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Superdiffusion induced by a long-correlated external random force

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We consider a particle immersed in a thermal reservoir and simultaneously subjected to an external random force that drives the system to a nonequilibrium situation. Starting from a Langevin equation description, we derive exact expressions for the mean-square displacement and the velocity autocorrelation function of the diffusing particle. An effective temperature is introduced to characterize the deviation from the internal equilibrium situation. Using a power-law force autocorrelation function, the mean-square displacement and the velocity autocorrelation function are analytically obtained in terms of Mittag-Leffler functions. In this case, we show that the present model exhibits a superdiffusive regime as a consequence of the competition between passive and active processes.

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I. INTRODUCTION

When a particle diffuses through a complex fluid, a disordered medium, or a biological material, usually displays an anomalous diffusive behavior [1–3]. A force-free stochastic process is said to exhibit anomalous diffusion when the mean-square displacement (MSD) deviates from the linear increase with time characteristic of the Brownian motion. In this case, the MSD adopts the asymptotic form:

$$\langle X^2(t)\rangle = 2\,D_\lambda\,t^\lambda, \tag{1}$$

where D_λ is the generalized diffusion coefficient and λ is the anomalous diffusion exponent, The process is called subdiffusive when $0 < \lambda < 1$ and superdiffusive when $\lambda > 1$. Normal Brownian diffusion corresponds to the case $\lambda = 1$. Then, subdiffusion disperses slower than Brownian motion, whereas superdiffusion disperses faster than it.

Even though the classification between subdiffusion and superdiffusion is based on the asymptotic behavior [Eq. (1)], it is usually not trivial to identify the underlying microscopic mechanism causing this behavior.

Subdiffusive dynamics has been observed in a broad variety of systems, such as driven granular fluids [4], protein conformational fluctuations [5], dynamics of proteins in intracellular media [6–9], and also in complex fluids such as colloidal glasses [10] and sheared granular materials [11]. In normal or subdiffusive behavior, the system is passively driven by collisions from the surrounding particles, and this input of energy from the environment is removed by friction. In this case, the fluctuation-dissipation theorem describes the balance between stochastic forces and friction.

On the other hand, a superdiffusive behavior is usually found in nonequilibrium systems driven by an external active mechanism. In particular, there are numerous examples in biological systems, such as cytoskeleton dynamics [12–15], intracellular transport of pigment organelles driven by myosin-V motors [16,17], the motion of individual proteins in plasma membranes [18], dynamics of fatty acid vesicles driven by adhesion gradients of a liquid substrate [19], amoeboid locomotion [20], and cell migration [21], among many others. The common characteristic of these systems is the presence of an external active process, sometimes called active diffusion [22], which has its origin in the transduction of some energy into mechanical work. This mechanism pushes the system to an out-of-equilibrium situation, which implies that the fluctuation-dissipation theorem is no longer valid. Therefore, we can think that one possible origin of superdiffusion is the injection of energy from an external source. This energy must not be dissipated by the environment in order to be used by the particle to diffuse faster than in normal Brownian motion.

The aim of this paper is to introduce and analyze a simple stochastic model that displays the superdiffusive features mentioned above. For this purpose, in Sec. II, we introduce a nonequilibrium model of a particle under the influence of an external random noise. From the corresponding Langevin equation, formal expressions for the MSD and the velocity autocorrelation function (VACF) are obtained, as well as an expression for the effective temperature. In Sec. III, we specialize the previous results to the case of a long correlated external force considering a power-law autocorrelation function. In this case, we derive exact expressions for the MSD and the VACF of the particle. Their long-time asymptotics are obtained, showing a superdiffusive behavior. Finally, Sec. IV contains some concluding remarks.

II. SIMPLE NONEQUILIBRIUM MODEL

In what follows we consider a particle immersed in a thermal environment and simultaneously driven by an external random force. The resulting dynamics is modeled via a Langevin equation, namely,

$$m\ddot{X}(t) + \gamma\,\dot{X}(t) = F(t), \tag{2}$$

where $X(t)$ denotes the position of the particle of mass m and γ is the friction coefficient.

The total random force $F(t)$ is the sum of two contributions, i.e.,

$$F(t) = \xi(t) + \chi(t), \tag{3}$$

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where $\xi(t)$ corresponds to the standard internal Brownian noise due to thermal activity and $\chi(t)$ is a random force that represents an external process that gives rise to active transport.

We assume that the internal noise $\xi(t)$, responsible for the passive (thermal) motion, is a zero-centered Gaussian and stationary random force with a correlation function:

$$\langle \xi(t)\xi(t')\rangle = C_0\,\delta(t-t'). \tag{4}$$

The fluctuation dissipation theorem [23] relates the internal noise $\xi(t)$ to the friction coefficient as $C_0 = 2k_BT\,\gamma$, where T is the absolute temperature and k_B is the Boltzmann constant.

