduction point of view.

An exciting possible application where reduction by stages is involved is that of various complex fluids, such as liquid crystals, where there is a group of particle relabeling symmetries, as in fluids and plasmas as well as an internal order parameter group. See Holm [2000].

2 Preliminary Constructions

In this section we recall some results about bundles and connections that we will use later. As a general reference, see Kobayashi and Nomizu [1963]; however, the reader should be warned that various conventions and notations differ from this reference and we shall point these out as we proceed. In general, we follow the conventions of Abraham and Marsden [1978] and Abraham, Marsden

and Ratiu [1988]. We provide intrinsic proofs with a view to infinite dimensional generalizations and also because of the insight they provide. Coordinate expressions are important in many applications, and they can be readily provided (some of these may be found in Marsden and Scheurle [1993ab] and Bloch, Krishnaprasad, Marsden and Murray [1996]).

2.1 Notation and general assumptions.

Maps. If $f : A \to B$ is a map between sets and $C \subset A$, we shall often write $f : C \to B$ instead of $f|C : C \to B$, for short.

Manifolds. Unless otherwise noted, for simplicity, manifolds are assumed to be C^∞, as are maps between them. While the manifolds will be assumed to be finite dimensional, many results can be easily generalized for infinite dimensions in a straightforward manner.

For a manifold Q, we let TQ be the tangent bundle of Q. An element of T_qQ will be denoted v_q, $u_q, \ldots$ or (using a standard abuse of notation), by $(q, \dot{q})$ or simply $\dot{q}$.

Actions. Unless otherwise noted, an action $\rho : G \times Q \to Q$ of a Lie group G on a manifold Q is assumed to satisfy the additional condition that, with this action, the manifold Q becomes a principal bundle with structure group G, say $\pi_G(Q) : Q \to Q/G$. In other words, we assume that Q/G is a manifold, that $\pi_G(Q)$ is a submersion and that the action is free. As is well known (see, for example, Abraham and Marsden [1978] for the proof), if the action is both free and proper, then this holds.

We will often use the equivalent notations $\pi_G(Q)(q) = [q]_G$ for the equivalence class of $q \in Q$. We will work with *left actions* in this paper unless explicitly noted otherwise; however, generalizations for right actions are straightforward.

The assumption of freeness of the action is of course a strong one and it corresponds, in the Hamiltonian case, to eliminating the case of *singular reduction* (see Sjamaar and Lerman [1991], Bates and Lerman [1997], and Ortega and Ratiu [1997, 1999]). Similarly, on the Lagrangian side, this paper does not address the case of singular reduction. Of course such questions are very interesting, but there is plenty to do even omitting that topic.

We will often use the equivalent notations $\rho(g, q) \equiv \rho_g(q) \equiv \rho_q(g) \equiv g \cdot q \equiv gq$ for the action of the group element g on the point $q \in Q$. However, the *concatenation notation* gq will be used most commonly. The tangent lift of this action will be denoted gv_q, where $v_q \in T_qQ$. For any element ξ in the Lie algebra $\mathfrak{g}$ of G, the infinitesimal generator at $q \in Q$ is denoted $\xi q \equiv \xi_Q(q)$ and

it is defined as usual by

$$\xi q := \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)q.$$

Bundles. If $\pi : P \to Q$ is a fiber bundle and $q \in Q$, the fiber $\pi^{-1}(q)$ at q is sometimes denoted P_q. If $\tau_i : V_i \to Q_i$ are vector bundles for $i = 1, 2$ and $f : V_1 \to V_2$ is a vector bundle map, the induced map on the zero-sections, which are usually identified with Q_i, $i = 1, 2$, is denoted $f_0 : Q_1 \to Q_2$, or sometimes, by a slight abuse of notation, simply f.

Given two bundles $\pi_i : P_i \to Q$, $i = 1, 2$, the fiber product is the bundle $\pi_1 \times_Q \pi_2 : P_1 \times_Q P_2 \to Q$ where $P_1 \times_Q P_2$ is the set of all elements $(p_1, p_2) \in P_1 \times P_2$ such that $\pi_1(p_1) = \pi_2(p_2)$ and the projection $\pi_1 \times_Q \pi_2 : P_1 \times_Q P_2 \to Q$ is naturally defined by $\pi_1 \times_Q \pi_2(p_1, p_2) = \pi_1(p_1) = \pi_2(p_2)$. The fiber is given by $(\pi_1 \times_Q \pi_2)^{-1}(q) = \pi_1^{-1}(q) \times \pi_2^{-1}(q)$.

The Whitney sum of two vector bundles $\tau_i : V_i \to Q$, $i = 1, 2$, over the same base is their fiber product. It is a vector bundle over Q and is denoted $V_1 \oplus V_2$. This bundle is obtained by taking the fiberwise direct sum of the fibers of V_1 and V_2.

Sometimes, for purposes of uniformizing notation, we will consider a manifold Q as being identified with the vector bundle over Q whose fibers have dimension 0. Therefore if $\tau : V \to Q$ is a vector bundle, $Q \oplus V$ is simply V, and an element $q \oplus v$ of $Q \oplus V$ satisfies $\tau v = q$.

Connections. The word *connection* will be used in two different but standard senses in this paper. Sometimes it will mean a *principal connection* on a principal bundle and sometimes will mean a *connection on a vector bundle*, usually denoted ∇ with the addition, sometimes, of some indexes to clarify the spaces on which the connection is defined. In any case, the context will always make it clear the sense in which we are using the word. We will recall some of the key notions and conventions used in the following paragraphs.

2.2 Connections on Principal Bundles.

Horizontal and vertical spaces. Let $\pi : Q \to Q/G$ be a left principal bundle, where π is the canonical projection. Recall that a (principal) connection A on Q is a Lie algebra valued one form $A : TQ \to \mathfrak{g}$ with the properties

i $A(\xi q) = \xi$ for all $\xi \in \mathfrak{g}$; that is, A takes infinitesimal generators of a given Lie algebra element to that element, and

ii $A(T_q\rho_g \cdot v) = \mathrm{Ad}_g(A(v))$, where Ad_g denotes the adjoint action of G on $\mathfrak{g}$.

The restriction of a connection to the tangent space T_qQ is denoted A_q. Recall that connections may be characterized by giving their vertical and horizontal spaces defined at $q \in Q$ by

$$\mathrm{Ver}_q = \mathrm{Ker}\, T_q\pi, \qquad \mathrm{Hor}_q = \mathrm{Ker}\, A_q.$$

Thus, $\mathrm{Ver}(TQ) = \cup_{q\in Q} \mathrm{Ver}_q$ is the subbundle of vectors tangent to the group orbits. The vertical and horizontal components of a vector v_q will be denoted $\mathrm{Ver}(v_q)$ and $\mathrm{Hor}(v_q)$ respectively. By definition,

$$\mathrm{Ver}(v_q) = A(v_q)q \quad \text{and} \quad \mathrm{Hor}(v_q) = v_q - A(v_q)q.$$

This provides a decomposition $TQ = \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$ where $\mathrm{Hor}(TQ) = \cup_{q\in Q} \mathrm{Hor}_q$ and $\mathrm{Ver}(TQ)$ are the horizontal and vertical subbundles of TQ, which are invariant under the action of G. A vector is called *horizontal* if its vertical component is zero; *i.e.,* if $A(v_q) = 0$ and it is called *vertical* if its horizontal component is zero; *i.e.,* if $T_q\pi(v_q) = 0$. Note that $T_q\pi : \mathrm{Hor}_q \to T_{\pi(q)}(Q/G)$ is an isomorphism.

Curvature. The *curvature* of A will be denoted B^A or simply B. By definition, it is the Lie algebra valued two form on Q defined by

$$B(u_q, v_q) = \mathbf{d}A(\mathrm{Hor}_q(u_q), \mathrm{Hor}_q(v_q)),$$

where $\mathbf{d}$ denotes the exterior derivative.

The reader should be aware that, although the notion of the curvature of a given connection on a principal bundle is a well established and an essentially unique concept, there are many conventions related to various sign conventions and definitions of the wedge product and the exterior derivative. The one adopted here for left actions is consistent with the one given in Abraham and Marsden [1978].

Cartan Structure Equations. The Cartan structure equations for the curvature state that for vector fields u, v (not necessarily horizontal) on Q we have

$$B(u, v) = \mathbf{d}A(u, v) - [A(u), A(v)], \tag{2.2.1}$$

where the bracket on the right hand side is the Lie bracket in $\mathfrak{g}$. We write this equation for short as

$$B = \mathbf{d}A - [A, A].$$

Horizontal lifts. Given a vector $X \in T_x(Q/G)$, and $q \in \pi^{-1}(x)$, the ***horizontal lift*** X_q^h of X at q is the unique horizontal vector in T_qQ that projects via $T\pi$ to the vector $X(x)$; that is, $X_q^h = (T_q\pi)^{-1}(X)$. We denote by X^h the vector field along $\pi^{-1}(x)$ formed by all horizontal lifts of X at points of $\pi^{-1}(x)$.

For any curve $x(t)$ in Q/G, where $t \in [a, b]$, the family of horizontal lifts is denoted x^h. The definition is the following. For any point $q_0 \in \pi^{-1}(x_0)$, where $x_0 = x(t_0)$, for some $t_0 \in [a, b]$, the horizontal lift of $x(t)$, which at $t = t_0$ coincides with q_0, is uniquely determined by requiring its tangent to be a horizontal vector. This curve is denoted $x_{q_0}^h$ and is defined on $[a, b]$.

Consider a curve $q(t)$, where $t \in [a, b]$, and choose $t_0 \in [a, b]$. Then there is a unique horizontal curve $q_h(t)$ such that $q_h(t_0) = q(t_0)$ and $\pi(q_h(t)) = \pi(q(t))$ for all $t \in [a, b]$. Therefore, we can define a curve $g_q(t), t \in [a, b]$ in G by the decomposition

$$q(t) = g_q(t)q_h(t) \tag{2.2.2}$$

for all $t \in [a, b]$. Evidently $g_q(t_0)$ is the identity. Also, notice that if $x(t) = \pi(q(t))$ and $q_0 = q(t_0)$ then $q_h(t) = x_{q_0}^h(t)$.

Lemma 2.2.1 *For any curve $q(t), t \in [a, b]$ in Q we have*

$$A(q, \dot{q}) = \dot{g}_q g_q^{-1}.$$

Proof. We start with the equality $g_q(t)q_h(t) = q(t)$. Differentiating this with respect to t gives

$$\dot{g}_q(t)q_h(t) + g_q(t)\dot{q}_h(t) = \dot{q}(t).$$

Here, the notation is interpreted in an obvious way; for example, for $u_g \in T_gG$ and $q \in Q$, $u_g q$ means the derivative of the orbit map $g \mapsto gq$ in the direction of u_g to give an element of $T_{gq}Q$.

By definition of a horizontal vector, $A(g_q(t)\dot{q}_h(t)) = 0$. Recall also that the connection reproduces the Lie algebra element on infinitesimal generators (*i.e.*, $A(\xi q) = \xi$ for $\xi \in \mathfrak{g}$ and $q \in Q$). In particular, for $\xi = \dot{g}_q g_q^{-1}$ and $q(t) = g_q(t)q_h(t)$ we get

$$A(\dot{g}_q(t)q_h(t)) = A(\dot{g}_q(t)g_q^{-1}(t)g_q(t)q_h(t)) = \dot{g}_q g_q^{-1},$$

from which the result follows. ■

2.3 Associated Bundles.

Besides the principal bundle $\pi : Q \to Q/G$ discussed above, consider a left action $\rho : G \times M \to M$ of the Lie group G on a manifold M. The ***associated bundle*** with standard fiber M is, by definition,

$$Q \times_G M = (Q \times M)/G,$$

where the action of G on $Q \times M$ is given by $g(q, m) = (gq, gm)$. The class (or orbit) of (q, m) is denoted $[q, m]_G$ or simply $[q, m]$. The ***projection*** $\pi_M : Q \times_G M \to Q/G$ is defined by $\pi_M([q, m]_G) = \pi(q)$ and it is easy to check that it is well defined and is a surjective submersion.

Parallel transport. Let $[q_0, m_0]_G \in Q \times_G M$ and let $x_0 = \pi(q_0) \in Q/G$. Let $x(t), t \in [a, b]$ be a curve in Q/G and let $t_0 \in [a, b]$ be such that $x(t_0) = x_0$. The ***parallel transport*** of this element $[q_0, m_0]_G$ along the curve $x(t)$ is defined to be the curve

$$[q, m]_G(t) = [x^h_{q_0}(t), m_0]_G.$$

Let us check that this curve is well defined. In fact, for any $g \in G$, the equivariance property of the connection gives $x^h_{gq_0}(t) = gx^h_{q_0}(t)$ for all t and hence

$$[x^h_{gq_0}(t), gm_0]_G = [gx^h_{q_0}(t), gm_0]_G = [x^h_{q_0}(t), m_0]_G$$

for all t.

Consider a curve $x(t), t \in [a, b]$ in Q/G, as before. For $t, t + s \in [a, b]$, we adopt the notation

$$\tau^t_{t+s} : \pi_M^{-1}(x(t)) \to \pi_M^{-1}(x(t + s))$$

for the parallel transport map along the curve $x(s)$ of any point

$$[q(t), m(t)]_G \in \pi_M^{-1}(x(t))$$

to the corresponding point

$$\tau^t_{t+s}[q(t), m(t)]_G \in \pi_M^{-1}(x(t + s)).$$

Thus,

$$\tau^t_{t+s}[q(t), m(t)]_G = [x^h_{q(t)}(t + s), m(t)]_G.$$

Associated vector bundles. Now we concentrate on the particular case when M is a vector space and ρ is a *linear representation.* (These will be the only associated bundles needed in the present work.) In this case, the associated bundle with standard fiber M is a vector bundle in a natural way. We now recall what the vector bundle structure is. If $[q,m]_G, [q,m_1]_G, [q,m_2]_G \in \pi_M^{-1}([q]_G)$, then the vector space structure in this fiber is defined by

$$a[q,m]_G = [q,am]_G \quad \text{and} \quad [q,m_1]_G + [q,m_2]_G = [q,m_1+m_2]_G.$$

We shall sometimes use the notation $\rho'(\xi)$ for the second component of the infinitesimal generator of an element $\xi \in \mathfrak{g}$, that is, $\xi m = (m, \rho'(\xi)m)$. Here we are using the identification $TM = M \times M$, appropriate for vector spaces. Thus, we are thinking of the infinitesimal generator as a map $\rho' : \mathfrak{g} \to \mathrm{End}(M)$ (the linear vector fields on M are identified with the space of linear maps of M to itself). Thus, we have a linear representation of the Lie algebra $\mathfrak{g}$ on the vector space M.

Definition 2.3.1 *Let $[q(t), m(t)]_G$, $t \in [a,b]$ be a curve in $Q \times_G M$, denote by*

$$x(t) = \pi_M\left([q(t), m(t)]_G\right) = \pi(q(t))$$

its projection on the base Q/G, and let, as above, τ_{t+s}^t, where $t, t+s \in [a,b]$, denote parallel transport along $x(t)$ from time t to time $t+s$. The ***covariant derivative*** *of $[q(t), m(t)]_G$ along $x(t)$ is defined as follows*

$$\frac{D[q(t), m(t)]_G}{Dt} = \lim_{s \to 0} \frac{\tau_t^{t+s}\left([q(t+s), m(t+s)]_G\right) - [q(t), m(t)]_G}{s}.$$

Thus, the covariant derivative of $[q(t), m(t)]_G$ is an element of $\pi_M^{-1}(x(t))$.

Notice that if $[q(t), m(t)]_G$ is a vertical curve, then its base point is constant; that is, for each $t \in [a,b]$,

$$x(t+s) = \pi_M\left([q(t+s), m(t+s)]_G\right) = x(t),$$

so that $x^h_{q(t)}(t+s) = q(t)$ for all s. Therefore,

$$\tau_t^{t+s}[q(t+s), m(t+s)]_G = [x^h_{q(t+s)}(t), m(t+s)]_G = [q(t), m(t+s)]_G$$

and so we get the well known fact that *the covariant derivative of a vertical curve in the associated bundle is just the fiber derivative.* That is,

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), m'(t)]_G,$$

where $m'(t)$ is the time derivative of m.

Affine Connections. The notion of covariant derivative can be defined from a different, more axiomatic, point of view, which will be useful later in this paper. By definition (see Kobayashi and Nomizu [1963]), a ***connection*** (sometimes called an ***affine connection*** to distinguish it from a principal connection) ∇ on a vector bundle $\tau : V \to Q$ is a map $\nabla : \mathfrak{X}^\infty(Q) \times \Gamma(V) \to \Gamma(V)$, say $(X, v) \mapsto \nabla_X v$, having the following two properties:

1. First, we require that

$$\nabla_{f_1 X_1 + f_2 X_2} v = f_1 \nabla_{X_1} v + f_2 \nabla_{X_2} v$$

   for all $X_i \in \mathfrak{X}^\infty(Q)$ (the space of smooth vector fields on Q), $f_i \in C^\infty(Q)$ (the space of smooth real valued functions on Q), $i = 1, 2$, and all $v \in \Gamma(V)$ (the space of smooth sections of the vector bundle V),

2. and secondly,

$$\nabla_X (f_1 v_1 + f_2 v_2) = X[f_1] v_1 + f_1 \nabla_X v_1 + X[f_2] v_2 + f_2 \nabla_X v_2$$

   for all $X \in \mathfrak{X}^\infty(Q)$, $f_i \in C^\infty(Q)$, and $v_i \in \Gamma(V)$, $i = 1, 2$.

Here, $X[f]$ denotes the derivative of f in the direction of the vector field X.

Given a connection on V, the ***parallel transport*** of a vector $v_0 \in \tau^{-1}(q_0)$ along a curve $q(t)$ in Q, $t \in [a, b]$ such that $q(t_0) = q_0$ for a fixed $t_0 \in [a, b]$, is the unique curve $v(t)$ such that $v(t) \in \tau^{-1}(q(t))$ for all t, $v(t_0) = v_0$ and which satisfies $\nabla_{\dot{q}(t)} v(t) = 0$ for all t. The operation of parallel transport establishes, as before, for each $t, s \in [a, b]$, a linear map

$$T^t_{t+s} : \tau^{-1}(q(t)) \to \tau^{-1}(q(t+s))$$

associated to each curve $q(t)$ in Q. Then we can define the operation of ***covariant derivative*** on curves $v(t)$ in V similar to that in the previous definition; that is,

$$\frac{Dv(t)}{Dt} = \frac{d}{ds} T^{t+s}_t v(t+s) \bigg|_{s=0}.$$

Observe that the connection ∇ can be recovered from the covariant derivative (and thus from the parallel transport operation). Indeed, ∇ is given by

$$\nabla_X v(q_0) = \frac{D}{Dt} v(t) \bigg|_{t=t_0},$$

where, for each $q_0 \in Q$, each $X \in \mathfrak{X}^\infty(Q)$, and each $v \in \Gamma(V)$, we have, by definition, that $q(t)$ is any curve in Q such that $\dot{q}(t_0) = X(q_0)$ and $v(t) = v(q(t))$ for all t. This property establishes in particular the uniqueness of the connection associated to the covariant derivative D/Dt.

A Formula for the Covariant Derivative Induced by a Principal Connection. Now we return to the study of the affine connection induced on an associated bundle. The following formula gives the relation between the covariant derivative of the affine connection and the principal connection.

Lemma 2.3.2

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), -\rho'\left(A\left(q(t), \dot{q}(t)\right)\right) m(t) + \dot{m}(t)]_G \,.$$

Proof. For fixed t and for any s we have

$$[q(t+s), m(t+s)]_G = [g_q(t+s)q_h(t+s), m(t+s)]_G,$$

where we assume $q_h(t) = q(t)$ and $g_q(t) = e$, the identity element of G. Then

$$\begin{aligned}\tau_s^{t+s}[q(t+s), m(t+s)]_G &= \tau_s^{t+s}[g_q(t+s)q_h(t+s), m(t+s)]_G \\ &= \tau_s^{t+s}[q_h(t+s), g_q(t+s)^{-1}m(t+s)]_G \\ &= [q_h(t), g_q(t+s)^{-1}m(t+s)]_G.\end{aligned}$$

Therefore, differentiating with respect to s at $s = 0$ we obtain

$$\frac{D[q(t), m(t)]_G}{Dt} = [q(t), -\dot{g}_q(t)m(t) + \dot{m}(t)]_G \,.$$

Since $g_q(t) = e$ we have, using lemma 2.2.1, that $\dot{g}_q(t) = A\left(q(t), \dot{q}(t)\right)$. Therefore,

$$\dot{g}_q(t)m(t) = \rho'\left(A\left(q(t), \dot{q}(t)\right)\right) m(t). \quad \blacksquare$$

Induced Connections on Associated Bundles. The previous definition of the covariant derivative of a curve in the associated vector bundle $Q \times_G M$ thus leads to a connection on $Q \times_G M$. Let us call this connection $\tilde{\nabla}^A$ or simply $\tilde{\nabla}$.

We now describe this connection $\tilde{\nabla}^A$ in more detail. Let $\varphi : Q/G \to Q \times_G M$ be a section of the associated bundle and let $X(x) \in T_x(Q/G)$ be a given vector tangent to Q/G at x. Let $x(t)$ be a curve in Q/G such that $\dot{x}(0) = X(x)$; thus, $\varphi\left(x(t)\right)$ is a curve in $Q \times_G M$. The covariant derivative of the section φ with respect to X at x is then, by definition,

$$\tilde{\nabla}^A_{X(x)}\varphi = \left.\frac{D\varphi\left(x(t)\right)}{Dt}\right|_{t=0}. \tag{2.3.1}$$

Notice that we only need to know φ along the curve $x(t)$ in order to calculate the covariant derivative.

The notion of a ***horizontal curve*** $[q(t), m(t)]_G$ on $Q \times_G M$ is defined by the condition that its covariant derivative is 0. A vector tangent to $Q \times_G M$ is called ***horizontal*** if it is tangent to a horizontal curve. Correspondingly, the ***horizontal space*** at a point $[q, m]_G$ belonging to $Q \times_G M$ is the space of all horizontal vectors at $[q, m]_G$.

The Adjoint Bundle. The case that interests us most in this paper occurs when $M = \mathfrak{g}$ and ρ_g is the adjoint action Ad_g.

Definition 2.3.3 *The associated bundle with standard fiber $\mathfrak{g}$, where the action of G on $\mathfrak{g}$ is the adjoint action, is called the* ***adjoint bundle****, and is sometimes denoted $\mathrm{Ad}(Q)$. We will use the notation $\tilde{\mathfrak{g}} := \mathrm{Ad}(Q)$ in this paper. We let $\tilde{\pi}_G : \tilde{\mathfrak{g}} \to Q/G$ denote the projection given by $\tilde{\pi}_G([q,\xi]_G) = [q]_G$.*

Lemma 2.3.4 *Let $[q(s), \xi(s)]_G$ be any curve in $\tilde{\mathfrak{g}}$. Then*

$$\frac{D[q(s),\xi(s)]_G}{Ds} = \Big[q(s), -\left[A\left(q(s),\dot{q}(s)\right),\xi(s)\right] + \dot{\xi}(s)\Big]_G .$$

Proof. Use the previous lemma and the fact that $\rho'(\xi) = \mathrm{ad}_\xi$. ■

The next Lemma says that the adjoint bundle is a ***Lie algebra bundle***.

Lemma 2.3.5 *Each fiber $\tilde{\mathfrak{g}}_x$ of $\tilde{\mathfrak{g}}$ carries a natural Lie algebra structure defined by*

$$[[q,\xi]_G, [q,\eta]_G] = [q, [\xi,\eta]]_G .$$

Proof. We must show that the bracket is well defined, which is done in a straightforward way as follows:

$$\begin{aligned}
[[gq, \mathrm{Ad}_g\,\xi]_G, [gq, \mathrm{Ad}_g\,\eta]_G] &= [gq, [\mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta]]_G \\
&= [gq, \mathrm{Ad}_g[\xi,\eta]]_G \\
&= [q, [\xi,\eta]]_G \\
&= [[q,\xi]_G, [q,\eta]_G] . \quad ■
\end{aligned}$$

2.4 The Bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$

Let $\pi : Q \to Q/G$ be a principal bundle with structure group G, as before. The tangent lift of the action of G on Q defines an action of G on TQ and so we can form the quotient $(TQ)/G =: TQ/G$. There is a well defined map $\tau_Q/G : TQ/G \to Q/G$ induced by the tangent of the projection map $\pi : Q \to Q/G$ and given by $[v_q]_G \mapsto [q]_Q$. The vector bundle structure of TQ is inherited by this bundle.

Lemma 2.4.1 *The rules*

$$[v_q]_G + [u_q]_G = [v_q + u_q]_G \quad \textit{and} \quad \lambda[v_q]_G = [\lambda v_q]_G,$$

where $\lambda \in \mathbb{R}$, $v_q, u_q \in T_qQ$ and $[v_q]_G$ and $[u_q]_G$ are their equivalence classes in the quotient TQ/G, define a vector bundle structure on TQ/G having base Q/G. The fiber $(TQ/G)_x$ is isomorphic, as a vector space, to T_qQ, for each $x = [q]_G$.

Proof. If $[q_0]_G = x$ is given, the isomorphism between the fiber $(TQ/G)_x$and $T_{q_0}Q$ is given by the map $[u_q]_G \mapsto g^{-1}u_q$, where $g \in G$ is uniquely determined by the relation $gq_0 = q$. ■

The bundle TQ/G is a fundamental object in the present paper. One can state reduced variational principles in a natural way in terms of this bundle without any reference to a connection on Q, which we shall describe in the next section.

It is, however, also interesting to introduce an (arbitrarily chosen) connection on Q relative to which one can realize the space TQ/G in a convenient way as well as writing the Lagrange-Poincaré equations in an interesting form. The next lemma is one of the main tools needed for doing this.

Lemma 2.4.2 *The map $\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ defined by*

$$\alpha_A\left([q,\dot{q}]_G\right) = T\pi(q,\dot{q}) \oplus [q, A(q,\dot{q})]_G$$

is a well defined vector bundle isomorphism. The inverse of α_A is given by

$$\alpha_A^{-1}\left((x,\dot{x}) \oplus [q,\xi]_G\right) = [(x,\dot{x})_q^h + \xi q]_G.$$

Proof. To show that α_A is well defined, observe that for any $g \in G$ we have $T\pi(gq, g\dot{q}) = T\pi(q,\dot{q})$ and also

$$[gq, A(gq, g\dot{q})]_G = [gq, \operatorname{Ad}_g A(q,\dot{q})]_G = [q, A(q,\dot{q})]_G.$$

Then we see that

$$\alpha_A\left([gq, g\dot{q}]_G\right) = \alpha_A\left([q,\dot{q}]_G\right).$$

To show that α_A^{-1} is well defined, notice that $(x,\dot{x})_{gq}^h = g(x,\dot{x})_q^h$ and that $(\operatorname{Ad}_g \xi)gq = g\xi q$. Therefore,

$$\alpha_A^{-1}\left((x,\dot{x}) \oplus [gq, \operatorname{Ad}_g \xi]_G\right) = \alpha_A^{-1}\left((x,\dot{x}) \oplus [q,\xi]_G\right). \quad ■$$

Remark. The bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$ do not depend on the connection A, but of course α_A does.

3 The Lagrange-Poincaré Equations

In this section we use the constructions from the previous section to show how to write the Lagrange-Poincaré equations in an intrinsic way. This is done in terms of an (arbitrarily chosen) connection A. The resulting equations are

given on the bundle $T^{(2)}(Q/G) \oplus \tilde{\mathfrak{g}}$, where $T^{(2)}(Q/G)$ is the second order tangent bundle (also called the 2-jet bundle) of Q/G, and whose general definition is recalled in §3.2 and where $\tilde{\mathfrak{g}} = Q \times_G \mathfrak{g}$ is the associated adjoint bundle to Q. A key point to doing this is to decompose arbitrary variations of curves in Q into vertical and horizontal components. This gives rise, correspondingly, to two reduced equations, namely, *vertical Lagrange-Poincaré equations* corresponding to vertical variations and Euler-Lagrange equations on Q/G with an additional term involving the curvature B of A, corresponding to horizontal variations, which we will call *horizontal Lagrange-Poincaré equations.* (Our conventions for the curvature were given in §2.2.)

3.1 The Geometry of Variations

Spaces of curves. Fix a time interval $I = [t_0, t_1]$. The space of all (smooth) curves from I to Q will be denoted $\Omega(Q)$. We shall not include the interval I explicitly in the notation for spaces of curves for simplicity; it will be understood from the context and explicitly stated when necessary.

Given a map $f : Q_1 \to Q_2$, the map $\Omega(f) : \Omega(Q_1) \to \Omega(Q_2)$ is defined by

$$\Omega(f)(q)(t) = f(q(t)),$$

for $q(t)$ an element of $\Omega(Q_1)$. For given $q_i \in Q$, $i = 0, 1$, by definition, $\Omega(Q; q_0)$ and $\Omega(Q; q_0, q_1)$ are, respectively, the spaces of curves $q(t)$ on Q such that $q(t_0) = q_0$ and $q(t_i) = q_i$, $i = 0, 1$.

If $\pi : Q \to S$ is a bundle, $q_0 \in Q$ and $\pi(q_0) = x_0$, then $\Omega(Q; x_0)$ denotes the space of all curves in $\Omega(Q)$ such that $\pi(q(t_0)) = x_0$. Similarly, $\Omega(Q; x_0, q_1)$ is the space of all curves in $\Omega(Q)$ such that $\pi(q(t_0)) = x_0$ and $q(t_1) = q_1$. The spaces of curves $\Omega(Q; q_0, x_1)$, $\Omega(Q; x_0, x_1)$, etc. are defined in a similar way.

If $V \to Q$ and $W \to Q$ are vector bundles then $\Omega(V) \to \Omega(Q)$ and $\Omega(W) \to \Omega(Q)$ are vector bundles in a natural way and there is a natural identification $\Omega(V \oplus W) \equiv \Omega(V) \oplus \Omega(W)$.

Deformations of Curves. A ***deformation*** of a curve $q(t)$ on a manifold Q is, by definition, a (smooth) function $q(t, \lambda)$ such that $q(t, 0) = q(t)$ for all t. The corresponding ***variation*** is defined by

$$\delta q(t) = \left. \frac{\partial q(t, \lambda)}{\partial \lambda} \right|_{\lambda=0}.$$

Variations of curves $q(t)$ belonging to $\Omega(Q; q_0)$ or $\Omega(Q; q_0, q_1)$ satisfy the corresponding fixed endpoints conditions, namely, $\delta q(t_0) = 0$ or $\delta q(t_i) = 0$ for $i = 0, 1$, respectively.

Let $\tau : V \to Q$ be a vector bundle and let $v(t, \lambda)$ be a deformation in V of a curve $v(t)$ in V. If $\tau(v(t, \lambda)) = q(t)$ does not depend on λ we will call $v(t, \lambda)$

a V-***fiber deformation*** of $v(t)$, or simply, a ***fiber deformation*** of $v(t)$. For each t, the variation

$$\delta v(t) = \left.\frac{\partial v(t,\lambda)}{\partial \lambda}\right|_{\lambda=0}$$

may be naturally identified with an element, also called $\delta v(t)$, of $\tau^{-1}(q(t))$. In this case, the curve δv in V is, by definition, a V-***fiber variation*** of the curve v, or, simply, a ***fiber variation*** of the curve v.

Horizontal and Vertical Variations. We now break up the variation of a curve into horizontal and vertical parts. Thus, we consider a curve $q \in \Omega(Q; q_0)$ (again, it is understood that the curves are defined on a fixed time interval $[t_0, t_1]$), where, as before, $Q \to Q/G$ is a principal bundle with a connection A.

A ***vertical variation*** δq of q satisfies, by definition, the condition $\delta q(t) = \mathrm{Ver}(\delta q(t))$ for all t. Similarly, a ***horizontal variation*** satisfies $\delta q(t) = \mathrm{Hor}(\delta q(t))$ for all t.

Clearly, any variation δq can be uniquely decomposed as follows:

$$\delta q(t) = \mathrm{Hor}(\delta q(t)) + \mathrm{Ver}(\delta q(t))$$

for all t, where $\mathrm{Ver}(\delta q(t)) = A(q(t), \delta q(t))q(t)$ and where $\mathrm{Hor}(\delta q(t)) = \delta q(t) - \mathrm{Ver}(\delta q(t))$.

The Structure of Vertical Variations. Given a curve $q \in \Omega(Q; q_0, q_1)$, let $v = A(q, \dot{q}) \in \mathfrak{g}$. Variations δq of $q(t)$ induce corresponding variations $\delta v \in \mathfrak{g}$ in the obvious way:

$$\delta v = \left.\frac{\partial A(q(t,\lambda), \dot{q}(t,\lambda))}{\partial \lambda}\right|_{\lambda=0}.$$

Consider the decomposition $q = g_q q_h$ introduced in equation (2.2.2). A ***vertical deformation*** $q(t,\lambda)$ can be written as $q(t,\lambda) = g_q(t,\lambda)q_h(t)$. The corresponding variation $\delta q(t) = \delta g_q(t) q_h(t)$ is of course also vertical.

Now we introduce some important notation. Define the curve

$$\eta(t) = \delta g_q(t) g_q(t)^{-1}$$

in $\mathfrak{g}$. The fixed endpoint condition gives $\eta(t_i) = 0$, $i = 1, 2$.

Notice that, by construction,

$$\delta q(t) = \delta g_q(t) q_h(t) = \eta(t) g_q(t) q_h(t) = \eta(t) q(t).$$

Lemma 3.1.1 *For any vertical variation $\delta q = \eta q$ of a curve $q \in \Omega(Q; q_0, q_1)$ the corresponding variation δv of $v = A(q, \dot{q})$ is given by $\delta v = \dot{\eta} + [\eta, v]$ with $\eta_i = 0$, $i = 0, 1$.*

Proof. We will give the proof in the case that G is a matrix group. The more general case can be treated using the appendix to Bloch, Krishnaprasad, Marsden and Ratiu [1996].

By Lemma 2.2.1, we have $v = \dot{g}_q g_q^{-1}$. Then

$$\begin{aligned}
\delta v &= (\delta \dot{g}_q) g_q^{-1} - \dot{g}_q g_q^{-1} \delta g_q g_q^{-1} \\
&= (\delta g_q)^{\cdot} g_q^{-1} - v\eta \\
&= (\dot{\eta} g_q + \eta \dot{g}_q) g_q^{-1} - v\eta \\
&= \dot{\eta} + [\eta, v]. \quad \blacksquare
\end{aligned}$$

Note. In the variational approach to the Euler–Poincaré equations (see Marsden and Ratiu [1994], Chapter 13), there is a class of constrained variations $\delta\xi = \dot{\eta} + [\xi, \eta]$ introduced for computing the corresponding variational principle. The above construction of v, η is *not* computing the same objects. These constrained variations are, instead, special instances of the construction of covariant variations, to be introduced shortly in Definition 3.1.3. In the second remark following Lemma 3.1.4, we shall explicitly remark on how the constructions of variations for the Euler-Poincaré equations and those for the Lagrange-Poincaré case are related.

The Structure of Horizontal Variations. Now we calculate variations δv corresponding to horizontal variations δq of a curve $q \in \Omega(Q; q_0, q_1)$.

Lemma 3.1.2 *Let δq be a horizontal variation of a curve $q \in \Omega(Q; q_0, q_1)$. Then the corresponding variation δv of $v = A(q, \dot{q})$ satisfies $\delta v = B(q)(\delta q, \dot{q})$.*

Proof. Let $q(t, \lambda)$ be a horizontal deformation of $q(t)$, that is, $\lambda \mapsto q(t, \lambda)$ is a horizontal curve for each t. Now we work locally in a local trivialization of the bundle and write the connection A in the following way:

$$v = A(q, \dot{q}) = \langle \mathcal{A}(q), \dot{q} \rangle .$$

Then, we compute using the chain rule:

$$\delta v = \left.\frac{\partial v}{\partial \lambda}\right|_{\lambda=0} = \left.\left\langle D\mathcal{A}(q) \cdot \frac{\partial q}{\partial \lambda},\ \dot{q} \right\rangle\right|_{\lambda=0} + \left.\left\langle \mathcal{A}(q),\ \frac{\partial^2 q}{\partial \lambda \partial t} \right\rangle\right|_{\lambda=0} .$$

On the other hand, since $\lambda \mapsto q(t, \lambda)$ is horizontal,

$$\left\langle \mathcal{A}(q), \frac{\partial q}{\partial \lambda} \right\rangle = 0,$$

and so, by differentiating with respect to t,

$$0 = \left\langle D\mathcal{A}(q)\cdot\dot{q},\ \frac{\partial q}{\partial\lambda}\right\rangle\Bigg|_{\lambda=0} + \left\langle \mathcal{A}(q),\ \frac{\partial^2 q}{\partial t\partial\lambda}\right\rangle\Bigg|_{\lambda=0}.$$

Then we obtain, by subtraction,

$$\delta v = \mathbf{d}\mathcal{A}(q)\left(\frac{\partial q}{\partial\lambda},\frac{\partial q}{\partial t}\right)\Bigg|_{\lambda=0}.$$

Since $\partial q/\partial\lambda$ is horizontal, Cartan's structure equation (2.2.1) implies

$$\delta v = B(q)\left(\frac{\partial q}{\partial\lambda},\frac{\partial q}{\partial t}\right)\Bigg|_{\lambda=0}$$

or, in other words, $\delta v = B(q)(\delta q,\dot{q})$. ■

The Covariant Variation on the Adjoint Bundle. Any curve in Q, $q\in\Omega(Q;q_0,q_1)$ induces a curve in $\tilde{\mathfrak{g}}$ in a natural way, namely,

$$[q,v]_G(t) = [q(t),v(t)]_G\,,$$

where $v(t)=A(q,\dot{q})$. Observe that, for each t, $[q,v]_G(t)\in\tilde{\mathfrak{g}}_{x(t)}$ (the fiber over $x(t)$), where $x(t)=\pi\left(q(t)\right)$ for all t. We want to study variations $\delta[q,v]_G$ corresponding to vertical and also to horizontal variations δq of q.

While vertical variations δq give rise to vertical variations $\delta[q,v]_G$, horizontal variations δq need not give rise to horizontal variations $\delta[q,v]_G$. The deviation of any variation $\delta[q,v]_G$ from being horizontal is measured by the *covariant variation* $\delta^A[q,v]_G(t)$, defined as follows

Definition 3.1.3 *For any given deformation $q(t,\lambda)$ of $q(t)$, the* ***covariant variation*** *$\delta^A[q,v]_G(t)$ is defined by*

$$\delta^A[q,v]_G(t) = \frac{D\left[q(t,\lambda),v(t,\lambda)\right]_G}{D\lambda}\Bigg|_{\lambda=0}.$$

Vertical Variations and the Adjoint Bundle. We first consider the case of vertical variations.

Lemma 3.1.4 *The covariant variation $\delta^A[q,v]_G(t)$ corresponding to a vertical variation $\delta q=\eta q$ is given by*

$$\delta^A[q,v]_G(t) = \frac{D[q,\eta]_G}{Dt} + [q,[v,\eta]]_G\,.$$

Proof. Let $q(t,\lambda)$ be a vertical deformation of $q(t)$ with, as usual, $\delta q = \partial q/\partial\lambda|_{\lambda=0}$. As we saw before, $q(t,\lambda) = g_q(t,\lambda)q_h(t)$ and $\delta g_q = \eta g_q$, where $\eta = \delta g_q g_q^{-1} = A(q,\delta q)$. If we let $v = A(q,\dot{q})$, by Lemma 3.1.1, we have $\delta v = \dot{\eta} + [\eta, v]$. From this and Lemma 2.3.4, we obtain

$$\begin{aligned}\delta^A[q,v]_G(t) &= [q, -[A(q,\delta q), v] + \delta v]_G \\ &= [q, -[\eta, v] + \dot{\eta} + [\eta, v]]_G \\ &= [q, \dot{\eta}]_G.\end{aligned}$$

Again, by lemma 2.3.4, we have

$$\begin{aligned}\frac{D[q,\eta]_G}{Dt} &= [q, -[A(q,\dot{q}), \eta] + \dot{\eta}]_G \\ &= [q, -[v, \eta] + \dot{\eta}]_G \\ &= [q, \dot{\eta} + [\eta, v]]_G\,.\end{aligned}$$

Therefore,

$$\delta^A[q,v]_G(t) = \frac{D[q,\eta]_G}{Dt} + [q,[v,\eta]]_G\,. \quad \blacksquare$$

Remarks.