In addition to the thermal noise, we introduce a random external force to model the action of active driving forces generated by an external energy source. For this purpose we consider a zero-centered Gaussian and stationary random external force $\chi(t)$, with a correlation function given by

$$\langle \chi(t)\chi(t')\rangle = \Lambda(|t-t'|), \tag{5}$$

and uncorrelated with $\xi(t)$, i.e., $\langle \chi(t)\xi(t')\rangle = 0$.

Notice that the dissipation and the external force come from different sources. Then, the correlation $\Lambda(t)$ is not related to the friction coefficient γ. As a consequence, an *overall* fluctuation-dissipation theorem is no longer valid and the system is driven out of equilibrium by the stochastic external force. As we will show below, the resulting active transport due to the action of $\chi(t)$ can be responsible for a transition to a superdiffusive regime.

It is possible to find the solutions of the Langevin equation (2) by means of the Laplace transformation technique. Formal expressions for the displacement $X(t)$ and the velocity $V(t)$ can be written as

$$X(t) = \langle X(t)\rangle + \int_0^t dt'G(t-t')F(t'), \tag{6}$$

$$V(t) = \langle V(t)\rangle + \int_0^t dt'g(t-t')F(t'), \tag{7}$$

where

$$\langle X(t)\rangle = x_0 + m\,v_0\,G(t), \tag{8}$$

$$\langle V(t)\rangle = m\,v_0\,g(t) \tag{9}$$

are its mean values, and $x_0 = X(t=0)$ and $v_0 = \dot{X}(t=0)$ are its deterministic initial values, respectively. For simplicity, we assume that $x_0 = 0$.

The relaxation function $G(t)$ is the Laplace inversion of

$$\widehat{G}(s) = \frac{1}{ms^2 + s\,\gamma}, \tag{10}$$

which implies that

$$G(t) = \frac{1}{\gamma}(1 - e^{-t/\tau_0}), \tag{11}$$

where $\tau_0 \equiv m/\gamma$ is the characteristic relaxation time related to the balance between the inertial term and the frictional force. The relaxation function $g(t)$ is the derivative of $G(t)$, i.e.,

$$g(t) = \frac{1}{m}\,e^{-t/\tau_0}. \tag{12}$$

From an experimental viewpoint, the evaluation of the MSD is often used for extracting information about the underlying dynamic [16,24–27]. The MSD is defined as

$$\rho(t,\tau) \equiv \langle [X(t+\tau) - X(t)]^2\rangle, \tag{13}$$

where $|X(t+\tau) - X(t)|$ is the particle displacement between two time points, t denotes the *absolute time*, and τ is the so-called *lag time* [15,24].

Alternative information about the experimentally observed behavior can be extracted from the normalized velocity autocorrelation function (VACF), defined as [28–30]

$$C_V(t,\tau) \equiv \frac{\langle V(t+\tau)V(t)\rangle}{\langle V^2(t)\rangle}. \tag{14}$$

Note that, in principle, both quantities depend on t and τ. However, as we will show below, under certain conditions the dynamics exhibited by our model becomes independent of the absolute time t.

It is interesting to note that a usually employed definition of the VACF is obtained by setting $t = 0$ in Eq. (14). This implies that $C_V(t=0,\tau) = \langle V(\tau)\,v_0\rangle/v_0^2 = g(\tau)$, which is independent of the external random force $\chi(t)$. Then, the role of the external agent cannot be studied solely by the computation of $C_V(t=0,\tau)$.

From Eqs. (13) and (14), one can realize that the two-time correlations $\langle X(t+\tau)X(t)\rangle$ and $\langle V(t+\tau)V(t)\rangle$ are the relevant quantities that must be evaluated to obtain explicit expressions for the MSD and the VACF. Their behavior can be obtained, for instance, using the double Laplace transform technique. After some algebra, and following a similar procedure as the one given in Refs. [17,24,29], the two-time correlation functions can be written as

$$\begin{aligned}\langle X(t+\tau)X(t)\rangle = {} & m^2\left(v_0^2 - \frac{k_BT}{m}\right)G(t)G(t+\tau) \\ & + k_BT[I(t) + I(t+\tau) - I(\tau)] + \Omega(t,\tau),\end{aligned} \tag{15}$$

$$\begin{aligned}\langle V(t+\tau)V(t)\rangle = {} & k_BT\,g(\tau) + m^2\left(v_0^2 - \frac{k_BT}{m}\right)g(t)g(t+\tau) \\ & + \Delta(t,\tau),\end{aligned} \tag{16}$$

where the kernel $I(t)$ reads

$$I(t) = \int_0^t dt_1 G(t_1) = \frac{1}{\gamma}[t - \tau_0(1 - e^{-t/\tau_0})], \tag{17}$$

and $G(t)$ and $g(t)$ are given by Eqs. (11) and (12), respectively. The contributions of the external random force to the two-time correlation functions are given by the last term of Eqs. (15) and (16), which reads

$$\Omega(t,\tau) = \int_0^{t+\tau} dx\,G(x)\int_0^t dy\,G(y)\,\Lambda(|y+\tau-x|), \tag{18}$$

$$\Delta(t,\tau) = \int_0^{t+\tau} dx\,g(x)\int_0^t dy\,g(y)\,\Lambda(|y+\tau-x|). \tag{19}$$

In Appendix A, we show a convenient way to write these functions as

$$\Omega(t,\tau) = \int_0^t dy\,[G(y)H(y+\tau) + G(y+\tau)H(y)]\,, \quad (20)$$

$$\Delta(t,\tau) = \int_0^t dy\,[g(y)J(y+\tau) + g(y+\tau)J(y)]\,, \quad (21)$$

where

$$H(t) = \int_0^t dy\, G(y)\Lambda(t-y), \quad (22)$$

$$J(t) = \int_0^t dy\, g(y)\Lambda(t-y) \quad (23)$$

are the convolutions of the kernels $G(t)$ and $g(t)$ with the external noise correlation function, respectively.

Notice that the two-time correlation functions (15) and (16) are exact and valid to all absolute time t and time lag τ. Moreover, the processes $X(t)$ and $V(t)$ are Gaussian because they are related to the Gaussian total force $F(t)$ through linear Eqs. (6) and (7).

In particular, by setting $\tau = 0$ in Eq. (15), we get

$$\langle X^2(t)\rangle = m^2\left(v_0^2 - \frac{k_B T}{m}\right) G^2(t) + 2k_B T\, I(t) + \Omega(t,0). \quad (24)$$

Then, inserting Eqs. (15) and (24) in the expression for the MSD Eq. (13) yields

$$\rho(t,\tau) = 2\,k_B T\, I(\tau) + (v_0^2 - k_B T)\,[G(t+\tau) - G(t)]^2 + \Omega(t+\tau,0) + \Omega(t,0) - 2\Omega(t,\tau). \quad (25)$$

On the other hand, the second moment of the velocity can be obtained setting $\tau = 0$ in Eq. (16) and using Eq. (12). This leads to

$$\langle V^2(t)\rangle = \frac{k_B T}{m} + \left(v_0^2 - \frac{k_B T}{m}\right) e^{-2\frac{t}{\tau_0}} + \Delta(t,0). \quad (26)$$

Notice that the long-time limit of Eq. (26) enables us to define an effective temperature $T_{\rm eff}$ as $k_B T_{\rm eff} = m\langle V^2(t\to\infty)\rangle$ [31–33]. Thus, from Eq. (21) and assuming that the involved integral in $\Delta(t,0)$ converges, we get

$$k_B T_{\rm eff} = k_B T + 2\,m\int_0^\infty dy\, g(y)J(y), \quad (27)$$

which quantifies the deviation from the internal equilibrium situation. This effective temperature is directly related to the energy exerted by the external random force to the particle, being $m\, g(t)J(t)$ the flux of energy into the particle (injected power) [34].

Finally, the formal expression for the VACF is obtained by inserting Eqs. (16) and (26) in Eq. (14).

III. SUPERDIFFUSION DRIVEN BY A POWER-LAW CORRELATED FORCE

The kernels (11), (12), and (17) are independent of the external noise, being equal to those obtained in the standard internal white-noise case. The contribution of this internal noise is responsible for the passive (thermal) transport, giving rise to a normal diffusive regime.

On the other hand, to model a transition to an anomalous diffusive regime, we must introduce an adequate form for the autocorrelation function (5) of the external random force. It is well known that an anomalous diffusive behavior is indicative of the presence of long-time tail correlations [35,36]. Then, to model the active transport, we assume a power-law correlation function $\Lambda(t)$ of the form [16,30,37]

$$\Lambda(t) = \frac{\Lambda_\alpha}{\Gamma(1-\alpha)}\, t^{-\alpha}, \qquad 0 < \alpha < 1, \quad (28)$$

where $\Gamma(z)$ is the gamma function and Λ_α is a proportionality coefficient dependent on the exponent α but independent of time. The Laplace transform of the correlation function (28) is given by

$$\widehat{\Lambda}(s) = \Lambda_\alpha\, s^{\alpha-1}. \quad (29)$$

Note that $\alpha \to 1$ corresponds to a white-noise limit and represents a series of instantaneous infinite force pulses. On the other hand, $\alpha \to 0$ corresponds to an indefinitely large memory case. Then, an intermediate exponent $0 < \alpha < 1$ can be considered as a smoothing of discontinuities in the instantaneous force pulses [13,16,38].