1. In view of lemma 2.3.5, we can write

$$[q,[v,\eta]]_G = [[q,v]_G, [q,\eta]_G]\,.$$

2. Let us now show that the formula $\delta v = \dot{\eta} + [v,\eta]$ for the constrained variations for the Euler–Poincaré equations (see Marsden and Ratiu [1994] and references therein) coincides with the construction of the covariant variation given in Definition 3.1.3. Given a Lie group G, we regard it as a principal bundle over a point; that is, we take $G = Q$. The identification of $\mathfrak{g}$ with $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in this case is given by $v \mapsto [e,v]_G$. Then this equivalence defines $\delta v \equiv \delta^A[e,v]_G$ and then the preceding lemma shows that $\delta v = \dot{\eta} + [v,\eta]$, which is the same type of variation one has for the Euler–Poincaré equations.

The Reduced Curvature Form. In preparation for the consideration of variations $\delta^A[q,v]_G(t)$ corresponding to horizontal variations, we prove the following result of independent interest.

Lemma 3.1.5 *The curvature 2-form $B \equiv B^A$ of the connection A induces a $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B} \equiv \tilde{B}^A$ on Q/G given by*

$$\tilde{B}(x)(\delta x, \dot{x}) = [q, B(q)(\delta q, \dot{q})]_G\,, \tag{3.1.1}$$

where for each $(x,\dot{x})$ and $(x,\delta x)$ in $T_x(Q/G)$, $(q,\dot{q})$ and $(q,\delta q)$ are any elements of T_qQ such that $\pi(q) = x$, $T\pi(q,\dot{q}) = (x,\dot{x})$ and $T\pi(q,\delta q) = (x,\delta x)$.

Proof. We show that the right hand side does not depend on the choice of $(q, \dot{q})$ and $(q, \delta q)$. For any $g \in G$ we have

$$\begin{aligned}[gq, B(gq)(g\delta q, g\dot{q})]_G &= [gq, \mathrm{Ad}_g\, B(q)(\delta q, \dot{q})]_G \\ &= [q, B(q)(\delta q, \dot{q})]_G\,. \quad \blacksquare\end{aligned}$$

Definition 3.1.6 *The $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B}$ on Q/G will be called the* ***reduced curvature form****.*

Horizontal Variations and the Adjoint Bundle. Now we are ready to describe covariant variations $\delta^A[q, v]_G(t)$ corresponding to horizontal variations δq.

Lemma 3.1.7 *Variations $\delta^A[q, v]_G(t)$ corresponding to horizontal variations δq are given by*

$$\delta^A[q, v]_G(t) = \tilde{B}(x)(\delta x, \dot{x})(t),$$

where $T\pi(q, \dot{q}) = (x, \dot{x})$, $T\pi(q, \delta q) = (x, \delta x)$, and $v = A(q, \dot{q})$.

Proof. By lemma 2.3.4, we have $\delta^A[q, v]_G(t) = [q, -[A(q, \delta q), v] + \delta v]_G$. Since δq is horizontal, we have $A(q, \delta q) = 0$. Using this and lemmas 3.1.2 and 3.1.5, we obtain $\delta^A[q, v]_G(t) = \tilde{B}(x)(\delta x, \dot{x})$. $\blacksquare$

3.2 The Euler-Lagrange and Euler-Poincaré Operators

The purpose of the next three sections is to carry out the reduction of the Euler-Lagrange equations by means of reduction of Hamilton's principle using the geometric set up in the preceding section. We will begin in this section with some geometric preliminaries and treat the standard case of the Euler-Lagrange and the Euler-Poincaré equations.

Reduced Spaces of Curves. In what follows we shall often identify the bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$, using the isomorphism α_A of lemma 2.4.2. This leads to other natural identifications as well. For instance, the reduced set of curves $[\Omega(Q; q_0, q_1)]_G$ is the set of curves $[q]_G(t) = [q(t)]_G$ on Q/G such that the curve $q(t)$ belongs to $\Omega(Q; q_0, q_1)$. This reduced set of curves is naturally identified with the set of curves $[q(t), \dot{q}(t)]_G$ in TQ/G such that $q(t_i) = q_i$, for $i = 0, 1$, and in turn, this is identified, via the map

$$\Omega(\alpha_A) : [\Omega(Q; q_0, q_1)]_G \to \Omega\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right),$$

with the set of curves

$$T\pi(q(t),\dot{q}(t)) \oplus [q(t), A\,(q(t),\dot{q}(t))]_G,$$

for all t, in $T(Q/G) \oplus \tilde{\mathfrak{g}}$, such that $q(t_i) = q_i$, for $i = 0, 1$, which we shall therefore denote $\Omega(\alpha_A)\,([\Omega(Q; q_0, q_1)]_G)$.

The Reduced Lagrangian. Let $L : TQ \to \mathbb{R}$ be an invariant Lagrangian, that is, $L\,(g(q,\dot{q})) = L(q,\dot{q})$ for all $(q,\dot{q}) \in TQ$ and all $g \in G$. Because of this invariance, we get a well defined ***reduced Lagrangian*** $l : TQ/G \to \mathbb{R}$ satisfying

$$l\,([q,\dot{q}]_G) = L(q,\dot{q}).$$

As we will see in detail in this section, the evolution of the reduced system will be a critical point, say a curve $[q]_G$ in the reduced set of curves $[\Omega(Q; q_0, q_1)]_G$, of the reduced action

$$\int_{t_0}^{t_1} l\,([q,\dot{q}]_G)\,dt$$

for suitable types of variations.

However, variations of curves in the reduced family of curves are not of the usual sort found in Hamilton's principle, and so the equations of motion in the bundle TQ/G cannot be written in a direct way.

In this section we use the description of vertical and horizontal variations given in the preceding section to derive equations of motion in the bundle $T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}}$ defined below. Equations corresponding to vertical variations will be called the *vertical Lagrange-Poincaré equations*, and equations corresponding to horizontal variations will be called the *horizontal Lagrange-Poincaré equations*.

Identification of Bundles. We shall allow a slight abuse of notation, namely we will consider l as a function defined on $T(Q/G) \oplus \tilde{\mathfrak{g}}$ or TQ/G interchangeably, using the isomorphism α_A. Also we shall often use a slight abuse of the variable-notation for a function, namely we will write $l(x,\dot{x},\bar{v})$ to emphasize the dependence of l on $(x,\dot{x}) \in T(Q/G)$ and $\bar{v} \in \tilde{\mathfrak{g}}$. However one should keep in mind that x, $\dot{x}$ and $\bar{v}$ cannot be considered as being independent variables unless $\tilde{\mathfrak{g}}$ and $T\,(Q/G)$ are trivial bundles. Even in those cases in which $\tilde{\mathfrak{g}}$ and $T(Q/G)$ are trivial, and therefore x, $\dot{x}$ and $\bar{v}$ can be considered as being independent variables in a natural way, it is sometimes convenient to proceed using the general theory.

The kth Order Tangent Bundle. Now we shall recall the definition of the kth-order tangent bundle $\tau_Q^{(k)} : T^{(k)}Q \to Q$. For $\bar{q} \in Q$, elements of $T_{\bar{q}}^{(k)}Q$ are equivalence classes of curves in Q, namely, two given curves $q_i(t), i = 1, 2$ such that $q_1(\bar{t_1}) = q_2(\bar{t_2}) = \bar{q}$ are equivalent, by definition, if and only if in any local chart we have $q_1^{(l)}(\bar{t_1}) = q_2^{(l)}(\bar{t_2})$, for $l = 1, 2, \ldots, k$, where $q^{(l)}$ denotes the derivative of order l. The equivalence class of the curve $q(t)$ at $\bar{q} = q(\bar{t})$ will be denoted $[q]_{\bar{q}}^{(k)}$. The projection

$$\tau_Q^{(k)} : T^{(k)}Q \to Q \quad \text{is given by} \quad \tau_Q^{(k)}\left([q]_{\bar{q}}^{(k)}\right) = \bar{q}.$$

It is well known that $T^{(1)}Q = TQ$ and that, for $l < k$, there is a well defined fiber bundle structure

$$\tau_Q^{(l,k)} : T^{(k)}Q \to T^{(l)}Q, \quad \text{given by} \quad \tau_Q^{(l,k)}\left([q]_{\bar{q}}^{(k)}\right) = [q]_{\bar{q}}^{(l)}.$$

It is also easy to see that for any map $f : M \to N$ we have a naturally induced map

$$T^{(k)}f : T^{(k)}M \to T^{(k)}N \quad \text{given by} \quad T^{(k)}f\left([q]_{\bar{q}}^{(k)}\right) = [f \circ q]_{f(\bar{q})}^{(k)}.$$

In particular, a group action $\rho : G \times Q \to Q$ can be naturally lifted to a group action

$$\rho^{(k)} : G \times T^{(k)}Q \to T^{(k)}Q \quad \text{given by} \quad \rho_g^{(k)}\left([q]_{\bar{q}}^{(k)}\right) = [\rho_g \circ q]_{\rho(g,\bar{q}}^{(k)}.$$

We will often denote $\rho_g^{(k)}\left([q]_{\bar{q}}^{(k)}\right) = \rho^{(k)}\left(g, [q]_{\bar{q}}^{(k)}\right) = g[q]_{\bar{q}}^{(k)}$.

Let $M \times N$ be the cartesian product of the manifolds M and N. Then, for any $(\bar{m}, \bar{n}) \in M \times N$ there is a natural identification $T_{(\bar{m},\bar{n})}^{(k)}((M \times N) \equiv T_{\bar{m}}^{(k)}M \times T_{\bar{n}}^{(k)}N$, which induces an identification $T^{(k)}(M \times N) \equiv T^{(k)}M \times T^{(k)}N$.

The natural action of G on the fiber bundle $T^{(k)}Q$ gives to $T^{(k)}Q$ a principal bundle structure with structure group G. The quotient $T^{(k)}Q/G$ can be easily shown to be a fiber bundle over the base Q/G. The bundle $T^{(2)}Q/G$ is the one that interests us most in this paper, because the Lagrange-Poincaré operator is a 1-form on $T^{(2)}Q/G$, of a special kind. The class of the element $[q]_{\bar{q}}^{(k)}$ in the quotient $T^{(k)}Q/G$ will be denoted $\left[[q]_{\bar{q}}^{(k)}\right]_G$, as usual. Since we have the projection $\pi_G(Q) : Q \to Q/G$ we obtain a bundle map

$$T^{(k)}\pi_G(Q) : T^{(k)}Q \to T^{(k)}(Q/G).$$

Moreover, it can be easily shown that this bundle map induces a well defined bundle map

$$T^{(k)}Q/G \to T^{(k)}(Q/G) \quad \text{given by} \quad \left[[q]_{\bar{q}}^{(k)}\right]_G \to T^{(k)}\pi_G(Q)\left([q]_{\bar{q}}^{(k)}\right).$$

Let $[x]^{(k)}_{\bar{x}} \in T^{(k)}(Q/G)$ and also $\bar{q} \in Q$ such that $\pi(\bar{q}) = [\bar{q}]_G = \bar{x}$ be given. Let $x(t)$ be any curve belonging to the class of $[x]^{(k)}_{\bar{x}}$. Then there is a unique horizontal lift $x^h_{\bar{q}}$ of $x(t)$. We define the horizontal lift of $[x]^{(k)}_{\bar{x}}$ at $\bar{q}$ by

$$[x]^{(k),h}_{\bar{x},\bar{q}} := \left[x^h_{\bar{q}}\right]^{(k)}_{\bar{q}}.$$

We must also remark that $T^{(k)}G$ carries a natural Lie group structure.[1] If $[g]^{(k)}_{\bar{g}}$, and $[h]^{(k)}_{\bar{h}}$ are classes of curves g and h in G, we define the product $[g]^{(k)}_{\bar{g}}[h]^{(k)}_{\bar{h}}$ as being the class $[gh]^{(k)}_{\bar{g}\bar{h}}$ at the point $\bar{g}\bar{h}$ of the curve gh. The Lie algebra $T_eT^{(k)}G$ of $T^{(k)}G$ can be naturally identified, as a vector space, with $(k+1)\mathfrak{g}$, which, therefore, carries a unique Lie algebra structure such that this identification becomes a Lie algebra isomorphism. There is also a natural identification of $T^{(k)}_eG$ with $k\mathfrak{g}$.

Also, for $k = 1, 2, \ldots,$ $T^{(k)}Q$ is a principal bundle with structure group $T^{(k)}G$ in a natural way. More precisely, if $[g]^{(k)}_{\bar{g}} \in T^{(k)}G$ is the class of a curve g in G and $[q]^{(k)}_{\bar{q}} \in T^{(k)}_{\bar{q}}(Q)$ is the class of a curve q in Q we let $[g]^{(k)}_{\bar{g}}[q]^{(k)}_{\bar{q}} \in T^{(k)}_{\bar{g}\bar{q}}Q$ denote the class $[gq]^{(k)}_{\bar{g}\bar{q}}$ of the curve gq at the point $\bar{g}\bar{q}$. In particular, if $\xi \in k\mathfrak{g}$ and $[q]^{(k)}_{\bar{q}} \in T^{(k)}Q$ are given, there is a well defined element $\xi[q]^{(k)}_{\bar{q}} \in T^{(k)}Q$.

Connection-like Structures on Higher Order Tangent Bundles. For $q \in \Omega(Q)$, we have the curve $[q(t), v(t)]_G$ in $\tilde{\mathfrak{g}}$, where $v(t) = A(q(t), \dot{q}(t))$. Lemma 2.3.4 shows that the covariant derivative of $[q(t), v(t)]_G$ is given by

$$\begin{aligned}\frac{D[q(t), v(t)]_G}{Dt} &= [q(t), -[A(q(t), \dot{q}(t)), v(t)] + \dot{v}(t)]_G \\ &= [q(t), \dot{v}(t)]_G.\end{aligned}$$

The second covariant derivative of $[q(t), v(t)]_G$, again by Lemma 2.3.4, is given by

$$\frac{D^2[q(t), v(t)]_G}{Dt^2} = [q(t), -[v(t), \dot{v}(t)] + \ddot{v}(t)]_G.$$

More generally, for each $k = 1, 2, \ldots,$ we can find, by induction, a curve $v_k(t)$ in $\mathfrak{g}$, having an expression that involves $v(t)$ and the derivatives $v^{(l)}(t)$, $l = 1, 2, \ldots, k-1$, such that

$$\frac{D^{k-1}[q(t), v(t)]_G}{Dt^{k-1}} = [q(t), v_k(t)]_G.$$

[1] Recall that $T^{(1)}G = TG$ is the semidirect product $G \circledS \mathfrak{g}$ and its Lie algebra is $2\mathfrak{g}$ which as a vector space is $\mathfrak{g} \oplus \mathfrak{g}$ and it carries the Lie algebra structure of the semidirect product $\mathfrak{g} \circledS \mathfrak{g}$ where the second factor is regarded as the representation space of the adjoint action. We will not need, or study, the Lie group structure of $T^{(k)}G$ in this paper, although this would be interesting to do.

More precisely, we have

$$v_1 = v \quad \text{and} \quad v_{k+1} = -[v, v_k] + \dot{v}_k,$$

for $k = 1, 2, \ldots$. In particular, we obtain

$$v_2(t) = \dot{v}(t), \quad v_3(t) = -\left[v(t),\ \dot{v}(t)\right] + \ddot{v}(t),$$

etc. In addition, we shall write, by definition, $v_0(t) = 0$. Using the fact that $v(t) = A\left(q(t), \dot{q}(t)\right)$, we can also find expressions for $v_k(t)$ in coordinates in terms of $q^{(l)}(t)$, $l = 1, 2, \ldots, k$. We state the following lemma, which is readily proved.

Lemma 3.2.1 *Let $q(t)$ be a given curve in Q such that $q(\bar{t}) = \bar{q}$. For each $k = 1, 2, \ldots$ the formula*

$$\bar{A}_k\left([q]_{\bar{q}}^{(k)}\right) = v_k(\bar{t})$$

gives a well defined map $\bar{A}_k : T^{(k)}Q \to \mathfrak{g}$. Therefore there is also a well defined map $A_k : T^{(k)}Q \to k\mathfrak{g}$, given, for each $k = 1, 2, \ldots,$ by

$$A_k\left([q]_{\bar{q}}^{(k)}\right) = \oplus_{l=1}^{k} v_l,$$

where we have written $k\mathfrak{g}$ to stand for the vector space direct sum $\oplus_{l=1}^{k}\mathfrak{g}$ of k copies of $\mathfrak{g}$.

Let $g \in G$ and $[q]_{\bar{q}}^{(k)} \in T_{\bar{q}}^{(k)}Q$ be given. Then we can easily prove that

$$A_k\left(g[q]_{\bar{q}}^{(k)}\right) = \mathrm{Ad}_g\, A_k\left([q]_{\bar{q}}^{(k)}\right),$$

using induction, the definition of A_k, and taking into account the formulas

$$\mathrm{Ad}_g\, \dot{v}_k = \frac{d}{dt}\,\mathrm{Ad}_g\, v_k$$

and $\mathrm{Ad}_g[v, v_k] = [\mathrm{Ad}_g\, v, \mathrm{Ad}_g\, v_k]$. According to Lemma 2.2.1, for any curve $q(t)$ in Q such that $q(\bar{t}) = \bar{q}$ we have

$$A(q, \dot{q}) = \dot{g}_q g_q^{-1}.$$

Using this and the inductive definition of v_k one can inductively find an expression for $v_k(\bar{t})$ in terms of $g_q^{(l)}(\bar{t})$, for $l = 1, 2, \ldots$. For instance, for the case of matrix groups, we can see directly that $v_1(\bar{t}) = v(\bar{t}) = \dot{g}_q(\bar{t})$, $v_2(\bar{t}) = \ddot{g}_q(\bar{t}) - \dot{g}_q(\bar{t})^2$, etc. It is not difficult to see that the expression of $v_k(\bar{t})$ is the sum of $g_q^{(k)}(\bar{t})$ plus terms involving only the lower order derivatives

$g_q^{(l)}(\bar{t})$, $l = 1, 2, \ldots, k-1$. Using this one can easily see that, given any element $\xi = (\xi_1, \ldots \xi_k)$ belonging to $k\mathfrak{g}$ and any element $\bar{q} \in Q$ one can find a unique $[q]_{\bar{q}}^{(k)}$ of the form $[g\bar{q}]_{\bar{q}}^{(k)}$ where $g(t)$ is a curve in G such that

$$g(\bar{t}) = e \quad \text{and} \quad A_k\left([q]_{\bar{q}}^{(k)}\right) = \xi.$$

In fact, since we obviously have $g_q(t) = g(t)$ it is enough to find $g(t)$ such that the derivatives $g^{(l)}(\bar{t})$, $l = 1, 2, \ldots, k$ satisfy the appropriate conditions as explained above. We shall call this unique element $\xi\bar{q}$, as before, and the set of all such elements will be called $k\mathfrak{g}\bar{q}$. Moreover, it is not difficult to see that the restriction $A_k : k\mathfrak{g}\bar{q} \to k\mathfrak{g}$ is a diffeomorphism, and therefore it naturally defines a unique vector space structure on $k\mathfrak{g}\bar{q}$ such that the restriction of A_k becomes a linear isomorphism. By construction we see that there is a natural identification between $k\mathfrak{g}\bar{q}$ and $T_{\bar{q}}^{(k)}(G\bar{q})$. Note that we have $A_k(\xi\bar{q}) = \xi$ for all $\xi \in k\mathfrak{g}$, analogous to what one has for connections.

Let us define, for each $k = 1, 2, \ldots$, the vector bundle $k\tilde{\mathfrak{g}}$ as being the Whitney sum of k copies of $\tilde{\mathfrak{g}}$. Define a map

$$T^{(k)}Q \to k\tilde{\mathfrak{g}} \quad \text{by} \quad [q]_{\bar{q}}^{(k)} \mapsto \left[\bar{q}, A_k\left([q]_{\bar{q}}^{(k)}\right)\right]_G,$$

where the last term is defined by

$$\left[\bar{q}, A_k\left([q]_{\bar{q}}^{(k)}\right)\right]_G = \oplus_{l=1}^{k}\left[\bar{q}, \bar{A}_l\left([q]_{\bar{q}}^{(l)}\right)\right]_G.$$

The definitions show that, given any curve $q(t)$ in Q such that $q(\bar{t}) = \bar{q}$, we have, at $t = \bar{t}$,

$$\left[\bar{q}, A_k\left([q]_{\bar{q}}^{(k)}\right)\right]_G = \bigoplus_{l=1}^{k} \frac{D^{(l)}[q(t), v(t)]_G}{Dt^{(l)}}.$$

We have essentially proven the following lemma, which generalizes Lemma 2.4.2

Lemma 3.2.2 *The map*

$$\alpha_{A_k} : T^{(k)}Q/G \to T^{(k)}(Q/G) \times_{Q/G} k\tilde{\mathfrak{g}}$$

defined by

$$\alpha_{A_k}\left(\left[[q]_{\bar{q}}^{(k)}\right]_G\right) = T^{(k)}\pi_G(Q)\left([q]_{\bar{q}}^{(k)}\right) \times_{Q/G} \left[\bar{q}, A_k\left([q]_{\bar{q}}^{(k)}\right)\right]_G$$

is a well defined bundle isomorphism. The inverse of α_A is given by

$$\alpha_{A_k}^{-1}\left([x]_{\bar{x}}^{(k)} \times_{Q/G} [\bar{q}, \xi]_G\right) = \xi[x]_{\bar{x},\bar{q}}^{(k),h}.$$

Remark. As we have said earlier, the bundles $T^{(k)}Q$, for $k = 1, 2$ are the only ones that interest us in this paper. The cases $k = 2, 3, \ldots$ are needed, for instance, to deal with higher order Lagrangians $L : T^{(k)}Q \to \mathbb{R}$. Then the Euler-Lagrange operator will be a 1-form $\mathcal{EL}(L)$ of a special kind on $T^{(k+1)}Q$. There are also several interesting structures in the bundles $T^{(k)}Q$ which we will not study in the present paper.

The Euler-Lagrange Operator. Next we shall introduce some more notation and recall some basic results concerning Euler-Lagrange operators. The fundamental connection between the variational and differential-equation description of the evolution of a given system is given by the following well known result.

Theorem 3.2.3 (Euler-Lagrange) *Let $L : TQ \to \mathbb{R}$ be a given Lagrangian on a manifold Q and let*

$$\mathfrak{S}(L)(q) = \int_{t_0}^{t_1} L(q, \dot{q})\, dt$$

be the action of L defined on $\Omega(Q; q_0, q_1)$. Let $q(t, \lambda)$ be a deformation of a curve $q(t)$ in $\Omega(Q; q_0, q_1)$ and let $\delta q(t)$ be the corresponding variation. Then, by definition, $\delta q(t_i) = 0$ for $i = 0, 1$.

There is a unique bundle map

$$\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$$

such that, for any deformation $q(t, \lambda)$, keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}(L)(q) \cdot \delta q = \int_{t_0}^{t_1} \mathcal{EL}(L)(q, \dot{q}, \ddot{q}) \cdot \delta q,$$

where, as usual,

$$\mathbf{d}\mathfrak{S}(L)(q) \cdot \delta q = \frac{d}{d\lambda} \mathfrak{S}(L)\left(q(t, \lambda)\right)\Big|_{\lambda=0}$$

with

$$\delta q(t) = \frac{\partial q(t, \lambda)}{\partial \lambda}\Big|_{\lambda=0}.$$

*The 1-form valued map $\mathcal{EL}(L)$ is called the **Euler-Lagrange operator**.*

In local coordinates $\mathcal{EL}(L)$ has the following well known expression:

$$\mathcal{EL}(L)_i(q, \dot{q}, \ddot{q})\, dq^i = \left(\frac{\partial L}{\partial q^i}(q, \dot{q}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) \right) dq^i$$

in which it is understood that one regards the second term on the right hand side as a function on the second order tangent bundle by formally applying the chain rule and then replacing everywhere dq/dt by $\dot{q}$ and $d\dot{q}/dt$ by $\ddot{q}$. The Euler-Lagrange equations can of course be written simply as $\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = 0$.

The Euler-Poincaré Operator. Analogous to the Euler-Lagrange operator, the Euler-Poincaré theorem (Marsden and Scheurle [1993b] and Bloch, Krishnaprasad, Marsden, and Ratiu [1996]; see also Marsden and Ratiu [1994], section 13.5) induces an operator, called the ***Euler-Poincaré operator***. It is defined, as before, by the variational principle.

Theorem 3.2.4 (Euler-Poincaré) *Let G be a Lie group, $L : TG \to \mathbb{R}$ a left G-invariant Lagrangian,*

$$\mathfrak{S}(L)(g) = \int_{t_0}^{t_1} L(g, \dot{g})\, dt$$

the ***action functional*** *of L defined on $\Omega(G; g_0, g_1)$, $l = L|\mathfrak{g}$ the reduced Lagrangian, and*

$$\mathfrak{S}_{red}(l)(v) = \int_{t_0}^{t_1} l(v)\, dt$$

the ***reduced action functional*** *defined on $\Omega(\mathfrak{g})$. Let $g(t, \lambda)$ be a deformation of a curve $g(t)$ in $\Omega(G; g_0, g_1)$, keeping the endpoints fixed, and let $\delta g(t)$ be the corresponding variation; thus, by definition, $\delta g(t_i) = 0$, for $i = 0, 1$. Let $v(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$.*

The following are equivalent:

a *the curve $g(t)$ satisfies the Euler-Lagrange equations $\mathcal{EL}(L)(g, \dot{g}, \ddot{g}) = 0$ on G;*

b *the curve $g(t)$ is a critical point of the action functional $\mathfrak{S}(L)$ for variations δg vanishing at the endpoints;*

c *the curve $v(t)$ solves the Euler-Poincaré equations*

$$\frac{d}{dt}\frac{\partial l}{\partial v} = \mathrm{ad}_v^* \frac{\partial l}{\partial v}.$$

d *the curve $v(t)$ is a critical point of the reduced action functional*

$$\mathfrak{S}_{red}(l)(v) = \int_{t_0}^{t_1} l(v(t))dt,$$

for variations of the form

$$\delta v = \dot{\eta} + [v, \eta], \tag{3.2.1}$$

where $\eta(t) \in \mathfrak{g}$ is an arbitrary curve that vanishes at the endpoints. These variations δv are exactly the variations induced by left translation of arbitrary deformations $g(t, \lambda)$ of the curve $g(t) = g(t, \lambda)$ such that $\delta g(t_i) = 0$, for $i = 0, 1$.

In addition, there is a unique bundle map

$$\mathcal{EP}(l) : 2\mathfrak{g} \to \mathfrak{g}^*$$

where $2\mathfrak{g} := \mathfrak{g} \oplus \mathfrak{g}$*, according with the definition introduced in the statement of lemma 3.2.1, such that, for any deformation* $v(t,\lambda) = g(t,\lambda)^{-1}\dot{g}(t,\lambda) \in \mathfrak{g}$ *induced on* $\mathfrak{g}$ *by a deformation* $g(t,\lambda) \in G$ *of* $g(t) \in \Omega(G; g_0, g_1)$ *keeping the endpoints fixed, and thus* $\delta g(t_i) = 0$*, for* $i = 0, 1$*, we have*

$$\mathbf{d}\mathfrak{S}_{red}(l)(v) \cdot \delta v = \int_{t_0}^{t_1} \mathcal{EP}(l)(v, \dot{v}) \cdot \eta \, dt,$$

where, as usual,

$$\mathbf{d}\mathfrak{S}_{red}(l)(v) \cdot \delta v = \frac{d}{d\lambda}\mathfrak{S}_{red}(l)\left(v(t,\lambda)\right)\Big|_{\lambda=0}$$

and $\delta v(t) = \partial v(t,\lambda)/\partial t|_{\lambda=0} = \dot{\eta}(t) + [v(t), \eta(t)]$.

The map $\mathcal{EP}(l)$ *is called the* ***Euler-Poincaré operator*** *and its expression is given by*

$$\mathcal{EP}(l)(v, \dot{v}) = \mathrm{ad}_v^* \frac{\delta l}{\delta v} - \frac{d}{dt}\frac{\delta l}{\delta v}$$

where, as before, it is to be understood that the time derivative on the second term is performed formally using the chain rule and that the expression dv/dt *is replaced throughout by* $\dot{v}$*.*

The Euler-Poincaré equations can be written simply as $\mathcal{EP}(l)(v, \dot{v}) = 0$. Formula (3.2.1) represents the most general variation δv of v induced by an arbitrary variation δg via left translation. The precise relationship is $\eta = g^{-1}\delta g$ and so the condition $\delta g = 0$ at the endpoints is equivalent to the condition $\eta = 0$ at the endpoints.

3.3 The Lagrange-Poincaré Operator

In this section we introduce the Lagrange-Poincaré operator using the same type of technique of reduction of variational principles that was used in the preceding section to define the Euler-Lagrange and the Euler-Poincaré operators.

Reducing the Euler-Lagrange Operator. The map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$, being G invariant, induces a quotient map

$$[\mathcal{EL}(L)]_G : T^{(2)}Q/G \to T^*Q/G,$$

which depends only on the reduced Lagrangian $l : TQ/G \to \mathbb{R}$; that is, we can identify $[\mathcal{EL}(L)]_G$ with an operator $\mathcal{EL}(l)$. This is called the ***reduced Euler-Lagrange operator*** and it does not depend on any extra structure on the principal bundle Q. However, to write the explicit expressions, which are also physically meaningful, we use the additional structure of a principal connection A on the principal bundle $Q \to Q/G$ to identify the quotient bundle

$$T^{(2)}Q/G \quad \text{with} \quad T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}}$$

and

$$T^*Q/G \quad \text{with} \quad T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

using the bundle isomorphisms α_{A_2} from Lemma 3.2.2 and α_A from Lemma 2.4.2, and also a connection ∇ on Q/G to concretely realize the reduced Euler-Lagrange operator; this will naturally lead us to the Lagrange-Poincaré operator.

The Geometry of Reduced Variations. A general variation $\delta\bar{v}(t)$ of a given curve $\bar{v}(t)$ in $\tilde{\mathfrak{g}}$ is constructed as follows: choose a family of curves $\bar{v}(t,s)$ in $\tilde{\mathfrak{g}}$ such that $\bar{v}(t,0) = \bar{v}(t)$ and define

$$\delta\bar{v}(t) = \left.\frac{\partial \bar{v}(t,s)}{\partial s}\right|_{s=0}.$$

This $\delta\bar{v}(t)$ is, for each t, an element of $T\tilde{\mathfrak{g}}$. However, it turns out that we will not need these kinds of general variations $\delta\bar{v}$ subsequently. Instead, we are interested in the special kind of deformations $\bar{v}(t,s)$ of the curve $\bar{v}(t)$ in which the projection $\tilde{\pi}_G(\bar{v}(t,s)) = x(t,s)$ does not depend on s, that is, deformations that take place only in the fiber of $\tilde{\mathfrak{g}}$ over $x(t) = \tilde{\pi}_G(\bar{v}(t))$; thus, for each fixed t, the curve $s \mapsto \bar{v}(t,s)$ is a curve in the fiber over $x(t)$. Then, since $\tilde{\mathfrak{g}}$ is a vector bundle, the variation $\delta\bar{v}(t)$ induced by such a deformation $\bar{v}(t,s)$, is naturally identified with a curve, also called $\delta\bar{v}(t)$, in $\tilde{\mathfrak{g}}$, a $\tilde{\mathfrak{g}}$-***fiber variation***, according to the notation introduced in the paragraph **Deformation of Curves** of §3.1. We remark that, in the rest of this paper, $\delta\bar{v}$ will always mean a $\tilde{\mathfrak{g}}$-variation, unless explicitly stated otherwise. Also, the meaning of $\delta\bar{v}$ as an element of $T\tilde{\mathfrak{g}}$ will never be used without an explicit previous warning.

Examples of $\tilde{\mathfrak{g}}$-fiber variations $\delta\bar{v}$ are, for instance, the covariant variations $\delta^A\bar{v}$ considered in Definition 3.1.3, but, of course, there are more general variations of this type. We have already encountered an easy example of such variations when reviewing the the Euler-Poincaré equations. In that case, $Q = G$, the connection $A : TG \to \mathfrak{g}$ is given by right translation, and $\delta^A v(t) = \dot{\eta}(t) + [v(t), \eta(t)]$, for $\eta(t)$ a curve in $\mathfrak{g}$ vanishing at the endpoints.

Of course, in the Euler-Poincaré case, it is obviously true that *any* deformation of a curve $v(t)$ is a deformation along the fiber, because the base of $\tilde{\mathfrak{g}}$ is a point. However, as we have seen in the Euler-Poincaré Theorem 3.2.4, it is *not* true that *any* curve in $\mathfrak{g}$ is induced by a variation δg that vanishes at the endpoints; the latter are only the curves of the type $\dot{\eta}(t) + [v(t), \eta(t)]$, for $\eta(t)$ an arbitrary curve in $\mathfrak{g}$ vanishing at the endpoints.

In the study of the Lagrange-Poincaré operator and Lagrange-Poincaré equation we will use variations of curves in $Q/G \oplus \tilde{\mathfrak{g}}$. (As explained in paragraph **Bundles** of §2.1, the first summand means the vector bundle over Q/G with zero dimensional fiber). For a given curve $x(t) \oplus \bar{v}(t)$ in $Q/G \oplus \tilde{\mathfrak{g}}$, and a given arbitrary deformation $x(t, \lambda) \oplus \bar{v}(t, \lambda)$, with $x(t, 0) \oplus \bar{v}(t, 0) = x(t) \oplus \bar{v}(t)$, of it, the corresponding *covariant variation* $\delta x(t) \oplus \delta^A \bar{v}(t)$ is, by definition,

$$\delta x(t) \oplus \delta^A \bar{v}(t) = \left.\frac{\partial x(t,s)}{\partial s}\right|_{s=0} \oplus \left.\frac{D\bar{v}(t,s)}{Ds}\right|_{s=0}.$$

It is clear that $\delta^A \bar{v}$ is a $\tilde{\mathfrak{g}}$-fiber variation of $\bar{v}$. The most important example of covariant variation $\delta x(t) \oplus \delta^A \bar{v}(t)$ is the one to be described next. Let $q(t, s)$ be a deformation of a curve $q(t) = q(t, 0)$ in Q. This induces a deformation $x(t, s) \oplus \bar{v}(t, s)$ of the curve $x(t) \oplus \bar{v}(t)$ by taking $x(t, s) = [q(t, s)]_G$ and $\bar{v}(t, s) = [q(t, s), A(q(t, s), \dot{q}(t, s))]_G$, where $\dot{q}(t, s)$ represents the derivative with respect to t. Using Lemma 2.3.4 and Definition 3.1.3, it follows that the covariant variation corresponding to this deformation of $x(t) \oplus \bar{v}(t)$ is $\delta x(t) \oplus \delta^A \bar{v}(t)$, where

$$\delta^A \bar{v}(t) = \frac{D[q(t), \eta(t)]_G}{Dt} + [q(t), [A(q(t), \dot{q}(t)), \eta(t)]]_G + \tilde{B}(\delta x(t), \dot{x}(t)),$$

is an element of $\tilde{\mathfrak{g}}$ for each t, with $\eta(t) \in \mathfrak{g}$ an arbitrary curve vanishing at the endpoints. This is a special kind of covariant variation. It is precisely to these kinds of variations that we will apply the usual techniques of the calculus of variations in the next theorems to derive the Lagrange-Poincaré operator and equation. The previous formula may be rewritten as follows, which emphasizes the similarity with the Euler-Poincaré case,

$$\delta^A \bar{v}(t) = \frac{D\bar{\eta}}{Dt}(t) + [\bar{v}(t), \bar{\eta}(t)] + \tilde{B}(\delta x(t), \dot{x}(t)),$$

where $\bar{\eta} = [q(t), \eta(t)]_G$.

The Lagrange-Poincaré Operator. We are now ready to state a theorem that introduces the Lagrange-Poincaré operator. Its proof will be contained in the proof of Theorem 3.3.4.

Theorem 3.3.1 *Let $L : TQ \to \mathbb{R}$ be an invariant Lagrangian on the principal bundle Q as before. Choose a principal connection A on Q and identify the bundles TQ/G and $T(Q/G) \oplus \tilde{\mathfrak{g}}$ using the isomorphism α_A and also the bundles $T^{(2)}Q/G$ and $T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}}$ using the isomorphism α_{A_2}, as before. Thus an element $[q, \dot{q}]_G$ of TQ/G can be written, equivalently, as an element $(x, \dot{x}, \bar{v})$ of $T(Q/G) \oplus \tilde{\mathfrak{g}}$. Let $l : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ be the reduced Lagrangian. Then there is a unique bundle map*

$$\mathcal{LP}(l) : T^{(2)}(Q/G) \oplus 2\tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

such that for any curve $q \in \Omega(Q; q_0, q_1)$ and any variation δq of q vanishing at the endpoints, the corresponding reduced curve $[q, \dot{q}]_G = (x, \dot{x}, \bar{v})$, where $\bar{v} = [q, A(q, \dot{q})]_G$, and covariant variation $\delta x \oplus \delta^A \bar{v}$, where

$$\delta^A \bar{v}(t) = \frac{D\bar{\eta}}{Dt}(t) + [\bar{v}(t), \bar{\eta}(t)] + \tilde{B}(\delta x(t), \dot{x}(t)),$$

with $\bar{\eta}(t) = [q(t), \eta(t)]_G$ and

$$\delta x(t) = T\pi(\delta q(t)),$$

satisfy

$$\mathcal{EL}(L)(q(t), \dot{q}(t), \ddot{q}(t)) \cdot \delta q(t) = \mathcal{LP}(l)(x(t), \dot{x}(t), \bar{v}(t)) \cdot (\delta x(t) \oplus \bar{\eta}(t)).$$

Notice that, after all the identifications described at the beginning of the present paragraph, the operator $\mathcal{LP}(l)$ coincides with the operator $[\mathcal{EL}(L)]_G$.

Definition 3.3.2 *The 1-form valued bundle map*

$$\mathcal{LP}(l) : T^{(2)}(Q/G) \ \oplus \ 2\tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

*defined in the preceding theorem will be called the **Lagrange-Poincaré operator**. The decomposition of the range space for $\mathcal{LP}(l)$ as a direct sum naturally induces a decomposition of the Lagrange-Poincaré operator*

$$\mathcal{LP}(l) = \mathrm{Hor}(\mathcal{LP})(l) \oplus \mathrm{Ver}(\mathcal{LP})(l)$$

*which define the **horizontal Lagrange-Poincaré operator** and the **vertical Lagrange-Poincaré operator**.*

*The **Lagrange-Poincaré equations** are, by definition, the equations $\mathcal{LP}(l) = 0$. The **horizontal Lagrange-Poincaré equation** and **vertical Lagrange-Poincaré equation** are, respectively, the equations $\mathrm{Hor}(\mathcal{LP})(l) = 0$ and $\mathrm{Ver}(\mathcal{LP})(l) = 0$.*

In the following paragraph we introduce some additional structure, namely, an arbitrary connection ∇ on the manifold Q/G. This will also help us write explicit expressions of $\mathrm{Hor}(\mathcal{LP})(l)$ and $\mathrm{Ver}(\mathcal{LP})(l)$.

The problem of computing the Lagrange-Poincaré equations can be done using any connection, as we remarked earlier and, in addition, the problem can be localized to any local trivialization of the bundle $Q \to Q/G$. Because of this, one may choose the vector space or trivial connection associated with such a local trivialization of the bundle. Of course we are not assuming that the bundle has a global flat connection.

Explicit formulas for $\mathrm{Hor}(\mathcal{LP})(l)$ and $\mathrm{Ver}(\mathcal{LP})(l)$ in coordinates using any connection can be calculated from what we have developed and are given in §4.2. Doing so, one arrives at the coordinate formulas given in Marsden and Scheurle [1993b]. We also mention that it is possible to derive these equations from Cendra, Ibort and Marsden [1987] in a straightforward way.

Reduced Covariant Derivatives. The question of calculating formulas for $\mathrm{Hor}(\mathcal{LP})(l)$ and $\mathrm{Ver}(\mathcal{LP})(l)$ rests on giving meaning to the partial derivatives

$$\frac{\partial l}{\partial x}, \quad \frac{\partial l}{\partial \dot{x}} \quad \text{and} \quad \frac{\partial l}{\partial \bar{v}}.$$

Since $\tilde{\mathfrak{g}}$ and $T(Q/G)$ are vector bundles, we may interpret the last two derivatives in a standard (fiber derivative) way as being elements of the dual bundles $T^*(Q/G)$ and $\tilde{\mathfrak{g}}^*$, for each choice of $(x, \dot{x}, \bar{v})$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$. In other words, for given $(x_0, \dot{x}_0, \bar{v}_0)$ and $(x_0, x', \bar{v}')$ we define

$$\frac{\partial l}{\partial \dot{x}}(x_0, \dot{x}_0, \bar{v}_0) \cdot x' = \frac{d}{ds}\bigg|_{s=0} l(x_0, \dot{x}_0 + sx', \bar{v}_0)$$

and

$$\frac{\partial l}{\partial \bar{v}}(x_0, \dot{x}_0, \bar{v}_0) \cdot \bar{v}' = \frac{d}{ds}\bigg|_{s=0} l(x_0, \dot{x}_0, \bar{v}_0 + s\bar{v}').$$

To define the derivative $\partial l/\partial x$, one uses the chosen connection ∇ on the manifold Q/G, as we will explain next. Let $(x_0, \dot{x}_0, \bar{v}_0)$ be a given element of $T(Q/G) \oplus \tilde{\mathfrak{g}}$. For any given curve $x(s)$ on Q/G, let $(x(s), \bar{v}(s))$ be the horizontal lift of $x(s)$ with respect to the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$ (see (2.3.1)) such that $(x(0), \bar{v}(0)) = (x_0, \bar{v}_0)$ and let $(x(s), u(s))$ be the horizontal lift of $x(s)$ with respect to the connection ∇ such that $(x(0), u(0)) = (x_0, \dot{x}_0)$. (Notice that in general, $(x(s), u(s))$ is *not* the tangent vector $(x(s), \dot{x}(s))$ to $x(s)$.)