Inserting Eq. (29) in Eqs. (22) and (23), and using Eqs. (10) and (12), we get

$$\widehat{J}(s) = \frac{\Lambda_\alpha}{m}\,\frac{s^{\alpha-1}}{s+\frac{1}{\tau_0}}, \quad (30)$$

$$\widehat{H}(s) = \frac{\Lambda_\alpha}{m}\,\frac{s^{\alpha-2}}{s+\frac{1}{\tau_0}}\,. \quad (31)$$

It is possible to Laplace invert the above functions using the two-parameter Mittag-Leffler function $E_{\mu,\nu}(y)$, defined through the series [39]

$$E_{\mu,\nu}(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\mu j + \nu)}\,, \qquad \mu\,,\nu > 0. \quad (32)$$

Taking into account that $E_{\mu,\nu}(y)$ satisfies [40]

$$\int_0^\infty e^{-st}\, t^{\nu-1}\, E_{\mu,\nu}(-at^\mu)\, dt = \frac{s^{\mu-\nu}}{s^\mu + a}, \quad (33)$$

the kernels (30) and (31) can be inverted as

$$J(t) = \frac{\Lambda_\alpha}{m}\, t^{1-\alpha}\, E_{1,2-\alpha}\left(-\frac{t}{\tau_0}\right), \quad (34)$$

$$H(t) = \frac{\Lambda_\alpha}{m}\, t^{2-\alpha}\, E_{1,3-\alpha}\left(-\frac{t}{\tau_0}\right). \quad (35)$$

Then, the use of the kernels (34) and (35) in expressions (20) and (21) fully determines the behavior of the contributions of the external force to the VACF and MSD. In what follows, we will show that the involved integrals can be analytically evaluated for any $0 < \alpha < 1$ coefficient and that the particle behavior exhibits a superdiffusive regime.

A. Effective temperature

From Eq. (26), we see that the contribution of the external noise to the second moment of the velocity is given by the term $\Delta(t,0)$, defined through Eq. (21). Using Eqs. (12), (30), and

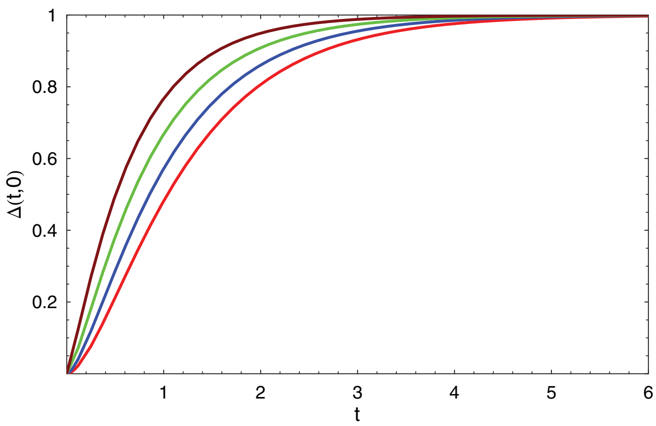

FIG. 1. (Color online) The term $\Delta(t,0)$ given by Eq. (37) for $\alpha = 0.2$, 0.4, 0.6, and 0.8 from bottom to top. The used parameters are $\Lambda_\alpha/m^2 = 1$ and $\tau_0 = 1$.

(33), the expression of $\Delta(t,0)$ can be rewritten as proportional to a convolution. In this way, we get

$$\Delta(t,0) = \frac{2}{m}\, e^{-\frac{t}{\tau_0}} \int_0^t dy\, e^{\frac{t-y}{\tau_0}}\, J(y) = 2\frac{\Lambda_\alpha}{m^2}\, e^{-\frac{t}{\tau_0}}\, \mathfrak{L}^{-1}\left(\frac{s^{\alpha-1}}{s^2 - \left(\frac{1}{\tau_0}\right)^2}\right), \tag{36}$$

which can be Laplace inverted using Eq. (33) as

$$\Delta(t,0) = 2\frac{\Lambda_\alpha}{m^2}\, e^{-\frac{t}{\tau_0}}\, t^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{t}{\tau_0}\right)^2\right]. \tag{37}$$

Using Eq. (32) one can deduce that for short times $\Delta(t,0)$ behaves as a power-law:

$$\Delta(t,0) \approx 2\frac{\Lambda_\alpha}{m^2}\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}. \tag{38}$$

On the other hand, the asymptotic behavior of Eq. (37) can be calculated using [41]

$$E_{2,\nu}(t^2) \approx t^{1-\nu}\frac{1}{2}(e^t + e^{-t}) - \frac{t^{-2}}{\Gamma(\nu-2)} + \dots, \qquad t \gg 1, \tag{39}$$

which leads to

$$\Delta(t,0) = \frac{\Lambda_\alpha}{m^2}\left[\tau_0^{2-\alpha}\left(1 - e^{-\frac{2t}{\tau_0}}\right) - 2\frac{\tau_0^2}{\Gamma(1-\alpha)}\, e^{-\frac{t}{\tau_0}}\, t^{-\alpha}\right]. \tag{40}$$

Figure 1 displays the contribution of Eq. (37) to $\langle V^2(t)\rangle$ for different values of α. This quantity is always positive, which means that the contribution to the internal energy increases due to the energy supply of the external random force despite the fact that it has zero mean value. Furthermore, note that the velocity takes more time to thermalize as α decreases because the random force is more correlated. However, the thermalization time is much shorter than the involved times in the evolution of the VACF and MSD (see below), due to the presence of the decaying exponential $e^{-\frac{t}{\tau_0}}$ in Eq. (37).

Using Eq. (40), the attained effective temperature of Eq. (27) can be written as

$$k_B T_{\text{eff}} = k_B T\left(1 + \varepsilon\, \tau_0^{1-\alpha}\right), \tag{41}$$

where

$$\varepsilon = \frac{\Lambda_\alpha}{C_0} \tag{42}$$

is a positive parameter that measures the relative intensity among the external random force and the thermal force. Then, this is the parameter indicative of the deviation from the equilibrium situation.

B. Velocity autocorrelation function

Note that to obtain an expression for the VACF Eq. (14) we need to evaluate the expression (21). Using a similar procedure used to obtain Eq. (37), and after some algebra, we find that

$$\begin{aligned}\Delta(t,\tau) = \frac{\Lambda_\alpha}{m^2}\Bigg\{ & e^{-\frac{t+\tau}{\tau_0}}\, t^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{t}{\tau_0}\right)^2\right] \\ & + e^{-\frac{t+2\tau}{\tau_0}}\, (t+\tau)^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{t+\tau}{\tau_0}\right)^2\right] \\ & - \tau^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right]\Bigg\}.\end{aligned} \tag{43}$$

Using the asymptotic expansion (39), one can verify that $\Delta(t,\tau)$ is independent of the absolute time for $t \gg \tau_0$. In this case, we get

$$\begin{aligned}\Delta(\tau) = \frac{\Lambda_\alpha}{m^2}\, \tau_0^{2-\alpha}\Bigg\{ & \cosh\left(\frac{\tau}{\tau_0}\right) \\ & - \left(\frac{\tau}{\tau_0}\right)^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right]\Bigg\}.\end{aligned} \tag{44}$$

Then, inserting Eq. (44) into Eq. (16) and using Eq. (41), we finally arrive at

$$C_V(\tau) = \frac{e^{-\frac{\tau}{\tau_0}} + \varepsilon\, \tau_0^{1-\alpha}\left\{\cosh\left(\frac{\tau}{\tau_0}\right) - \left(\frac{\tau}{\tau_0}\right)^{2-\alpha} E_{2,3-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right]\right\}}{1 + \varepsilon\, \tau_0^{1-\alpha}}. \tag{45}$$

In Fig. 2, the VACF Eq. (45) is displayed for different values of the exponent α. Notice that the VACF is always positive for any α, which implies a persistent behavior, a signature of a superdiffusive behavior [17]. As α diminishes, the velocity is more correlated because the sequences of active directed movements spend more time trying to push the particle in the same direction. This behavior is opposite to what happens in the subdiffusive case, where the whip-back effect is present [30,36,37]. In particular, the large time lag behavior of Eq. (45) can be obtained using the expansion (39). Hence,

$$C_V(\tau) \approx \frac{\varepsilon\, \tau_0}{1 + \varepsilon\, \tau_0^{1-\alpha}}\, \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)}, \tag{46}$$

showing explicitly that the VACF possesses a positive tail, decaying as a pure power law.

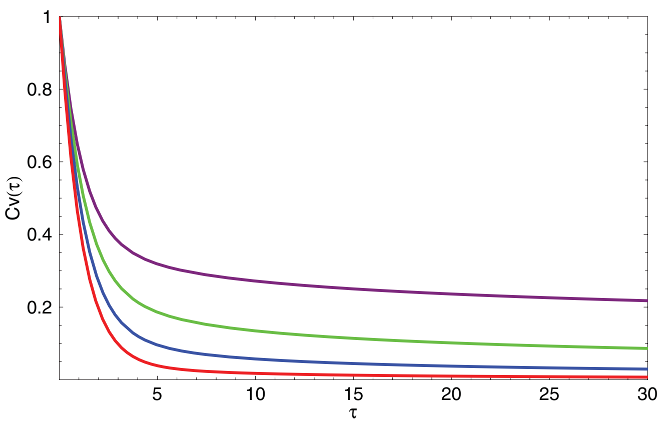

FIG. 2. (Color online) VACF vs time lag τ for $\alpha = 0.2$, 0.4, 0.6, and 0.8 from top to bottom, with $\varepsilon = 1$ and $\tau_0 = 1$.