Thus, $(x(s), u(s), \bar{v}(s))$ is a horizontal curve with respect to the connection $C = \nabla \oplus \tilde{\nabla}^A$ naturally defined on $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in terms of the connection ∇ on $T(Q/G)$ and the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$.

Definition 3.3.3 *The* ***covariant derivative*** *of l with respect to x at $(x_0, \dot{x}_0, \bar{v}_0)$ in the direction of $(x(0), \dot{x}(0))$ is defined by*

$$\frac{\partial^C l}{\partial x}(x_0, \dot{x}_0, \bar{v}_0)\,(x(0), \dot{x}(0)) = \left.\frac{d}{ds}\right|_{s=0} l\,(x(s), u(s), \bar{v}(s))\,.$$

We shall often write

$$\frac{\partial^C l}{\partial x} \equiv \frac{\partial l}{\partial x},$$

whenever there is no danger of confusion.

The covariant derivative on a given vector bundle, for instance $\tilde{\mathfrak{g}}$, induces a corresponding covariant derivative on the dual bundle, in our case $\tilde{\mathfrak{g}}^*$. More precisely, let $\alpha(t)$ be a curve in $\tilde{\mathfrak{g}}^*$. We define the covariant derivative of $\alpha(t)$ in such a way that for any curve $\bar{v}(t)$ on $\tilde{\mathfrak{g}}$ such that both $\alpha(t)$ and $\bar{v}(t)$ project on the same curve $x(t)$ on Q/G, we have

$$\frac{d}{dt}\langle \alpha(t), \bar{v}(t)\rangle = \left\langle \frac{D\alpha(t)}{Dt}, \bar{v}(t)\right\rangle + \left\langle \alpha(t), \frac{D\bar{v}(t)}{Dt}\right\rangle.$$

Likewise we can define the covariant derivative in the vector bundle $T^*(Q/G)$. Then we obtain a covariant derivative on the vector bundle $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

It is in the sense of this definition that terms like

$$\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}$$

in the second equation (which defines the horizontal Lagrange-Poincaré operator) and

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}$$

in the first equation (which defines the vertical Lagrange-Poincaré equation) of the following theorem should be interpreted. In this case D/Dt means the covariant derivative in the bundle $T^*(Q/G)$. In the first equation D/Dt is the covariant derivative in the bundle $\tilde{\mathfrak{g}}^*$.

Reduced Variational Principles and The Lagrange-Poincaré Equations. The main result in this section is the following theorem. Its proof also contains the proof of the preceding theorem.

Theorem 3.3.4 *Under the hypothesis of Theorem* 3.3.1 *we have the following:*

The ***vertical Lagrange-Poincaré operator*** *is given by*

$$\mathrm{Ver}(\mathcal{LP})(l)\cdot\bar{\eta}=\left(-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})+\mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})\right)\cdot\bar{\eta}$$

or simply,

$$\mathrm{Ver}(\mathcal{LP})(l)=\left(-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})+\mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})\right)$$

and the ***horizontal Lagrange-Poincaré operator*** *is given by*

$$\mathrm{Hor}(\mathcal{LP})(l)\cdot\delta x=\left(\frac{\partial l}{\partial x}(x,\dot{x},\bar{v})-\frac{D}{Dt}\frac{\partial l}{\partial\dot{x}}(x,\dot{x},\bar{v})\right)\delta x-\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})\tilde{B}(x)(\dot{x},\delta x).$$

or simply,

$$\mathrm{Hor}(\mathcal{LP})(l)=\left(\frac{\partial l}{\partial x}(x,\dot{x},\bar{v})-\frac{D}{Dt}\frac{\partial l}{\partial\dot{x}}(x,\dot{x},\bar{v})\right)-\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})\tilde{B}(x)(\dot{x},.).$$

Proof. In order to compute the vertical and horizontal Lagrange-Poincaré operator, it suffices to consider variations $\delta^A\bar{v}$ of a curve $x(t)\oplus\bar{v}(t)$ corresponding to vertical and horizontal variations δq of a curve $q\in\Omega(Q,q_0,q_1)$. The computations below will show that these variations suffice to give us the variational principle in the directions of the two summands in $\delta x\oplus\bar{\eta}\in T(Q/G)\oplus\tilde{\mathfrak{g}}$.

First, consider variations $\delta^A\bar{v}$ of a curve $x(t)\oplus\bar{v}(t)$ corresponding to vertical variations δq of a curve q. We have

$$0=\delta\int_{t_0}^{t_1}l(x,\dot{x},\bar{v})dt=\int_{t_0}^{t_1}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})\delta^A\bar{v}dt.$$

According to lemma 3.1.4 with $\bar{v}=[q,v]_G$ we obtain, for all curves $\eta(t)\in\mathfrak{g}$ such that $\eta(t_i)=0$ for $i=1,2$

$$\begin{aligned}0&=\int_{t_0}^{t_1}\left\langle\frac{\partial l}{\partial\bar{v}},\frac{D[q,\eta]_G}{Dt}+[q,[v,\eta]]_G\right\rangle dt\\&=\int_{t_0}^{t_1}\left\langle-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}+\mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial\bar{v}},[q,\eta]_G\right\rangle dt.\end{aligned}$$

Arbitrariness of η then yields arbitrariness of $\bar{\eta}=[q,\eta]_G$ and we obtain

$$\mathrm{Ver}(\mathcal{LP})(l)=-\frac{D}{Dt}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v})+\mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial\bar{v}}(x,\dot{x},\bar{v}).$$

Now consider variations $\delta x\oplus\delta^A\bar{v}$ corresponding to horizontal variations δq. Then we have, for all δx with $\delta x(t_i)=0$, for $i=0,1$

$$\delta\int_{t_0}^{t_1}l(x,\dot{x},\bar{v})dt=\int_{t_0}^{t_1}\left(\frac{\partial l}{\partial x}\delta x+\frac{\partial l}{\partial\dot{x}}\delta\dot{x}+\frac{\partial l}{\partial\bar{v}}\delta^A\bar{v}\right)dt.$$

Integration by parts and lemma 3.1.7 with $\bar{v} = [q, v]_G$ gives

$$\delta \int_{t_0}^{t_1} l(x, \dot{x}, \bar{v}) dt = \int_{t_0}^{t_1} \left[\left(\frac{\partial l}{\partial x} - \frac{D}{Dt} \frac{\partial l}{\partial \dot{x}} \right) (x, \dot{x}, \bar{v}) \delta x - \frac{\partial l}{\partial \bar{v}} (x, \dot{x}, \bar{v}) \tilde{B}(x)(\dot{x}, \delta x) \right] dt.$$

We need to justify the integration by parts of the term

$$\frac{\partial l}{\partial \dot{x}} \delta \dot{x}.$$

This is done by showing that

$$\delta \dot{x} = \frac{D}{D\lambda} \frac{\partial x}{\partial t} = \frac{D}{Dt} \frac{\partial x}{\partial \lambda},$$

which can be done, for example, by using Gaussian coordinates relative to the connection ∇ at each point $x(t)$. Arbitrariness of δx then yields

$$\mathrm{Hor}(\mathcal{LP})(l)(x, \dot{x}, \bar{v}) = \frac{\partial l}{\partial x}(x, \dot{x}, \bar{v}) - \frac{D}{Dt} \frac{\partial l}{\partial \dot{x}}(x, \dot{x}, \bar{v}) - \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \mathbf{i}_{\dot{x}} \tilde{B}(x). \quad \blacksquare$$

3.4 The Reduced Variational Principle

Now we turn to the reduction of the variational principle. As we stated at the beginning of this section, Hamilton's principle for a G-invariant L is equivalent to a reduced variational principle for l with respect to a reduced set of curves $[\Omega(Q; q_0, q_1)]_G$. Translated to the concrete realizations of our reduced bundles using the map α_A, this reads as follows:

The following conditions are equivalent:

(i) Hamilton's principle holds: The curve $q(t)$ is a critical point of the action functional

$$\int_{t_0}^{t_1} L(q, \dot{q}) dt$$

on $\Omega(Q; q_0, q_1)$

(ii) The ***reduced variational principle*** holds: The curve $(x(t), \bar{v}(t))$ is a critical point of the action functional

$$\int_{t_0}^{t_1} l\left(x(t), \dot{x}(t), \bar{v}(t)\right) dt$$

on the reduced family of curves $\Omega\left(\alpha_A\left([\Omega(Q; q_0, q_1)]_G\right)\right)$.

Now comes a main theorem which summarizes what we have done so far.

Theorem 3.4.1 *The following conditions are equivalent:*

(i) *Hamilton's principle holds: The curve $q(t)$ is a critical point of the action functional*

$$\int_{t_0}^{t_1} L(q,\dot{q})dt$$

on $\Omega(Q;q_0,q_1)$, that is,

$$\delta\int_{t_0}^{t_1} L(q,\dot{q})dt = 0$$

for arbitrary variations δq of the curve q such that $\delta q(t_i)=0$, for $i=0,1$.

(ii) *The **reduced variational principle** holds: The curve $x(t)\oplus\bar{v}(t)$ is a critical point of the action functional*

$$\int_{t_0}^{t_1} l\left(x(t),\dot{x}(t),\bar{v}(t)\right)dt$$

on the reduced family of curves $\alpha_A\left([\Omega(Q;q_0,q_1)]_G\right)$, that is,

$$\delta\int_{t_0}^{t_1} l\left(x(t),\dot{x}(t),\bar{v}(t)\right)dt = 0,$$

for variations $\delta x\oplus\delta^A\bar{v}$ of the curve $x(t)\oplus\bar{v}(t)$, where $\delta^A\bar{v}$ has the form

$$\delta^A\bar{v} = \frac{D\bar{\eta}}{Dt} + [\bar{v},\bar{\eta}] + \tilde{B}(\delta x,\dot{x}),$$

with the boundary conditions $\delta x(t_i)=0$ and $\bar{\eta}(t_i)=0$, for $i=0,1$. If $\bar{v}=[q,v]_G$ with $v=A(q,\dot{q})$ then $\bar{\eta}$ can be always written $\bar{\eta}=[q,\eta]_G$, and the condition $\bar{\eta}(t_i)=0$ for $i=0,1$ is equivalent to the condition $\eta(t_i)=0$ for $i=0,1$.

Also, if $x(t)=[q]_G$ and $\bar{v}=[q,v]_G$ where $v=A\left(q,\dot{q}\right)$, then variations $\delta x\oplus\delta^A\bar{v}$ such that

$$\begin{aligned}\delta^A\bar{v} &= \frac{D\bar{\eta}}{Dt} + [\bar{v},\bar{\eta}]\\ &\equiv \frac{D[q,\eta]_G}{Dt} + [q,[v,\eta]]_G\end{aligned}$$

with $\bar{\eta}(t_i)=0$ (or, equivalently, $\eta(t_i)=0$) for $i=0,1$ correspond exactly to vertical variations δq of the curve q such that $\delta q(t_i)=0$ for $i=0,1$, while variations $\delta x\oplus\delta^A\bar{v}$ such that

$$\delta^A\bar{v} = \tilde{B}(\delta x,\dot{x})$$

with $\delta x(t_i)=0$ for $1=0,1$, correspond exactly to horizontal variations δq of the curve q such that $\delta q(t_i)=0$.

(iii) *The following* ***vertical Lagrange-Poincaré equations****, corresponding to vertical variations, hold:*

$$\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x,\dot{x},\bar{v}) = \mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}}(x,\dot{x},\bar{v})$$

and the ***horizontal Lagrange-Poincaré equations****, corresponding to horizontal variations, hold:*

$$\frac{\partial l}{\partial x}(x,\dot{x},\bar{v}) - \frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\dot{x},\bar{v}) = \left\langle \frac{\partial l}{\partial \bar{v}}(x,\dot{x},\bar{v}), \mathbf{i}_{\dot{x}}\tilde{B}(x)\right\rangle.$$

Remarks.

1. The operators $\mathcal{EL}(l)$, $\mathrm{Hor}(\mathcal{LP})(l)$ and $\mathrm{Ver}(\mathcal{LP})(l)$ depend on the (principal) connection A on the principal bundle Q but not on the connection ∇ on Q/G. It is only the explicit expressions of $\mathrm{Hor}(\mathcal{LP})(l)$ and $\mathrm{Ver}(\mathcal{LP})(l)$ that appear in Theorem 3.3.4 that depend on ∇. As we have remarked previously, in local coordinates it is often convenient to choose ∇ to be simply the usual Euclidean, or vector space connection.

2. Important particular cases of Theorems 3.3.1 and 3.3.4 occur when $G = Q$ and also when $G = \{e\}$. If $G = Q$ then the operator $\mathrm{Hor}(\mathcal{LP})(l)$ is 0 and $\mathrm{Ver}(\mathcal{LP})(l)$ is the Euler–Poincaré operator, as we saw before. Thus, in a sense, the vertical Lagrange–Poincaré operator in the bundle $\tilde{\mathfrak{g}}$ is a covariant version of the usual Euler–Poincaré operator on a Lie algebra. If $G = \{e\}$ then $\mathrm{Ver}(\mathcal{LP})(l)$ is 0, $L = l$ and $\mathrm{Hor}(\mathcal{LP})(l) = \mathcal{EL}(L)$ is the usual Euler-Lagrange operator.

4 Wong's Equations and Coordinate Formulas

To illustrate the Lagrange-Poincaré theory that we have developed, we first consider an interesting example, that of Wong's equations. Secondly, in this section we give coordinate formulas for the Lagrange-Poincaré equations. Wong's equations are first done intrinsically and then are used to illustrate the coordinate formulas.

4.1 Wong's Equations

The Context of Wong's Equations. Wong's equations arise in at least two different interesting contexts. The first of these, in the the work of Wong [1970], Sternberg [1977], Weinstein [1978] and Montgomery [1984], concerns the dynamics of a colored particle in a Yang-Mills field. The second context is that of the falling cat theorem of Montgomery [1990, 1993]. For a direct proof of the falling cat theorem using the ideas of Lagrangian reduction, see Koon and Marsden [1997a] and Cendra, Holm, Marsden and Ratiu [1998].

The Abstract Setting. Let (X, g) be a given Riemannian manifold and let ∇ be the corresponding Levi-Civita connection. Let G be a compact group with a bi-invariant Riemannian metric κ. Let $\pi : Q \to X$ be a principal bundle with structure group G acting on the left, let A be a principal connection on Q and let B be the curvature of A. Now define the Lagrangian $L : TQ \to \mathbb{R}$ by

$$L(q, \dot{q}) = \frac{1}{2}\kappa\left(A(q, \dot{q}), A(q, \dot{q})\right) + \frac{1}{2}g\left(\pi(q)\right)\left(T\pi(q, \dot{q}), T\pi(q, \dot{q})\right).$$

This Lagrangian is G-invariant and our object is to carry out the constructions for Lagrangian reduction as described in the preceding sections to this situation.

We note that in the special case of $G = S^1$, this Lagrangian is the Kaluza-Klein Lagrangian for the motion of a particle in a magnetic field. In this case, the constructions are done directly in Marsden and Ratiu [1994]. More generally, this Lagrangian is the Kaluza-Klein Lagrangian for particles in a Yang-Mills field A.

Construction of the Reduced Bundle. An element of $\tilde{\mathfrak{g}}$ has the form $\bar{v} = [q, v]_G$ where $q \in Q$ and $v \in \mathfrak{g}$. Since κ is bi-invariant, its restriction to $\mathfrak{g}$ is Ad-invariant, and so we can define the fiber metric k on $\tilde{\mathfrak{g}}$ by

$$k\left([q, u]_G, [q, v]_G\right) = \kappa(u, v).$$

The reduced bundle is $T(Q/G) \oplus \tilde{\mathfrak{g}} \equiv TX \oplus \tilde{\mathfrak{g}}$ and a typical element of it is denoted $(x, \dot{x}, \bar{v})$. The reduced Lagrangian is given by

$$l(x, \dot{x}, \bar{v}) = \frac{1}{2}k(\bar{v}, \bar{v}) + \frac{1}{2}g(x)(x, \dot{x}).$$

Calculation of the Reduced Equations. Now we will write the vertical and horizontal Lagrange-Poincaré equations. We start by writing the vertical Lagrange-Poincaré equation from Theorem 3.3.1 as follows:

$$\left(-\frac{D}{Dt}\frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) + \mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v})\right) \cdot \bar{\eta} = 0 \tag{4.1.1}$$

for all $\bar{\eta} \in \tilde{\mathfrak{g}}$. We first note that

$$\frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) = k(\bar{v}, \cdot)$$

and hence

$$\left(\mathrm{ad}^*_{\bar{v}}\frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v})\right)\bar{\eta} = k\left(\bar{v}, [\bar{v}, \bar{\eta}]\right) = 0,$$

since κ and hence k are bi-invariant. Thus, the vertical Lagrange-Poincaré equation is

$$\frac{D}{Dt}k(\bar{v},\cdot)=0,$$

which is one of Wong's equations, namely the charge equation. We will see this explicitly in coordinates later.

From Theorem 3.4.1, the horizontal Lagrange-Poincaré equation is

$$\frac{\partial l}{\partial x}(x,\dot{x},\bar{v})-\frac{D}{Dt}\frac{\partial l}{\partial \dot{x}}(x,\dot{x},\bar{v})=\left\langle \frac{\partial l}{\partial \bar{v}}(x,\dot{x},\bar{v}),\mathbf{i}_{\dot{x}}\tilde{B}(x)\right\rangle .$$

Perhaps the easiest way to work out this expression is to do so in a local trivialization of the principal bundle, which induces a trivialization of $\tilde{\mathfrak{g}}$. In such a local trivialization, the metric k is independent of the base point x and so the left side of the preceding equation, and making use of the vertical equation, becomes the usual Euler-Lagrange expression. Note that this expression is independent of which affine connection is used on X. It is well known that the Euler-Lagrange expression for the kinetic energy on X gives the covariant acceleration $\nabla_{\dot{x}}\dot{x}$ using the Levi-Civita connection for the metric on X. Therefore, the horizontal Lagrange-Poincaré equation becomes

$$(\nabla_{\dot{x}}\dot{x})^{\flat}=-k\left(\bar{v},\tilde{B}(x)(\dot{x},\,\cdot)\right),$$

which is the second Wong equation.

4.2 The Local Vertical and Horizontal Equations

In this section we shall derive local forms, that is for a local trivialization of a principal bundle, of both the vertical and the horizontal Lagrange-Poincaré operator. The expressions that we obtain coincide with or can be easily derived from the ones obtained in Cendra and Marsden [1987], Cendra, Marsden and Ibort [1987] and Marsden and Scheurle [1993b], with some changes in the notation. We start with the covariant formulas for the vertical and horizontal Lagrange-Poincaré operators described in the previous theorems and the local expressions are then easily derived.

The Local Vertical Lagrange-Poincaré Equation. We shall now derive local coordinate expressions for the vertical Lagrange–Poincaré equations.

Suppose that Q has dimension n, so that Q/G has dimension $r=n-\dim G$. We choose a local trivialization of the principal bundle $Q\to Q/G$ to be $X\times G$, where X is an open set in $\mathbb{R}^r$. Thus, we consider the trivial principal bundle $\pi:X\times G\to X$ with structure group G acting only on the second factor by

left multiplication. Let e be the neutral element of G, and let A be a given principal connection on the bundle $Q \to Q/G$, or, in local representation, on the bundle $X \times G \to X$. We shall also assume that there are local coordinates $x^\alpha, \alpha = 1, \dots, r$ in X and that *we choose the standard flat connection on* X. Then, at any tangent vector $(x, g, \dot{x}, \dot{g}) \in T_{(x,g)}(X \times G)$ we have

$$A(x, g, \dot{x}, \dot{g}) = \mathrm{Ad}_g \left(A_e(x) \cdot \dot{x} + v\right)$$

where A_e is the $\mathfrak{g}$-valued 1-form on X defined by $A_e(x) \cdot \dot{x} = A(x, e, \dot{x}, 0)$ and $v = g^{-1}\dot{g}$. The vector bundle isomorphism α_A in this case becomes

$$\alpha_A \left([x, g, \dot{x}, \dot{g}]_G\right) = (x, \dot{x}) \oplus \bar{v}$$

where $\bar{v} = (x, A_e(x) \cdot \dot{x} + v)$. We will often write $(x, \dot{x}, \bar{v})$ instead of $(x, \dot{x}) \oplus \bar{v}$, and sometimes, simply $\bar{v} = A_e(x) \cdot \dot{x} + v$, for simplicity. Let us choose maps

$$\mathbf{e}_b : X \to \mathfrak{g}$$

where $b = 1, ..., \dim(G)$, such that, for each $x \in X$, $\{\mathbf{e}_b(x) \mid b = 1, ..., \dim(G)\}$ is a basis of $\mathfrak{g}$. For each $b = 1, .., \dim(G)$, let $\bar{\mathbf{e}}_b(x)$ be the section of $\tilde{\mathfrak{g}}$ given by $\bar{\mathbf{e}}_b(x) = [x, e, \mathbf{e}_b(x)]_G \equiv (x, \mathbf{e}_b(x))$. Let us call $p = p(x, \dot{x}, \bar{v})$ the vertical body momentum of the reduced system, that is, by definition,

$$p(x, \dot{x}, \bar{v}) = \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}). \tag{4.2.1}$$

We denote the components of p by $p_b = p(\bar{\mathbf{e}}_b) \equiv \langle p, \bar{\mathbf{e}}_b \rangle$. We want to find an equation for the evolution of p_b. We have

$$\begin{aligned} \frac{d}{dt} p_b &= \frac{d}{dt} \langle p, \bar{\mathbf{e}}_b \rangle \\ &= \left\langle \frac{D}{Dt} p, \bar{\mathbf{e}}_b \right\rangle + \left\langle p, \frac{D}{Dt} \bar{\mathbf{e}}_b \right\rangle . \end{aligned} \tag{4.2.2}$$

Using the vertical Lagrange-Poincaré equation we obtain, immediately,

$$\begin{aligned} \left\langle \frac{D}{Dt} p, \bar{\mathbf{e}}_b \right\rangle &= \langle p, [\bar{v}, \bar{\mathbf{e}}_b] \rangle \\ &= \langle p, [A_e(x) \cdot \dot{x} + v, \mathbf{e}_b] \rangle . \end{aligned} \tag{4.2.3}$$

Lemma 2.3.4 gives the general formula for calculating the covariant derivative of a given curve $[q(t), \xi(t)]_G$ in $\tilde{\mathfrak{g}}$,

$$\frac{D[q(t), \xi(t)]_G}{Dt} = \Big[q(t), -[A\left(q(t), \dot{q}(t)\right), \xi(t)] + \dot{\xi}(t)\Big]_G . \tag{4.2.4}$$

We are going to apply this formula for the case of the curve

$$\bar{\mathbf{e}}_b\left(x(t)\right) = \left[x(t), e, \mathbf{e}_b\left(x(t)\right)\right]_G \equiv \left(x(t), \mathbf{e}_b\left(x(t)\right)\right)$$

in $\tilde{\mathfrak{g}}$. Note that the tangent vector to the curve $q(t) \equiv (x(t), e)$ is $(q(t), \dot{q}(t)) \equiv (x(t), e, \dot{x}(t), 0)$, and therefore $A\left(q(t), \dot{q}(t)\right) \equiv A_e\left(x(t)\right) \cdot \dot{x}$. Using equation (4.2.4) we obtain

$$\begin{aligned} \frac{D}{Dt}\bar{\mathbf{e}}_b &= \left[x, e, -[A_e(x) \cdot \dot{x}, \mathbf{e}_b] + \dot{\mathbf{e}}_b\right]_G \\ &\equiv \left(x, -[A_e(x) \cdot \dot{x}, \mathbf{e}_b] + \dot{\mathbf{e}}_b\right). \end{aligned} \tag{4.2.5}$$

Using equations (4.2.2), (4.2.3) and (4.2.5) we obtain the equation

$$\frac{dp_b}{dt} = \langle p, [v, \mathbf{e}_b] + \dot{\mathbf{e}}_b \rangle . \tag{4.2.6}$$

Using this equation we can easily find an expression in coordinates for the vertical Lagrange-Poincaré equation. Let us choose the functions $\mathbf{e}_b(x)$ to be constant functions, therefore, we have $\dot{\mathbf{e}}_b = 0$ and the equation for the evolution of p_b becomes

$$\frac{dp_b}{dt} = \langle p, [v, \mathbf{e}_b] \rangle . \tag{4.2.7}$$

Recall that $\bar{v} - A(x) \cdot \dot{x} = v$, and thus the equation (4.2.7) becomes

$$\frac{dp_b}{dt} = \langle p, [\bar{v} - A(x) \cdot \dot{x}, \mathbf{e}_b] \rangle . \tag{4.2.8}$$

Let C^a_{bd} be the structure constants of the Lie algebra $\mathfrak{g}$. For the given local coordinates x^α in X let $A^a_\alpha(x)$ be the coefficients of A_e, that is, by definition, $(A_e(x) \cdot \dot{x})^a \mathbf{e}_a = A^a_\alpha(x)\dot{x}^\alpha \mathbf{e}_a$. Then equation (4.2.8) becomes the "***Poincaré part***" of the Lagrange-Poincaré equations:

$$\frac{dp_b}{dt} = p_a \left(C^a_{db}\bar{v}^d - C^a_{db}A^d_\alpha \dot{x}^\alpha\right), \tag{4.2.9}$$

This equation coincides with equation (5.3.3) of Bloch, Krishnaprasad, Marsden and Murray [1996] and equation (3.2) in Koon and Marsden [1997a]. It also agrees, up to some sign problems, with that in Marsden and Scheurle [1993b] and is also implicit in Cendra, Ibort and Marsden [1987].

Equation (4.2.9) reduce to the Euler-Poincaré equations in the case that the base is a point. If we think of the variables evolving as $(x, \dot{x}, \bar{v})$, and, thinking of p_b as a function of these variables using the definition of p_b, we can write the equation as

$$\frac{dp_b(x, \dot{x}, \bar{v})}{dt} = p_a(x, \dot{x}, \bar{v}) \left(C^a_{db}\bar{v}^d - C^a_{db}A^d_\alpha(x)\dot{x}^\alpha\right). \tag{4.2.10}$$

Notice that using the variable v, which is obtained by $\bar{v} = A(x) \cdot \dot{x} + v$, we can write the equation as

$$\frac{dp_b(x, \dot{x}, A(x) \cdot \dot{x} + v)}{dt} = p_a(x, \dot{x}, A(x)\dot{x} + v) C^a_{db} v^d. \tag{4.2.11}$$

Observe that one can calculate p_b in equation (4.2.1) by taking the derivative of l with respect to either $\bar{v}^b$ or v^b.

In Bloch, Krishnaprasad, Marsden and Murray [1996], the variable $\bar{v}^b$ is called Ω^b and is interpreted as the locked body angular velocity. This variable is intrinsic, given the choice of a connection, whereas v^b depends on the local trivialization. In fact, the form of equation (4.2.11) is dependent on choosing a local trivialization.

The Local Horizontal Lagrange-Poincaré Equation. To calculate the local horizontal Lagrange-Poincaré equation we shall first calculate

$$\frac{\partial^C l}{\partial x}(x, \dot{x}, \bar{v}) \cdot \delta x.$$

By Definition 3.3.3 of the notation $\partial^C l / \partial x$, we have

$$\frac{\partial^C l}{\partial x}(x, \dot{x}, \bar{v}) \cdot \delta x = \frac{d}{d\lambda} l\left(x + \lambda \delta x, \dot{x}, \bar{w}(\lambda)\right)\Big|_{\lambda=0},$$

where $\bar{w}(\lambda)$ is a curve such that $\bar{w}(\lambda) \in \tilde{\mathfrak{g}}_{x+\lambda\delta x}$ for each λ, $\bar{w}(0) = \bar{v}$ and

$$\frac{D\bar{w}(\lambda)}{D\lambda} = 0.$$

If $\bar{w}(\lambda) = (x + \lambda\delta x, e, w(\lambda))$, we can deduce from equation (4.2.4)

$$\frac{D\bar{w}(\lambda)}{D\lambda} = \left(x(\lambda), e, -\left[A\left(x(\lambda)\right) \cdot \delta x, w(\lambda)\right] + \frac{dw(\lambda)}{d\lambda}\right). \tag{4.2.12}$$

Therefore we must have

$$\frac{dw(\lambda)}{d\lambda} = \left[A\left(x(\lambda)\right) \cdot \delta x, w(\lambda)\right].$$

We then obtain

$$\frac{\partial^C l}{\partial x}(x, \dot{x}, \bar{v}) \cdot \delta x = \frac{\partial l}{\partial x}(x, \dot{x}, \bar{v}) \cdot \delta x + \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \cdot \left[A\left(x(\lambda)\right) \cdot \delta x, \bar{w}(\lambda)\right].$$

On the other hand, it is easy to see that $\tilde{B}(x)(\dot{x}, \delta x) = (x, e, B(x, e)(\dot{x}, \delta x))$. Then the horizontal Lagrange-Poincaré operator is

$$\left(\frac{\partial l}{\partial x}(x, \dot{x}, \bar{v}) - \frac{d}{dt}\frac{\partial l}{\partial \dot{x}}(x, \dot{x}, \bar{v})\right) \delta x = \frac{\partial l}{\partial \bar{v}}(x, \dot{x}, \bar{v}) \left(B(x, e)(\dot{x}, \delta x) + [\bar{v}, A(x) \cdot \delta x]\right). \tag{4.2.13}$$

As we did with the vertical Lagrange-Poincaré operator, it is convenient to rewrite this equation explicitly in coordinates and we easily obtain

$$\frac{\partial l}{\partial x^\alpha}(x,\dot{x},\bar{v}) - \frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha}(x,\dot{x},\bar{v}) = \frac{\partial l}{\partial \bar{v}^a}(x,\dot{x},\bar{v})\left(B^a_{\beta\alpha}(x,e)\dot{x}^\beta + C^a_{db}\bar{v}^d A^b_\alpha(x)\right), \tag{4.2.14}$$

where a fixed basis $\bar{\mathbf{e}}_a$ of $\tilde{\mathfrak{g}}$ has been chosen and, in this basis, $\bar{v} = \bar{v}^a\bar{\mathbf{e}}_a$. This equation coincides with equation (5.3.2) of Bloch, Krishnaprasad, Marsden and Murray [1996] and equation (3.1) of Koon and Marsden [1997a]. We remark that in these papers the convention for the sign of the curvature $B^a_{\alpha\beta}$ is the opposite to the one used in this paper.

Summary. The Lagrange-Poincaré equations in coordinates have been shown to be (dropping the independent variables from the notation)

$$\frac{dp_b}{dt} = p_a\left(C^a_{db}\bar{v}^d - C^a_{db}A^d_\alpha\dot{x}^\alpha\right) \tag{4.2.15}$$

$$\frac{\partial l}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} = \frac{\partial l}{\partial \bar{v}^a}\left(B^a_{\beta\alpha}\dot{x}^\beta + C^a_{db}\bar{v}^d A^b_\alpha\right), \tag{4.2.16}$$

where, as usual, a summation is implied over repeated indices.

In the special case when the bundle $Q \to Q/G$ is endowed with a trivial connection in local representation, i.e., $A = 0$, these equations simply become

$$\frac{dp_b}{dt} - p_a C^a_{db}\bar{v}^d = 0 \tag{4.2.17}$$

$$\frac{\partial l}{\partial x^\alpha} - \frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} = 0, \tag{4.2.18}$$

which are the ***Hamel equations*** (Hamel [1904]).

In many concrete applications, involving, for instance, stability theory, the general form of the Lagrange-Poincaré equations given in (4.2.15) and (4.2.16) are more useful than Hamel's equations, as explained by an example in Marsden and Scheurle [1993b].

Coordinate Version of Wong's Equations. Locally, the expression of the Lagrangian l in the variables $(x,\dot{x},\bar{v})$ is

$$l(x,\dot{x},\bar{v}) = \frac{1}{2}\kappa_{ab}\bar{v}^a\bar{v}^b + \frac{1}{2}g_{\alpha\beta}(x)\dot{x}^\alpha\dot{x}^\beta.$$

The local expression of the vertical Lagrange-Poincaré equation is given by (4.2.15), where

$$p_a = \frac{\partial l}{\partial \bar{v}^a} = \kappa_{ab}\bar{v}^b.$$

However, the first term on the right hand side of (4.2.15) equals

$$p_a C^a_{db} \bar{v}^d = \kappa_{ae} \bar{v}^e C^a_{db} \bar{v}^d.$$

However, bi-invariance of κ means that

$$\kappa_{ae} C^a_{db} = \kappa_{ab} C^a_{ed}$$

and so we get

$$p_a C^a_{db} \bar{v}^d = \kappa_{ae} \bar{v}^e C^a_{db} \bar{v}^d = \kappa_{ab} \bar{v}^e C^a_{ed} \bar{v}^d = \kappa_{ab} [\bar{v}, \bar{v}]^a = 0.$$

Therefore, the vertical equation becomes

$$\frac{dp_b}{dt} = -p_a C^a_{db} A^d_\alpha \dot{x}^\alpha.$$

The local expression of the horizontal Lagrange-Poincaré equation is given by (4.2.16), where

$$\frac{\partial l}{\partial x^\alpha} = \frac{1}{2} \frac{\partial g_{\beta\gamma}(x)}{\partial x^\alpha} \dot{x}^\beta \dot{x}^\gamma.$$

The second term on the right hand side of (4.2.16) vanishes as in the case of the vertical equation. Therefore the horizontal Lagrange-Poincaré equation is given by

$$\frac{d}{dt}(g_{\alpha\beta}(x)\dot{x}^\beta) = -p_a B^a_{\beta\alpha} \dot{x}^\beta + \frac{1}{2} \frac{\partial g_{\beta\gamma}(x)}{\partial x^\alpha} \dot{x}^\beta \dot{x}^\gamma,$$

or equivalently, with $p_\alpha := g_{\alpha\beta}(x)\dot{x}^\beta$,

$$\frac{d}{dt} p_\alpha = -p_a B^a_{\beta\alpha} \dot{x}^\beta - \frac{1}{2} \frac{\partial g^{\beta\gamma}}{\partial x^\alpha} p_\beta p_\gamma,$$

which is the second Wong equation.

5 The Lie Algebra Structure on Sections of the Reduced Bundle

The main result of this section is the establishment of a natural Lie algebra structure on the space $\Gamma(T(Q/G) \oplus \tilde{\mathfrak{g}})$ of sections of the bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$, which will be used for reduction in the next section. This quotient Lie algebra structure is defined in Definition 5.2.3 and it is computed in Theorem 5.2.4. This Lie algebra is, roughly speaking, a synthesis of the Jacobi-Lie bracket of vector fields on shape space Q/G with the Lie algebra structure on the bundle $\tilde{\mathfrak{g}}$. However, as we shall see, this Lie algebra structure involves the reduced curvature form as well.

5.1 The Bundle $T(Q/G) \oplus \tilde{\mathfrak{g}}$ Revisited

In this subsection, we assume that we have the following set up: a manifold Q, a group action of G on Q, say $\rho : G \times Q \to Q$ and a connection A on the principal bundle $\pi : Q \to Q/G$.

The Vertical and Horizontal Invariant Bundles. Let $\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ be the vector bundle isomorphism defined in lemma 2.4.2 and let $TQ = \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$ be the decomposition into horizontal and vertical parts. Since the bundles $\mathrm{Hor}(TQ)$ and $\mathrm{Ver}(TQ)$ are G-invariant we have

$$TQ/G = \mathrm{Hor}(TQ)/G \oplus \mathrm{Ver}(TQ)/G.$$

It is easy to check that $\alpha_A(\mathrm{Hor}(TQ)/G) = T(Q/G)$ and that $\alpha_A(\mathrm{Ver}(TQ)/G) = \tilde{\mathfrak{g}}$.

Definition 5.1.1 *Let*

$$\iota_G(TQ) : I_G(TQ) \to Q/G$$

be the vector bundle whose fiber $(\iota_G(TQ))^{-1}(x)$ at an element $x = [q]_G$ in Q/G is the vector space of all invariant vector fields on Q along $\pi^{-1}(x)$. That is, an element of $I_G(TQ)$ is a vector field, say Z, defined only at points $q \in \pi^{-1}(x)$, that is, $Z(q) \in TQ$ for all $q \in \pi^{-1}(x)$, such that $gZ = Z$ for all $g \in G$.

We also let

$$\iota_G(TQ)^V : I_G^V(TQ) \to Q/G$$

be the vector bundle whose fiber $(\iota_G(TQ)^V)^{-1}(x)$ at an element $x = [q]_G$ in Q/G is the vector space of all vertical invariant vector fields on $\pi^{-1}(x)$. That is, an element of $I_G^V(TQ)$ is a vector field, say Y, on the manifold $\pi^{-1}(x)$ such that $gY = Y$ for all $g \in G$. We call $\iota_G(TQ)^V : I_G^V(TQ) \to Q/G$ the ***vertical invariant bundle.***

Likewise, we define

$$\iota_G(TQ)^H : I_G^H(TQ) \to Q/G$$

to be the vector bundle whose fiber $(\iota_G(TQ)^H)^{-1}(x)$ at an element $x = [q]_G$ in Q/G is the vector space of all horizontal invariant vector fields on Q along $\pi^{-1}(x)$. That is, an element of $I_G^H(TQ)$ is an horizontal vector field, say X defined only at points $q \in \pi^{-1}(x)$, such that $X(q) \in \mathrm{Hor}(TQ)$ for all $q \in \pi^{-1}(x)$ and such that $gX = X$ for all $g \in G$. We call $\iota_G(TQ)^H : I_G^H(TQ) \to Q/G$ the ***horizontal invariant bundle.***

Brackets of Invariant Vector Fields. Recall that the Lie bracket of two invariant vector fields, say X and Y, is again an invariant vector field. Also, if X and Y are both vertical invariant vector fields, that is, $X, Y \in I_G^V(TQ)$, then $[X, Y]$ is also a vertical invariant vector field, that is, $[X, Y] \in I_G^V(TQ)$.

For any $q \in Q$, the map $\rho_q : G \to \pi^{-1}(x)$, where $x = [q]_G$ is given by $\rho_q(g) = gq$, is a diffeomorphism that commutes with the action of G, that is, $\rho_q(hg) = h\rho_q(g)$, for all $h, g \in G$. Using this, it follows that $I_G^V(TQ)_x$ is isomorphic to the Lie algebra of left invariant vector fields on G, which in turn, is naturally identified with T_eG in the usual way (we shall often use the same symbol $\mathfrak{g}$ to denote both T_eG with the usual Lie algebra structure and, also, the space of left invariant vector fields on G, with the bracket given by the Lie bracket of vector fields). This identification is given by the push-forward of vector fields $\rho_{q*} : \mathfrak{g} \to I_G^V(TQ)_x$. Notice that the tangent map of ρ_q at e is given by $T_e\rho_q\xi = \xi q$. This implies in particular that for any given $\xi \in \mathfrak{g}$, this time considered as being a left invariant vector field on G, we have $\rho_{q*}\xi(q) = \xi q$.

On the other hand the map $T\pi$ establishes an isomorphism between $I_G^H(TQ)$ and $T(Q/G)$.

The Bundles $I_G^V(TQ)$ and $\tilde{\mathfrak{g}}$ are Isomorphic. The next lemma shows that there is a natural isomorphism between these bundles.

Lemma 5.1.2 *The map*

$$\beta_A : I_G^V(TQ) \to \tilde{\mathfrak{g}}$$

given by

$$\beta_A(Y) = [q, A(Y(q))]_G,$$

where Y is an element of the fiber of $I_G^V(TQ)$ at $x \in Q/G$ and $q \in \pi^{-1}(x)$ is arbitrary, is a well defined Lie algebra bundle isomorphism. The inverse of β_A is defined by the following condition: $\beta_A^{-1}([q, \xi]_G)$ is the (unique) left invariant vector field Y on $\pi^{-1}(x)$ such that $Y(q) = \xi q$.