C. Mean-square displacement

We now analyze the behavior of the MSD Eq. (25). From Eq. (21) and using Eqs. (10), (11), and (31), we write

$$\Omega(t,0) = \frac{2}{\gamma}\int_0^t dy\, H(y) - \frac{2}{\gamma}\frac{\Lambda_\alpha}{m^2}\, e^{-\frac{t}{\tau_0}}\,\mathfrak{L}^{-1}\left[\frac{s^{\alpha-2}}{s^2-\left(\frac{1}{\tau_0}\right)^2}\right]. \tag{47}$$

Then, using the property [40]

$$\int_0^t dx\, x^{\nu-1}E_{\mu,\nu}(\lambda x^\mu) = t^\nu E_{\mu,\nu+1}(\lambda t^\mu) \qquad (\nu > 0) \tag{48}$$

and Eq. (33), we get

$$\Omega(t,0) = \frac{2}{\gamma}\frac{\Lambda_\alpha}{m}\, t^{3-\alpha}\left\{E_{1,4-\alpha}\left(-\frac{t}{\tau_0}\right) - e^{-\frac{t}{\tau_0}}E_{2,4-\alpha}\left[\left(\frac{t}{\tau_0}\right)^2\right]\right\}. \tag{49}$$

Taking into account the asymptotic expression [40]

$$E_{\alpha,\beta}(-y) \simeq -\sum_{j=1}^{p}\frac{(-y)^{-j}}{\Gamma(\beta-\alpha j)} + O(y^{-1-p}), \qquad y \gg 1, \tag{50}$$

and Eq. (39), one finds that

$$\Omega(t,0) = \frac{\Lambda_\alpha}{\gamma^2}\left[\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{\tau_0\, t^{1-\alpha}}{\Gamma(2-\alpha)}\right] + O(t^{-\alpha}), \tag{51}$$

for $t \gg \tau_0$.

Then, inserting Eq. (51) in the expression (24), and using Eqs. (11) and (17), we realize that the second moment $\langle X^2(t)\rangle$ attains a superdiffusive regime with a scaling exponent $2-\alpha$.

Notice that to explicitly calculate the MSD Eq. (25) we need to evaluate the complete term $\Omega(t,\tau)$ given by Eq. (21). This calculation involves some straightforward but lengthy algebra, some details of which are sketched in Appendix B. Then, using Eq. (B3), it can be seen that

$$\rho(\tau) = \frac{2k_BT}{\gamma}[\tau - \tau_0(1-e^{-\tau/\tau_0})] + 2\frac{\Lambda_\alpha}{m}\frac{\tau_0^{3-\alpha}}{\gamma} \times\left\{\left(\frac{\tau}{\tau_0}\right)^{3-\alpha}E_{1,4-\alpha}\left(-\frac{\tau}{\tau_0}\right) - \left(\frac{\tau}{\tau_0}\right)^{3-\alpha} \times E_{2,4-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right] + \cosh\left(\frac{\tau}{\tau_0}\right) - 1\right\}. \tag{52}$$

Notice that the MSD Eq. (52), which is one of the main results of this work, is exact and valid for all time lags τ provided that $t \gg \tau_0$.

Introducing the series expansion (32) in Eq. (52), the short-time behavior of the MSD can be written as

$$\rho(\tau) \approx \frac{k_BT_{\text{eff}}}{m}\tau^2, \tag{53}$$

where we use the definition (41). Hence, the MSD exhibits a ballistic behavior with a coefficient proportional to T_{eff} instead of T.

On the other hand, using the asymptotic expansions (39) and (50) in Eq. (52), the long-time behavior of the MSD emerges as

$$\rho(\tau) \approx \frac{2k_BT}{\gamma}\left[\tau + \frac{\varepsilon}{\Gamma(3-\alpha)}\tau^{2-\alpha}\right]. \tag{54}$$

In particular, for lag times $\tau \gg \tau_* = [\Gamma(3-\alpha)/\varepsilon]^{\frac{1}{1-\alpha}}$ the MSD Eq. (54) exhibits a pure superdiffusive behavior of the form

$$\rho(\tau) = 2\, D_\alpha \tau^{2-\alpha}, \tag{55}$$

where the generalized diffusion coefficient is given by

$$D_\alpha = \frac{\Lambda_\alpha}{\gamma^2}\frac{1}{\Gamma(3-\alpha)}. \tag{56}$$

It is worth mentioning that, although $0 < \alpha < 1$, the MSD always exhibits a superdiffusive regime with an exponent $1 < 2-\alpha < 2$. The expression (54) coincides with a previous result obtained in Ref. [16], where a particle immersed in a viscoelastic environment and subjected to an external random force is considered in the case that the inertial effects are negligible. Moreover, the asymptotic Eq. (54) is in accordance with those results obtained in [35,42–44], where only an external noise contribution is considered.