We will extend the domain of β_A as follows: we define the vector bundle isomorphism

$$\beta_A : I_G^H(TQ) \oplus I_G^V(TQ) \to T(Q/G) \oplus \tilde{\mathfrak{g}}$$

in such a way that it coincides with the isomorphism given by $T\pi$ on the summand $I_G^H(TQ)$, and with the isomorphism β_A defined above on the summand $I_G^V(TQ)$.

Proof. To see that β_A is well defined we simply check that for any $g \in G$ we have

$$\begin{aligned}[gq, A(Y(gq))]_G &= [gq, A(gY(q))]_G \\ &= [gq, \mathrm{Ad}_g Y(q)]_G \\ &= [q, A(Y(q))]_G.\end{aligned}$$

It is easy to check that β_A is linear on each fiber. Now we will show that β_A is a Lie algebra bundle isomorphism. Let us fix $q \in Q$. Then the pull-back $\rho_q^* : I_G^V(TQ)_x \to \mathfrak{g}$ is a Lie algebra isomorphism and we can easily check that, for all $Y \in I_G^V(TQ)_x$, and any $q \in \pi^{-1}(x)$ we have $\rho_q^*(Y) = A(Y(q))$. Therefore we have, for all $X, Y \in I_G^V(TQ)_x$,

$$\begin{aligned}A([X,Y](q)) &= \rho_q^*([X,Y]) \\ &= [\rho_q^*(X), \rho_q^*(Y)] \\ &= [A(X(q)), A(Y(q))].\end{aligned}$$

Using this and the definition of the Lie bracket on $\tilde{\mathfrak{g}}$ (see lemma 2.3.5) we can write

$$\begin{aligned}\beta_A([X,Y]) &= [q, A([X,Y](q))]_G \\ &= [q, [A(X(q)), A(Y(q))]]_G \\ &= [\beta_A(X), \beta_A(Y)].\end{aligned}$$

To check that β_A^{-1} is given by the rule described in the statement of the lemma, we must simply check that $A(Y(q)) = \xi$. The rest of the proof, namely, to show that the extended β_A is a vector bundle isomorphism is easy. ■

5.2 The Lie Algebra of Sections of $T(Q/G) \oplus \tilde{\mathfrak{g}}$

Quotient Vector Bundles. We begin with some preliminaries concerning quotient vector bundles with some additional structures.

Let $\tau : V \to Q$ be a given vector bundle and let the group G act on V and let $\rho : G \times V \to V$ denote the action (see §2.1).

Recall that, by definition, the action ρ is a ***vector bundle action*** if for each $g \in G$ the map $\rho_g : V \to V$ is a vector bundle isomorphism. This implies, in particular, that there is an action $\rho_0 : G \times Q \to Q$ such that for each $q \in Q$ the restriction of ρ_g to $\tau^{-1}(q)$ is a linear isomorphism $\rho_g : \tau^{-1}(q) \to \tau^{-1}(\rho_{0g}q)$. We will often use the simpler notation gv and also gq instead of $\rho_g(v)$ and $\rho_0 g$ respectively. As we said in §2.1, throughout this work we are going to assume that the action ρ_0 of G on Q described above is free and, moreover, that with this action Q is a principal bundle. Then also V is a principal bundle

with structure group G. Although this assumption is not strictly needed for the validity of some properties we will still take it for granted in this work, to simplify the exposition. An immediate consequence of this is that the quotient V/G carries a naturally defined vector bundle structure over the base Q/G.

More precisely we have the following lemma, whose proof, which is standard, we include for the sake of completeness.

Lemma 5.2.1 *The quotient V/G carries a naturally defined vector bundle structure over the base Q/G, say $\tau/G : V/G \to Q/G$, where $\tau/G([v]_G)$ is defined by $\tau/G([v]_G) = [\tau v]_G$.*

The projection $\pi_G(V) : V \to V/G$ is a surjective vector bundle homomorphism and the restriction $\pi_G(V) : \tau^{-1}(q) \to (\tau/G)^{-1}([q]_G)$ is a linear isomorphism for each $q \in Q$. Assume that $\tau' : V' \to Q'$ is another vector bundle and that there is an action $G \times V' \to V'$ having the same properties as the action of G on V. Let $f : V \to V'$ be an equivariant vector bundle map. Then the naturally induced quotient map $[f]_G : V/G \to V'/G$ is a vector bundle map. If f is an isomorphism, so is $[f]_G$.

Proof. First we show that τ/G is well defined, namely, for any $g \in G$ and any $v \in V$ we have $\tau/G([gv]_G) = [\tau gv]_G = [g\tau v]_G = [\tau v]_G$. Now we define the vector space structure on each fiber $\tau/G^{-1}([q]_G)$. Let $[v_i]_G \in V/G$, for $i = 1, 2$ be such that $\tau/G[v_1]_G = \tau/G[v_2]_G = [q]_G$, for some $q \in Q$. This implies that there are $g_i \in G$, $i = 1, 2$ such that $g_i\tau v_i = q$ for $i = 1, 2$. The g_i are uniquely determined, for a fixed q as before, because the action ρ_0 is free. Define $[v_1]_G + [v_2]_G = [g_1v_1 + g_2v_2]_G$. We must show that this gives a well defined additive structure. Elements of V equivalent to v_i are of the type h_iv_i with $h_i \in G$ for $i = 1, 2$. For given h_iv_i with $h_i \in G$ for $i = 1, 2$, the only elements $d_i \in G$ such that $d_i\tau h_iv_i = q$ for $i = 1, 2$ are $d_i = g_ih_i^{-1}$. Then our definition gives

$$\begin{aligned}[h_1v_1]_G + [h_2v_2]_G = [d_1h_1v_1 + d_2h_2v_2]_G &= [g_1v_1 + g_2v_2]_G \\ &= [v_1]_G + [v_2]_G.\end{aligned}$$

This shows that the definition does not depend on the choice of the element belonging to $[v_i]_G$ for $i = 1, 2$, for given q. Now if we choose an arbitrary element say $hq \in [q]_G$, then the only elements $d_i \in G$ such that $d_i\tau v_i = hq$ are $d_i = hg_i$, for $i = 1, 2$. Then

$$\begin{aligned}[v_1]_G + [v_2]_G &= [hg_1v_1 + hg_2v_2]_G \\ &= [h(g_1v_1 + g_2v_2)]_G \\ &= [g_1v_1 + g_2v_2]_G\end{aligned}$$

as before.

We can define $\lambda[v]_G = [\lambda v]_G$ and check that it is well defined in a similar way. To finish the proof, it only remains to show that the restriction of π_G to each fiber $\pi_G(V) : \tau^{-1}(q) \to (\tau/G)^{-1}([q]_G)$ is a linear isomorphism. This can be easily established using the definition of the linear structure on $(\tau/G)^{-1}([q]_G)$. We omit the rest of the proof, which can also be easily performed using standard techniques. ■

Spaces of Sections. For a vector bundle $\tau : V \to Q$, the vector space of sections of V is denoted by $\Gamma(V)$, which is also a $C^\infty(Q)$-module. Let $f : V_1 \to V_2$ be a vector bundle homomorphism where $\tau^i : V_i \to Q_i$ is a vector bundle for $i = 1, 2$. This implies in particular that there is a map $f_0 : Q_1 \to Q_2$ such that, for each $q \in Q_1$, the restriction $f : (\tau^1)^{-1}(q) \to (\tau^2)^{-1}(f_0(q))$ is linear. Assume, in addition, that this restriction is a linear isomorphism. Then f induces a linear map $f^* : \Gamma(V_2) \to \Gamma(V_1)$ defined by $f^*(s)(q) = f^{-1}(s(f_0(q)))$, where f^{-1} is the inverse of the restriction of f to $(\tau^2)^{-1}(q)$. The properties $(f \circ h)^* = h^* \circ f^*$ and $\mathrm{id}_V^* = \mathrm{id}_{\Gamma(V)}$, where both f and h satisfy the condition that their restriction to each fiber is a linear isomorphism, can be easily checked. It follows in particular that if f is an isomorphism, then $f^{-1*} = f^{*-1}$. In this case we write $f^{-1*} = f_*$.

Sections $s_i \in \Gamma(V_i)$, $i = 1, 2$ are said to be ***f-related*** if for all $q \in Q_1$ we have $f(s_1(q)) = s_2(f_0(q))$. We can easily show that if for each $q \in Q_1$ the restriction $f : (\tau^1)^{-1}(q) \to (\tau^2)^{-1}(f_0(q))$ is a linear isomorphism as before, then for any given $s_2 \in \Gamma(V_2)$, we have $s_1 = f^*(s_2)$ is the only section of V_1 which is f-related to s_2.

If the group G acts on the vector bundle $\tau : V \to Q$ as before, a section $s : Q \to V$ is called an ***invariant section*** if for all $g \in G$ and all $q \in Q$ we have $gs(q) = s(gq)$. The set of invariant sections of V is a subspace of $\Gamma(V)$ denoted $\Gamma^G(V)$.

Lemma 5.2.2 *Let the group G act on the vector bundle V and let $\pi_G(V) : V \to V/G$ be the vector bundle homomorphism described in lemma* 5.2.1. *Then $(\pi_G(V))^* : \Gamma(V/G) \to \Gamma^G(V)$ is a linear isomorphism.*

Proof. Let $\bar{s} \in \Gamma(V/G)$. Then for each $q \in Q$ and each $g \in G$, using the fact that the restriction of $\pi_G(V)$ to each fiber is an isomorphism, we have

$$\begin{aligned}
\pi_G(V)^* \bar{s}(gq) &= \pi_G(V)^{-1}(\bar{s}([gq]_G)) \\
&= g\left(\pi_G(V)^{-1}(\bar{s}([q]_G))\right) \\
&= g\pi_G(V)^* \bar{s}(q),
\end{aligned}$$

where in the first equation $\pi_G(V)$ is restricted to $\tau^{-1}(gq)$ and in the second $\pi_G(V)$ is restricted to $\tau^{-1}(q)$. The second equality is then an easy consequence of the definition of a vector bundle action and, also, the definition of quotient

vector bundle. Thus, we have have shown that $\pi_G(V)^*$ is an injective map into $\Gamma^G(V)$. Now let $s \in \Gamma^G(V)$. Define $\bar{s} \in \Gamma(V/G)$ by $\bar{s}([q]_G) = [s(q)]_G$. The element $\bar{s}$ is well defined because, since s is invariant, for any $g \in G$ and any $q \in Q$ we have $[s(gq)]_G = [gs(q)]_G = [s(q)]_G$. We can easily check that $s = \pi_G(V)^*\bar{s}$ which finishes the proof. ■

Quotient Lie Algebras. If $v \in \Gamma^G(V)$ we will denote $[v]_G$ or, sometimes, $\bar{v}$ the corresponding section of $\Gamma(V/G)$ via the isomorphism $\pi_G(V)^*$ of the previous lemma. Let ρ be an action of the group G on the vector bundle $\tau : V \to Q$ as before. Then we obtain a linear action $\rho_* : G \times \Gamma(V) \to \Gamma(V)$ given by the operation, defined before the lemma, $\rho_{g*}s$ for each $g \in G$ and each $s \in \Gamma(V)$. We will often write simply g_*s instead of $\rho_{g*}s$. It is clear that $\Gamma^G(V)$ is an invariant subspace of $\Gamma(V)$. Now assume that there is a Lie algebra structure on $\Gamma(V)$ which is invariant under the action ρ_*, that is, $g_*[s_1, s_2] = [g_*s_1, g_*s_2]$ for all $s_i \in \Gamma(V)$, $i = 1, 2$ and all $g \in G$. In particular, $\Gamma^G(V)$ is a Lie subalgebra of $\Gamma(V)$.

Since $\pi_G(V)^* : \Gamma(V/G) \to \Gamma^G(V)$ is a linear isomorphism we can define a Lie algebra structure on $\Gamma(V/G)$ in the following way.

Definition 5.2.3 *The* ***quotient Lie algebra structure*** *on* $\Gamma(V/G)$ *is defined by*

$$[\bar{s}_1, \bar{s}_2] = (\pi_G(V)^*)^{-1} [\pi_G(V)^*\bar{s}_1, \pi_G(V)^*\bar{s}_2] .$$

The most important case of the situation described above is the case of the vector bundle TQ on which G acts by the tangent lift of the action of G on Q and the Lie bracket on $\Gamma(TQ) \equiv \mathfrak{X}^\infty(Q)$ is the usual Lie bracket of vector fields. According to the previous results we obtain a quotient Lie algebra structure on $\Gamma(TQ/G)$.

Let us denote $\pi \equiv \pi_G(Q)$ the natural projection of the principal bundle $\pi : Q \to Q/G$, for simplicity. Recall that for each choice of a principal connection A on Q we have vector bundle isomorphisms

$$\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$$

and

$$\beta_A : I_G^H(TQ) \oplus I_G^V(TQ) \to T(Q/G) \oplus \tilde{\mathfrak{g}}.$$

The linear isomorphism α_A^* defines a Lie algebra structure on $\Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$ by declaring that it be a Lie algebra isomorphism. However, to calculate this Lie bracket on $\Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$ we shall make use of the equivalent condition that β_A^* is a Lie algebra isomorphism. We do this in the next theorem.

The map $T\pi$ establishes a well defined isomorphism between $\Gamma\left(I_G^H(TQ)\right)$ and $\mathfrak{X}^\infty(Q/G)$. More precisely, for any $X \in \Gamma\left(I_G^H(TQ)\right)$ the vector field $\pi_* X \in \mathfrak{X}^\infty(Q/G)$ given by $\pi_* X(x) = T\pi X(q)$, where $q \in \pi^{-1}(x)$ is arbitrary, is well defined. It is also known that if $X, Y \in \Gamma\left(I_G^H(TQ)\right)$ then $\pi_*[X,Y] = [\pi_* X, \pi_* Y]$. Observe that the inverse of $\pi_* : \Gamma\left(I_G^H(TQ)\right) \to \mathfrak{X}^\infty(Q/G)$ is given by the horizontal lift of vector fields, ${}^h : \mathfrak{X}^\infty(Q/G) \to \Gamma\left(I_G^H(TQ)\right)$, which also coincides with the restriction of β_A^* to $\mathfrak{X}^\infty(Q/G)$. Also we recall that if $X, Y \in \Gamma\left(I_G^H(TQ)\right)$ then $A([X,Y]) = -B(X,Y)$ where B is the curvature of A. Finally we should remark that, from what we have said before, we can deduce that there are natural identifications

$$\Gamma^G(TQ) \equiv \Gamma(TQ/G) \equiv \Gamma\left(I_G^H(TQ)\right) \oplus \Gamma\left(I_G^V(TQ)\right),$$

where the last identification involves the choice of a connection.

Calculation of the Lie Algebra Structure. The main result of this section is a formula, given in the next theorem, for the Lie bracket on the Lie algebra $\Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right) \equiv \mathfrak{X}^\infty(Q/G) \oplus \Gamma(\tilde{\mathfrak{g}})$ which involves the Lie bracket on $\tilde{\mathfrak{g}}$, the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$ and the $\tilde{\mathfrak{g}}$-valued curvature $\tilde{B}^A$.

Theorem 5.2.4 *Let $X_i \oplus \bar{\xi}_i \in \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$, $i = 1, 2$ be given sections of $T(Q/G) \oplus \tilde{\mathfrak{g}}$. Then*

$$[X_1 \oplus \bar{\xi}_1, X_2 \oplus \bar{\xi}_2] = [X_1, X_2] \oplus \tilde{\nabla}^A_{X_1} \bar{\xi}_2 - \tilde{\nabla}^A_{X_2} \bar{\xi}_1 - \tilde{B}^A(X_1, X_2) + [\bar{\xi}_1, \bar{\xi}_2].$$

A remark is in order to avoid any confusion in the interpretation of the bracket notation in the preceding and in several other formulas. Namely, the bracket on the left-hand-side is the bracket in $\Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$ given in Definition 5.2.3, while the bracket that appears immediately before the sign $\oplus$ is the usual bracket of vector fields. This caution is needed to avoid the apparently contradictory statement

$$[X_1, X_2] \equiv [X_1 \oplus 0, X_2 \oplus 0] = [X_1, X_2] \oplus -\tilde{B}^A(X_1, X_2).$$

Proof of the Theorem. There are elements $Y_i \in \Gamma\left(I_G^V(TQ)\right)$ such that $\beta_A^* \bar{\xi}_i = Y_i$ for $i = 1, 2$. On the other hand $\beta_A^* X_i = X_i^h$ for $i = 1, 2$. According to the definition of the bracket given before the theorem, we have

$$[X_1 \oplus \bar{\xi}_1, X_2 \oplus \bar{\xi}_2] = \beta_{A*}[X_1^h + Y_1, X_2^h + Y_2].$$

For any $q \in Q$ let $x = \pi(q) = [q]_G$. Then we have

$$[X_1^h, X_2^h](q) = [X_1, X_2]^h(q) - B\left(X_1^h(q), X_2^h(q)\right) q.$$

By applying β_{A*} to this we obtain, in the Lie algebra $\Gamma\left(T(Q/G)\oplus\tilde{\mathfrak{g}}\right)$, for any $x\in Q/G$,

$$\begin{aligned}[X_1,X_2](x) &= [X_1,X_2](x)\oplus\left[q,-B\left(X_1^h(q),X_2^h(q)\right)\right]_G\\ &= [X_1,X_2](x)\oplus -\tilde{B}^A(x)(X_1,X_2).\end{aligned}$$

Using lemma 5.1.2 we can deduce that for any $x\in Q/G$,

$$\beta_{A*}[Y_1,Y_2](x)=[\bar{\xi}_1,\bar{\xi}_2](x).$$

It only remains to calculate $[X_1,\bar{\xi}_2]=\beta_{A*}[X_1^h,Y_2]$ and $[X_2,\bar{\xi}_1]=\beta_{A*}[X_2^h,Y_1]$. For this, we first recall that, since X_1^h is an invariant horizontal vector field, its flow X_{1t}^h, which is the horizontal lift of the flow X_{1t} of X_1, transforms a point $q\in\pi^{-1}(x)$ into a point $X_{1t}^h(q)\in\pi^{-1}\left(X_{1t}(x)\right)$. We deduce that for any vertical vector field Y on Q, $X_{1t}^{h*}Y$ is also vertical. Thus, the Lie bracket

$$[X_1^h,Y]=\frac{d}{dt}X_{1t}^{h*}Y\bigg|_{t=0}$$

is also a vertical vector field.

Using this, we see that for any $q\in Q$ we have

$$[X_1^h,Y_2](q)=A\left([X_1^h,Y_2](q)\right)q.$$

We can also easily see, using the invariance of X_1^h and Y_2, that $[X_1^h,Y_2]$ is also invariant, more precisely, it is a vertical invariant vector field. Next we should calculate $\beta_{A*}[X_1^h,Y_2]$.

For any $x\in Q/G$ and any $q\in\pi^{-1}(x)$, we have

$$\beta_{A*}[X_1^h,Y_2](x)=\left[q,A\left([X_1^h,Y_2](q)\right)\right]_G.$$

By Cartan's structure equation we have

$$\mathbf{d}A(X_1^h,Y_2)=\left[A(X_1^h),A(Y_2)\right]+B(X_1^h,Y_2).$$

Since $A(X_1^h)=0$ (because X_1^h is horizontal) and $B(X_1^h,Y_2)=0$ (because Y_2 is vertical), we obtain $dA(X_1^h,Y_2)=0$. On the other hand we have the formula

$$\mathbf{d}A(X_1^h,Y_2)=X_1^hA(Y_2)-Y_2A(X_1^h)-A\left([X_1^h,Y_2]\right).$$

We conclude that $A\left([X_1^h,Y_2]\right)=X_1^hA(Y_2)$. This shows in particular that, at a given point $q\in Q$, $A\left([X_1^h,Y_2](q)\right)$ only depends on $X_1^h(q)$ and not on the behavior of X_1^h in a neighborhood of q. Now it is clear that, for any $q\in Q$, we have

$$\left(X_1^hA(Y_2)\right)(q)=\frac{d}{dt}A\left(Y_2\left(X_{1t}^h(q)\right)\right)\bigg|_{t=0}.$$

Let $q \in Q$ and $x = \pi(q)$, thus $X_{1t}(x) = \pi X_{1t}^h(q)$, for all t. Let $\xi_2(t) = A\left(Y_2\left(X_{1t}^h(q)\right)\right)$ for all t. Then we see that $\bar{\xi}_2\left(x(t)\right) = \left[X_{1t}^h(q), \xi_2(t)\right]_G$ for all t and therefore

$$\tilde{\nabla}^A_{X_1(x)}\bar{\xi}_2(x) = \frac{D}{Dt}\left[X_{1t}^h(q), \xi_2(t)\right]_G\Big|_{t=0}.$$

Now using lemma 2.3.4 to calculate the covariant derivative of the curve $[X_{1t}^h(q), \xi(t)]_G$ we obtain

$$\frac{D}{Dt}\left[X_{1t}^h(q), \xi_2(t)\right]_G\Big|_{t=0} = [q, \dot{\xi}_2(0)]_G.$$

But $\dot{\xi}_2(0) = \left(X_1^h A(Y_2)\right)(q)$ which implies $\dot{\xi}_2(0) = A\left([X_1^h, Y_2](q)\right)$. We conclude that

$$\tilde{\nabla}^A_{X_1}\bar{\xi}_2 = \beta_{A*}[X_1^h, Y_2].$$

The relation

$$\tilde{\nabla}^A_{X_2}\bar{\xi}_1 = \beta_{A*}[X_2^h, Y_1]$$

follows in a similar way.

Collecting these results, we obtain

$$[X_1 \oplus \bar{\xi}_1, X_2 \oplus \bar{\xi}_2] = [X_1, X_2] \oplus \tilde{\nabla}^A_{X_1}\bar{\xi}_2 - \tilde{\nabla}^A_{X_2}\bar{\xi}_1 - \tilde{B}(X_1, X_2) + [\bar{\xi}_1, \bar{\xi}_2],$$

as desired. ■

The following corollary is a consequence of Theorem 5.2.4.

Corollary 5.2.5 *Let $p_{Q/G} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to T(Q/G)$ and $p_{\tilde{\mathfrak{g}}} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ be the natural projections. Then the naturally induced maps $p_{(Q/G)*} : \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right) \to \Gamma\left(T(Q/G)\right)$ given by $p_{(Q/G)*}(X \oplus \bar{\xi}) = X$ and $p^*_{\tilde{\mathfrak{g}}} : \Gamma\left(\tilde{\mathfrak{g}}\right) \to \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$ given by $p^*_{\tilde{\mathfrak{g}}}(\bar{\xi}) = 0 \oplus \bar{\xi}$ are Lie algebra homomorphisms. Naturally, the Lie algebra structure on $\Gamma\left(\tilde{\mathfrak{g}}\right)$ is defined pointwise, that is, for given $\bar{\xi}$ and $\bar{\eta}$ in $\Gamma\left(\tilde{\mathfrak{g}}\right)$ we have, $[\bar{\xi}, \bar{\eta}](x) = [\bar{\xi}(x), \bar{\eta}(x)]$ for all $x \in Q/G$.*

6 Reduced Tangent Bundles

The results of the preceding sections may be viewed as a geometrized and intrinsic way of writing the results of Cendra, Ibort and Marsden [1987] and of Marsden and Scheurle [1993b]. Next we turn to our main task *finding a context in which the Lagrangian reduction process can be iterated*. In other words, we develop a context in which the objects are stable under Lagrangian reduction. These objects will be called *reduced tangent bundles.*

Lagrange-Poincaré Bundles and Reduced Tangent Bundles. We begin by describing the geometric objects on which reduced Lagrangians are naturally defined. They form a category of vector bundles denoted $\mathfrak{LP}$, objects of which will be called *Lagrange-Poincaré bundles*. Important special Lagrange-Poincaré bundles are the *reduced tangent bundles*, which form a subcategory denoted $\mathfrak{RT}$, which is the *smallest* subcategory that contains the tangent bundles TQ and which remains stable under reduction, and each bundle carries some additional structure needed to describe the reduction of given Lagrangians defined on them and the corresponding variational principles.

For example, objects of $\mathfrak{RT}$ are vector bundles of the type $T(Q/G) \oplus \tilde{\mathfrak{g}}$ where, as we have seen, $\tilde{\mathfrak{g}}$ is a Lie algebra bundle, there is a covariant derivative D^A on $\tilde{\mathfrak{g}}$ and a $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{B}$, and in addition, there is a Lie algebra structure on the space of sections of $T(Q/G) \oplus \tilde{\mathfrak{g}}$ which is related to D^A, $\tilde{B}$ and the Lie algebra structure on fibers of $\tilde{\mathfrak{g}}$ by the formula of Theorem 5.2.4.

In this section we show how to reduce further these kinds of objects. They are, in a sense, special cases of *Lie algebroids* (see Mackenzie [1987], Weinstein [1996, 1998] and Cannas da Silva and Weinstein [1999]) although we consider some extra structure on them. We also recall (see Weinstein [1996] and Marsden, Pekarsky and Shkoller [2000]) that these notions are very useful in discrete Lagrangian mechanics. Further exploration of the link between our work, that of groupoids and algebroids as well as noncommutative differential geometry would of course be very interesting.

In the next section, we will show that the reduction process can be performed by stages, and also how to write explicit expressions for Lagrangians and variational principles reduced by stages.

Even though the most important objects of $\mathfrak{LP}$ seem to be those of $\mathfrak{RT}$, it is natural to first deal with the larger category $\mathfrak{LP}$ rather than the category $\mathfrak{RT}$.

The objects in the category $\mathfrak{LP}$ of *Lagrange-Poincaré bundles* are vector bundles of the type $TQ \oplus V$ where V is a Lie algebra bundle over the base Q and which carry

1. a covariant derivative D on V
2. a V-valued 2-form on Q and
3. a Lie algebra structure on $\Gamma(TQ \oplus V)$ that satisfies an equality formally similar to the formula of Theorem 5.2.4.

We shall make this more explicit and describe the mappings in this category shortly.

We will show that the category $\mathfrak{RT}$ is strictly contained in the category $\mathfrak{LP}$. Moreover, the simplest way of describing $\mathfrak{RT}$ is by defining it as the smallest subcategory of $\mathfrak{LP}$ that contains the tangent bundles TQ. The special

importance of the category $\mathfrak{RT}$ however, is that every object of it is a reduced tangent bundle, after some number of reduction stages, and reduced versions of Euler-Lagrange equations corresponding to invariant Lagrangians on tangent bundles can be written in terms of these reduced tangent bundles.

It is important to remark at this point that we can obtain a generalization of all this, that is a category bigger than $\mathfrak{LP}$, which contains in particular vector bundles which are some subbundles of bundles of $\mathfrak{LP}$, to describe reduction in non-holonomic mechanics. This as well as other interesting topics like a local study of objects of $\mathfrak{LP}$ will be the purpose of future work.

An important result in this section is an explicit expression for the reduction by stages of the Lie algebra of sections of bundles which are objects of $\mathfrak{LP}$. This, together with the results of §5.2, will be related in the next section to the Poisson bracket on the dual bundles of the bundles which are objects of $\mathfrak{LP}$. These dual bundles carry a Poisson bracket induced by the structure of the objects of $\mathfrak{LP}$. Therefore, by duality, we obtain a direct link to the topic of Poisson reduction by stages of at least some important examples of Poisson Manifolds. The special case of duals of elements of $\mathfrak{RT}$ gives cotangent Poisson reduction (see Montgomery, Marsden and Ratiu [1984] and Montgomery [1986]). A generalization of all this for more general Poisson manifolds is tied to the consideration of a category bigger than $\mathfrak{LP}$ as indicated above, and will be also the purpose of future work.

6.1 The Geometry of Lagrange-Poincaré Bundles

We will need to fix some notation for maps induced on quotients. Recall that if a Lie group G acts on manifolds E and F and $f : E \to F$ is a G-equivariant map, then there is a natural quotient map, which we denote by $f/G : E/G \to F/G$, defined by

$$f/G\left([a]_G\right) = \left[f(a)\right]_G .$$

Lagrange-Poincaré Bundles. We now give the details of the definition of a Lagrange-Poincaré bundle.

Definition 6.1.1 *The category $\mathfrak{LP}$ of **Lagrange-Poincaré bundles** is defined as follows:*

(A.) *The **objects** of $\mathfrak{LP}$ are vector bundles which are Whitney sums of the form $\tau_Q \oplus \tau : TQ \oplus V \to Q$ where $\tau_Q : TQ \to Q$ is the tangent bundle of a manifold Q and $\tau : V \to Q$ is a vector bundle, together with some extra structure given by*

(a) *A Lie algebra structure on each fiber of V, in such a way that V is a Lie algebra bundle. The Lie bracket of given elements v_1 and v_2 in $\tau^{-1}(q)$ is denoted $[v_1, v_2]$.*

(b) *A V-valued 2-form ω on Q.*

(c) *A covariant derivative D for curves in V, related in the standard way to a connection ∇ on V, namely, let $v(t)$ be any curve in V, consider the curve $q(t) = \tau(v(t))$ on Q and let*

$$T_t^{t+h} : \tau^{-1}q(t+h) \to \tau^{-1}q(t)$$

be the parallel transport along $q(t)$ defined by ∇. Then

$$\frac{Dv(t)}{Dt} = \frac{d}{dh} T_t^{t+h} v(t+h)\Big|_{h=0}.$$

(d) *In addition, the operation defined on sections $X \oplus u$ of $TQ \oplus V$ by*

$$[X_1 \oplus u_1, X_2 \oplus u_2] = [X_1, X_2] \oplus \nabla_{X1} u_2 - \nabla_{X2} u_1 - \omega(X_1, X_2) + [u_1, u_2]$$

is assumed to be a Lie bracket on $\Gamma(TQ \oplus V)$.

(B.) *A* ***morphism*** *$f : TQ_1 \oplus V_1 \to TQ_2 \oplus V_2$ between Lagrange-Poincaré bundles is a vector bundle map satisfying the following conditions, where $f_0 : Q_1 \to Q_2$ is the induced map on the base:*

(a) *$f(TQ_1) \subset TQ_2$ and, moreover, $f|TQ_1 = Tf_0$.*

(b) *$f(V_1) \subset V_2$ and the restricted vector bundle map $f|V_1$ commutes with the structure on V given by $[\,,]$, D and ω. More precisely, if $u_i \in \tau^{-1}(q)$ and $X_i \in \tau_{Q_1}^{-1}(q)$ are given for $i = 1, 2$ and if $v(t)$ is also given, then*

$$f([u_1, u_2]) = [f(u_1), f(u_2)],$$

$$f(\omega(X_1, X_2)) = \omega(f(X_1), f(X_2))$$

and

$$f\left(\frac{Dv(t)}{Dt}\right) = \frac{Df(v(t))}{Dt}.$$

Bundles of the type $T(Q/G) \oplus \tilde{\mathfrak{g}}$ as considered in previous sections are important examples of elements of $\mathfrak{LP}$.

Projections and Injections. The following lemma is an easy consequence of the previous definition.

Lemma 6.1.2 *Let $TQ \oplus V$ be an object of $\mathfrak{LP}$ and let ∇, ω, $[\,,]$ be the structure on V. Then*

(a) *The maps $\Gamma(TQ \oplus V) \to \Gamma(TQ)$ given by $X \oplus v \to X$ and $\Gamma(V) \to \Gamma(TQ \oplus V)$ given by $v \to 0 \oplus v$ are Lie algebra homomorphisms.*

(b) *Let $\varphi \in C^\infty(Q)$ be given and let $X_i \in \Gamma(TQ)$ and $v_i \in \Gamma(V)$ be also given, for $i = 1, 2$. Then we have*

$$\begin{aligned}
[v_1, \varphi v_2] &= \varphi[v_1, v_2] \\
[X_1, \varphi X_2] &= X_1[\varphi]X_2 + \varphi[X_1, X_2] \\
[X_1, \varphi v_2] &= X_1[\varphi]v_2 + \varphi[X_1, v_2] \\
[\varphi X_1, v_2] &= \varphi[X_1, v_2] \\
[X_1 \oplus v_1, \varphi(X_2 \oplus v_2)] &= X_1[\varphi](X_2 \oplus v_2) + \varphi[X_1 \oplus v_1, X_2 \oplus v_2].
\end{aligned}$$

Let us denote $W = TQ \oplus V$ and also $w_i = X_i \oplus v_i$, for $i = 1, 2$. Denote $w_1[\varphi] = X_1[\varphi]$. Then the previous equalities can be converted into a single equality

$$[w_1, \varphi w_2] = \varphi[w_1, w_2] + w_1[\varphi]w_2.$$

Morphisms between Lagrange-Poincaré Bundles. The nature of morphisms between Lagrange-Poincaré bundles is clarified by the following.

Lemma 6.1.3 (a) *Let $TQ \oplus V$ be an object of $\mathfrak{LP}$ and let ∇, ω, $[\,,]$ be the structure on V. Assume that there is an structure ∇', ω', $[\,,]'$ on V such that the Lie algebra on $\Gamma(TQ \oplus V)$ defined by ∇', ω', $[\,,]'$ is the same as the Lie algebra defined by ∇, ω, $[\,,]$. Then $\nabla' = \nabla$, $\omega' = \omega$ and $[\,,]' = [\,,]$.*

(b) *Let $TQ_i \oplus V_i$ be objects of $\mathfrak{LP}$ for $i = 1, 2$. Let $f : TQ_1 \oplus V_2 \to TQ_2 \oplus V_2$ be a vector bundle isomorphism. Assume that*

$$f_* : \Gamma(TQ_1 \oplus V_2) \to \Gamma(TQ_2 \oplus V_2)$$

is a Lie algebra homomorphism. Then $f(V_1) = V_2$. Assume, in addition, that $f(TQ_1) \subset TQ_2$. Then f is an isomorphism of $\mathfrak{LP}$, that is, of Lagrange-Poincaré bundles.

Proof. (a) By hypothesis, we have, for all $X_i \oplus u_i \in \Gamma(TQ \oplus V)$, $i = 1, 2$,

$$\begin{aligned}[X_1, X_2] \oplus \nabla_{X_1} u_2 - \nabla_{X_2} u_1 - \omega(X_1, X_2) + [u_1, u_2] \\ = [X_1, X_2] \oplus \nabla'_{X_1} u_2 - \nabla'_{X_2} u_1 - \omega'(X_1, X_2) + [u_1, u_2]'.\end{aligned}$$

Taking $u_1 = u_2 = 0$ and X_i arbitrary for $i = 1, 2$, we obtain $\omega' = \omega$. Taking $X_1 = X_2 = 0$ and u_i arbitrary for $i = 1, 2$, we obtain $[\,,]' = [\,,]$. Taking $u_1 = 0$, $X_2 = 0$ and, also, u_2 and X_1 arbitrary we obtain $\nabla' = \nabla$.
(b) First, we will prove that $f(V_1) = V_2$. Using contradiction, assume that there is an element $\bar{v}_1 \in V_1$ such that $f(\bar{v}_1) = \bar{X}_2 \oplus \bar{v}_2$, where $\bar{X}_2 \in TQ_2$, say $\bar{X}_2 \in T_{q_2}Q_2$, satisfies $\bar{X}_2 \neq 0$. Then there exists $\varphi_2 \in C^\infty(Q_2)$ such that $\bar{X}_2(\varphi_2)(q_2) \neq 0$. Let $q_1 = f_0^{-1}(q_2)$ and $\varphi_1 = f_0^* \varphi_2 \equiv \varphi_2 \circ f_0$. Let $v_1 \in \Gamma(V_1)$ be such that $v_1(q_1) = \bar{v}_1$ and let $f_* v_1 = X_2 \oplus v_2$. Thus, $X_2(q_2) \oplus v_2(q_2) = \bar{X}_2 \oplus \bar{v}_2$. We obviously have $[\varphi_1 v_1, v_1] = \varphi_1 [v_1, v_1] = 0$, and therefore $f_*[\varphi_1 v_1, v_1] = 0$.

On the other hand, using the fact that f_* is a Lie algebra isomorphism and also Lemma 6.1.2, we have

$$\begin{aligned}f_*[\varphi_1 v_1, v_1] &= [\varphi_2(X_2 \oplus v_2), X_2 \oplus v_2] \\ &= -X_2(\varphi_2)(X_2 \oplus v_2) + \varphi_2[X_2 \oplus v_2, X_2 \oplus v_2] \\ &= -X_2(\varphi_2)(X_2 \oplus v_2),\end{aligned}$$

which gives a contradiction since $X_2(\varphi_2)(q_2) \neq 0$ and $X_2(q_2) = \bar{X}_2 \neq 0$. We have proven that $f(V_1) \subset V_2$.

We can prove in a similar way, using the fact that f^{-1} is a vector bundle isomorphism and that $(f^{-1})_*$ is a Lie algebra isomorphism, that $f^{-1}(V_2) \subset V_1$. Therefore, $f(V_1) = V_2$.

Now let ∇_i, ω_i, $[\,,]_i$ be the structure on the bundle V_i for $i = 1, 2$ and assume that $f(TQ_1) \subset TQ_2$. Using this and the fact that $f(V_1) = V_2$, it follows that $f(TQ_1) = TQ_2$. Using this and the fact that f is a vector bundle isomorphism and also that f_* is a Lie algebra isomorphism, we have (with an obvious meaning for $[\,,]_i$, ∇_i and ω_i, $i = 1, 2$. for any $X \oplus v, X' \oplus v' \in \Gamma(TQ_1 \oplus V_1)$,

$$\begin{aligned}f_*[X \oplus v, X' \oplus v']_1 &= [f_*(X \oplus v), f_*(X' \oplus v')]_2 \\ &= [f_* X \oplus f_* v, f_* X' \oplus f_* v']_2 \\ &= [f_* X, f_* X'] \oplus \nabla_{2 f_* X} f_* v' - \nabla_{2 f_* X'} f_* v \\ &\quad - \omega_2(f_* X, f_* X') + [f_* v, f_* v']_2.\end{aligned}$$

Taking $X = X' = 0$, we get $f_*[v, v']_1 = [f_* v, f_* v']_2$. Next, taking $X' = 0$ and $v = 0$ we obtain $f_* \nabla_{1X} v' = \nabla_{2 f_* X} f_* v'$. Finally, taking $v = v' = 0$, we have $f_* \omega_1(X, X') = \omega_2(f_* X, f_* X')$ and also $f_*[X, X'] = [f_* X, f_* X']$.

To show that f is a morphism of Lagrange-Poincaré bundles (i.e., a morphism in the category $\mathfrak{LP}$), it remains to show that $f|TQ_1 = Tf_0$. Let

$X \in \Gamma(TQ_1)$ and $\varphi \in C^\infty(Q_1)$. Then we have

$$f_*[X, \varphi X] = f_*\left(X[\varphi]X\right) = f_*\left(X[\varphi]\right) f_* X.$$

On the other hand,

$$f_*[X, \varphi X] = [f_*X, f_*(\varphi X)] = [f_*X, f_*\varphi f_*X] = \left((f_*X)(f_*\varphi)\right)(f_*X).$$

Therefore we have proven that for any $X \in \Gamma(TQ_1)$ and $\varphi \in C^\infty(Q_1)$ we have $f_*\left(X[\varphi]\right) = \left((f_*X)[f_*\varphi]\right)$. In view of this, one proves that $f|TQ_1 = Tf_0$ by a standard argument. ■

Morphism-related Sections of Lagrange-Poincaré Bundles. Let $f : TQ_1 \oplus V_1 \to TQ_2 \oplus V_2$ be a given morphism of Lagrange-Poincaré bundles and let $X \oplus u \in \Gamma(TQ_1 \oplus V_1)$ and $Y \oplus v \in \Gamma(TQ_2 \oplus V_2)$ be given. Then $X \oplus u$ and $Y \oplus v$ are said to be ***f-related*** if

$$f\left(X(q) \oplus u(q)\right) = Y\left(f_0(q)\right) \oplus v\left(f_0(q)\right)$$

for all $q \in Q_1$. In particular, it is easy to prove that if $X \oplus u$ and $Y \oplus v$ are f-related then X and Y are f_0-related as vector fields according to the usual definition.

Lemma 6.1.4 *Let $f : TQ_1 \oplus V_1 \to TQ_2 \oplus V_2$ be a morphism of Lagrange-Poincaré bundles and let $X_i \oplus u_i \in \Gamma(TQ_1 \oplus V_1)$ for $i = 1, 2$ and also $Y_i \oplus v_i \in \Gamma(TQ_2 \oplus V_2)$ for $i = 1, 2$ be given. Assume that $X_i \oplus u_i$ and $Y_i \oplus v_i$ are f-related for $i = 1, 2$. Then $[X_1 \oplus u_1, X_2 \oplus u_2]$ and $[Y_1 \oplus v_1, Y_2 \oplus v_2]$ are also f-related.*

Proof. For all $q \in Q_1$ we have

$$\begin{aligned} & f\left([X_1 \oplus u_1, X_2 \oplus u_2](q)\right) \\ & \quad = f\left(\left([X_1, X_2] \oplus \nabla_{X1} u_2 - \nabla_{X2} u_1 - \omega(X_1, X_2) + [u_1, u_2]\right)(q)\right) \\ & \quad = [Y_1, Y_2](q) \oplus \nabla_{Y_1} v_2 - \nabla_{Y_2} v_1 - \omega(Y_1, Y_2) + [v_1, v_2]\left(f_0(q)\right), \end{aligned}$$

where the first equality is a consequence of Definition 6.1.1 and the second is a consequence of Definition 6.1.1, the fact that f is a morphism in the category $\mathfrak{LP}$ and the fact that $X_i \oplus u_i$, and $Y_i \oplus v_i$ are f-related for $i = 1, 2$. ■

Lemma 6.1.5 *Let $f : TQ_1 \oplus V_1 \to TQ_2 \oplus V_2$ be a morphism in $\mathfrak{LP}$ and assume, in addition, that f is a linear isomorphism on each fiber. Then $f^* : \Gamma(TQ_2 \oplus V_2) \to \Gamma(TQ_1 \oplus V_1)$ is a Lie algebra homomorphism.*

Proof. It is a consequence of the remarks following Lemma 5.2.1 that f^* is well defined and that, for given $Y_i \oplus v_i \in \Gamma(TQ_2 \oplus V_2)$, $i = 1, 2$, $f^*(Y_i \oplus v_i)$ is a well defined section in $\Gamma(TQ \oplus V)$ and, moreover, $f^*(Y_i \oplus v_i) = f^*Y_i \oplus f^*v_i$ and $Y_i \oplus v_i$ are f-related for $i = 1, 2$. Then, according to Lemma 6.1.4, $[f^*(Y_1 \oplus v_1), f^*(Y_2 \oplus v_2)]$ and $[Y_1 \oplus v_1, Y_2 \oplus v_2]$ are also f-related. Therefore, since under our hypothesis, there is one and only one section of $TQ_1 \oplus V_1$ which is f-related to a given section of $TQ_2 \oplus V_2$, we can conclude that

$$[f^*(Y_1 \oplus v_1), f^*(Y_2 \oplus v_2)] = f^*\left([Y_1 \oplus v_1, Y_2 \oplus v_2]\right). \quad \blacksquare$$

Group Actions on Lagrange-Poincaré Bundles. As we have already indicated, when tangent bundles are reduced, one enters the category of Lagrange-Poincaré bundles and so to continue the process of reduction, it is important to know how these objects themselves behave under reduction. The first job is to examine group actions on vector bundles with the extra structure given in Definition 6.1.1.

Definition 6.1.6 *Let $\tau^i : V_i \to Q_i$, $i = 1, 2$ be vector bundles with extra structure $[\,,]$, ω and D satisfying conditions* (a), (b) *and* (c) *of Definition* 6.1.1. *Recall that a vector bundle map $f : V_1 \to V_2$ is called a morphism if it commutes with the structure given by $[,]$, ω and D, that is, if conditions stated in Definition* 6.1.1B(b) *are satisfied.*

An ***action*** *$\rho : G \times V \to V$ of a Lie group G on a vector bundle $\tau : V \to Q$ with the extra structure $[\,,]$, ω and D as before, is a vector bundle action such that, for each $g \in G$, $\rho_g : V \to V$ is a morphism in the previous sense. More precisely, we assume*

(a) *$g[v_1, v_2] = [gv_1, gv_2]$ for all $g \in G$ and all elements v_1 and v_2 in V such that $\tau(v_1) = \tau(v_2)$.*

(b) *$g\omega(X, Y) = \omega(gX, gY)$, for all $g \in G$ and all X and Y in TQ such that $\tau_Q(X) = \tau_Q(Y)$.*

(c)

$$\frac{Dgv(t)}{Dt} = g\frac{Dv(t)}{Dt}$$

for any curve $v(t)$ on V and any $g \in G$.

Now we shall state the following lemma whose proof is straightforward.

Lemma 6.1.7 *Let $\tau^i : V_i \to Q_i$, $i = 1, 2$ be vector bundles with an extra structure given by $[\,,]$ and ω satisfying conditions* (a), (b) *and* (c) *of Definition* 6.1.1A. *Let $\rho : G \times V \to V$ be a vector bundle action. Then, assuming*

invariance under the action ρ of any one of the structures $[\,,]$ or ω, it gives rise to a corresponding quotient structure on the quotient vector bundle V/G. These quotient structures are naturally defined by the following conditions:

(a) $[[v_1]_G, [v_2]_G] = [[v_1, v_2]]_G$ *for all elements* v_1 *and* v_2 *in* V *such that* $\tau(v_1) = \tau(v_2)$.

(b) $[\omega]_G([X]_G, [Y]_G) = [\omega(X, Y)]_G$ *for all* X *and* Y *in* TQ *such that* $\tau_Q(X) = \tau_Q(Y)$.

We will sometimes use the isomorphism α_A of Lemma 2.4.2 as an identification and, consequently, write $[X]_G \equiv T\pi X \oplus \bar{\xi}$ and also $[Y]_G \equiv T\pi Y \oplus \bar{\eta}$ and therefore

$$\begin{aligned}
&[\omega]_G([X]_G, [Y]_G) \\
&\qquad \equiv [\omega]_G\left(T\pi X \oplus \bar{\xi}, T\pi Y \oplus \bar{\eta}\right) \\
&\qquad \equiv [\omega]_G(T\pi X, T\pi Y) + [\omega]_G(T\pi X, \bar{\eta}) + [\omega]_G\left(\bar{\xi}, T\pi Y\right) + [\omega]_G\left(\bar{\xi}, \bar{\eta}\right).
\end{aligned}$$

Invariance Properties of Covariant Derivatives. To pass covariant derivatives to quotient bundles, we will need some preparatory lemmas.

Lemma 6.1.8 *Let $\tau : V \to Q$ be a vector bundle and let D be a covariant derivative associated to an affine connection ∇ on V, as we explained before (see the note following Definition* 2.3.1 *and the remark following Lemma* 2.3.2*). Let $\rho : G \times V \to V$ be a vector bundle action. If ρ commutes with D in the sense of Definition* 6.1.6(c)*), then ∇ is invariant in the following sense:*

$$\nabla_{g_* X} g_* v = g_* \nabla_X v$$

for all $X \in \mathfrak{X}^\infty(Q)$, all $v \in \Gamma(V)$ and all $g \in G$. Conversely, if ∇ is invariant then D is invariant.

We will omit the proof of this lemma, which is more or less straightforward.

Horizontal and Vertical Components of Connections. To define the notion of quotient derivative or quotient connection, we will need some extra structure, namely, a principal connection on Q. Our first job is to synthesize this connection with a given connection on a vector bundle to obtain *horizontal covariant derivatives and connections.*

Definition 6.1.9 *Let $\tau : V \to Q$ be a vector bundle and let D be a covariant derivative associated to a connection ∇ on V. Let $\rho : G \times V \to V$ be a vector bundle action. Let A be a principal connection on the principal bundle Q with structure group G where $\rho_0 : G \times Q \to Q$ is the action. Let $v(t)$ be any curve*

in V and let $q(t) = \tau(v(t))$ for all t. Choose t_0 and let $q_0 = q(t_0)$. Let g_q and q_h be as in §2.1, thus $q(t) = g_q(t)q_h(t)$ and $g_q(t_0) = e$. Then we define

$$v_h(t) = g_q^{-1}(t)v(t)$$

$$\left.\frac{D^{(A,H)}v(t)}{Dt}\right|_{t=t_0} = \left.\frac{Dv_h(t)}{Dt}\right|_{t=t_0}$$

and

$$\left.\frac{D^{(A,V)}v(t)}{Dt}\right|_{t=t_0} = \left.\frac{Dv(t)}{Dt}\right|_{t=t_0} - \left.\frac{D^{(A,H)}v(t)}{Dt}\right|_{t=t_0}.$$

We will call

$$\frac{D^{(A,H)}v(t)}{Dt}$$

*the A-**horizontal covariant derivative** of v and*

$$\frac{D^{(A,V)}v(t)}{Dt}$$

*the A-**vertical covariant derivative** of v. We also define $\nabla_X^{(A,H)}v$ and $\nabla_X^{(A,V)}v$ by*

$$\nabla_X^{(A,H)}v(q_0) = \left.\frac{D^{(A,H)}}{Dt}v(t)\right|_{t=t_0}$$

and

$$\nabla_X^{(A,V)}v(q_0) = \left.\frac{D^{(A,V)}}{Dt}v(t)\right|_{t=t_0}$$

*where, for each $q_0 \in Q$, each $X \in \mathfrak{X}^\infty(Q)$ and each $v \in \Gamma(V)$, we have, by definition, that $q(t)$ is any curve in Q such that $\dot{q}(t_0) = X(q_0)$ and $v(t) = v(q(t))$ for all t. We will call $\nabla^{(A,H)}$ the A-**horizontal component** and $\nabla^{(A,V)}$ the A-**vertical component** of the connection ∇.*

We see from this definition that

$$\nabla_X v(q_0) = \nabla_X^{(A,H)}v(q_0) + \nabla_X^{(A,V)}v(q_0).$$

The following lemma gives, in particular, an alternative characterization of $\nabla_X^{(A,H)}v$ and $\nabla_X^{(A,V)}v$ for invariant v. It also shows that when restricted to invariant sections $v \in \Gamma^G(V)$, the operator $\nabla^{(A,H)}$ has the formal properties of a connection.

Lemma 6.1.10

(a) *Let* $\mathrm{Hor}^A(X) \equiv X_H$ *and* $\mathrm{Ver}^A(X) \equiv X_V$*, for short. Then we have, for each* $q_0 \in Q$*, each* $X \in \mathfrak{X}^\infty(Q)$ *and each* $v \in \Gamma^G(V)$*,*

$$\nabla_X^{(A,H)} v(q_0) = \nabla_{X_H} v(q_0)$$

and

$$\nabla_X^{(A,V)} v(q_0) = \nabla_{X_V} v(q_0).$$

In particular,

$$\nabla_X v(q_0) = \nabla_{X_H} v(q_0) + \nabla_{X_V} v(q_0).$$

(b) *Let* $v \in \Gamma^G(V)$ *and et* $q(t)$ *be any curve in* Q *such that* $\dot{q}(t_0) = X(q_0)$ *and let* $q_h(t)$ *and* $g_q(t)$ *be as in §2. Then*

$$\nabla_X^{(A,H)} v(q_0) = \frac{D}{Dt} g_q^{-1}(t) v(t)\bigg|_{t=t_0} = \frac{D}{Dt} v(q_h(t))\bigg|_{t=t_0}$$

and

$$\nabla_X^{(A,V)} v(q_0) = \frac{D}{Dt} g_q(t) v(t_0)\bigg|_{t=t_0}.$$

In particular, $\nabla_X^{(A,V)} v(q_0)$ *depends only on* $\xi_0 = \dot{g}_q(t_0)$ *and* $v(q_0) = v(t_0)$*.*

(c) *Let* $v \in \Gamma^G(V)$*, let* $q(t)$ *be any curve in* Q *and let* $v(t) = v\left(q(t)\right)$ *for all* t*. Then*

$$\frac{D^{(A,H)}}{Dt} v(t)\bigg|_{t=t_0} = \frac{D}{Dt} g_q^{-1}(t) v(t)\bigg|_{t=t_0} = \frac{D}{Dt} v(q_h(t))\bigg|_{t=t_0}$$

and

$$\frac{D^{(A,V)}}{Dt} v(t)\bigg|_{t=t_0} = \frac{D}{Dt} g_q(t) v(t_0)\bigg|_{t=t_0}.$$

Proof. (a) Let $q(t)$ be any curve in Q such that $\dot{q}(t_0) = X(q_0)$ and let $v(t) = v\left(q(t)\right)$ for all t. Then we can easily see that $\tau\left(v_h(t)\right) = q_h(t)$ for all t. Since v is invariant we have $v(t) = g_q(t) v\left(q_h(t)\right)$. This argument and the definition of ∇, $\nabla^{(A,H)}$ and $\nabla^{(A,V)}$ shows that part (a) of the lemma follows without difficulty.

(b) This part is basically a restatement of part (a). The first equality is a direct consequence of the definition of $\nabla^{(A,H)}$. Now Let $\xi_0 = \dot{g}_q(t_0)$. Then

$$X_V(q_0) = \xi_0 q_0 = \frac{d}{dt}(g_q(t)q_0)\bigg|_{t=t_0}.$$

Therefore the second equality is a consequence of the second equality of part (a) and the invariance of v.

(c) To prove this part we can proceed essentially as in the proof of parts (a) and (b). Alternatively, we can deduce it directly from part (b). ■

As a technical point, we note that in the proof of the previous lemma we are strongly using the fact that the covariant derivative D comes from a connection ∇ which, in particular, has the formal property $\nabla_{X+Y} = \nabla_X + \nabla_Y$.

The next two lemmas establish some basic properties of invariant covariant derivatives and invariant connections.

Lemma 6.1.11 (A) *Assume the same hypothesis as in Definition* 6.1.9 *and, in addition, that D is invariant in the sense of Definition* 6.1.6(c). *Then*

$$\frac{D^{(A,H)}}{Dt} \quad \textit{and} \quad \frac{D^{(A,V)}}{Dt}$$

are also invariant in the following sense

$$\frac{D^{(A,H)}gv(t)}{Dt} = g\frac{D^{(A,H)}v(t)}{Dt}$$

and

$$\frac{D^{(A,V)}gv(t)}{Dt} = g\frac{D^{(A,V)}v(t)}{Dt}$$

for all curves $v(t)$ on V and all $g \in G$.

(B) *Let $\nabla^{(A,H)}$ and $\nabla^{(A,V)}$ be the horizontal and vertical components of a given connection ∇ on V and assume that ∇ is invariant (see Lemma* 6.1.8*). Then $\nabla^{(A,H)}$ and $\nabla^{(A,V)}$ are also invariant in the following sense:*

$$\nabla^{(A,H)}_{g_*X} g_* v = g_* \nabla^{(A,H)}_X v$$

and

$$\nabla^{(A,V)}_{g_*X} g_* v = g_* \nabla^{(A,V)}_X v$$

for all $X \in \mathfrak{X}^\infty(Q)$, all $v \in \Gamma(V)$ and all $g \in G$.

Proof. (A) Let $a \in G$ and $v(t)$ be given. Let $\tau(v(t)) = q(t)$. Then we immediately have $\tau(av(t)) = aq(t)$. For any chosen t_0 we have $aq(t) = g_{aq}(t)(aq)_h(t)$ and $q(t) = g_q(t)q_h(t)$ with $g_{aq}(t_0) = g_q(t_0) = e$. We know that $(aq)_h(t) = aq_h(t)$ for all t. Using these equalities we can easily see that $g_{aq}(t) = ag_q(t)a^{-1}$ for all t. Then

$$\begin{aligned}
\frac{D^{(A,H)}}{Dt}av(t)\bigg|_{t=t_0} &= \frac{D}{Dt}g_{aq}^{-1}(t)av(t)\bigg|_{t=t_0} = \frac{D}{Dt}ag_q^{-1}(t)a^{-1}av(t)\bigg|_{t=t_0} \\
&= \frac{D}{Dt}ag_q^{-1}(t)v(t)\bigg|_{t=t_0} = \frac{D}{Dt}av_h(t)\bigg|_{t=t_0} \\
&= a\frac{D}{Dt}v_h(t)\bigg|_{t=t_0} = a\frac{D^{(A,H)}}{Dt}v(t)\bigg|_{t=t_0}.
\end{aligned}$$

To prove that

$$\frac{D^{(A,V)}av(t)}{Dt} = a\frac{D^{(A,V)}v(t)}{Dt}$$

for all curves $v(t)$ on V and all $a \in G$ we simply use Definition 6.1.9 and invariance of D and of $D^{(A,H)}$.

(B) To prove invariance of $\nabla^{(A,H)}$ and $\nabla^{(A,V)}$ we use part (A) and the definitions of $\nabla^{(A,H)}$ and $\nabla^{(A,V)}$. ■

To prove the preceding lemma for invariant v, we can use, alternatively, Lemma 6.1.10 and recall that for any $g \in G$ and any $X \in \mathfrak{X}^\infty(Q)$ we have $(g_*X)_H = g_*X_H$ and also $(g_*X)_V = g_*X_V$.

Lemma 6.1.12 *Assume the same hypothesis as in Lemma* 6.1.11. *Then for any curve $a(t)$ on G and any curve $v(t)$ on V we have*

$$\frac{D^{(A,H)}}{Dt}[a(t)v(t)] = a(t)\frac{D^{(A,H)}}{Dt}v(t).$$

Proof. Let $q(t) = \tau(v(t))$, then $a(t)q(t) = \tau(a(t)v(t))$. Choose t_0. Then we have, for all t, $g_{aq}(t)(aq)_h(t) = a(t)q(t)$ and also $g_q(t)q_h(t) = q(t)$, where $g_{aq}(t_0) = g_q(t_0) = e$. We can easily check that $(aq)_h(t) = a(t_0)q_h(t)$, for all t. Then $a(t)q(t) = g_{aq}(t)a(t_0)q_h(t)$, for all t. On the other hand $a(t)q(t) = a(t)g_q(t)q_h(t)$. Therefore $g_{aq}(t) = a(t)g_q(t)\left(a(t_0)\right)^{-1}$. Using this and invariance of D we obtain

$$\begin{aligned}
\frac{D^{(A,H)}}{Dt}a(t)v(t)\bigg|_{t=t_0} &= \frac{D}{Dt}a(t_0)\left(g_q(t)\right)^{-1}a^{-1}(t)a(t)v(t)\bigg|_{t=t_0} \\
&= a(t_0)\frac{D}{Dt}\left(g_q(t)\right)^{-1}v(t)\bigg|_{t=t_0} \\
&= a(t_0)\frac{D^{(A,H)}}{Dt}v(t)\bigg|_{t=t_0}. \quad ■
\end{aligned}$$

Properties of Horizontal Connections. The next two lemmas are included for the sake of completeness only, although they will not be used in an essential way in the present work.

Lemma 6.1.13 *Assume the same conditions as in Definition* 6.1.9. *Let* $v(t)$ *be a curve in* V, $q(t) = \tau(v(t))$, *and let* $f(t)$ *be a real valued function. Then*

$$\frac{D^{(A,H)}}{Dt}[f(t)v(t)] = \dot{f}(t)v(t) + f(t)\frac{D^{(A,H)}}{Dt}v(t)$$

for all t *and*

$$\frac{D^{(A,V)}}{Dt}[f(t)v(t)] = f(t)\frac{D^{(A,V)}}{Dt}v(t)$$

for all t.

Proof. It is not difficult to prove this lemma from the definitions. We will omit the details. ■

The following lemma generalizes Lemma 6.1.12

Lemma 6.1.14 *For each curve* $q(t)$ *in* Q *and each choice of* t_0 *define* $\xi(t_0) = \dot{g}_q(t_0)$. *Thus, as* t_0 *varies,* $\xi(t_0)$ *describes a curve in* $\mathfrak{g}$ *which we will denote* $\xi_q(t)$. *Assume the same conditions as in Definition* 6.1.9. *Now let* $v(t)$ *be a curve in* V *and let* $q(t) = \tau(v(t))$ *for all* t. *Let* $a(t)$ *be any curve in* G. *Then we have*

(A)

$$\left.\frac{D^{(A,H)}}{Dt}[a(t)v(t)]\right|_{t=t_0} = \left.\frac{D}{Dt}\left[a(t_0)\left(g_q(t)\right)^{-1}v(t)\right]\right|_{t=t_0}$$

(B)

$$\xi_{aq}(t) = \mathrm{Ad}_{a(t)}\,\xi_q(t) + \dot{a}(t)a^{-1}(t)$$

for all t. *For each* t,

$$\frac{D^{(A,V)}}{Dt}[a(t)v(t)]$$

only depends on $a(t)v(t)$ *and* $\xi_{aq}(t)$.

Proof. (A) The proof of this part is an immediate consequence of the last part of the proof of Lemma 6.1.12.

(B) From the proof of Lemma 6.1.12 it follows that $g_{aq}(t) = a(t)g_q(t)\left(a(t_0)\right)^{-1}$. Using this, we can easily prove that $\xi_{aq}(t) = \mathrm{Ad}_{a(t)}\,\xi_q(t) + \dot{a}(t)a^{-1}(t)$. ■

6.2 Reduction of Lagrange-Poincaré Bundles

Now we embark on the task of reducing Lagrange-Poincaré bundles. The goal is to see how the structures on these bundles pass to the quotient bundle so that the result is still a Lagrange-Poincaré bundle. Most of this structure passes in a fairly straightforward way. However, things are not quite so simple with the covariant derivative so that we need to pay special attention to this.

Quotient Covariant Derivatives and Quotient Connections. Now we are ready to pass covariant derivatives as well as horizontal and vertical covariant derivatives and connections to the quotient.

The notion of quotient connection may be defined without any reference to quotient covariant derivatives. However, here we prefer a parallel treatment of these two related topics, in view to further application to variational principles.

There are some interesting links between the constructions here and those that appear in the theory of geometric phases (see Marsden, Montgomery and Ratiu [1990], §13), but this will not be pursued in this paper.

First, we define the notion of reduced horizontal covariant derivative and reduced horizontal connection. The consistency of the definitions of $\left[D^{(A,H)}/Dt\right]_G$ and $\left[\nabla^{(A,H)}\right]_G$ below is a consequence of Lemmas 6.1.12 and 6.1.11(B). We omit the details of the proof, which can be obtained without difficulty.

Definition 6.2.1 *Let us assume the conditions of Definition* 6.1.9 *and, in addition, assume that D is invariant, and therefore, the associated connection ∇ on the vector bundle V is also invariant. We define the* ***reduced horizontal covariant derivative*** $\left[D^{(A,H)}/Dt\right]_G$ *on the vector bundle V/G by*

$$\left[\frac{D^{(A,H)}}{Dt}\right]_G [v(t)]_G = \left[\frac{D^{(A,H)}}{Dt} v(t)\right]_G$$

for any curve $v(t)$ on V.

We define the ***reduced horizontal connection*** $\left[\nabla^{(A,H)}\right]_G$ *by*

$$\left[\nabla^{(A,H)}\right]_{G[X]_G} [v]_G = \left[\nabla^{(A,H)}_X v\right]_G$$

for given $[X]_G$ in $\Gamma(TQ/G)$ and $[v]_G$ in $\Gamma(V/G)$ and corresponding $X \in \Gamma^G(TQ)$ and $v \in \Gamma^G(V)$. (recall that we can identify elements of $\Gamma^G(TQ)$ with elements of $\Gamma(TQ/G)$ and also elements of $\Gamma^G(V)$ with elements of $\Gamma(V/G)$ using Lemma 5.2.2). Then the connection associated to $\left[D^{(A,H)}/Dt\right]_G$ is precisely $\left[\nabla^{(A,H)}\right]_G$.

The reduced horizontal connection can be naturally interpreted as a connection, say $\bar{\nabla}$, on the vector bundle V/G in the usual sense (as in the note

following Definition 2.3.1), as follows. Let Y be any vector field on Q/G, let Y^h be the horizontal lift of Y. For any given $[v]_G \in \Gamma(V/G)$ as before let

$$\bar{\nabla}_Y [v]_G = \left[\nabla^{(A,H)}_{Y^h} v\right]_G .$$

Then we can check that the properties defining a connection hold for $\bar{\nabla}$. Moreover, we can easily see that if X is any vector field such that $X_H = Y^h$ then

$$\bar{\nabla}_Y [v]_G = \left[\nabla^{(A,H)}_{X} v\right]_G .$$

It is important for us to define some kind of quotient vertical covariant derivative and also quotient vertical connection. However, those will not be a covariant derivative and a connection in the usual sense. Let us begin with a simple case, namely, the case of a constant curve. This is instructive because the answer turns out to be nonzero.

Let $[v_0]_G \in V/G$ be given and choose $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$, where $x_0 = \tau/G\left([v_0]_G\right)$. We are going to think of $[v_0]_G$ as being a constant curve and we want to define the covariant derivative of it with respect to $[q_0, \xi_0]_G$. Let us choose $v_0 \in V$ and also $q_0 \in Q$ such that $\tau(v_0) = q_0$. Thus, since $[q_0, \xi_0]_G$ is given, ξ_0 is uniquely determined. Choose a curve $g_0(t)$ on G such that $g_0(t_0) = e$ and $\dot{g}_0(t_0) = \xi_0$. Let $v_0(t) = g_0(t)v_0$. Then it is easy to see that

$$\left.\frac{D^{(A,H)} v_0(t)}{Dt}\right|_{t=t_0} = 0$$

and therefore,

$$\left.\frac{D^{(A,V)} v_0(t)}{Dt}\right|_{t=t_0} = \left.\frac{D^{(A)} v_0(t)}{Dt}\right|_{t=t_0} .$$

It is clear that the right-hand side does not depend on the choice of the curve $g_0(t)$ satisfying the previous conditions, and now we shall see that it depends only on $[v_0]_G \in V/G$ and $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$ and not on the particular choice of v_0 and q_0 satisfying the previous assumptions. Let us choose $v_1 \in [v_0]_G$ and also $q_1 \in [q_0]_G$ such that $\tau(v_1) = q_1$. Then there exists $h \in G$ such that $q_1 = hq_0$ and $\xi_1 = h\xi_0 h^{-1}$, and we also know from Lemma 5.2.1 that $v_1 = hv_0$. If a curve $g_0(t)$ on G satisfies $g_0(t_0) = e$ and $\dot{g}_0(t_0) = \xi_0$ then we can choose, without loss of generality, the curve $g_1(t) = hg_0(t)h^{-1}$ that satisfies, $g_1(t_0) = e$ and $\dot{g}_1(t_0) = \xi_1$. Define, $v_1(t) = g_1(t)v_1$ then we have $v_1(t) = hg_0(t)h^{-1}hv_0 = hv_0(t)$. Then using Lemma 6.1.11, we obtain

$$\left[\left.\frac{D^{(A)} v_1(t)}{Dt}\right|_{t=t_0}\right]_G = \left[h \left.\frac{D^{(A)} v_0(t)}{Dt}\right|_{t=t_0}\right]_G = \left[\left.\frac{D^{(A)} v_0(t)}{Dt}\right|_{t=t_0}\right]_G .$$

Then we can consistently define the reduced vertical covariant derivative of the constant curve $[v_0]_G$ on V/G with respect to $[q_0, \xi_0]_G$ where $\tau/G[v_0]_G = x_0$ and $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$ as we explained before, by the formula

$$\left[\frac{D^{(A,V)}}{Dt}\right]_{G[q_0,\xi_0]_G} [v_0]_G \bigg|_{t=t_0} = \left[\frac{D^{(A,V)} v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G .$$

Similarly, we can define the reduced covariant derivative of the constant curve $[v_0]_G$ on V/G with respect to $[q_0, \xi_0]_G$ where $\tau/G[v_0]_G = x_0$ and $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$ by

$$\left[\frac{D^{(A)}}{Dt}\right]_{G[q_0,\xi_0]_G} [v_0]_G \bigg|_{t=t_0} = \left[\frac{D^{(A)} v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G .$$

We see from previous formulas that, for constant curves, the reduced vertical covariant derivative and the reduced covariant derivative coincide and of course in general this is not zero. Next we shall consider the general case.

Let $[v]_G(t)$ be a curve on V/G and $x(t) = \tau/G\left([v]_G(t)]_G\right)$. Choose $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$, where $x_0 = x(t_0) = \tau/G\left([v]_G(t_0)\right)$. We want to define the covariant derivative of $[v]_G(t)$ with respect to $[q_0, \xi_0]_G$ at $t = t_0$. Let us choose $v_0 \in V$ and also $q_0 \in Q$ such that $\tau(v_0) = q_0$. Thus, since $[q_0, \xi_0]_G$ is given, ξ_0 is uniquely determined. Choose a curve $g_0(t)$ on G such that $g_0(t_0) = e$ and $\dot{g}_0(t_0) = \xi_0$. We can easily see from Lemma 5.2.1 that there exists a unique curve $v^h_{q_0}(t)$ on V such that $\left[v^h_{q_0}(t)\right]_G = [v]_G(t)$ and $\tau\left(v^h_{q_0}(t)\right) = x^h_{q_0}(t)$ for all t. Let $v_0(t) = g_0(t) v^h_{q_0}(t)$. Also from Lemma 5.2.1 we can see easily that $v_0(t)$ is the only curve on V such that $[v_0(t)]_G = [v]_G(t)$ and $\tau\left(v_0(t)\right) = q_0(t)$ for all t. Then it is easy to see that

$$\frac{D^{(A,H)} v_0(t)}{Dt}\bigg|_{t=t_0} = \frac{D^{(A,H)} v^h_0(t)}{Dt}\bigg|_{t=t_0} = \frac{D^{(A)} v^h_0(t)}{Dt}\bigg|_{t=t_0}$$

does not depend on the choice of $[q_0, \xi_0]_G$ as before (in fact, one can prove that it only depends on $d[v]_G(t)/dt|_{t=t_0}$.) Then, we have

$$\frac{D^{(A,V)} v_0(t)}{Dt}\bigg|_{t=t_0} = \frac{D^{(A)} v_0(t)}{Dt}\bigg|_{t=t_0} - \frac{D^{(A)} v^h_0(t)}{Dt}\bigg|_{t=t_0} .$$

It is clear that the right-hand side does not depend on the choice of the curve $g_0(t)$ satisfying the previous conditions, and now we shall see that it depends only on the curve $[v]_G(t)$ on V/G be and on $[q_0, \xi_0]_G \in \tilde{\mathfrak{g}}_{x_0}$ and not on the particular choice of v_0 and q_0 satisfying the previous assumptions. Let us choose $v_1 \in [v_0]_G$ and also $q_1 \in [q_0]_G$ such that $\tau(v_1) = q_1$. Then there exists $h \in G$ such that $q_1 = hq_0$ and $\xi_1 = h\xi_0 h^{-1}$, and we also know from Lemma 5.2.1 that $v_1 = hv_0$. If a curve $g_0(t)$ on G satisfies $g_0(t_0) = e$ and $\dot{g}_0(t_0) = \xi_0$

then we can choose, without loss of generality, the curve $g_1(t) = hg_0(t)h^{-1}$ that satisfies, $g_1(t_0) = e$ and $\dot{g}_1(t_0) = \xi_1$. We can easily see from Lemma 5.2.1 that there exists a unique curve $v^h_{q_1}(t)$ on V such that $\left[v^h_{q_1}(t)\right]_G = [v]_G(t)$ and $\tau\left(v^h_{q_1}(t)\right) = x^h_{q_1}(t)$ for all t. Define, $v_1(t) = g_1(t)v^h_{q_1}(t)$, for all t, then we see that $v_1(t) = hg_0(t)h^{-1}hv_0 = hv_0(t)$ and, also from Lemma 5.2.1, we can see easily that $v_1(t)$ is the only curve on V such that $[v_1(t)]_G = [v]_G(t)$ and $\tau(v_1(t)) = q_1(t)$ for all t. Then using Lemma 6.1.11, we obtain

$$\left[\frac{D^{(A)}v_1(t)}{Dt}\bigg|_{t=t_0}\right]_G = \left[h\,\frac{D^{(A)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G = \left[\frac{D^{(A)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G.$$

This and the definition of the vertical covariant derivative shows that

$$\left[\frac{D^{(A,V)}v_1(t)}{Dt}\bigg|_{t=t_0}\right]_G = \left[h\,\frac{D^{(A,V)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G = \left[\frac{D^{(A,V)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G.$$

This justifies the following definition of the reduced covariant derivative and also the reduced vertical covariant derivative.

Definition 6.2.2 *Assume the same conditions as in Definition 6.1.9 and assume that the connection ∇ and therefore also the covariant derivative D/Dt is invariant. Let $[v]_G(t)$ be a given curve in V/G and choose $[q_0, \xi_0]_G \in \tau([v]_G(t_0))$.*

(A) *The **reduced covariant derivative** $[D^{(A)}/Dt]_{G[q_0,\xi_0]_G}[v]_G(t_0)$ of the curve $[v]_G(t)$ **with respect to** $[q_0, \xi_0]_G$ at $t = t_0$ is defined by*

$$\left[\frac{D^{(A)}}{Dt}\right]_{G[q_0,\xi_0]_G}[v]_G(t)\bigg|_{t=t_0} = \left[\frac{D^{(A)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G.$$

(B) *The **reduced vertical covariant derivative** $[D^{(A,V)}/Dt]_{G[q_0,\xi_0]_G}[v]_G(t_0)$ of the curve $[v]_G(t)$ **with respect to** $[q_0, \xi_0]_G$ at $t = t_0$ is defined by*

$$\left[\frac{D^{(A,V)}}{Dt}\right]_{G[q_0,\xi_0]_G}[v]_G(t)\bigg|_{t=t_0} = \left[\frac{D^{(A,V)}v_0(t)}{Dt}\bigg|_{t=t_0}\right]_G.$$

The following formula is a direct consequence of the previous definition and previous equalities

$$\begin{aligned}&\left[\frac{D^{(A)}}{Dt}\right]_{G[q_0,\xi_0]_G}[v]_G(t)\bigg|_{t=t_0}\\ &\quad= \left[\frac{D^{(A,V)}}{Dt}\right]_{G[q_0,\xi_0]_G}[v]_G(t)\bigg|_{t=t_0} + \left[\frac{D^{(A,H)}}{Dt}\right]_G[v]_G(t)\bigg|_{t=t_0}.\end{aligned}$$

Next we consider an important particular case in which the reduced vertical covariant derivative can be calculated with a formula entirely similar to the one derived before for the case of a constant curve.

Lemma 6.2.3 *Assume the conditions of the previous Definition, and in addition, assume that there is an invariant section $v' \in \Gamma^G(V)$ such that $v_0(t) = v'\left(g_0(t)x^h_{q_0}(t)(t)\right)$. Then*

$$\left[\frac{D^{(A,V)}}{Dt}\right]_{G[q_0,\xi_0]_G}[v]_G(t)\Bigg|_{t=t_0} = \left[\frac{D^{(A)}g_0(t)v_0}{Dt}\Bigg|_{t=t_0}\right]_G.$$

Proof. The proof is a direct consequence of Lemma 6.1.10(c) ■

Reduced Vertical Connections. Now we will describe the notion of reduced vertical connection and also reduced connection.

Definition 6.2.4 *Let $[q_0,\xi_0]_G \in \tilde{\mathfrak{g}}$ with $\tilde{\pi}_G[q_0,\xi_0]_G = [q_0]_G$ and let $[v]_G \in \Gamma(V/G)$, where $v \in \Gamma^G(V)$ according to Lemma 5.2.2. Let $Y_0 \in I^V_G(Q)$ such that $\beta_A(Y_0) = [q_0,\xi_0]_G$, according to Lemma 5.1.2. Then*

(A) *The **reduced vertical connection** $\left[\nabla^{(A,V)}\right]_{G,[q_0,\xi_0]_G}$ is defined by the condition*

$$\left[\nabla^{(A,V)}_{Y_0}v\right]_G = \left[\nabla^{(A,V)}\right]_{G,[q_0,\xi_0]_G}[v]_G.$$

(B) *Let, in addition, $\bar{X}_0 \in T_{[q_0]_G}(Q/G)$ and define $X_0 = \bar{X}^h_0$, therefore $X_0 \in I^H_G(Q)$, and also define $Z_0 = X_0 + Y_0$. The **reduced connection** $\left[\nabla^{(A)}\right]_G$ is defined by the condition*

$$[\nabla_{Z_0}v]_G = \left[\nabla^{(A)}\right]_{G,\bar{X}_0\oplus[q_0,\xi_0]_G}[v]_G.$$

The consistency of Definition 6.2.4 is easily proven using Lemma 6.1.11, which establishes invariance of $\nabla^{(A,V)}$ and $\nabla^{(A,H)}$, and also Lemma 5.2.2.

The following lemma establishes the link between the notions of reduced vertical covariant derivative (resp. reduced covariant derivative) and reduced vertical connection (resp. reduced connection)

Lemma 6.2.5 *Let $[q_0,\xi_0]_G \in \tilde{\mathfrak{g}}$ and let $[v]_G \in \Gamma(V/G)$. where $v \in \Gamma^G(V)$ according to Lemma 5.2.2. Let $[q]_G(t) = x(t)$ be any curve in Q/G such that $\tilde{\pi}_G([q_0,\xi_0]_G) = [q]_G(t_0)$ and let, with a convenient abuse of notation, $[v]_G(t) = [v]_G([q]_G(t))$. Assume that D is invariant. Then we have*

(A)

$$\left[\frac{D^{(A,V)}}{Dt}\right]_{G,[q_0,\xi_0]_G}[v]_G\Bigg|_{t=t_0} = \left[\nabla^{(A,V)}\right]_{G,[q_0,\xi_0]_G}[v]_G\left([q]_G(t_0)\right)$$

(B)

$$\left[\frac{D^{(A)}}{Dt}\right]_{G,[q_0,\xi_0]_G}[v]_G\Bigg|_{t=t_0} = \left[\nabla^{(A)}V\right]_{G,\dot{x}(t_0)\oplus[q_0,\xi_0]_G}[v]_G\left([q]_G(t_0)\right).$$

Proof. The proof is a direct consequence of Lemma 6.2.3 and the fundamental link between covariant derivatives and connections in a vector bundle explained before (see the paragraph **Affine Connections** in §2.3). ■

Group Actions on Lagrange-Poincaré Bundles. Now we shall define the notion of an action of a group G on an object of $\mathfrak{LP}$.

Definition 6.2.6 *An **action in the category** $\mathfrak{LP}$ of a group G on an object $TQ\oplus V$ of $\mathfrak{LP}$ is a vector bundle action $\rho : G\times TQ\oplus V \to TQ\oplus V$ such that, for each $g\in G$, $\rho_g : TQ\oplus V \to TQ\oplus V$ is an isomorphism of $\mathfrak{LP}$. Sometimes we will call an action in the category $\mathfrak{LP}$ simply an action, if the fact that it is an action in the category $\mathfrak{LP}$ is clear from the context.*

Corollary 6.2.7 *If $\rho : G\times TQ\oplus V \to TQ\oplus V$ is an action in the category $\mathfrak{LP}$ then we have*

(1) *$\rho_* : G\times\Gamma(TQ\oplus V) \to \Gamma(TQ\oplus V)$ is a Lie algebra representation of G, that is, a linear representation such that, for each $g\in G$, $\rho_{g*} : \Gamma(TQ\oplus V)\to\Gamma(TQ\oplus V)$ is a Lie algebra isomorphism.*

(2) *$\rho : G\times TQ \to TQ$ is the tangent lift of the action $\rho_0 : G\times Q\to Q$.*

Proof. (1) is a consequence of Lemma 6.1.5 and (2) is a consequence of the definition of morphism in the category $\mathfrak{LP}$.

Definition 6.2.8 *Let $\tau : V\to Q$ be a vector bundle and let $\rho : G\times V\to V$ be a vector bundle action. As usual, the action $\rho_0 : G\times Q\to Q$ is assumed to give Q the structure of a principal bundle. We will denote $\iota_G(V) : I_G(V)\to Q/G$ the vector bundle whose fiber at the point $[q]_G\in Q/G$ is the space of all invariant sections of V on the orbit $Gq\equiv[q]_G$. In addition, we will denote $\gamma_G(V)$ the vector bundle isomorphism $\gamma_G(V) : I_G(V)\to V/G$ given by*

$$\gamma_G(V)\left(v_{[q]_G}\right) = \left[v_{[q]_G}(q)\right]_G$$

(here, $v_{[q]_G}$ is a section of V on the orbit $Gq\equiv[q]_G$!).