It should be noted that a similar asymptotic behavior can be found starting from a generalized Langevin equation [45], fractional Langevin equation [37], or fractional Klein-Kramers equation [21,46]. However, the physical origin is very different in these models because they assume a temporal memory and a long-correlated *internal* noise

Another useful magnitude to characterize the underlying dynamics is the effective exponent $\beta(\tau)$, calculated via the logarithmic derivative of the MSD-versus-lag time [15,16,21]:

$$\beta(\tau) = \frac{d}{d\ln\tau}\ln\rho(\tau). \tag{57}$$

In Fig. 3, we show the effective exponent as a function of the time lag for different values of the exponent α. Note that all the curves decrease from $\beta(0) = 2$, which corresponds to the

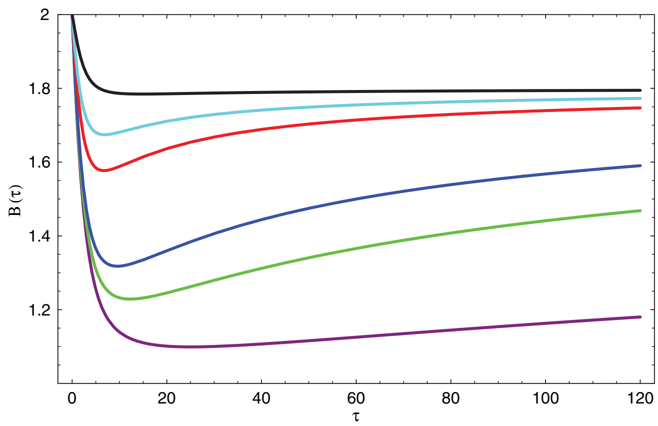

FIG. 3. (Color online) The logarithmic derivative of the MSD as a function of the time lag for $\alpha = 1/5$ and $\varepsilon = 0.01$, 0.05, 0.1, 0.5, 1, and 5 from bottom to top, with $\Lambda_\alpha/m^2 = 1$ and $\tau_0 = 1$. The corresponding values for τ_* are 603.26, 80.68, 33.92, 4.54, 1.91, and 0.26, respectively. In all cases, the asymptotic value is $\beta(\infty) = 1.8$.

initial ballistic regime (53) and shows a continuous transition to the estimated asymptotic exponent $\beta_\infty = 2 - \alpha$.

As τ increases, the behavior depends on the relation between τ_0 and τ_*. If $\tau_0 > \tau_*$, the curve displays a monotonic approach to β_∞. However, if $\tau_0 < \tau_*$, the function exhibits a minimum, which is closer to $\beta = 1$ as τ_* increases due to the relative weight of the linear term in the MSD Eq. (54). Next, a transition to a $\tau^{2-\alpha}$ behavior of the MSD is observed. Inserting Eq. (54) into Eq. (57), we see that $\beta(\tau)$ tends to $2 - \alpha$ for large times $\tau \gg \tau_*$ as

$$\beta(\tau) = (2 - \alpha) + \frac{\Gamma(3 - \alpha)}{\varepsilon}\,\tau^{\alpha-1}. \quad (58)$$

Then, as ε diminishes (or equivalently as τ_* increases), $\beta(\tau)$ takes more time to reach the asymptotic value β_∞.

IV. CONCLUSIONS

The aim of this paper is to clarify certain issues about the microscopic origin of the superdiffusive behavior in open systems. For this purpose, we present a simple nonequilibrium stochastic model for active systems, which exhibits a superdiffusive behavior. This model considers a particle immersed in a thermal reservoir and subjected to the action of a long-correlated external stochastic force. Using a power-law autocorrelation function, we have shown that the dynamics can be obtained completely in analytical form.

It should be noted that the present model does not include a dissipative memory kernel, as is usually included in the generalized Langevin equations [23], or any fractional derivative, as in the case of the fractional Langevin equations [37]. On the contrary, in our case the anomalous diffusion phenomenon comes only from the long tails in the correlation of external stochastic force. Despite the fact that the random force averages out to zero, the superdiffusion regime emerges naturally as a consequence of competition between passive and active transport. This active transport has its origin in the injection of energy from an external source which is not dissipated by the environment, implying that its temperature should increase. This mechanism is responsible for the fact that the particle spreads faster than in the normal Brownian case.