It is easy to see that $\gamma_G(V)$ is a well defined vector bundle isomorphism. Observe that Definition 6.2.8 is consistent with Definition 5.1.1; in fact, we can easily see that $I_G^H(TQ) = I_G(\mathrm{Hor}(TQ))$ and $I_G^V(TQ) = I_G(\mathrm{Ver}(TQ))$. Obviously we have a vector bundle isomorphism

$$\gamma : I_G^H(TQ) \oplus I_G^V(TQ) \oplus I_G(V) \to \mathrm{Hor}(TQ)/G \oplus \mathrm{Ver}(TQ)/G \oplus (V/G)$$

defined by

$$\begin{aligned} \gamma &\equiv \gamma_G(\mathrm{Hor}(TQ)) \oplus \gamma_G(\mathrm{Ver}(TQ)) \oplus \gamma_G(V) \\ &\equiv \gamma_G(\mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ) \oplus V). \end{aligned}$$

Isomorphisms Between Quotient Bundles. Using Lemma 5.1.2 and taking into account the natural identifications $TQ \equiv \mathrm{Hor}(TQ) \oplus \mathrm{Ver}(TQ)$ and also $TQ/G \equiv \mathrm{Hor}(TQ)/G \oplus \mathrm{Ver}(TQ)/G$, we obtain a vector bundle isomorphism

$$\beta_A \oplus \gamma_G(V) : I_G^H(TQ) \oplus I_G^V(TQ) \oplus I_G(V) \to T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G).$$

Also from Lemma 5.2.2 we see that there are linear isomorphisms

$$\pi_G(V)^* : \Gamma(V/G) \to \Gamma^G(V),$$

$$\pi_G(\mathrm{Hor}(TQ))^* : \Gamma(\mathrm{Hor}(TQ)/G) \to \Gamma^G(\mathrm{Hor}(TQ)),$$

and

$$\pi_G(\mathrm{Ver}(TQ))^* : \Gamma(\mathrm{Ver}(TQ)/G) \to \Gamma^G(\mathrm{Ver}(TQ)).$$

Definition 6.2.9 *Define the linear isomorphism*

$$\begin{aligned} \pi_* : \Gamma^G(\mathrm{Hor}(TQ)) \oplus \Gamma^G(\mathrm{Ver}(TQ)) \oplus \Gamma^G(V) \\ \to \Gamma(\mathrm{Hor}(TQ)/G) \oplus \Gamma(\mathrm{Ver}(TQ)/G) \oplus \Gamma(V/G) \end{aligned}$$

by

$$\pi_* = (\pi_G(\mathrm{Hor}(TQ))^* \oplus \pi_G(\mathrm{Ver}(TQ))^* \oplus \pi_G(V)^*)^{-1}.$$

Similarly, the linear isomorphism

$$\begin{aligned} \varphi \equiv \varphi_A(Q,G,V) \equiv \varphi_A : \Gamma^G(\mathrm{Hor}(TQ)) \oplus \Gamma^G(\mathrm{Ver}(TQ)) \oplus \Gamma^G(V) \\ \to \Gamma(T(Q/G)) \oplus \Gamma(\tilde{\mathfrak{g}}) \oplus \Gamma(V/G) \end{aligned}$$

is defined by

$$\varphi = (\beta_A \oplus \gamma_G(V))_* \circ \gamma^* \circ \pi_*.$$

If we consider $\gamma^* \circ \pi_*$ as being an identification, which is natural in the present context, we can write $\varphi \equiv (\beta_A \oplus \gamma_G(V))_*$. Likewise, if we consider γ^* and π_* as identifications we can write $\varphi : \Gamma(TQ/G) \oplus \Gamma(V/G) \to \Gamma((TQ/G) \oplus \tilde{\mathfrak{g}}) \oplus \Gamma(V/G)$ as $\varphi \equiv \alpha_A^* \oplus id_{(V/G)^*}$.

Structure of Reduced Lagrange-Poincaré Bundles. Now we state and prove our main result in this section.

Theorem 6.2.10 *Let $\tau_Q \oplus \tau : TQ \oplus V \to Q$ be an object of $\mathfrak{LP}$ and let ∇, ω, $[\,,]$ be the structure on V. Let $\rho : G \times (TQ \oplus V) \to TQ \oplus V$ be an action in the category $\mathfrak{LP}$. Choose a connection A on the principal bundle Q with structure group G. Consider the bundle*

$$T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G).$$

Define the structure $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$, $[\,,]^{\tilde{\mathfrak{g}}}$ on $\tilde{\mathfrak{g}} \oplus (V/G)$ by

$$\begin{aligned}
\nabla^{\tilde{\mathfrak{g}}}_X \left(\bar{\xi} \oplus [v]_G\right) &= \tilde{\nabla}^A_X \bar{\xi} \oplus [\nabla^{(A,H)}]_{GX}[v]_G - [\omega]_G(X, \bar{\xi}) \\
\omega^{\tilde{\mathfrak{g}}}(X_1, X_2) &= \tilde{B}^A(X_1, X_2) \oplus [\omega]_G(X_1, X_2) \\
\left[\bar{\xi}_1 \oplus [v_1]_G, \bar{\xi}_2 \oplus [v_2]_G\right]^{\tilde{\mathfrak{g}}} &= [\bar{\xi}_1, \bar{\xi}_2] \oplus [\nabla^{(A,V)}]_{G\bar{\xi}_1}[v_2]_G - [\nabla^{(A,V)}]_{G\bar{\xi}_2}[v_1]_G \\
&\quad - [\omega]_G(\bar{\xi}_1, \bar{\xi}_2) + [[v_1]_G, [v_2]_G]\,.
\end{aligned}$$

Then the bundle $T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$ with the structure on the bundle $\tilde{\mathfrak{g}} \oplus (V/G)$ given by $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$, $[,]^{\tilde{\mathfrak{g}}}$ is an object of the category $\mathfrak{LP}$.

Remark. Theorems 6.2.10 and 5.2.4 seem to shed light on the structure of Lie group extensions, namely that they necessarily have the structure, roughly speaking of a semidirect product with a curvature cocycle. It would be of interest, perhaps, to explore this technical point further.

Proof of the Theorem. It is easy to see that $\nabla^{\tilde{\mathfrak{g}}}$ is a connection on $\tilde{\mathfrak{g}} \oplus (V/G)$, that is, it satisfies the properties stated in the note following Definition 2.3.1. It is also easy to check that $\omega^{\tilde{\mathfrak{g}}}$ is a $\tilde{\mathfrak{g}} \oplus (V/G)$ -valued 2-form. We must now show that the expression of $[,]^{\tilde{\mathfrak{g}}}$ given in the statement of the theorem defines a Lie bracket on fibers of $\tilde{\mathfrak{g}} \oplus (V/G)$ that gives this bundle the structure of a Lie algebra bundle. It is clear that each one of the terms $[\bar{\xi}_1, \bar{\xi}_2]$, $-[\omega]_G(\bar{\xi}_1, \bar{\xi}_2)$ and $[[v_1]_G, [v_2]_G]$ defines a bilinear and skew symmetric operation on fibers of $\tilde{\mathfrak{g}} \oplus (V/G)$. Now using Lemma 6.1.10 and Definition 6.2.4 we can see that the term

$$[\nabla^{(A,V)}]_{G\bar{\xi}_1}[v_2]_G - [\nabla^{(A,V)}]_{G\bar{\xi}_2}[v_1]_G$$

also defines a bilinear and skew symmetric operation on fibers of $\tilde{\mathfrak{g}} \oplus (V/G)$. So far we have proven that the expression of $[,]^{\tilde{\mathfrak{g}}}$ defines a bilinear and skew symmetric operation on fibers of $\tilde{\mathfrak{g}} \oplus (V/G)$. Moreover, it is easy to see that for any function $f \in C^\infty(Q/G)$ we have

$$\left[\bar{\xi}_1 \oplus [v_1]_G, f\left(\bar{\xi}_2 \oplus [v_2]_G\right)\right]^{\tilde{\mathfrak{g}}} = f\left[\bar{\xi}_1 \oplus [v_1]_G, \bar{\xi}_2 \oplus [v_2]_G\right]^{\tilde{\mathfrak{g}}}.$$

It remains to prove that $[\,,]^{\tilde{\mathfrak{g}}}$ satisfies the Jacobi identity. For this we must first study the Lie algebra structure on

$$\Gamma\left(T(Q/G)\right) \oplus \Gamma\left(\tilde{\mathfrak{g}}\right) \oplus \Gamma(V/G) \equiv \Gamma\left(\left(T(Q/G)\right) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right).$$

According to Definition 6.1.1 the operation on sections $Z_i \oplus v_i \in \Gamma(TQ \oplus V)$, $1 = 1, 2$ given by

$$[Z_1 \oplus v_1, Z_2 \oplus v_2] = [Z_1, Z_2] \oplus \nabla_{Z1} v_2 - \nabla_{Z2} v_1 - \omega(Z_1, Z_2) + [v_1, v_2]$$

is a Lie bracket.

Invariance of ∇, ω and $[\,,]$ implies that

$$\Gamma^G(TQ \oplus V) \equiv \Gamma^G\left(\mathrm{Hor}(TQ)\right) \oplus \Gamma^G\left(\mathrm{Ver}(TQ)\right) \oplus \Gamma^G(V)$$

is a Lie subalgebra. Using this and the linear isomorphism φ we can define a Lie algebra on

$$\Gamma\left(T(Q/G)\right) \oplus \Gamma\left(\tilde{\mathfrak{g}}\right) \oplus \Gamma(V/G)$$

by

$$\begin{aligned}\left[X_1 \oplus \bar{\xi}_1 \oplus [v_1]_G, X_2 \oplus \bar{\xi}_2 \oplus [v_2]_G\right] = \varphi\big[&\varphi^{-1}\left(X_1 \oplus \bar{\xi}_1 \oplus [v_1]_G\right), \\ &\varphi^{-1}\left(X_2 \oplus \bar{\xi}_2 \oplus [v_2]_G\right)\big].\end{aligned}$$

We are going to show that this Lie bracket coincides with the one constructed on

$$\Gamma\left(T(Q/G)\right) \oplus \Gamma\left(\tilde{\mathfrak{g}}\right) \oplus \Gamma(V/G)$$

using the structures $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ and $[\,,]^{\tilde{\mathfrak{g}}}$, according to Definition 6.1.1. We know that $\varphi^{-1} X_i = X_i^h$, for $i = 1, 2$. Let $\varphi^{-1}\bar{\xi}_i = Y_i$ and $\varphi^{-1}[v_i]_G = v_i$, for $i = 1, 2$ (recall that there is a unique $v_i \in \Gamma^G(V)$ in each class $[v_i]_G$). By Definition 6.1.1 we have

$$\begin{aligned}&[X_1^h \oplus Y_1 \oplus v_1, X_2^h \oplus Y_2 \oplus v_2] \\ &\quad = [X_1^h + Y_1, X_2^h + Y_2] \oplus \nabla_{X_1^h + Y_1} v_2 - \nabla_{X_2^h + Y_2} v_1 \\ &\qquad - \omega(X_1^h + Y_1, X_2^h + Y_2) + [v_1, v_2].\end{aligned}$$

Since $\varphi \,|\, \Gamma^G(TQ) = \alpha_{A*} \,|\, \Gamma^G(TQ)$ we have, using Theorem 5.2.4, that

$$\varphi[X_1^h \oplus Y_1, X_2^h \oplus Y_2] = [X_1, X_2] \oplus \tilde{\nabla}^A_{X_1} \bar{\xi}_2 - \tilde{\nabla}^A_{X_2} \bar{\xi}_1 - \tilde{B}^A(X_1, X_2) + [\bar{\xi}_1, \bar{\xi}_2].$$

Using Definition 6.2.4 we have, for $\{i, j\} = \{1, 2\}$,

$$\varphi(\nabla_{X_i^h + Y_i} v_j) = \left[\nabla^{(A,H)}\right]_{GX_i} [v_j]_G + \left[\nabla^{(A,V)}\right]_{G\bar{\xi}_i} [v_j]_G.$$

Using Lemma 6.1.7 we can easily see that

$$\begin{aligned}\varphi\omega(X_1^h + Y_1, X_2^h + Y_2) \\ = [\omega]_G(X_1, X_2) + [\omega]_G(X_1, \bar{\xi}_2) + [\omega]_G(\bar{\xi}_1, X_2) + [\omega]_G(\bar{\xi}_1, \bar{\xi}_2)\end{aligned}$$

and also that

$$\varphi[v_1, v_2] = [[v_1]_G, [v_2]_G] \,.$$

Thus, we have proven that the operation on

$$\Gamma\left((T(Q/G)) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right)$$

using the structures $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ and $[,]^{\tilde{\mathfrak{g}}}$, according to Definition 6.1.1 is a Lie bracket, moreover we have a description of this bracket using the isomorphism φ. It follows that the restriction of this Lie bracket to $\Gamma\left(\tilde{\mathfrak{g}} \oplus (V/G)\right)$ is also a Lie bracket. This restriction is given by $[,]^{\tilde{\mathfrak{g}}}$, therefore the Jacobi identity for the bilinear and skew symmetric operation on fibers of $\Gamma\left(\tilde{\mathfrak{g}} \oplus (V/G)\right)$ defined by $[,]^{\tilde{\mathfrak{g}}}$ must be satisfied. This finishes the proof of the theorem. ■

Now we shall introduce some more notation.

Definition 6.2.11 *Assume the hypothesis of Theorem 6.2.10. The bundle*

$$T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$$

with the ***reduced structure*** $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ *and* $[\,,]^{\tilde{\mathfrak{g}}}$ *on* $\tilde{\mathfrak{g}} \oplus (V/G)$ *is called the* ***reduced bundle with respect to the group*** G ***and the connection*** A. *The* ***reduced Lie algebra structure*** *on*

$$\Gamma(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G))$$

is the one described in Theorem 6.2.10 using the reduced structure $\nabla^{\tilde{\mathfrak{g}}}$, $\omega^{\tilde{\mathfrak{g}}}$ *and* $[\,,]^{\tilde{\mathfrak{g}}}$, *according to Definition 6.1.1. Let* $W = TQ \oplus V$, *for short. We will denote* α_A^W *the natural map*

$$\alpha_A^W : (TQ \oplus V)/G \to T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G),$$

more precisely, $\alpha_A^W = \alpha_A \oplus id_{V/G}$, *where* α_A *is the map of Lemma 2.4.2. To make the notation consistent in the case* $V = 0$, *that is,* $W = TQ$, *we will write* $\alpha_A \equiv \alpha_A^{TQ}$.

Lemma 6.2.12 *Assume the hypothesis of Theorem 6.2.10. Then we have*

(a) *The push forward map*

$$\alpha^W_{A*} : \Gamma\left((TQ \oplus V)/G\right) \to \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right)$$

is a Lie algebra isomorphism. Here the Lie algebra on $\Gamma\left((TQ \oplus V)/G\right)$ *is the natural quotient Lie algebra structure, as defined in the comments following Lemma 5.2.2.*

(b) *Let* A' *be another connection on the principal bundle* Q *with structure group* G*. Then* $\alpha^W_{A'} \circ (\alpha^W_A)^{-1}$ *is an isomorphism in the category* $\mathfrak{LP}$ *if and only if* $A = A'$*.*

Proof. (a) The proof of this part is an easy consequence of the proof of Theorem 6.2.10.

(b) Let

$$X \equiv X \oplus 0 \oplus 0 \in \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right).$$

The proof of Theorem 6.2.10 shows that $(\alpha^W_A)^{-1}(X)$ is the horizontal lift of X with respect to the connection A and then that $(\alpha^W_{A'} \circ (\alpha^W_A)^{-1})(X)$ can be written

$$(\alpha^W_{A'} \circ (\alpha^W_A)^{-1})(X) = X \oplus \bar{\xi},$$

where $\bar{\xi} \in \tilde{\mathfrak{g}}$. Also, by the same argument from the proof of Theorem 6.2.10, we can see that $\bar{\xi} = 0$ if and only if $(\alpha^W_A)^{-1}(X)$ is horizontal with respect to the connection A'. On the other hand it is clear, also from the proof of Theorem 6.2.10, that for any

$$\bar{\xi} \oplus [v]_G \equiv 0 \oplus \bar{\xi} \oplus [v]_G \in \Gamma\left(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right)$$

we have

$$(\alpha^W_{A'} \circ (\alpha^W_A)^{-1})\left(\bar{\xi} \oplus [v]_G\right) \in \Gamma\left(\tilde{\mathfrak{g}} \oplus (V/G)\right).$$

Using what we have proven so far, we can deduce that the conditions

$$\left(\alpha^W_{A'} \circ (\alpha^W_A)^{-1}\right) T(Q/G) = T(Q/G)$$

and

$$\left(\alpha^W_{A'} \circ (\alpha^W_A)^{-1}\right)\left(\tilde{\mathfrak{g}} \oplus (V/G)\right) = \tilde{\mathfrak{g}} \oplus (V/G)$$

are satisfied if and only if $A = A'$. ■

6.3 Reduction by Stages of objects of $\mathfrak{LP}$

We shall begin by recollecting, without proof, some more or less standard facts. In this section G will be a Lie group and $N \subset G$ a normal Lie subgroup.

Lemma 6.3.1 *Assume that a group G acts on the left on a manifold Q (this time we do not need to assume that, with this action, Q becomes a principal bundle) and let $N \subset G$ be a normal subgroup. Then the rule $[g]_N[q]_N = [gq]_N$ defines an action of G/N on Q/N. The map*

$$i^Q_{G/N} : Q/G \to (Q/N)/(G/N)$$

given by

$$i^Q_{G/N}([q]_G) = [[q]_N]_{(G/N)}$$

is a bijection.

If the action of G on Q is free then the action of N on Q is free and also the action of G/N on Q/N is free. Conversely, if the action of N on Q is free and also the action of G/N on Q/N is free then the action of G on Q is free.

Now assume, in addition, that $\pi : Q \to Q/G$, is a principal bundle with structure group G. Then the map $i^Q_{G/N}$ is a well defined diffeomorphism. Moreover, Q/N is a principal bundle with structure group G/N and Q is a principal bundle with structure group N. Conversely, if Q/N is a principal bundle with structure group G/N and Q is a principal bundle with structure group N then Q is a principal bundle with structure group G.

We are interested in cases of the previous lemma in which the manifold Q carries some extra structures which remain invariant under the action of the group G and we also want to show how to obtain the corresponding quotient structures by stages. The next lemma considers the case of an invariant vector bundle structure. We will use some notation and results described in section §5.2, in particular Lemma 5.2.1

Lemma 6.3.2 *Let $\tau : V \to Q$ be a vector bundle and let $\rho : G \times V \to V$ be a vector bundle action of the group G on V such that the action $\rho_0 : G \times Q \to Q$ gives Q the structure of a principal bundle. Let $N \subset G$ be a normal subgroup. Then*

$$\tau/N : V/N \to Q/N \quad \text{and} \quad \tau/G : V/G \to Q/G$$

are vector bundles and G/N acts with a vector bundle action on V/N. The quotient

$$(\tau/N)/(G/N) : (V/N)/(G/N) \to (Q/N)/(G/N)$$

is a vector bundle. The isomorphism $i^V_{G/N}$ is a vector bundle isomorphism.

Now we state the following lemma whose proof is straightforward:

Lemma 6.3.3 *Assume the hypothesis of Lemma 6.1.7 and, in addition, assume that N is a normal subgroup of G. Thus, in particular, any one of the structures (a) or (b) defined in Lemma 6.1.7 which is invariant under the action of G is also invariant under the restricted action of N, then it gives rise to a corresponding quotient structure on the quotient vector bundle V/N. Then the quotient structures so defined are invariant under the action of G/N on V/N described in Lemma (6.3.2) and then define corresponding quotient structures on the bundle $(V/N)/(G/N)$. Moreover, the isomorphism $i^V_{G/N}$ described in Lemma (6.3.2) commutes with those structures, in other words, it is also an isomorphism with respect to them.*

Let us now consider an important example of this situation. Let us assume the hypothesis of Lemma (6.3.1). Let $\tilde{\mathfrak{n}}$ be as in Definition 2.3.3. Then G/N acts on $\tilde{\mathfrak{n}}$ as follows:

$$[g]_N[q,\xi]_N = [gq, \mathrm{Ad}_g\,\xi]_N.$$

We will prove that this action is well defined. Since $N \subset G$ is a normal subgroup, for any $g \in G$ and any $n \in N$ there exist $n'_g, n''_g \in N$ such that $gn = n'_g g$ and $ng = gn''_g$. Besides, for any $\xi \in \mathfrak{n}$ we have $\mathrm{Ad}_g\,\xi \in \mathfrak{n}$. Now let $n_i \in N$ be given for $i = 1, 2$. Then we have

$$\begin{aligned}
[n_1 g]_N[n_2 q, \mathrm{Ad}_{n_2}\,\xi]_N &= [n_1 g n_2 q, \mathrm{Ad}_{n_1 g}\,\mathrm{Ad}_{n_2}\,\xi]_N \\
&= [n_1 n'_{2g} g q, \mathrm{Ad}_{n_1 n'_{2g} g}\,\xi]_N \\
&= [n_1 n'_{2g} g q, \mathrm{Ad}_{n_1 n'_{2g}}\,\mathrm{Ad}_g\,\xi]_N \\
&= [gq, \mathrm{Ad}_g\,\xi]_N,
\end{aligned}$$

which shows that the action is well defined.

Now we will study invariance properties of the structures of $\tilde{\mathfrak{n}}$. We will omit the proof of the next lemma, which is straightforward.

Lemma 6.3.4 *Assume the same hypothesis as in Lemma 6.3.3 and, in addition, choose a connection A_N on the principal bundle Q with structure group N. Then the action of G/N on $\tilde{\mathfrak{n}}$ defined above commutes with the Lie algebra structure on fibers of $\tilde{\mathfrak{n}}$ defined in Lemma 2.3.5 and also with the $\tilde{\mathfrak{n}}$- valued curvature 2-form $\tilde{B}^{A_N}$ defined by equation (3.1.1). More precisely, for any $[q,\xi_i]_N \in \tilde{\mathfrak{n}}$, $i = 1, 2$, any $X_i \in T_{[q]_N}(Q/N)$, $i = 1, 2$ and any $g \in G$ we have*

$$[[g]_N[q,\xi_1]_N, [g]_N[q,\xi_2]_N] = [g]_N\,[[q,\xi_1]_N, [q,\xi_2]_N]$$

and

$$\tilde{B}^{A_N}\left([g]_N X_1, [g]_N X_2\right) = [g]_N \tilde{B}^{A_N}(X_1, X_2).$$

It is not generally true that for any connection A_N the covariant derivative $\tilde{D}^{A_N}$ or, equivalently, the connection $\tilde{\nabla}^{A_N}$ on the bundle $\tilde{\mathfrak{n}}$, is invariant under the action of G/N. However there is always a connection A_N having this property, that is, having the property that the action of G/N commutes with $\tilde{D}^{A_N}$ or, equivalently, with the connection $\tilde{\nabla}^{A_N}$, as we shall see next. Let $\langle , \rangle$ be any G-invariant metric on Q. Then for each $q \in Q$ we have a direct sum orthogonal decomposition

$$T_qQ = \mathrm{Ver}^N(T_qQ) + H_N(q),$$

where $H_N(q)$ is the orthogonal complement of $\mathrm{Ver}^N(TQQ)$. Given $v_q \in T_qQ$ we can then write $v_q = \xi q + v_{qH_N}$ where $v_{qH_N} \in H_N(q)$ and $\xi \in \mathfrak{n}$. We can easily see that the restriction

$$T_q\pi_N \mid H_N(q) : H_N(q) \to T_{[q]_N}(Q/N)$$

is a linear isomorphism and that the collection of all $H_N(q)$ is G- invariant, therefore, also N-invariant. Thus the collection of all $H_N(q)$ defines a connection on the principal bundle Q with structure group N. Let A_N be the corresponding connection 1-form. By definition, we have for $v_q = \xi q + v_{qH_N}$ as before, $A_N(v_q) = \xi$. The horizontal spaces of this connection are $\mathrm{Hor}_q^{A_N}(TQ) \equiv H_N(q)$.

Lemma 6.3.5 *Let A_N be the connection associated to a G-invariant metric $\langle , \rangle$ as explained above. Then for any $g \in G$ and any $v_q \in T_qQ$ we have $A_N(gv_q) = \mathrm{Ad}_g A_N(v_q)$.*

Proof. For any $g \in G$ and any $v_q \in T_qQ$ we have

$$A_N(gv_q) = A_N(g\xi q + gv_{qH_N}) = A_N(g\xi q) + A_N(gv_{qH_N}).$$

Since the collection of horizontal spaces is invariant, it follows that $gv_{qH_N} \in H_N(gq)$ and therefore $A_N(gv_{qH_N}) = 0$. On the other hand we have $A_N(g\xi q) = A_N(\mathrm{Ad}_g \xi gq)$. Since $\mathfrak{n}$ is normal, we have $\mathrm{Ad}_g \xi \in \mathfrak{n}$. Then we have

$$A_N(gv_q) = \mathrm{Ad}_g \xi = \mathrm{Ad}_g A_N(v_q) \quad \text{for all} \quad q \in Q. \quad \blacksquare$$

Lemma 6.3.6 (a) *Let A_N be a connection on the principal bundle Q with structure group N having the property that for any $g \in G$ and any $v_q \in T_qQ$ we have $A_N(gv_q) = \mathrm{Ad}_g A_N(v_q)$. Let $\tilde{D}^{A_N}$ be the covariant derivative on $\tilde{\mathfrak{n}}$ associated to A_N according to Definition 2.3.1. Then $\tilde{D}^{A_N}$ is G/N- invariant.*

(b) *Assume the same conditions as in a). Then the action of G/N on $T(Q/N) \oplus \tilde{\mathfrak{n}}$ defined by the natural actions of G/N on $T(Q/N)$ and on $\tilde{\mathfrak{n}}$ defined above commutes with the isomorphism $\alpha_{A_N} : TQ/N \to T(Q/N) \oplus \tilde{\mathfrak{n}}$ (see Lemma 2.4.2).*

Proof. (a) Using the previous lemmas the proof of this part is straightforward. According to Lemma 2.3.4 we have

$$\frac{\tilde{D}^{A_N}[q(s),\xi(s)]_G}{Ds} = \Big[q(s), -\left[A_N\left(q(s),\dot{q}(s)\right),\xi(s)\right] + \dot{\xi}(s)\Big]_N .$$

Given any $[g]_N \in G/N$ we have

$$\begin{aligned}\frac{\tilde{D}^{A_N}[gq(s),\operatorname{Ad}_g\xi(s)]_G}{Ds} &= \Big[gq(s), -\left[A_N\left(gq(s),g\dot{q}(s)\right),\operatorname{Ad}_g\xi(s)\right] + \operatorname{Ad}_g\dot{\xi}(s)\Big]_N \\ &= \Big[gq(s), -\operatorname{Ad}_g\left[A_N\left(q(s),\dot{q}(s)\right),\xi(s)\right] + \operatorname{Ad}_g\dot{\xi}(s)\Big]_N \\ &= [g]_N\Big[q(s), -\left[A_N\left(q(s),\dot{q}(s)\right),\xi(s)\right] + \dot{\xi}(s)\Big]_N .\end{aligned}$$

(b) Using the previous lemmas the proof of this part is straightforward. We shall omit the details. ■

Theorem 6.3.7 *Let Q be a principal bundle with structure group G and let N be a normal subgroup of G. Let A_N be a connection on the principal bundle Q with structure group N having the property that for any $g \in G$ and any $v_q \in T_qQ$ we have $A_N(gv_q) = \operatorname{Ad}_g A_N(v_q)$. Consider the action of G/N on $T(Q/N) \oplus \tilde{\mathfrak{n}}$ such that its restriction to $\tilde{\mathfrak{n}}$ coincides with the natural action, defined above, and its restriction to $T(Q/N)$ coincides with the tangent lift of the action of G/N on Q/N defined in Lemma 6.3.1. Then this action is an action in the category $\mathfrak{LP}$. The reduced bundle of $T(Q/N) \oplus \tilde{\mathfrak{n}}$ by the group G/N with respect to any connection $A_{G/N}$ on the bundle Q/N with structure group G/N is an object of $\mathfrak{LP}$.*

Proof. The proof is a direct consequence of Definition 6.2.4, Lemmas 6.3.4 and 6.3.6 and Theorem 6.2.10. ■

More Preparatory Lemmas. Before we can state and prove our main result on Lagrangian reduction by stages we need a few more lemmas.

Lemma 6.3.8 *Assume the conditions of Theorem 6.3.7 and, in addition, choose a connection A_G on the principal bundle Q with structure group G. Let $K = G/N$. Consider the following vector bundle isomorphisms:*

(a) *the quotient isomorphism (see Lemma 6.3.6 and Definition 6.2.11)*

$$[\alpha^{TQ}_{A_N}]_{G/N} : (TQ/N)/(G/N) \to (T(Q/N) \oplus \tilde{\mathfrak{n}})/(G/N),$$

(b) *the isomorphism (see Lemma 6.3.1)*

$$i^{TQ}_{G/N} : TQ/G \to (TQ/N)/(G/N),$$

(c) *the isomorphism (see Definition 6.2.11)*

$$\alpha^{TQ}_{A_G} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

(d) *the isomorphism (see Definition 6.2.11)*

$$\begin{aligned}\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}} &: (T(Q/N) \oplus \tilde{\mathfrak{n}})/(G/N) \\ &\to T((Q/N)/(G/N)) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}}/(G/N)),\end{aligned}$$

where $\mathfrak{k}$ is the Lie algebra of K, thus we have a Lie algebra isomorphism $\mathfrak{k} \equiv \mathfrak{g}/\mathfrak{n}$.

Then the linear isomorphisms, given by the push forward of maps corresponding to the previous vector bundle isomorphisms

(a')

$$[\alpha^{TQ}_{A_N}]_{G/N*} : \Gamma((TQ/N)/(G/N)) \to \Gamma((T(Q/N) \oplus \tilde{\mathfrak{n}})/(G/N))$$

(b')

$$i^{TQ}_{G/N*} : \Gamma(TQ/G) \to \Gamma((TQ/N)/(G/N))$$

(c')

$$\alpha^{TQ}_{A_G*} : \Gamma(TQ/G) \to (T(Q/G) \oplus \tilde{\mathfrak{g}})$$

(d')

$$\begin{aligned}\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}*} &: \Gamma((T(Q/N) \oplus \tilde{\mathfrak{n}})/(G/N)) \\ &\to \Gamma\Big(T((Q/N)/(G/N)) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}}/(G/N))\Big)\end{aligned}$$

are Lie algebra isomorphisms.

Consider the composition

(e)

$$\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)} = \alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}} \circ [\alpha^{TQ}_{A_N}]_{G/N} \circ i^{TQ}_{G/N} \circ (\alpha^{TQ}_{A_G})^{-1},$$

which is a vector bundle isomorphism from

$$T(Q/G) \oplus \tilde{\mathfrak{g}} \quad \textit{onto} \quad T\left((Q/N)/(G/N)\right) \oplus \tilde{\mathfrak{k}} \oplus \left(\tilde{\mathfrak{n}}/(G/N)\right).$$

Then $\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}$ *induces a Lie algebra isomorphism*

(e')

$$\begin{aligned}\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)*} &: \Gamma\left((T(Q/G) \oplus \tilde{\mathfrak{g}})\right) \\ &\to \Gamma\Big(T\left((Q/N)/(G/N)\right) \oplus \tilde{\mathfrak{k}} \oplus \left(\tilde{\mathfrak{n}}/(G/N)\right)\Big).\end{aligned}$$

Proof. The conclusion follows using Lemmas 5.2.1, (6.3.6), (6.2.12), the paragraph on quotient Lie algebras in section §5.2 and standard work with quotient manifolds. ■

It is not generally true that $\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}$ is an isomorphism in the category $\mathfrak{LP}$, for arbitrary choices of the connections A_N, $A_{G/N}$ and A_G. Next we will show that it is always possible to choose A_N, $A_{G/N}$ and A_G in such a way that $\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}$ is is an isomorphism in the category $\mathfrak{LP}$.

Lemma 6.3.9 *Assume the hypothesis of Lemma 6.3.5 and define the connection* A_N *as in that lemma. Define the connection* A_G *on the principal bundle* Q *with structure group* G *by the condition that, for all* $q \in Q$, $\mathrm{Hor}^{A_G}_q(TQ)$ *is the orthogonal complement of* $\mathrm{Ver}^G_q(TQ)$. *For each* $[q]_N \in Q/N$ *let*

$$H_{[q]_N}((T(Q/N)) = T\pi_N \,\mathrm{Hor}^{A_G}_q(TQ),$$

where $\pi_N : Q \to Q/N$ *is the natural principal bundle projection. Then the collection of all* $H_{[q]_N}((T(Q/N))$ *as* $[q]_N$ *varies in* Q/N *defines a connection* $A_{G/N}$ *on the principal bundle* Q/N *with structure group* G/N, *by the condition*

$$H_{[q]_N}((T(Q/N)) \equiv \mathrm{Hor}^{A_{G/N}}_{[q]_N}((T(Q/N)).$$

The following property is satisfied: for all $v_q \in TQ$ *and all* $q \in Q$ *we have* $A_G(v_q) = 0$ *if and only if* $A_N(v_q) = 0$ *and* $A_{G/N}\left(T\pi_N(v_q)\right) = 0$.

We shall omit the proof of this lemma which can be carried out in a standard way using the preceding results.

The next theorem, together with its generalization for arbitrary objects of $\mathfrak{LP}$ given in Theorem 6.3.14 constitute our main result on Lagrangian reduction by stages.

Theorem 6.3.10 *Assume the hypothesis of Theorem 6.3.7 and, in addition, choose the connections A_N, $A_{G/N}$ and A_G in such a way that the following property is satisfied: for all $v_q \in TQ$ and all $q \in Q$ we have $A_G(v_q) = 0$ if and only if $A_N(v_q) = 0$ and $A_{G/N}(T\pi_N(v_q)) = 0$ Then $\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}$ (see Lemma 6.3.8 is an isomorphism in the category $\mathfrak{LP}$.*

Proof. Taking into account Lemmas 6.3.9 and 6.1.3 we see that we only need to show that

$$\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}(T(Q/G)) = T((Q/N)/(G/N)).$$

However, this can be established easily from the definition of $\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}$. We shall omit the details. ■

The following corollary gives an structure theorem for Lie algebras.

Corollary 6.3.11 *Let G be a Lie group, let N be a normal subgroup and let $K = G/N$. Let $\mathfrak{g}$, $\mathfrak{n}$ and $\mathfrak{k}$ be the Lie algebras of G, N and K respectively. Choose an identification $\mathfrak{g} \equiv \mathfrak{k} \oplus \mathfrak{n}$ as linear spaces. Choose any connection A_N on the principal bundle G with structure group N such that $A_N(gv_q) = \mathrm{Ad}_g A_N(v_q)$ for all $g \in G$, all $v_q \in T_qQ$ and all $q \in Q$. Then the Lie algebra structure on $\mathfrak{g}$ can be written in terms of the Lie algebra structure on $\mathfrak{n}$ and $\mathfrak{k}$ and also $\nabla^{(A_N,V)}$ and $\tilde{B}^{A_N}$ as follows:*

$$\begin{aligned}[\kappa_1 \oplus \eta_1, \kappa_2 \oplus \eta_2] = [\kappa_1, \kappa_2] \oplus [\nabla^{(A_N,V)}]_{G/N\kappa_1}\eta_2 - [\nabla^{(A_N,V)}]_{G/N\kappa_2}\eta_1 \\ - [\tilde{B}^{A_N}]_{G/N}(\kappa_1, \kappa_2) + [\eta_1, \eta_2].\end{aligned}$$

Proof. The proof is a straightforward application of Theorems 6.2.10, 6.3.7 and 6.3.10. ■

Now we shall generalize Theorem 6.3.7, Lemma 6.3.8 and Theorem 6.3.10 for arbitrary objects $TQ \oplus V$ rather than the particular case $V = 0$ considered in those results. We will give precise statements, however, we shall omit the proofs, which can be performed in a way entirely similar to the proofs of Theorem 6.3.7, Lemma 6.3.8 and Theorem 6.3.10.

Theorem 6.3.12 *Let $W = TQ \oplus V$ be an object of $\mathfrak{LP}$ and let $G \times W \to W$ be an action in the category $\mathfrak{LP}$ of a Lie group G on W, in particular, this action induces a principal bundle structure on Q with structure group G. Let N be a normal subgroup of G and let A_N be a connection on the principal bundle Q with structure group N having the property that for any $g \in G$ and any $v_q \in T_qQ$ we have $A_N(gv_q) = \mathrm{Ad}_g A_N(v_q)$. Consider the action of G/N on $T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N)$ such that its restriction to $T(Q/N) \oplus \tilde{\mathfrak{n}}$ coincides with*

the action defined in Theorem 6.3.7 and its restriction to V/N is the quotient action of the action of G on V by N (see Lemma 5.2.1. Then this is an action in the category $\mathfrak{LP}$. The reduced bundle of $T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N)$ by the group G/N with respect to any connection $A_{G/N}$ on the bundle Q/N with structure group G/N is an object of $\mathfrak{LP}$.

Lemma 6.3.13 *Assume the conditions of Theorem 6.3.12 and, in addition, choose a connection A_G on the principal bundle Q with structure group G. Let $K = G/N$. Consider the following vector bundle isomorphisms:*

(a) the quotient isomorphism (see Lemma 6.3.6)

$$[\alpha^W_{A_N}]_{G/N} : (W/N)/(G/N) \to (T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N))/(G/N),$$

(b) the isomorphism (see Lemma 6.3.1)

$$i^W_{G/N} : W/G \to (W/N)/(G/N),$$

(c) the isomorphism (see Definition 6.2.11)

$$\alpha^W_{A_G} : W/G \to T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G),$$

(d) the isomorphism (see Definition 6.2.11)

$$\begin{aligned}\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}\oplus(V/N)}_{A_{G/N}} &: (T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N))/(G/N) \\ &\to T((Q/N)/(G/N)) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}} \oplus (V/N))/(G/N),\end{aligned}$$

where $\tilde{\mathfrak{k}}$ is the Lie algebra of K.

Then the corresponding linear isomorphisms

(a')

$$[\alpha^W_{A_N}]_{G/N*} : \Gamma((W/N)/(G/N)) \to \Gamma((T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N))/(G/N)),$$

(b')

$$i^W_{G/N*} : \Gamma(W/G) \to \Gamma((W/N)/(G/N)),$$

(c')

$$\alpha^W_{A_G*} : \Gamma(W/G) \to \Gamma(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)),$$

(d')

$$\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}\oplus(V/N)}_{A_{G/N}*} : \Gamma\left((T(Q/N)\oplus\tilde{\mathfrak{n}}\oplus(V/N))/(G/N)\right)$$
$$\to \Gamma\left(T\left((Q/N)/(G/N)\right)\oplus\tilde{\mathfrak{k}}\oplus\left(\tilde{\mathfrak{n}}\oplus(V/N)\right)/(G/N)\right)$$

are Lie algebra isomorphisms. Consider the composition

(e)

$$\epsilon^W_{(A_N,A_{G/N},A_G)} = \alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}\oplus(V/N)}_{A_{G/N}} \circ [\alpha^W_{A_N}]_{G/N} \circ i^W_{G/N} \circ (\alpha^W_{A_G})^{-1},$$

which is a vector bundle isomorphism from $T(Q/G)\oplus\tilde{\mathfrak{g}}\oplus(V/G)$ *onto*

$$T\left((Q/N)/(G/N)\right)\oplus\tilde{\mathfrak{k}}\oplus\left(\tilde{\mathfrak{n}}\oplus(V/N)\right)/(G/N).$$

Then $\epsilon^W_{(A_N,A_{G/N},A_G)}$ *induces a Lie algebra isomorphism*

(e')

$$\epsilon^W_{(A_N,A_{G/N},A_G)*} : \Gamma((T(Q/G)\oplus\tilde{\mathfrak{g}}\oplus(V/G))$$
$$\to \Gamma\left(T\left((Q/N)/(G/N)\right)\oplus\tilde{\mathfrak{k}}\oplus\tilde{\mathfrak{n}}/(G/N)\oplus((V/N)/(G/N))\right).$$

Theorem 6.3.14 *Assume the hypothesis of Theorem 6.3.12 and, in addition, choose the connections* A_N, $A_{G/N}$ *and* A_G *in such a way that the following property is satisfied: for all* $v_q \in TQ$ *and all* $q \in Q$ *we have* $A_G(v_q) = 0$ *if and only if* $A_N(v_q) = 0$ *and* $A_{G/N}(T\pi_N(v_q)) = 0$ *Then* $\epsilon^W_{(A_N,A_{G/N},A_G)}$ *is an isomorphism in the category* $\mathfrak{LP}$.

6.4 The Subcategory $\mathfrak{RT}$ and Reduction by Stages of Variational Principles on TQ.

Let $\mathcal{T}$ be the category of tangent bundles of manifolds and tangent lifts of maps, which is a subcategory of $\mathfrak{LP}$. The category $\mathfrak{RT}$ is defined as the smallest subcategory of $\mathfrak{LP}$ that contains $\mathcal{T}$ and is closed under reduction. Thus, as vector bundles, elements of $\mathfrak{RT}$ are constructed by an inductive procedure of the type $V_0 = TQ$, $V_{i+1} = V_i/G_i$, where for each $i = 1, 2, \ldots$, G_i is a group acting on V_i. Using this and the fact that tangent bundles are always orientable manifolds we can easily prove that objects of $\mathfrak{RT}$ are orientable manifolds. This shows in particular that $\mathfrak{RT}$ does not coincides with $\mathfrak{LP}$ since there are simple examples of objects of $\mathfrak{LP}$ that are nonorientable manifolds. For instance, let $Q \equiv S^1$ and consider the vector bundle $TQ \oplus M$.

where M is the Möbius band, thus, as a manifold $TQ \oplus M$ is nonorientable. The Lie algebra structure on fibers of M must be 0 because the fibers are 1-dimensional. The M- valued 2- form ω must be 0 because Q is 1-dimensional. Now choose a vector bundle metric on M. Then define the connection ∇ on the vector bundle M by the condition that a curve in M is horizontal if and only if its distance to the 0 section is constant. Then we can check that, with this structure, $TQ \oplus M$ is an object of $\mathfrak{LP}$.

It is immediate that all the results of the previous sections involving generic objects of $\mathfrak{LP}$ are valid for generic objects of $\mathfrak{RT}$. A study of the local structure of objects of $\mathfrak{RT}$ or $\mathfrak{LP}$ and the study of categories bigger that $\mathfrak{LP}$, in connection with nonholonomic mechanics, Poisson geometry and other topics is being planned for a future work.

In this section, we shall expose the theory of Lagrangian reduction by stages for the bundle TQ, where Q is a principal bundle with structure group G having a normal subgroup N. This theory explains how the Hamilton's variational principle corresponding to a given Lagrangian $L : TQ \to \mathbb{R}$ invariant under the action of G, and therefore also under the action of N, is reduced by stages, first under the action of N and then under the action of G/N. The reduced bundles for these reduced variational principles and corresponding Lagrange-Poincaré operators are elements of $\mathfrak{RT}$, as we saw in section §3.2 where the case of 1-stage reduction was studied. A more general theory of reduction by stages of variational principles for any given object of $\mathfrak{LP}$ including the reduction of the corresponding variational principles is being planned for a future work.

We shall begin with some remarks on the geometry of variations, using a notation slightly more precise than the one used in sections §2.1, §3.1 and §3.2. In this section, we will often use, for short,the notation $TQ \oplus V$ to denote an element of $\mathfrak{RT}$, meaning that this bundle is isomorphic to a bundle of the type $T(M/K) \oplus \tilde{\mathfrak{k}}$, where M is a principal bundle with structure group G.

Let Q be a manifold. The ***tangent lift*** of a given curve $q \in \Omega(Q)$ is the curve $\ell(q)$ on $q \in \Omega(TQ)$ defined by $\ell(q)(t) = (q(t), \dot{q}(t))$ for all t, thus we have a map $\ell : \Omega(Q) \to \Omega(TQ)$.

The ***space of allowed variations*** of a curve q is the tangent space $\Delta_q = T_q\Omega(Q; q_0, q_1)$, that is, elements of Δ_q are variations δq such that $\delta q(t_i) = 0$ for $i = 1, 2$, where, as usual, δq is the derivative of some deformation $q_\lambda(t)$, that is, for all t, we have

$$\delta q(t) = \frac{d}{d\lambda} q_\lambda(t)\bigg|_{\lambda=0}.$$

We have a canonical inclusion map $T_q\Omega(Q) \to \Omega(TQ)$. The ***space of lifted allowed variations*** of a curve $\ell(q)$ is the subspace Δ_q^ℓ of $T_{\ell(q)}\Omega(TQ)$ defined

by

$$\Delta_q^\ell = T_q\ell\left(T_q\Omega(Q;q_0,q_1)\right),$$

where

$$T_q\ell : T_q\Omega(Q) \to T_{\ell(q)}\Omega(TQ)$$

is the tangent map of ℓ. A generic element of Δ_q^ℓ can therefore be written $T_q\ell.\delta q$, for some variation δq of q such that $\delta q(t_i) = 0$ for $i = 1, 2$. It is easy to prove that the restriction of $T_q\ell$ to $T_q\Omega(Q;q_0,q_1) = \Delta_q$ is a linear bijective map onto Δ_q^ℓ.

The following definition is inspired by Lemmas 3.1.4 and 3.1.7.

Definition 6.4.1 *Let $TQ \oplus V$ be an object of $\mathfrak{RT}$ and let $\tau : V \to Q$ be the projection. We denote $\ell\Omega(Q) \oplus \Omega(V)$ the set of all $\ell(q) \oplus v \in \Omega(TQ \oplus V)$. Likewise $\ell\Omega(Q;q_0) \oplus \Omega(V;q_0)$ is the subset of all $\ell(q) \oplus v \in \Omega(TQ \oplus V)$ such that $q(t_0) = q_0$ and $\ell\Omega(Q;q_0,q_1) \oplus \Omega(V;q_0,q_1)$ is the subset of all $\ell(q) \oplus v \in \Omega(TQ \oplus V)$ such that $q(t_0) = q_0$, and $q(t_1) = q_1$, (see §2.1). Let v be a curve in V and let $q = \tau v$. We will sometimes think of the manifold Q as being identified with the vector bundle having base Q and 0-dimensional fiber (see §2). Thus if $q = \tau v$ we can write $v \equiv q \oplus v$. The **space of allowed variations** of $q \oplus v$ is the subspace $\Delta_{q\oplus v}$ of $T_q\Omega(Q) \oplus \Omega(V)$ of all curves of the type*

$$\delta q \oplus \delta v = \delta q \oplus \frac{Dw}{Dt} + [v,w] - \omega(q)(\dot{q},\delta q),$$

*where δq is a variation of q with $\delta q(t_i) = 0$ for $i = 1, 2$, that is, $\delta q \in \Delta q$, and w is a curve in V such that $\tau(w(t)) = q(t)$ for all t and $w(t_i) = 0$ for $i = 1, 2$. The **space of lifted allowed variations** of a curve $\ell(q) \oplus v \in \Omega(TQ \oplus V)$ is the subspace*

$$\Delta_{q\oplus v}^\ell = \Delta_q^\ell \oplus \Omega(V) \quad \textit{of} \quad T_{\ell(q)}\Omega(TQ) \oplus \Omega(V)$$

of all variations of the type

$$T_q\ell.\delta q \oplus \frac{Dw}{Dt} + [v,w] - \omega(q)(\dot{q},\delta q),$$

where δq is a variation of q with $\delta q(t_i) = 0$ for $i = 1, 2$ and w is a curve in V such that $\tau(w(t)) = q(t)$ for all t and $w(t_i) = 0$ for $i = 1, 2$.

Assume that the group G acts on the manifold Q. Then we have maps (see Lemma 5.2.1 and Lemma 2.4.2)

$$\pi_G(TQ) : TQ \to TQ/G \quad \text{and} \quad \alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}.$$

Let $\ell(q) \in \Omega(TQ)$. Then we have

$$\Omega\left(\alpha_A \circ \pi_G(TQ)\right)\left(\ell(q)\right) = \ell(x) \oplus \bar{\xi},$$

where

$$x(t) = [q]_G(t) \quad \text{and} \quad \bar{\xi}(t) = [q(t), A\left(q(t), \dot{q}(t)\right)]_G$$

for all t.

Lemma 6.4.2 *Assume the hypothesis of Lemma 2.4.2. Then the map*

$$\Omega\left(\alpha_A \circ \pi_G(TQ)\right) : \Omega(TQ) \to \Omega\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$$

restricted to $\ell\Omega(Q; q_0)$ is bijective. The image of this restriction is

$$\ell\Omega\left(Q/G; x_0\right) \oplus \Omega\left(\tilde{\mathfrak{g}}; x_0\right).$$

Proof. Let $\dot{x}(t) \oplus \bar{\xi}(t) = \ell(x)(t) \oplus \bar{\xi}(t)$ be given such that $x(t_0) = x_0$. We must show that there is a unique curve $q \in \Omega(Q; q_0)$ such that

$$\Omega(\alpha_A \circ \pi_G(TQ))\ell(q) = \ell(x) \oplus \bar{\xi}.$$

We can always write $\bar{\xi}(t) = [x^h_{q_0}(t), \xi(t)]_G$. Let $q(t) = g(t)x^h_{q_0}(t)$ where $g(t)$ is a curve in G. Then we can see that

$$\Omega(\alpha_A \circ \pi_G(TQ))\ell(q) = \left(\dot{x}(t) \oplus [x^h_{q_0}(t), g^{-1}(t)\dot{q}(t)]_G\right).$$

Thus $g(t)$ must satisfy $\dot{g}(t) = g(t)\xi(t)$ for all t and $g(t_0) = e$. ■

Corollary 6.4.3 *The map*

$$\Omega\left(\alpha_A \circ \pi_G(TQ)\right) : \Omega(TQ) \to \Omega\left(T(Q/G) \oplus \tilde{\mathfrak{g}}\right)$$

restricted to $\ell\Omega(Q; q_0, q_1)$ is injective.

Lemma 6.4.4 *Assume the hypothesis of lemma 6.4.2. Then the restriction of the map $T_q\Omega\left(\alpha_A \circ \pi_G(TQ)\right)$ to Δ^ℓ_q is a linear isomorphism onto $\Delta^\ell_{x\oplus\bar{\xi}}$ where*

$$\dot{x} \oplus \bar{\xi} = \ell(x) \oplus \bar{\xi} = \Omega(\alpha_A \circ \pi_G(TQ))\ell(q)$$

and $\Delta^\ell_{x\oplus\bar{\xi}}$ is the space of lifted allowed variations of $\ell(x) \oplus \bar{\xi}$ (see Definition 6.4.1).

Proof. Using Lemmas 3.1.4, 3.1.7 and 6.4.2 and considering that an element of Δ_q^ℓ can be represented as the derivative of a deformation $\ell\left((q_\lambda(t)\right)$ of a curve $q(t) \equiv q_0(t)$ at $\lambda = 0$, it follows easily that

$$T_q\Omega\left(\alpha_A \circ \pi_G(TQ)\right)\Delta_q^\ell \subset \Delta_{x\oplus\bar{\xi}}^\ell.$$

Injectiveness and also surjectiveness of the restriction $T_q\Omega\left(\alpha_A \circ \pi_G(TQ)\right) \mid \Delta_q^\ell$ can be proved using a reconstruction procedure as in the proof of lemma 6.4.2. Namely, let be given a deformation

$$\ell(x_\lambda) \oplus \bar{\xi}_\lambda \quad \text{of} \quad \ell(x) \oplus \bar{\xi},$$

where $\bar{\xi}_\lambda(t) = [x^h_{q_0\lambda}(t), \xi_\lambda(t)]_G$. This gives a unique deformation $g_\lambda(t)$ of g satisfying $\dot{g}_\lambda(t) = g_\lambda(t)\xi_\lambda(t)$, and this gives a unique deformation $q_\lambda(t) = g_\lambda(t)x^h_{q_0\lambda}(t)$ of q. Therefore by differentiation with respect to λ at $\lambda = 0$ we obtain the inverse of the tangent map $T_q\Omega\left(\alpha_A \circ \pi_G(TQ)\right)$. ■

Lemma 6.4.2 and corollary (6.4.3) can be easily generalized for any object of $\mathfrak{LP}$ instead of TQ. However, a generalization of Lemma 6.4.4 involves a careful study of the geometry of spaces of variations and will be the purpose of a future work on general Lagrangian reduction in the category $\mathfrak{LP}$, as we have said above. More precisely, we have

Lemma 6.4.5 *Assume the hypothesis of lemma 6.4.2 and, in addition, let $W = TQ \oplus V$ be an element of $\mathfrak{LP}$. Then the map*

$$\Omega\left(\alpha_A^W \circ \pi_G(W)\right) : \Omega(W) \to \Omega\left(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right)$$

restricted to $\ell\Omega(Q; q_0) \oplus \Omega(V; q_0)$ is bijective. The image of this restriction is

$$\ell\Omega(Q/G; x_0) \oplus \Omega(\tilde{\mathfrak{g}}; x_0) \oplus \Omega(V/G; x_0).$$

Proof. For a given curve

$$\dot{x}(t) \oplus \bar{\xi}(t) \oplus [v]_G(t) \quad \text{with} \quad x(t_0) = x_0$$

we find, using the curve $\dot{x}(t) \oplus \bar{\xi}(t)$, the curve $q \in \Omega(Q; q_0)$ as in lemma 6.4.2. Then using Lemma 5.2.1 we see that there is a unique curve $v \in \Omega(V)$ such that $\tau v(t) = q(t)$, where $\tau : V \to Q$ is the projection of V, and its class is $[v]_G(t)$ for all t. ■

We shall now state the generalization of corollary (6.4.3) However, we will omit the proofs which is similar to the proofs of that result, but using Lemma 6.4.5 instead of lemma 6.4.2.

Corollary 6.4.6 *The map*

$$\Omega\left(\alpha_A^W \circ \pi_G(W)\right) : \Omega(W) \to \Omega\left(T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)\right)$$

restricted to $\ell\left(\Omega(Q; q_0, q_1)\right) \oplus \Omega(V)$ *is injective.*

Next we will see how the space of lifted allowed variations is transformed under reduction by stages.

Lemma 6.4.7 *Assume the hypothesis of Theorem 6.3.14. Then we have*

$$\begin{aligned}
&\epsilon^W_{(A_N, A_{G/N}, A_G)} \circ \alpha^W_{A_G} \circ \pi_G(W) \\
&\quad = \alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}\oplus(V/N)}_{A_{G/N}} \circ \pi_{G/N}\left(T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N)\right) \circ \alpha_{A_N} \circ \pi_N(W).
\end{aligned}$$

Proof. We can prove in a straightforward manner that

$$\begin{aligned}
i^W_{G/N} \circ \pi_G(W) &= \pi_{G/N}(W/N) \circ \pi_N(W) \\
[\alpha^W_{A_N}]_{G/N} \circ \pi_{G/N}(W/N) &= \pi_{G/N}\left(T(Q/N) \oplus \tilde{\mathfrak{n}} \oplus (V/N)\right) \circ \alpha^W_{A_N}.
\end{aligned}$$

Using this and the definition of $\epsilon^W_{(A_N, A_{G/N}, A_G)}$ we can prove the lemma without difficulty. ■

Theorem 6.4.8 *Assume the hypothesis of Theorem 6.3.10. Then we have*

$$\begin{aligned}
&T\Omega\left(\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}\right) \circ T\Omega\left(\alpha^{TQ}_{A_G} \circ \pi_G(TQ)\right) \\
&\quad = T\Omega\left(\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}} \circ \pi_{G/N}\left(T(Q/N) \oplus \tilde{\mathfrak{n}}\right)\right) \circ T\Omega\left(\alpha^{TQ}_{A_N} \circ \pi_N(TQ)\right).
\end{aligned}$$

Let $q \in \Omega(Q)$ *and let*

$$\begin{aligned}
\Omega\left(\alpha^{TQ}_{A_G} \circ \pi_G(TQ)\right)(\ell(q)) &= [q]_G \oplus \bar{\xi}, \\
\Omega\left(\alpha^{TQ}_{A_N} \circ \pi_N(TQ)\right)(\ell(q)) &= [q]_N \oplus \bar{\eta}, \\
\Omega\left(\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}} \circ \pi_{G/N}\left(T(Q/N) \oplus \tilde{\mathfrak{n}}\right)\right)(\ell([q]_N \oplus \bar{\eta})) &= [[q]_N]_{G/N} \oplus \bar{\kappa} \oplus [\bar{\eta}]_{G/N}.
\end{aligned}$$

Then

$$\begin{aligned}
T\Omega\left(\alpha^{TQ}_{A_G} \circ \pi_G(TQ)\right) &: \Delta^\ell_q \to \Delta^\ell_{[q]_G \oplus \bar{\xi}}, \\
T\Omega\left(\alpha^{TQ}_{A_N} \circ \pi_N(TQ)\right) &: \Delta^\ell_q \to \Delta^\ell_{[q]_N \oplus \bar{\eta}}, \\
T\Omega\left(\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}} \circ \pi_{G/N}\left(T(Q/N) \oplus \tilde{\mathfrak{n}}\right)\right) &: \Delta^\ell_{[q]_N \oplus \bar{\eta}} \to \Delta^\ell_{[[q]_N]_{G/N} \oplus \bar{\kappa} \oplus [\bar{\eta}]_{G/N}}, \\
T\Omega\left(\epsilon^{TQ}_{(A_N, A_{G/N}, A_G)}\right) &: \Delta^\ell_{[q]_G \oplus \bar{\xi}} \to \Delta^\ell_{[[q]_N]_{G/N} \oplus \bar{\kappa} \oplus [\bar{\eta}]_{G/N}}
\end{aligned}$$

are linear isomorphisms.

Proof. The first equality in the statement of the theorem is a consequence of Lemma 6.4.7. The fact that

$$T\Omega\left(\alpha^{TQ}_{A_G}\circ\pi_G(TQ)\right):\Delta^\ell_q\to\Delta^\ell_{[q]_G\oplus\bar\xi},$$
$$T\Omega\left(\alpha^{TQ}_{A_N}\circ\pi_N(TQ)\right):\Delta^\ell_q\to\Delta^\ell_{[q]_N\oplus\bar\eta},$$
$$T\Omega\left(\epsilon^{TQ}_{(A_N,A_{G/N},A_G)}\right):\Delta^\ell_{[q]_G\oplus\bar\xi}\to\Delta^\ell_{[[q]_N]_{G/N}\oplus\bar\kappa\oplus[\bar\eta]_{G/N}}$$

are linear isomorphisms is a consequence of lemma 6.4.4 and Theorem 6.3.10. Then it follows from the first equality in the statement of the theorem that also

$$T\Omega\left(\alpha^{T(Q/N)\oplus\tilde{\mathfrak{n}}}_{A_{G/N}}\circ\pi_{G/N}\left(T(Q/N)\oplus\tilde{\mathfrak{n}}\right)\right):\Delta^\ell_{[q]_N\oplus\bar\eta}\to\Delta^\ell_{[[q]_N]_{G/N}\oplus\bar\kappa\oplus[\bar\eta]_{G/N}}$$

is a linear isomorphism. ■

The previous theorem says that, in the process of reducing the bundle TQ by stages, the spaces of lifted allowed variations can also be reduced by stages. This leads to reducing variational principles by stages, as we will see next.

First we need to generalize the notion of action. Let $TQ\oplus V$ be an object of $\mathfrak{RT}$, where where $\tau:V\to Q$ is the projection of V, and the structure on V is given by ∇, ω, $[\,,]$. Let

$$L:TQ\oplus V\to\mathbb{R}$$

be a given Lagrangian, let $v\in\Omega(V)$ be a given curve and let $q=\tau v$. The ***action of*** L ***at the curve*** $v\in\Omega(V)$ is, by definition,

$$J(L)(q)=\int_{t_0}^{t_1}L(q,\dot q,v)dt.$$

We will also call this quantity the ***action of*** L ***at the curve*** $\ell(q)\oplus v$ ***of*** $\Omega(TQ\oplus V)$ and denote it, also, $J(L)\left(\ell(q)\right)$.

The following definitions are obviously inspired in the results of §3.2. The ***Lagrange-Poincaré operator*** is a bundle map

$$\mathcal{LP}(L):T^{(2)}Q\oplus 2V\to T^*Q\oplus V^*$$

defined by

$$(\mathcal{LP})(L)(\delta q\oplus\delta v)=\mathrm{Hor}(\mathcal{LP})(L)\delta q+\mathrm{Ver}(\mathcal{LP})(L)\delta v,$$

where the ***vertical and horizontal Lagrange-Poincaré operators*** are given by

$$\mathrm{Ver}(\mathcal{LP})(L)\delta v = \left(-\frac{D}{Dt}\frac{\partial L}{\partial v}(q,\dot{q},v) + \mathrm{ad}_v^* \frac{\partial L}{\partial v}(q,\dot{q},v)\right)\delta v,$$

$$\mathrm{Hor}(\mathcal{LP})(L)\delta q = \left(\frac{\partial L}{\partial q}(q,\dot{q},v) - \frac{D}{Dt}\frac{\partial L}{\partial \dot{q}}(q,\dot{q},v)\right)\delta q - \frac{\partial L}{\partial v}(q,\dot{q},v)\omega(q)(\dot{q},\delta q).$$

Here we must choose an arbitrary affine connection ∇ on Q to make sense of the covariant derivatives, in a similar way as we explained in §3.2 for the case $V = \tilde{\mathfrak{g}}$. For instance, in local coordinates, we can choose the Euclidean connection.

Now assume that there is an action of the group G on $TQ \oplus V$ in the category $\mathfrak{LP}$ and that the Lagrangian L is invariant under the action. Then for any choice of a connection A_G on the principal bundle Q we can identify

$$(TQ \oplus V)/G \quad \text{with} \quad T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$$

via the isomorphism $\alpha_{A_G}^{TQ\oplus V}$ and we obtain an induced Lagrangian

$$L^{(G)} : T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G) \to \mathbb{R},$$

called the ***reduced Lagrangian*** (to make the notation consistent with the one used in the particular case $V = 0$ considered in Theorem 3.3.4 we should write $l \equiv L^{(G)}$, in that particular case). If, furthermore, $L^{(G)}$ is invariant under the action of another group K on the object

$$T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$$

in the category $\mathfrak{LP}$, and we choose a connection A_K on the principal bundle Q/G with structure group K, then we will denote the reduced Lagrangian simply $L^{(G,K)}$ instead of $\left(L^{(G)}\right)^{(K)}$, for short. We have

$$L^{(G,K)} : T\left((Q/G)/K\right) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{g}}/K) \oplus (V/G)/K) \to \mathbb{R}.$$

As we said before, in this work we will explain how to perform reduction by stages only for the case $V = 0$, leaving a more general theory for a future work.

Assume the hypothesis of Theorem 6.4.8, let $K = G/N$ and let $L : TQ \to \mathbb{R}$ be a given Lagrangian. Then we have reduced Lagrangians

$$L^{(G)} : T(Q/G) \oplus \tilde{\mathfrak{g}} \to \mathbb{R},$$

$$L^{(N)} : T(Q/N) \oplus \tilde{\mathfrak{n}} \to \mathbb{R}$$

and

$$L^{(N,K)} : T\left((Q/N)/(K)\right) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}}/K \to \mathbb{R}.$$

For any given curve $q \in \Omega(Q)$ let

$$\Omega\left(\alpha_{A_G}^{TQ} \circ \pi_G(TQ)\right)(\ell(q)) = x \oplus \bar{\xi},$$
$$\Omega\left(\alpha_{A_N}^{TQ} \circ \pi_N(TQ)\right)(\ell(q)) = y \oplus \bar{\eta},$$
$$\Omega\left(\alpha_{A_K}^{T(Q/N)\oplus\tilde{\mathfrak{n}}} \circ \pi_K\left(T(Q/N) \oplus \tilde{\mathfrak{n}}\right)\right)(\ell(y \oplus \bar{\eta})) = z \oplus \bar{\kappa} \oplus [\bar{\eta}]_K,$$

as in Theorem 6.4.8. The next theorem is the main result of this section

Theorem 6.4.9 *Assume the situation explained above. Then the following conditions are equivalent:*

(i) The curve $q(t)$ is a critical point of the action functional

$$\int_{t_0}^{t_1} L(q, \dot{q})dt$$

with restrictions on the variations given by Δ_q^ℓ.

(ii) The Euler-Lagrange equations

$$(\mathcal{EL})(L)(q) = 0$$

are satisfied.

(iii) The curve $x(t) \oplus \bar{\xi}(t)$ is a critical point of the action functional

$$\int_{t_0}^{t_1} L^{(G)}\left(x(t), \dot{x}(t), \bar{\xi}(t)\right) dt$$

with restrictions on the variations given by $\Delta_{x\oplus\bar{\xi}}^\ell$.

(iv) The Euler-Lagrange equations

$$(\mathcal{EL})(L^{(G)})(x \oplus \bar{\xi}) = 0$$

are satisfied.

(v) The curve $y(t) \oplus \bar{\eta}(t)$ is a critical point of the action functional

$$\int_{t_0}^{t_1} L^{(N)}\left(y(t), \dot{y}(t), \bar{\eta}(t)\right) dt$$

with restrictions on the variations given by $\Delta_{y\oplus\bar{\eta}}^\ell$.

(vi) The Euler-Lagrange equations

$$(\mathcal{EL})(L^{(N)})(y \oplus \bar{\eta}) = 0$$

are satisfied.

(vii) The curve $z \oplus \bar{\kappa} \oplus [\bar{\eta}]_K$ is a critical point of the action functional

$$\int_{t_0}^{t_1} L^{(N,K)} \left(z \oplus \bar{\kappa} \oplus [\bar{\eta}]_K \right) dt$$

with restrictions on the variations given by $\Delta_{z \oplus \bar{\kappa} \oplus [\bar{\eta}]_K}$.

(viii) The Euler-Lagrange equations

$$(\mathcal{EL})(L^{(N,K)})(z \oplus \bar{\kappa} \oplus [\bar{\eta}]_K) = 0$$

are satisfied.

We will omit the proof of this theorem. It can be easily performed using an argument similar to the one used in the proof of Theorem 3.3.4 and also using Theorem 6.4.8.

Remark. The main point of Theorem 6.4.9 is the equivalence with the last two statements (*vii*) and (*viii*), since the equivalence between the statements (*i*) to (*vi*) has ben already established in Theorem 3.3.4. This shows that one can write the Euler-Lagrange equations for the Lagrangian

$$L^{(N,K)} : T\left((Q/N)/K\right) \oplus \tilde{\mathfrak{k}} \oplus (\tilde{\mathfrak{n}}/K) \to \mathbb{R}$$

using the reduced structures given by the formulas of Theorem 6.2.10. More precisely, the structure on the bundle $\tilde{\mathfrak{n}}$ is the one given by $\tilde{\nabla}^{A_N}$, $\tilde{B}^{A_N}$, $[\,,]^{A_N}$. Then from Theorem 6.2.10 we obtain the formulas for the structure ∇, ω, $[\,,]$ on the bundle $\tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{n}}$, with $\nabla \equiv \tilde{\nabla}^{A_N}$, $\omega \equiv \tilde{B}^{A_N}$, $[\,,] \equiv [\,,]^{A_N}$.

All this can be generalized for several stages, that is, for the case where we have a chain

$$\{e\} \equiv N_0 \subset N_1 \subset N_2 \subset N_3 \ldots \subset N_r \equiv G,$$

where for each $i = 0, 1, 2, \ldots, r-1$, N_i is a normal subgroup of N_{i+1}. This together with some applications will be the purpose of a future work.

7 Further Examples

We have already seen one example of Lagrange-Poincaré reduction, namely the study of Wong's equations. Now we turn to some examples that involve reduction by stages. The first one concerns the classical setting of semidirect products in which one is given a Lagrangian that is invariant under the action of a semidirect product, for example, as in underwater vehicle dynamics (see Leonard and Marsden [1997]). Here we show how this theory fits into the framework of the present paper.

Following this, we consider central extensions from the Lagrangian viewpoint, the sort of example that is well known in the Hamiltonian context of the KdV equation, as in Marsden, Misiolek, Perlmutter and Ratiu [1997, 2000].

The third example is that of the spacecraft with internal rotors. The group in this case is just a direct product, but it is nonetheless interesting. See, for example, Bloch, Krishnaprasad,Marsden, and Sánchez de Alvarez [1992].

The Lagrangian version of systems with a Lagrangian depending on parameters, in which semidirect products are, in a sense, created, is studied in the fourth example. These systems include, for instance, the classical heavy top. This sort of theory was studied in Holm, Marsden and Ratiu [1998] and in Cendra, Holm, Marsden and Ratiu [1998].

7.1 Semidirect Products

In this subsection we show that the reduction of system with a symmetry group that is a semidirect product of a Lie group G with a vector space V can be done in two stages, reducing by the normal subgroup V first and by the group G second.

Let G be a group and let $\rho : G \times V \to V$ be a linear representation of G on a vector space V. We will write, equivalently, $\rho(g, v) \equiv \rho_g v \equiv \rho(g)v \equiv gv$. Let $S = G \circledS V$ be the semidirect product. Thus, by definition,

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, g_1 v_2 + v_1)$$

and $(g, v)^{-1} = (g^{-1}, -g^{-1}v)$. The Lie algebra of S is $\mathfrak{s} = \mathfrak{g} \circledS V$ with the Lie bracket given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, , \xi_2], \rho'(\xi_1)v_2 - \rho'(\xi_2)v_1)\,.$$

We will think of V as being a normal subgroup of S as usual, via the inclusion map $V \to G \circledS V$ given by $v \to (e, v)$. Next we will perform the reduction of $TS = T(G \circledS V)$ in the category $\mathfrak{LP}$ in 2 stages, first we reduce by V and then by $G \equiv (G \circledS V)/V$.

Stage 1. We can identify in a natural way $T(G \circledS V) = TG \times V \times V$. Let us choose the trivial connection A_V on the principal bundle $G \circledS V$ with structure group V, that is $A_V(g, \dot{g}, v, \dot{v}) = \dot{v}$. Then the curvature $B^{A_V} = 0$. Also by definition V is Abelian and we can identify its Lie algebra $\mathfrak{v} \equiv V$, where the Lie bracket is 0. We will also identify $(G \circledS V)/V \equiv G$ by $[g, v]_V \equiv g$. The bundle $\tilde{\mathfrak{v}}$ is a trivial bundle with base $(G \circledS V)/V \equiv G$ and we can identify $\tilde{\mathfrak{v}} \equiv G \times V$ via $[(g, v), \xi]_V \equiv (g, \xi)$. The covariant derivative on $\tilde{\mathfrak{v}}$ is given by

$$\frac{\tilde{D}^{A_V}}{Dt}\left[(g(t), v(t)), \xi(t)\right]_V \equiv \frac{\tilde{D}^{A_V}}{Dt}(g(t), \xi(t)) = (g(t), \dot{\xi}(t)).$$

We also have $\tilde{B}^{A_V} = 0$, and $[\,,]^{\tilde{\mathfrak{v}}} = 0$. The reduced bundle that we obtain after the first stage of reduction is then

$$T\left((G \circledS V)/V\right) \oplus \tilde{\mathfrak{v}} \equiv TG \times V$$

with the structure given above.

Stage 2. The action of G on $\tilde{\mathfrak{v}}$ commutes with the structure $\nabla \equiv \tilde{\nabla}^{A_V}$, $\omega \equiv \tilde{B}^{A_V} = 0$, $[\,,] \equiv [\,,]^{\tilde{\mathfrak{v}}} = 0$. Now consider the principal bundle G over a point with structure group G and connection $A_G(g, \dot{g}) = \dot{g}g^{-1}$. We are in position to apply Theorems 6.2.10, 6.3.10 and 6.4.9. Reduction by G of the bundle $TG \times V$ obtained in the first stage gives, using Theorem 6.2.10, a reduced bundle which is clearly isomorphic, in the category $\mathfrak{LP}$, to the bundle $\mathfrak{g} \oplus V$ whose base has dimension 0. It is clear that $\omega^{\tilde{\mathfrak{g}}} = 0$, $\nabla^{\tilde{\mathfrak{g}}} = 0$, because the base has dimension 0. Besides we can check very easily that the formula for $[\,,]^{\tilde{\mathfrak{g}}}$ gives the usual semidirect product Lie algebra structure on $\mathfrak{g} \circledS V$. We remark that even if the bundle obtained in the first stage depends on the connection A, the bundle obtained in the second stage does not depend on the connection A. This is, of course, also a direct consequence of Theorem 6.3.10.

7.2 Central Extensions

We are going to consider the particular case of $\mathbb{R}$ or S^1-group extensions. This already includes the Bott-Virasoro central extension of $\mathrm{Diff}(S^1)$ as an interesting infinite dimensional example. More general central extensions by Abelian Lie groups, relative to an arbitrary action (see, for instance, De Azcárraga and Izquierdo [1995]), can also be dealt with using the methods of this paper. For instance, the examples studied in Marsden, Misiolek, Perlmutter and Ratiu [1998], [2000], such as the Heisenberg group and the Bott-Virasoro group treated there from a Hamiltonian point of view, can also be approached using our Lagrangian techniques.

Let G be a Lie group and consider central extensions of G of the type $\hat{G} = G \times \mathbb{R}$ or $\hat{G} = G \times S^1$, where the composition law is given by

$$(g, \alpha)(h, \beta) = (gh, \alpha + \beta + \Sigma(g, h)) .$$

If we deal with $G \times S^1$, the sum in second component is understood to be modulo 2π. Here, $\Sigma : G \times G \to \mathbb{R}$ is a group 2-cocycle relative to the trivial action of G on $\mathbb{R}$ and hence, Σ satisfies the cocycle identity

$$\Sigma(g, h) + \Sigma(gh, k) = \Sigma(g, hk) + \Sigma(h, k).$$

As is well known, the second group cohomology of G with values in an Abelian group classifies the extensions of G by this Abelian group. Therefore, in the definition of the composition law for $\hat{G}$ we can modify Σ by the addition of a 2-coboundary such that $\Sigma(e, e) = 0$. The cocycle identity implies that $\Sigma(h, e) = \Sigma(e, h) = \Sigma(e, e)$ and $\Sigma(h, h^{-1}) = \Sigma(h^{-1}, h)$ for all $h \in G$. It can be directly checked that the cocycle identity and the condition $\Sigma(e, e) = 0$, are necessary and sufficient for the composition law defined above to satisfy the group axioms. It is easily verified that the neutral element of $\hat{G}$ is $(e, 0)$, where e is the neutral element of G. The inverse of (g, α) is given by

$$(g, \alpha)^{-1} = \left((g^{-1}, -\alpha - \Sigma(g, g^{-1})\right) .$$

An element of the tangent space $T_{(g,\alpha)}\hat{G}$ is often denoted by $(g, \alpha, \dot{g}, \dot{\alpha})$ or $(\dot{g}, \dot{\alpha})$. The Lie algebra $\hat{\mathfrak{g}}$ is the space of all tangent vectors at $(e, 0)$, that is, vectors $(e, 0, \xi, a)$, where ξ is an element of the Lie algebra $\mathfrak{g}$. With this basic formulas in place we can calculate the Lie bracket in $\hat{\mathfrak{g}}$ which is given by

$$[(\xi, a), (\eta, b)] = ([\xi, \eta], \partial_1\partial_2\Sigma(e, e).\eta.\xi - \partial_2\partial_1\Sigma(e, e).\eta.\xi) .$$

Here we define

$$\partial_1\partial_2\Sigma(e, e).\eta.\xi = \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \Sigma(\exp t\xi, \exp s\eta)$$

and

$$\partial_2\partial_1\Sigma(e, e).\eta.\xi = \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \Sigma(\exp s\eta, \exp t\xi).$$

We will define the Lie algebra cocycle σ by

$$\sigma(\xi, \eta) = \partial_1\partial_2\Sigma(e, e).\eta.\xi - \partial_2\partial_1\Sigma(e, e).\eta.\xi$$

and then the Lie bracket is written $[(\xi, a), (\eta, b)] = ([\xi, \eta], \sigma(\xi, \eta))$.

The set $N = \{(e, \alpha) \mid \alpha \in \mathbb{R}\}$ is a normal subgroup of $\hat{G}$ isomorphic to $\mathbb{R}$ or S^1. Moreover, N is contained in the center of $\hat{G}$. We can therefore consider $\hat{G}$ as a principal bundle with structure group N. Now consider any metric on $\hat{\mathfrak{g}}$, for instance one given by the simple formula

$$\langle (\xi, a), (\eta, b) \rangle = \langle \xi, \eta \rangle + ab,$$

where (by a slight abuse of notation), $\langle \xi, \eta \rangle$ is a given inner product on $\mathfrak{g}$. This generates a left invariant metric on $\hat{G}$ in the following way. Let $(\dot{g}_i, \dot{\alpha}_i) \in T_{(g,\alpha)}\hat{G}$, $i = 1, 2$. Then define

$$\begin{aligned}\langle (\dot{g}_1, \dot{\alpha}_1), (\dot{g}_2, \dot{\alpha}_2) \rangle = &\left\langle g^{-1}\dot{g}_1, g^{-1}\dot{g}_2 \right\rangle \\ &+ \left(\dot{\alpha}_1 + \partial_2\Sigma(g^{-1}, g) \cdot \dot{g}_1\right)\left(\dot{\alpha}_2 + \partial_2\Sigma(g^{-1}, g) \cdot \dot{g}_2\right).\end{aligned}$$

Now we can proceed to perform the construction of the mechanical connection, that is, the connection whose horizontal spaces are the orthogonal complements of the vertical spaces of the bundle $\hat{G}$ with structure group N. Then the conclusion of 6.3.5 holds, that is, the corresponding connection 1-form given by

$$A_N(\dot{g}, \dot{\alpha}) = \dot{\alpha} + \partial_2\Sigma(g^{-1}, g) \cdot \dot{g}.$$

is $\hat{G}$-equivariant and not just N-equivariant.

Now we apply the reduction by stages process given by Theorems 6.3.14 and 6.2.10. The first stage is reduction by N and we obtain the reduced bundle $TG \oplus \tilde{\mathfrak{n}}$, since $\hat{G}/N \equiv G$. We obtain a structure $\nabla \equiv \tilde{\nabla}^{A_N}$, $B \equiv \tilde{B}^{A_N}$, and $[\,,]$ on the bundle $\tilde{\mathfrak{n}}$. Then by applying Theorems 6.3.14 and 6.2.10 we see that we obtain an equivalent expression for the Lie bracket on $\hat{\mathfrak{g}} \equiv \mathfrak{g} \oplus [\mathfrak{n}]_G$. We can easily check that, in the expression of the Lie bracket of Theorem 6.2.10, the terms $[\nabla^{(A,V)}]_{G\bar{\xi}_1}[v_2]_G$, $[\nabla^{(A,V)}]_{G\bar{\xi}_2}[v_1]_G$, as well as the term $[[v_1]_G, [v_2]_G]$ are all 0. Thus, in particular, *the cocycle σ is minus the reduced curvature.*

7.3 Rigid Body with Rotors

We will use the description of the rigid body with rotors given in Marsden and Scheurle [1993b].

The configuration space for a rigid body with three rotors aligned with, say, the principal axes, is

$$Q = SO(3) \times S^1 \times S^1 \times S^1$$

with elements denoted

$$(R, \theta_1, \theta_2, \theta_3),$$

where the angles are relative to the carrier. The group structure on Q is the direct product structure. Then in particular $N = SO(3)$ is a normal subgroup. The body angular velocity is $\Omega = R^{-1}\dot{R}$. We also denote

$$\Omega_r = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3).$$

We now think of Q as a principal bundle with structure group N and choose the mechanical connection A_N given by

$$A_N(R, \theta_1, \theta_2, \theta_3) = R^{\mathrm{1}}\dot{R} + (I + K)^{-1}K\Omega_r.$$

The first stage of reduction is reduction by N which gives the reduced bundle

$$T(S^1 \times S^1 \times S^1) \oplus \tilde{\mathfrak{n}} \equiv T(S^1 \times S^1 \times S^1) \times \mathfrak{n}.$$

The second stage is reduction by the group $S^1 \times S^1 \times S^1 \equiv Q/N$. We can easily see that the reduced bundle is

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \tilde{\mathfrak{n}}/S^1 \times S^1 \times S^1 \equiv \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathfrak{n}.$$

We can also easily see that, in our case, several terms of the expression of the Lie bracket given in Theorem 6.2.10 are 0, more precisely, $[\bar{\xi}_1, \bar{\xi}_2] = 0$, $[\nabla^{(A,V)}]_{G\bar{\xi}_i}[v_j]_G = 0$ for $\{i, j\} = \{1, 2\}$ and $[\omega]_G(\bar{\xi}_1, \bar{\xi}_2) = 0$. We can also easily check that the term $[[v_1]_G, [v_2]_G]$ is, in our case, simply $[\Omega, \Omega]$. Thus the Lie algebra structure on $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathfrak{n}$ obtained by reduction by stages coincides with the direct product Lie algebra, as expected according to Theorems 6.3.14 and 6.2.10.

7.4 Systems Depending on a Parameter

In this subsection, we consider Lagrangian systems depending on a parameter and show how they fit into the framework developed in this paper. In the process we shall recover the Euler-Poincaré equation in Holm, Marsden and Ratiu [1998]; see equation (7.4.1) below.

Recall (see Holm, Marsden and Ratiu [1998] for additional explanation) that on the Hamiltonian side of this same problem, the semidirect product $G \circledS V$ appears as a symmetry group of an enlarged system and symplectic reduction by stages is relevant. The situation on the Lagrangian side of the same problem is somewhat different.

The fundamental difference between this subsection and §7.1 is that in §7.1, one imagines having a system whose given symmetry group is a semidirect product from the outset, as, for example, the Euclidean group is the symmetry group of an underwater vehicle. Here, on the other hand, the Lagrangian has a symmetry group G, but in a way that includes the dependence on a parameter

$a \in V^*$, where V is a space on which G acts. The goal is then to show that the more general Euler-Poincaré equations referred to above, can be obtained by using Lagrangian reduction with respect to the action of G.

There are interesting links between the set up described below and the topic of Clebsch variables and Lin constraints; these are described in Cendra and Marsden [1987]. We also refer to Cendra, Holm, Marsden and Ratiu [1998] and Cendra, Holm, Hoyle and Marsden [1998] for interesting connections with degenerate Legendre transformations.