Starting from a Langevin equation, we have found analytical expressions for the two-time dynamics of the process, valid for all absolute times and times lags. We have demonstrated that the resulting dynamics presents a superdiffusive asymptotic regime for all values of the exponent α. In particular, we have shown that the MSD and the VACF can be expressed in terms of Mittag-Leffler functions, which reaffirms the importance of these functions in the description of anomalous diffusion phenomena [45,47]. Considering that the absolute time t is greater than the inertial time τ_0, we have demonstrated that the MSD and the VACF can be expressed as a simple expression depending only on the time lag τ. We have also shown that as the stochastic force becomes more correlated (α tends to zero) the system becomes more superdiffusive; i.e., the anomalous diffusion exponent is closer to 2 (ballistic limit).

In a related work [34], we show how this simple model allows us to analytically analyze the stochastic properties of energy and the fluctuation relations [48,49] associated to the specific case of a superdiffusive behavior.

Finally, we believe that this proposed model can be useful for a variety of physical and biological applications in order to analyze different stochastic processes leading to an anomalous superdiffusion.

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APPENDIX A: EXPRESSIONS FOR $\Omega(t,\tau)$ and $\Delta(t,\tau)$

To obtain the last term of Eqs. (15) and (16), we use the symmetry property of the correlation function, i.e., $\Lambda(x) = \Lambda(-x)$. Then, and changing the order of integration, it can be demonstrated that

$$\int_0^{t+\tau} dx\, f(x) \int_0^t dy\, f(y)\Lambda(y + \tau - x) = \int_0^t dx\,[f(x)R(x+\tau) + f(x+\tau)R(x)], \quad (A1)$$

where

$$R(t) = \int_0^t dt'\, f(t')\Lambda(t - t') \quad (A2)$$

is the convolution between $f(t)$ and $\Lambda(t)$. Then, the Laplace transform of $R(t)$ is

$$\widehat{R}(s) = \widehat{f}(s)\widehat{\Lambda}(s). \quad (A3)$$

In particular, for $\tau = 0$, we get

$$\int_0^t dx\, f(x) \int_0^t dy\, f(y)\Lambda(y - x) = 2\int_0^t dx f(x)\Lambda(x). \quad (A4)$$

APPENDIX B: TOTAL CONTRIBUTION TO THE MSD

To evaluate the expression $\Omega(t,\tau)$ given by Eq. (20), we use the properties of the exponential function and the expressions (11) and (35). In this way, one finds that

$$\int_0^t dt_1 G(t_1+\tau)H(t_1) = \frac{\Lambda_\alpha}{m^2}\frac{1}{\gamma}\, t^{3-\alpha}\left\{E_{1,4-\alpha}\left(-\frac{t}{\tau_0}\right) - e^{-\frac{t+\tau}{\tau_0}} E_{2,4-\alpha}\left[\left(\frac{t}{\tau_0}\right)^2\right]\right\}. \tag{B1}$$

Similarly, making a change of variables and splitting the involved integral into two terms, one gets

$$\int_0^t dt_1 G(t_1)H(t_1+\tau) = \frac{\Lambda_\alpha}{m^2}\frac{1}{\gamma}\left((t+\tau)^{3-\alpha}\left\{E_{1,4-\alpha}\left(-\frac{t+\tau}{\tau_0}\right) - e^{-\frac{t}{\tau_0}} E_{2,4-\alpha}\left[\left(\frac{t+\tau}{\tau_0}\right)^2\right]\right\} - \tau^{3-\alpha}\left\{E_{1,4-\alpha}\left(-\frac{\tau}{\tau_0}\right) - E_{2,4-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right]\right\}\right). \tag{B2}$$

Finally, inserting Eqs. (B1) and (B2) in Eq. (21), and considering that $t/\tau_0 \gg 1$, the contribution of the external random force to the MSD is

$$\Omega(t+\tau,0) + \Omega(t,0) - 2\,\Omega(t,\tau) = 2\frac{\Lambda_\alpha}{m^2}\frac{\tau_0^{3-\alpha}}{\gamma}\left\{\left(\frac{\tau}{\tau_0}\right)^{3-\alpha} E_{1,4-\alpha}\left(-\frac{\tau}{\tau_0}\right) - \left(\frac{\tau}{\tau_0}\right)^{3-\alpha} E_{2,4-\alpha}\left[\left(\frac{\tau}{\tau_0}\right)^2\right] + \cosh\left(\frac{\tau}{\tau_0}\right) - 1\right\}, \tag{B3}$$

where we used the approximations (39) and (50).

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