Consider a Lagrangian

$$L : T(G \times Q) \times V^* \to \mathbb{R},$$

where G is a group, Q is a manifold and V^* is the dual of the vector space V. The value of L at the point $(g, q, \dot{g}, \dot{q}, a_0) \in T(G \times Q) \times V^*$ will be denoted $L(g, q, \dot{g}, \dot{q}, a_0)$, as usual, and we will think of a_0 as being a parameter that remains fixed along the evolution of the system. Assume that there is a linear action $\rho : G \times V \to V$ of G on V, so there is also an induced action $\rho_* : G \times V^* \to V^*$ such that $\langle ga_0, gb_0 \rangle = \langle a_0, b_0 \rangle$ for all $a_0 \in V^*$, all $b_0 \in V$ and all $g \in G$, where, as usual, we write $ga_0 = \rho(g, a_0)$ and $gb_0 = \rho_*(g, b_0)$. We will often write $\langle a_0, b_0 \rangle = \langle b_0, a_0 \rangle$, due to the identification $V^{**} \equiv V$. Assume that L has the following invariance property:

$$L(hg, q, h\dot{g}, \dot{q}, ha_0) = L(g, q, \dot{g}, \dot{q}, a_0)$$

for all $a_0 \in V^*$, all $q \in Q$ and all $h, g \in G$. Let

$$L(e, q, \xi, \dot{q}, a) = l(\xi, q, \dot{q}, a),$$

for all $\xi \in \mathfrak{g}$, all $q \in Q$ and all $a \in V^*$. Then the invariance property implies, for all $g \in G$, all $q \in Q$ and all $a \in V^*$,

$$L(g, q, \dot{g}, \dot{q}, a_0) = l(\xi, q, \dot{q}, a),$$

where $\xi = g^{-1}\dot{g}$ and $a = g^{-1}a_0$.

By direct calculation we can show that the following conditions are equivalent:

(i) The curve $(g(t), q(t), a_0)$ is a critical point of the action

$$\int_{t_0}^{t_1} L(g, q, \dot{g}, \dot{q}, a_0)dt$$

with restrictions on variations given by $\delta g(t_i) = 0$ for $i = 0, 1$, $\delta q(t_i) = 0$ for $i = 0, 1$, and $\delta a_0 = 0$.

(ii) The curve $(\xi(t), q(t), a(t))$*, where* $a(t) = g^{-1}(t)a_0$ *for all* t *and* $\xi(t) = g^{-1}(t)\dot{g}$*, is a critical point of the action*

$$\int_{t_0}^{t_1} l(\xi, q, \dot{q}, a)dt$$

with restrictions on variations given by

$$\delta\xi = \dot{\eta} + [\xi, \eta],$$

where η *is any curve in* $\mathfrak{g}$ *such that* $\eta(t_i) = 0$ *for* $i = 0, 1$*,*

$$\delta q(t_i) = 0$$

for $i = 0, 1$*,*

$$\delta a = -\eta a$$

and

$$\dot{a} + \xi a = 0,$$

for all t*. We remark that this last condition comes from the condition* $\dot{a}_0 = 0$ *together with* $a_0 = ga$*.*

A direct application of (ii) leads to the Euler-Lagrange equations

$$-\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial a} \diamond a = 0 \tag{7.4.1}$$

$$\frac{\partial l}{\partial q} - \frac{d}{dt}\frac{\partial l}{\partial \dot{q}} = 0,$$

where for all $\eta \in \mathfrak{g}$, all $a \in V^*$ and all $b \in V$ we have, by definition,

$$(b \diamond a)(\eta) = -\langle \eta a, b \rangle .$$

The first of this equations is called the ***Euler–Poincaré*** equation (see Holm, Marsden and Ratiu [1998]). This equation together with the Euler-Lagrange equations in the variable $(q, \dot{q})$ and the equation $\dot{a} + \xi a = 0$ form the complete set of equations of the system in terms of the variables (ξ, q, a).

Now we shall recast conditions (i) and (ii) into an equivalent form. The idea is to introduce the condition that a_0 is a constant of the motion, that is $\dot{a}_0 = 0$, with a Lagrange multiplier. Thus let us define the Lagrangian

$$\bar{L} : T(G \times Q \times V^* \times V) \to \mathbb{R}$$

by

$$\bar{L}(g, q, a_0, b_0, \dot{g}, \dot{q}, \dot{a}_0, \dot{b}_0) = L(g, q, \dot{g}, \dot{q}) + \langle \dot{a}_0, b_0 \rangle .$$

Now we observe that $G \times Q \times V^* \times V$ is a principal bundle with structure group G acting as before, that is, $h(g, q, a_0.b_0) = (hg, q, ha_0, hb_0)$. Moreover, $G \times Q \times V^* \times V$ is isomorphic, as a principal bundle, to the trivial bundle $G \times Q \times V^* \times V$ where the action of G is given by

$$h \cdot (g, q, a, b) = (hg, q, a, b),$$

for all $h, g \in G$, all $a \in V^*$ and all $b \in V$. More precisely, we have the isomorphism

$$\psi : G \times Q \times V^* \times V \to G \times Q \times V^* \times V$$

given by

$$\psi(g, q, a_0, b_0) = (g, q, g^{-1}a_0, g^{-1}b_0) \equiv (g, q, a, b).$$

We can check that

$$\psi\left(h(g, q, a_0, b_0)\right) = h \cdot \psi(g, q, a_0, b_0) \equiv h \cdot (g, q, a, b)$$

for all $h, g \in G$, all $a_0 \in V^*$ and all $b_0 \in V$. We can easily check that the composition $\bar{L} \circ T\psi^{-1} =: L^V$ is given by

$$L^V(g, q, a, b, \dot{g}, \dot{q}, \dot{a}, \dot{b}) = L(g, q, \dot{g}, \dot{q}, a) + \left\langle \dot{a} + g^{-1}\dot{g}a, b \right\rangle .$$

We will use the trivial bundle, where the action of G is given by the $\cdot$ operation, from now on. Thus, the base is

$$(G \times Q \times V^* \times V)/G \equiv Q \times V^* \times V$$

in a natural way. The Lagrangian on this bundle is L^V. Using techniques as in Cendra and Marsden[1987], which gives a version of the Lagrange multiplier theorem, we can show that conditions (i) and (ii) are equivalent to any of the following conditions

(iii) The curve $(g(t), q(t), a_0, b_0)$ is a critical point of the action

$$\int_{t_0}^{t_1} \bar{L}(g, q, a_0, b_0, \dot{g}, \dot{q}, \dot{a}_0, \dot{b}_0)dt$$

with restrictions on the variations given by

$$\delta g(t_i) = 0, \quad \delta q(t_i) = 0, \quad \delta a_0(t_i) = 0 \quad \textit{and} \quad \delta b_0(t_i) = 0$$

for $i = 0, 1$.

(iv) The curve $(g(t), q(t), a(t), b(t))$ is a critical point of the action

$$\int_{t_0}^{t_1} L^V(g, q, a, b, \dot{g}, \dot{q}, \dot{a}, \dot{b})dt$$

with restriction on the variations given by

$$\delta g(t_i) = 0, \quad \delta q(t_i) = 0, \quad \delta a(t_i) = 0 \quad \textit{and} \quad \delta b(t_i) = 0$$

for $i = 0, 1$.

However, this time we do not want to use that version of the Lagrange multiplier theorem. Instead, we want to use directly Theorem 3.3.4 to obtain a reduced system, starting with the Lagrangian L^V, equivalent to the system

$$\begin{aligned} -\frac{d}{dt}\frac{\partial l}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi} + \frac{\partial l}{\partial a} \diamond a &= 0 \\ \frac{\partial l}{\partial q} - \frac{d}{dt}\frac{\partial l}{\partial \dot{q}} &= 0 \\ \dot{a} + \xi a &= 0 \end{aligned} \tag{7.4.2}$$

obtained before.

We should calculate the reduced bundle $T(Q \times V^* \times V) \oplus \tilde{\mathfrak{g}}$. We can easily check that the bundle $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{g}} \equiv Q \times V^* \times V \times \mathfrak{g}$. The Lie algebra structure on $\tilde{\mathfrak{g}}$ is given by

$$[(q, a, b, \xi_1), (q, a, b, \xi_2)] = (q, a, b, [\xi_1, \xi_2]).$$

Now let us choose the trivial principal connection A on $G \times Q \times V^* \times V$, that is, the connection given by

$$A(g, q, a, b, \dot{g}, \dot{q}, \dot{a}, \dot{b}) = g^{-1}\dot{g}.$$

Using Lemma 2.3.4 we can see that the covariant derivative $\tilde{D}^A$ on $\tilde{\mathfrak{g}}$ is given by

$$\frac{D}{Dt}(q(t), a(t), b(t), \xi(t)) = \left(q(t), a(t), b(t), \dot{\xi}(t)\right).$$

Therefore the connection $\tilde{\nabla}^A$ on $\tilde{\mathfrak{g}}$ is given by

$$\tilde{\nabla}^A_{(q,a,b,\dot{q},\dot{a},\dot{b})}(q, a, b, \xi) = \left(q, a, b, \frac{\partial \xi}{\partial a}\dot{a} + \frac{\partial \xi}{\partial b}\dot{b}\right),$$

for any section $(q, a, b) \mapsto (q, a, b, \xi(q, a, b))$ of $\tilde{\mathfrak{g}}$ and any tangent vector

$$(q, a, b, \dot{q}, \dot{a}, \dot{b})$$

at the point (q, a, b) of the base $Q \times V^* \times V$ of $\tilde{\mathfrak{g}}$. Since the curvature of A is $B^A = 0$ we have $\tilde{B}^A = 0$. A generic element of the bundle

$$T(Q \times V^* \times V) \oplus \tilde{\mathfrak{g}}$$

will be written $(q, a, b, \dot{q}, \dot{a}, \dot{b}, \xi)$. Let l^V be the reduced Lagrangian

$$l^V : T(Q \times V^* \times V) \oplus \tilde{\mathfrak{g}} \to \mathbb{R}.$$

Thus, $l^V(q, a, b, \dot{q}, \dot{a}, \dot{b}, \xi)$ is a function of the independent variables

$$(q, a, b, \dot{q}, \dot{a}, \dot{b}, \xi).$$

More precisely we can easily see that

$$l^V(q, a, b, \dot{q}, \dot{a}, \dot{b}, \xi) = l(q, \xi, \dot{q}, a) + \langle \dot{a} + \xi a, b \rangle ,$$

where, as before,

$$l(\xi, q, \dot{q}, a) = L(e, q, \xi, \dot{q}, a).$$

We can read off directly from the expression of the Lagrangian l^V above that the condition $\dot{a} + \xi a = 0$ has been imposed with the Lagrangian multiplier b. We can write (see Cendra and Marsden[1987])

$$l^V(q, a, b, \dot{q}, \dot{a}, \dot{b}, \xi) = l(q, \xi, \dot{q}, a) + J(a, b)(\xi) + \theta_0(a, b)(\dot{a}, \dot{b}),$$

where $J : V^* \times V \to \mathfrak{g}^*$ is the momentum map of the action ρ lifted to the cotangent bundle $V^* \times V \equiv V^* \times V^{**}$, and θ_0 is the canonical 1-form on $V^* \times V \equiv V^* \times V^{**}$. Using Theorem 3.3.4 we can obtain the vertical and also the horizontal Lagrange-Poincaré equations. The vertical Lagrange-Poincaré equation is

$$-\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}(\xi, q, \dot{q}, a) + J(a, b)\right) + \mathrm{ad}^*_\xi \left(\frac{\partial l}{\partial \xi}(\xi, q, \dot{q}, a) + J(a, b)\right) = 0.$$

The horizontal Euler-Lagrange equation is

$$\begin{aligned} \frac{\partial l}{\partial q}(\xi, q, \dot{q}, a) - \frac{d}{dt}\frac{\partial l}{\partial \dot{q}}(\xi, q, \dot{q}, a) &= 0 \\ \frac{\partial l}{\partial a}(\xi, q, \dot{q}, a) + \frac{\partial J(a, b)(\xi)}{\partial a} - \frac{db}{dt} &= 0 \\ \dot{a} + \xi a &= 0. \end{aligned}$$

Using the property $\langle \xi a, b \rangle + \langle a, \xi b \rangle = 0$, we can see that

$$\frac{\partial J(a, b)(\xi)}{\partial a} = -\xi b.$$

The last two horizontal equations can be rewritten

$$\begin{aligned}\dot{b} &= -\xi b + \frac{\partial l}{\partial a}\\ \dot{a} &= -\xi a.\end{aligned}$$

It is also clear that

$$\frac{dJ(a,b)}{dt} = dJ(a,b)(\dot{a},\dot{b}).$$

Therefore we obtain

$$\frac{dJ(a,b)}{dt} = dJ(a,b)\left(-\xi a, -\xi b + \frac{\partial l}{\partial a}\right).$$

Using directly the formula $J(a,b)(\nu) = \langle \nu a, b\rangle = -\langle a, \nu b\rangle$ we can prove in a straightforward manner that

$$dJ(a,b)\left(-\xi a, -\xi b + \frac{\partial l}{\partial a}\right) = \mathrm{ad}^*_\xi J(a,b) - \frac{\partial l}{\partial a} \diamond a.$$

Using this together with the vertical Lagrange-Poincaré equation we obtain, at each point $(\xi, q, \dot{q}, a)$,

$$-\frac{d}{dt}\left(\frac{\partial l}{\partial \xi}\right) - \frac{\partial l}{\partial a} \diamond a + \mathrm{ad}^*_\xi\left(\frac{\partial l}{\partial \xi}\right) = 0.$$

which is the Euler–Poincaré equation. We can easily conclude from all this that any solution to the reduced system of equations is a solution of the system (7.4.2).

We can also prove the converse, namely, starting with a solution $(\xi(t), a(t))$ of the system (7.4.2) we can see that the curve $b(t) = g(t)b_0$, for any given $b_0 \in V$, is such that $(\xi(t), a(t), b(t))$ is a solution of the system formed by the reduced vertical and horizontal Euler–Poincaré equations. This is done by simply reversing the previous procedure.

8 The Category $\mathfrak{LP}^*$ and Poisson Geometry

In this section we will define the dual $\mathfrak{LP}^*$, in some sense, of the category $\mathfrak{LP}$. The objects of $\mathfrak{LP}^*$ are dual bundles and carry a Poisson structure which is dual to the Lie bracket structure on sections of objects of the category $\mathfrak{LP}$. Cotangent bundles are important examples of objects of $\mathfrak{LP}^*$. The parallelism between Lagrangian mechanics and Poisson mechanics is a consequence of the Legendre transformation. Interesting aspects of this parallelism can be

viewed under the light of the categorical duality. For instance, reduction in the category $\mathfrak{LP}^*$ is dual to reduction in the category $\mathfrak{LP}$. This includes cotangent bundle reduction as a special case (see Montgomery [1986]). Thus, $\mathfrak{LP}^*$ is an interesting class of Poisson manifolds which is stable under reduction.

In this section we establish the basic link between Lagrangian and Poisson mechanics given by this duality and show how reduction in the category $\mathfrak{LP}^*$, which is a subcategory of the category of Poisson manifolds and Poisson maps, can be viewed as being dual to reduction in the category $\mathfrak{LP}$. A more complete study, including a precise description of the symplectic leaves of objects of $\mathfrak{LP}^*$ and several other related topics will be the purpose of a future work.

8.1 The Poisson Bracket on Duals of Objects of $\mathfrak{LP}$

We shall begin with some notation and definitions. Let $W = TQ \oplus V$ be an object of $\mathfrak{LP}$ and let $\pi : W \to Q$, $p_Q : W \to TQ$ and $p_V : W \to V$ be the natural projections. Let $W^* = TQ^* \oplus V^* \equiv T^*Q \oplus V^*$ be the dual of W and let

$$\bar{\pi} : W^* \to Q, \qquad \bar{p}_Q : W^* \to T^*Q \quad \text{and} \quad \bar{p}_V : W^* \to V^*$$

be the naturally induced projections. A section $w \in \Gamma(W)$ will be sometimes written more explicitly as $w = X \oplus v$ or $w = \dot{q} \oplus v$ Likewise, a section $\mu \in \Gamma(W^*)$ will be sometimes written $\mu = \gamma \oplus \nu$. We have well defined maps

$$\Gamma(p_Q) : \Gamma(W) \to \Gamma(TQ) \quad \text{and} \quad \Gamma(p_V) : \Gamma(W) \to \Gamma(V)$$

and also

$$\Gamma(\bar{p}_Q) : \Gamma(W^*) \to \Gamma(T^*Q) \quad \text{and} \quad \Gamma(\bar{p}_V) : \Gamma(W^*) \to \Gamma(V^*).$$

We will often denote

$$\Gamma(p_Q)(w) \equiv p_Q w \quad \text{and} \quad \Gamma(p_V)(w) \equiv p_V w$$

and also

$$\Gamma(\bar{p}_Q)\mu \equiv \bar{p}_Q \mu \quad \text{and} \quad \Gamma(\bar{p}_V)(\mu) \equiv \bar{p}_V \mu.$$

Thus, if we write a section $w \in \Gamma(W)$ as $w = X \oplus v$ and a section $\mu \in \Gamma(W^*)$ as $\mu = \gamma \oplus \nu$, we have

$$\Gamma(p_Q)(w) \equiv p_Q w = X \quad \text{and} \quad \Gamma(p_V)(w) \equiv p_V w = v$$

and also

$$\Gamma(\bar{p}_Q)\mu \equiv \bar{p}_Q \mu = \gamma \quad \text{and} \quad \Gamma(\bar{p}_V)(\mu) \equiv \bar{p}_V \mu = \nu.$$

Definition 8.1.1 *For each $w \in \Gamma(W)$ we define the function $P(w) \in C^\infty(W^*)$ by*

$$P(w)(\mu) = \mu(w)$$

for all $\mu \in W^*$. In addition, for each $f \in C^\infty(Q)$ we have the function $f \circ \bar{\pi} \in C^\infty(W^*)$.

The space $\mathrm{A}(W^*) \subset C^\infty(W^*)$ is the vector space generated by the set of all the functions that are affine along the fibers of $C^\infty(W^*)$, that is, equivalently, the space generated by the set of all functions of the type $P(w)$, $w \in \Gamma(W)$ together with $f \circ \bar{\pi}$, $f \in C^\infty(Q)$.

Lemma 8.1.2 *Let $w_i \in \Gamma(W)$ and $f_i \in C^\infty(Q)$, for $i = 1, 2$. Define*

$$\begin{aligned}
\{P(w_i), P(w_j)\} &= -P\left([w_i, w_j]\right) \\
\{P(w_i), f_j \circ \bar{\pi}\} &= -\left(\Gamma(p_Q)(w_i)\right)(f_j) \circ \bar{\pi} \\
\{f_i \circ \bar{\pi}, f_j \circ \bar{\pi}\} &= 0.
\end{aligned}$$

We will sometimes write, for given $f \in C^\infty(Q)$ and given $w \in \Gamma(W)$, $f \equiv f \circ \bar{\pi}$ and $w(f) \equiv (\Gamma(p_Q)(w))(f) \circ \bar{\pi}$, for short.

Then $\{\,,\}$ extends to a uniquely determined skew-symmetric bilinear map

$$\{\,,\} : \mathrm{A}(W^*) \times \mathrm{A}(W^*) \to \mathrm{A}(W^*).$$

The operation $\{\,,\}$ satisfies the Jacobi identity, that is

$$\{f_1, \{f_2, f_3\}\} = \{\{f_1, f_2\}, f_3\} + \{f_2, \{f_1, f_3\}\}$$

for all $f_i \in \mathrm{A}(W^*)$, $i = 1, 2$.

Proof. Every element of $\mathrm{A}(W^*)$ can be written $f \circ \bar{\pi} + P(w)$ for some uniquely determined $f \in C^\infty(Q)$ and $w \in \Gamma(W)$. Using this we can prove in a straightforward manner that $\{\,,\}$ extends to a uniquely determined skew-symmetric bilinear form on $\mathrm{A}(W^*)$. The verification that $\{\,,\}$ satisfies the Jacobi identity is also a straightforward calculation. ■

Theorem 8.1.3 *The operation $\{\,,\}$ on $\mathrm{A}(W^*)$ defined in the previous lemma can be uniquely extended to a Poisson bracket*

$$\{\,,\} : C^\infty(W^*) \times C^\infty(W^*) \to C^\infty(W^*).$$

Proof. Let $f_i \in C^\infty(W^*)$, $i = 1, 2$ be given. Let $\sigma_0 W^*$ be also given and let $\bar{\pi}(\sigma_0) = q_0$. We are going to define $\{f_1, f_2\}(\sigma_0)$. Consider any section $\sigma \in \Gamma(W^*)$ such that $\sigma(q_0) = \sigma_0$. Then for each $q \in Q$ we have the Taylor expansion, for $i = 1, 2$,

$$f_i(\mu) = f_i(\sigma(q)) + \frac{\partial f_i}{\partial \mu_F}(\sigma(q))(\mu - \sigma(q)) + \epsilon_i(\sigma(q), \mu) \cdot (\mu - \sigma(q)),$$

where for each $q \in Q$ we have $\mu \in W_q^*$ and

$$\frac{\partial f_i}{\partial \mu_F}(\sigma(q))$$

is the derivative of f_i at $\sigma(q)$ along the fiber W_q^*, and besides, we have $\epsilon_i(\sigma(q), \mu) \to 0$ as $\mu \to \sigma(q)$. Define $w_i \in \Gamma(W)$ and $a_i \in C^\infty(Q)$ for $i = 1, 2$ by

$$w_i(q) = \frac{\partial f_i}{\partial \mu_F}(\sigma(q))$$

and

$$a_i(q) = f_i(\sigma(q)) - \frac{\partial f_i}{\partial \mu_F}(\sigma(q))\,\sigma(q)$$

for all $q \in Q$. Thus, for $i = 1, 2$, $a_i + P(w_i)$ is the affine approximation of f_i along the fibers of $\Gamma(W^*)$. To emphasize the dependence on σ we will sometimes write $a_i^\sigma \equiv a_i$, $w_i^\sigma \equiv w_i$ and $f_i^\sigma \equiv a_i^\sigma + P(w_i^\sigma)$, for $i = 1, 2$. The functions f_i^σ are elements of $\mathrm{A}(W^*)$. Then we can define, for each choice of σ and $\sigma_0 = \sigma(q_0)$,

$$\{f_1, f_2\}^{\sigma,\sigma_0}(\mu) = \{f_1^\sigma, f_2^\sigma\}(\mu),$$

where the bracket on the right hand side is the bracket on $\mathrm{A}(W^*)$ defined before. We want to show that

$$\{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0)$$

only depends on σ_0 and not on σ. Next we will show that the following property holds:

If $df_i(\sigma_0) = 0$ for $i = 1$ or $i = 2$ then $\{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0) = 0$.

We will work in a local bundle chart of $W = TQ \oplus V$ such that the restriction of it to TQ is a natural tangent bundle chart. This can be done by first choosing a vector bundle chart of V. This induces a chart of Q which has a naturally associated tangent bundle chart. The chart on W induces a chart on $W^* =$

$T^*Q \oplus V^*$. An element of W will be represented in the local chart by $(q, w) \equiv (q, \dot{q} \oplus v)$ and an element of W^* will be represented in the corresponding local chart by $(q, \mu) \equiv (q, p \oplus \nu)$. Thus we will write, with a slight abuse of notation,

$$\sigma_0 = (q_0, \mu_0) = (q_0, p_0 \oplus \nu_0)$$

and

$$\sigma(q) = (q, \mu(q)) = (q, p(q) \oplus \nu(q))$$

and also

$$w_i^\sigma(q) \equiv (q, w_i^\sigma(q)) \equiv (q, \dot{q}_i^\sigma(q) \oplus v_i^\sigma(q)).$$

A straightforward calculation shows that, for $i = 1, 2$, we have $df_i(\sigma_0) = df_i^\sigma(\sigma_0)$. It is also easy to see that $df_i^\sigma(\sigma_0) = 0$ for $i = 1$ or $i = 2$ if and only if the following equalities hold for $i = 1$ or $i = 2$ respectively :

$$\begin{aligned} \frac{\partial a_i^\sigma}{\partial q}(q_0) + \mu_0 \frac{\partial w_i^\sigma}{\partial q}(q_0) &= 0 \\ w_i^\sigma(q_0) &= 0 \end{aligned}$$

or, equivalently for $i = 1, 2$,

$$\begin{aligned} \frac{\partial a_i^\sigma}{\partial q}(q_0) + p_0 \frac{\partial \dot{q}_i^\sigma}{\partial q}(q_0) + \nu_0 \frac{\partial v_i^\sigma}{\partial q}(q_0) &= 0 \\ \dot{q}_i^\sigma(q_0) &= 0 \\ v_i^\sigma(q_0) &= 0. \end{aligned}$$

This is valid for *any* chart as before. From now on we will assume that the previous conditions are satisfied for $i = 1$. The case $i = 2$ can be established in an entirely analogous way. By definition we have

$$\begin{aligned} \{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0) &= \{f_1^\sigma, f_2^\sigma\}(\sigma_0) \\ &= \{a_1^\sigma + P(w_1^\sigma), a_2^\sigma + P(w_2^\sigma)\}(\sigma_0) \\ &= w_2^\sigma(a_1^\sigma)(q_0) - w_1^\sigma(a_2^\sigma)(q_0) - P\left([w_1^\sigma, w_2^\sigma]\right)(q_0). \end{aligned}$$

Since $w_1^\sigma(q_0) = 0$ we have $w_1^\sigma(q_0)(a_2^\sigma)(q_0) = 0$. On the other hand, by definition, we have

$$\begin{aligned} w_2^\sigma(q_0)(a_1^\sigma)(q_0) &= (p_Q w_2^\sigma)(a_1^\sigma)(q_0) \\ &= \frac{\partial a_1^\sigma}{\partial q}(q_0)(p_Q w_2^\sigma)(q_0) \\ &= \frac{\partial a_1^\sigma}{\partial q}(q_0)\dot{q}_2^\sigma(q_0). \end{aligned}$$

By Definition 6.1.1 the Lie bracket in the category $\mathfrak{LP}$ is given by

$$\begin{aligned}[w_1^\sigma, w_2^\sigma] &= [\dot{q}_1^\sigma \oplus v_1^\sigma, \dot{q}_2^\sigma \oplus v_2^\sigma] \\ &= [\dot{q}_1^\sigma, \dot{q}_2^\sigma] \oplus \nabla_{\dot{q}_1^\sigma} v_2^\sigma - \nabla_{\dot{q}_2^\sigma} v_1^\sigma - \omega(\dot{q}_1^\sigma, \dot{q}_2^\sigma) + [v_1^\sigma, v_2^\sigma].\end{aligned}$$

Since $\dot{q}_1^\sigma = 0$ and $v_1^\sigma(q_0) = 0$ we have

$$[w_1^\sigma, w_2^\sigma](q_0) = [\dot{q}_1^\sigma, \dot{q}_2^\sigma](q_0) \oplus -\nabla_{\dot{q}_2^\sigma} v_1^\sigma(q_0).$$

We can easily show that

$$[\dot{q}_1^\sigma, \dot{q}_2^\sigma](q_0) = -\frac{\partial \dot{q}_1^\sigma}{\partial q}(q_0).$$

Now we choose the chart on W in such a way that the Christoffel symbol of the connection ∇ on V vanishes at q_0, which does not implies any loss of generality. Then we obtain

$$\nabla_{\dot{q}_2^\sigma} v_1^\sigma(q_0) = \frac{\partial v_1^\sigma}{\partial q}(q_0)\dot{q}_2^\sigma(q_0).$$

Therefore, using previous equalities, we can conclude that

$$\begin{aligned}\{f_1^\sigma, f_2^\sigma\}(\sigma_0) &= \frac{\partial a_1^\sigma}{\partial q}(q_0)\dot{q}_2^\sigma(q_0) + p_0\frac{\partial \dot{q}_1^\sigma}{\partial q}(q_0)\dot{q}_2^\sigma(q_0) + \nu_0\frac{\partial v_1^\sigma}{\partial q}(q_0)\dot{q}_2^\sigma(q_0) \\ &= \left(\frac{\partial a_1^\sigma}{\partial q}(q_0) + p_0\frac{\partial \dot{q}_1^\sigma}{\partial q}(q_0) + \nu_0\frac{\partial v_1^\sigma}{\partial q}(q_0)\right)\dot{q}_2^\sigma(q_0) \\ &= 0.\end{aligned}$$

We have therefore proved the aforementioned property that if $df_i(\sigma_0) = 0$ for $i = 1$ or $i = 2$ then $\{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0) = 0$. It follows by a standard argument that for given $f_i, g_i \in C^\infty(W^*)$, $i = 1, 2$ such that $df_i(\sigma_0) = dg_i(\sigma_0)$ and for any section $\sigma \in \Gamma(W^*)$ such that $\sigma(q_0) = \sigma_0$ we have

$$\{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0) = \{g_1, g_2\}^{\sigma,\sigma_0}(\sigma_0).$$

We can easily check that if $f_i \in \mathrm{A}(W^*)$, for $i = 1, 2$, then for any section $\sigma \in \Gamma(W^*)$ and any choice of σ_0 such that $\sigma(q_0) = \sigma_0$ we have $f_i = f_i^\sigma$, for $i = 1, 2$ and also

$$\{f_1, f_2\}(\sigma_0) = \{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0).$$

On the other hand we can show using the Taylor expansion that for any choice of a section $\sigma' \in \Gamma(W^*)$ such that $\sigma'(q_0) = \sigma_0$ and any $f \in C^\infty(W^*)$ we have $df^{\sigma'}(\sigma_0) = df(\sigma_0) = df^\sigma(\sigma_0)$. Using all this we can see that

$$\begin{aligned}\{f_1, f_2\}^{\sigma,\sigma_0}(\sigma_0) &= \{f_1^\sigma, f_2^\sigma\}^{\sigma,\sigma_0}(\sigma_0) \\ &= \{f_1^{\sigma'}, f_2^{\sigma'}\}^{\sigma,\sigma_0}(\sigma_0) = \{f_1^{\sigma'}, f_2^{\sigma'}\}^{\sigma',\sigma_0}(\sigma_0) = \{f_1, f_2\}^{\sigma',\sigma_0}(\sigma_0).\end{aligned}$$

So far we have proved that the bracket $\{f_1, f_2\}(\sigma_0)$ is well defined, that is, it does not depends on the choice of the section $\sigma' \in \Gamma(W^*)$ as far as $\sigma'(q_0) = \sigma_0$.

Now we must prove that for given $f_i \in C^\infty(W^*)$, $i = 1, 2, 3$ and any given $\sigma_0 \in W^*$ we have

$$\{f_1, \{f_2, f_3\}\}(\sigma_0) = \{\{f_1, f_2\}, f_3\}(\sigma_0) + \{f_2, \{f_1, f_3\}\}(\sigma_0).$$

Choose $\sigma \in \Gamma(W^*)$ such that $\sigma(q_0) = \sigma_0$. Using what we have proven so far we can see that the question reduces itself to proving that

$$\{f_1^\sigma, \{f_2^\sigma, f_3^\sigma\}\}(\sigma_0) = \{\{f_1^\sigma, f_2^\sigma\}, f_3^\sigma\}(\sigma_0) + \{f_2^\sigma, \{f_1^\sigma, f_3^\sigma\}\}(\sigma_0).$$

We can prove that this last equality is true using Lemma 8.1.2 because $f_i^\sigma \in \mathrm{A}(W^*)$, for $i = 1, 2, 3$.

Finally we must prove that

$$\{f_1, f_2 f_3\}(\sigma_0) = \{f_1, f_2\}(\sigma_0) f_3(\sigma_0) + f_2(\sigma_0)\{f_1, f_3\}(\sigma_0).$$

Since $f_i^\sigma \in \mathrm{A}(W^*)$, for $i = 1, 2, 3$ we can write $f_i = a_i + P(w_i)$, for $i = 1, 2, 3$. Using the bilinearity and skew-symmetry of $\{\,,\}$ we can easily see that it is sufficient to consider the following particular cases:

$$\begin{aligned}
&f_1 \equiv a_1, f_2 \equiv a_2, f_3 \equiv a_3;\\
&f_1 \equiv a_1, f_2 \equiv a_2, f_3 \equiv P(w_3);\\
&f_1 \equiv a_1, f_2 \equiv P(w_2), f_3 \equiv P(w_3);\\
&f_1 \equiv P(w_1), f_2 \equiv a_2, f_3 \equiv a_3;\\
&f_1 \equiv P(w_1), f_2 \equiv a_2, f_3 \equiv P(w_3);\\
&f_1 \equiv P(w_1), f_2 \equiv P(w_2), f_3 \equiv P(w_3).
\end{aligned}$$

The first, second, fourth and fifth cases can be established using directly the definition of the bracket on $\mathrm{A}(W^*)$ and the fact that for any $a \in C^\infty(Q)$ and any $w \in \Gamma(W)$ we have $aP(w) = P(aw)$. To prove the third case we first write the Taylor expansion of $P(w_2)$ and $P(w_3)$ namely

$$P(w_i)(q, \mu) = \sigma(q)(w_i)(q) + (\mu - \sigma(q))\,(w_i)(q)$$

for $i = 2, 3$. Then, since

$$d\left((\mu - \sigma(q))\,(w_2)(q)\,(\mu - \sigma(q))\,(w_3)(q)\right) = 0$$

at $(q, \mu) = \sigma_0$ we see that

$$\begin{aligned}
\{a_1, P(w_2)P(w_3)\}(\sigma_0) &= \{a_1, \sigma(q)(w_2)(q)P(w_3) + P(w_2)\sigma(q)(w_3)(q)\}(\sigma_0)\\
&= \{a_1, P\left(\sigma(q)(w_2)(q)w_3\right) + P\left(\sigma(q)(w_3)(q)w_2\right)\}(\sigma_0)\,.
\end{aligned}$$

Using this the property that we want to prove can be established using the definition of the bracket $\mathrm{A}(W^*)$. To prove the sixth case we use the Taylor expansion of $P(w_2)$ and $P(w_3)$ and we obtain

$$\begin{aligned}
&\{P(w_1), P(w_2)P(w_3)\}(\sigma_0) \\
&\quad = \{P(w_1), \sigma(q)(w_2)(q)P(w_3) + P(w_2)\sigma(q)(w_3)(q)\}(\sigma_0) \\
&\quad = \{P(w_1), P\left(\sigma(q)(w_2)(q)w_3\right) + P\left(\sigma(q)(w_3)(q)w_2\right)\}(\sigma_0)\,.
\end{aligned}$$

Using this the property can be established using the definition of the bracket on $\mathrm{A}(W^*)$. ■

Remark. In the extreme case $V = 0$, that is, $W \equiv TQ$ we have $W^* = T^*Q$ and the Poisson structure on W^* coincides with the standard Poisson structure on the symplectic manifold T^*Q. In the extreme case in which Q is a point, that is, $W \equiv V$ we have $W^* = V^*$ and the Poisson structure on W^* coincides with the standard Lie-Poisson structure on the dual of a Lie algebra.

8.2 Poisson Reduction in the Category $\mathfrak{LP}^*$ Viewed as Dual to Reduction in the Category $\mathfrak{LP}$

Let $W = TQ \oplus V$ be an object of $\mathfrak{LP}$ and assume the hypothesis of Theorem 6.2.10. The composition of the natural map $\pi_G(W) : W \to W/G$ and the diffeomorphism

$$\alpha_A^W : (TQ \oplus V)/G \to T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G),$$

of Definition 6.2.11 gives a map $P_G(W) = \alpha_A^W \circ \pi_G(W)$. Introducing the notation

$$W_G = T(Q/G) \oplus \tilde{\mathfrak{g}} \oplus (V/G)$$

for the target space, W_G is an object of $\mathfrak{LP}$, and

$$P_G(W) : W \to W_G$$

is a morphism of $\mathfrak{LP}$. The restriction of $P_G(W)$ to the zero section coincides with the natural projection $\pi_G(Q) : Q \to Q/G$ and the restriction of $P_G(W)$ to each fiber is a linear isomorphism. Thus, $P_G(W)$ induces a vector bundle morphism $\bar{P}(W)_G : W^* \to W_G^*$. The restriction of $\bar{P}(W)_G$ to the zero section coincides with the natural projection $\pi_G(Q) : Q \to Q/G$. The restriction of $\bar{P}(W)_G$ to each fiber is a linear isomorphism, which is precisely the dual of the inverse of the restriction of $P_G(W)$ to the same fiber.

According to Theorem 8.1.3, W^* and W_G^* carry a Poisson structure. We are going to show that $\bar{P}(W)_G$ *is a Poisson map.*

First, we will introduce some notation. In short, reduced objects under the action of G will be denoted with a subindex $_G$. Let us give more precise definitions. For given $w \in W$ we will denote $w_G = P_G(W)(w)$. Likewise, for $\mu \in W^*$ we will denote $\mu_G = \bar{P}(W)_G(\mu)$. If $w \in \Gamma^G(W)$ or $\mu \in \Gamma^G(W^*)$ then, respectively, $w_G \in \Gamma(W_G)$ is the unique section of W_G such that

$$P_G(W)\,(w(q)) = w_G\,(\pi(q))$$

for all $q \in Q$ and $\mu_G \in \Gamma(W_G^*)$ is the unique section of W_G^* such that

$$\bar{P}(W)_G(W)\,(\mu(q)) = \mu_G\,(\pi(q)) \quad \text{for al} \quad q \in Q.$$

If $f \in C^\infty(W^*)$ is invariant or $a \in C^\infty(Q)$ is invariant then, respectively, $f_G \in C^\infty(W_G^*)$ is the unique function such that $f = f_G \circ \bar{P}(W)_G$ and $a_G \in C^\infty(Q/G)$ is the unique function such that $a = a_G \circ \pi_G(Q)$. We can easily show that, for any $w \in W$ and any $\mu \in W^*$ we have $\mu_G(w_G) = \mu(w)$. For given $w_i \in \Gamma^G(W)$, $i = 1, 2$, we have proven before (this is part of the content of Theorem 6.2.10) that $[w_{1G}, w_{2G}] = [w_1, w_2]_G$. It is also easy to see that for given $w \in \Gamma^G(W)$ and $a \in C^\infty(Q)$ invariant we have $w_G(a_G) = (w(a))_G$. Using the last two assertions we can easily show that for any given $w_i \in \Gamma^G(W)$, $i = 1, 2$ we have

$$\{P(w_{1G}), P(w_{2G})\} = \{P(w_1), P(w_2)\}_G$$

that is, for any $\sigma \in W^*$ we have

$$\{P(w_{1G}), P(w_{2G})\}(\sigma_G) = \{P(w_1), P(w_2)\}_G(\sigma_G).$$

Using the previous notation the assertion that $\bar{P}(W)_G$ is a Poisson map can be restated as follows: for any given invariant $f_i \in C^\infty(W^*)$, $i = 1, 2$, we have $\{f_{1G}, f_{2G}\} = \{f_1, f_2\}_G$.

First, we will consider the case in which $f_i \in \mathrm{A}(W^*)$, $i = 1, 2$. Then $f_i = a_i + P(w_i)$, $i = 1, 2$. Invariance of f_i, $i = 1, 2$ implies invariance of a_i and of w_i, that is, in particular, $w_i \in \Gamma^G(W)$, $i = 1, 2$. Using this and the previous formulas we can show in a straightforward manner that $\{f_{1G}, f_{2G}\} = \{f_1, f_2\}_G$.

Now we will consider the general case. Let $f_i \in C^\infty(W^*)$, $i = 1, 2$ be invariant. Let $\sigma_0 \in W^*$, say $\sigma_0 \in W^*_{q_0}$. We can always find $\sigma \in \Gamma^G(W^*)$ such that $\sigma(q_0) = \sigma_0$. We can easily show that $f_i^\sigma \in C^\infty(W^*)$ is invariant, for $i = 1, 2$. Let $f_i^\sigma = a_i^\sigma + P(w_i^\sigma)$, $i = 1, 2$. We can show easily, for $i = 1, 2$, that $(a_i^\sigma)_G = (a_{iG})^{\sigma_G}$ and also that $(w_i^\sigma)_G = (w_{iG})^{\sigma_G}$. Then we have

$$\begin{aligned}\{f_1, f_2\}(\sigma_0) &= \{f_1^\sigma, f_2^\sigma\}(\sigma_0) = \{f_1^\sigma, f_2^\sigma\}_G(\sigma_{0G}) = \{(f_1^\sigma)_G, (f_2^\sigma)_G\}(\sigma_{0G}) \\ &= \{(f_{1G})^{\sigma_G}, (f_{2G})^{\sigma_G}\}(\sigma_{0G}) = \{f_{1G}, f_{2G}\}(\sigma_{0G}).\end{aligned}$$

We have proved the following

Theorem 8.2.1 *$\bar{P}(W)_G : W^* \to W^*_G$ is a Poisson map.*

This theorem establishes that Poisson reduction in the category $\mathfrak{LP}^*$, in particular cotangent bundle reduction, is dual to reduction in the category $\mathfrak{LP}$. The decomposition of the bracket as a sum of three brackets given in Montgomery [1986] should be compared, via duality, with the decomposition of the reduced Lie bracket on sections of the reduced object of $\mathfrak{LP}$ given in Theorem 6.2.10. As we said before, a more detailed study of all this is being planned for a future work.

Acknowledgments. We thank Darryl Holm, Gerard Misiolek, Matt Perlmutter and Jürgen Scheurle for several helpful suggestions. The research of JEM was partially supported by the California Institute of Technology, NSF grant DMS-9802106 and that of TSR by NSF Grant DMS-9802378 and FNS Grant 21-54138.98.

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