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Cold Fermions with Pairing Interactions: New Results Based on Fluiddynamical Descriptions

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Abstract We present a rigorous derivation of the moment hierarchy of the density and pair density matrices of a two species fermion superfluid in coordinate representation. We discuss the tools to truncate at any desired level and present the derivation of the Extended Superfluid Thomas-Fermi (ESTF) fluiddynamical scheme. In order to establish the equation of state in equilibrium to be incorporated in the truncation, we extend the method of Papenbrock and Bertsch. We examine the dynamics of fluctuations in homogeneous fermion matter and show that it is consistent with the ordinary Random-Phase-approximation. We discuss some numerical results for equilibrium profiles and collective fluctuations of trapped cold gases.

Keywords Fermion superfluid · Moment hierarchy · Collective spectrum

1 Introduction

Cold atom physics is a fertile source for studies of tunable fermion superfluidity [1–3]. The origin of theoretical studies of superfluidity of fermion matter is the Bardeen-Cooper-Schrieffer (BCS) theory, originally developed for the electron gas [4] and widely applied later to superfluid nuclei, a paradigm of finite systems [5, 6]. The latter indicate that in spite of being a mean field description, the BCS method is

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essentially microscopic, and demands lengthy calculations in a single-fermion basis, both for the equilibrium properties and for the low energy excitations.

The analysis of the collective spectrum of homogeneous superfluids has been put forward by Anderson [7] and Bogoliubov [8]. The Random-Phase-Approximation (RPA) formalism for the superconducting electron gas [7] is based on the linearization of the mean field equations of motion (EOM's) for the quasiparticles and permits to identify, in the case of a neutral system, two well separated modes: a low energy oscillation involving potential flow of the condensate [9], corresponding to the collisionless sound of a normal fluid, and a gapped fluctuation of the pair density, acknowledged as the pairing vibration. For trapped gases, the simplest local density approximation (LDA) that resorts to a Thomas-Fermi (TF) description of the fermion cloud plus a local BCS prescription for the superfluid gap, has been extensively applied to study the structure and the collective modes of these systems. Although the results could be regarded as relatively satisfactory as compared with i.e., Quantum Monte Carlo calculations [10–15], the impossibility of reaching the pairing vibration might be viewed as a limitation of the approach. Oscillations of the magnitude of the pairing gap at the so–called unitary limit have been recently examined in Ref. [16], employing a local energy–density functional theory and solving the mean field EOM's with an appropriate renormalization procedure, to remove the ultraviolet divergences in the integrated quantities (see i.e., Ref. [17] for details). The analysis indicates that the small amplitude dynamics of these modes is very similar to the earlier results in Ref. [18].

In contrast with microscopic approaches of many body systems, in various fields of physics a description in terms of macroscopic fields—typically particle, momentum and energy densities—has proven useful for insights of equilibrium profiles and collective spectrum [19]. Fermion fluiddynamics (FD) that explicitly incorporates distorsions of the Fermi surface can overcome this difficulty. Several versions of FD have been presented in the literature; similarly to classical hydrodynamics, the philosophy is to derive EOM's for the moments of the particle density operator and close the hierarchy by a convenient truncation. For instance, in Ref. [20] and a series of papers with applications to nuclear physics, the closed system of EOM's involves the particle and momentum density of a normal fluid with a local equation of state (EOS) corresponding to a TF approximation.

Recently, we derived a scheme for FD of fermion superfluids with two spin species, starting from the EOM's of the particle field operators with a Hamiltonian containing a zero range interaction between the different spins and decoupling products of field operators in a mean field approach. The truncation of the momentum hierarchy up to the momentum density complemented by various choices of the local EOS, named the Superfluid-Thomas-Fermi (STF) frame, was employed in preliminary calculations of equilibrium profiles of the particle density and superfluid gap [21], and particle current density [22], as well as of low energy collective modes [23]. The latter are restricted to sound-like modes, and the impossibility of reproducing high energy, gapped modes resembling the pairing vibrations of superfluids can be regarded as a weakness of the lowest level of truncation of the moment hierarchy. In a brief communication [24] we outlined a scheme that resembles classical hydrodynamics more closely by including the kinetic energy and its fluctuation; a somehow crude estimate indicates that the latter is the responsible of the gapped branch of the collective spectrum.

In this paper we present a rigorous derivation of the moment hierarchy together with the tools to truncate at any desired level. For this sake, in Sect. 2 we present the derivation of the Extended Superfluid Thomas-Fermi (ESTF) scheme for the particle and pair density together with their first and second moment. In Sect. 3 we review the method originally proposed by Papenbrock and Bertsch [25] (hereafter referred to as PB) for the EOS of a homogeneous superfluid, with the necessary modifications and extensions for the case of interest. In Sect. 4, we examine the dynamics of fluctuations in homogeneous fermion matter and compare our results with the RPA ones in Ref. [7]. Some numerical results are presented and discussed in Sects. 5, and 6 contains the summary and perspectives.

2 General Formalism for ESTF

The starting point for our formulation is a zero-temperature grand potential operator for fermions interacting with a zero range force if their spin projections $\sigma = \pm$ differ. The populations N_σ are subject to external potentials $V_\sigma(\mathbf{r})$

$$\begin{aligned}\hat{\Omega} &= H - \mu_+ N_+ - \mu_- N_- \\ &= \int d\mathbf{r} \sum_\sigma \left[-\frac{\hbar^2}{2m} \Psi_\sigma^\dagger(\mathbf{r}) \nabla^2 \Psi_\sigma(\mathbf{r}) + [V_\sigma(\mathbf{r}) - \mu_\sigma] \Psi_\sigma^\dagger(\mathbf{r}) \Psi_\sigma(\mathbf{r}) \right] \\ &\quad + g \int d\mathbf{r} \Psi_+^\dagger(\mathbf{r}) \Psi_-^\dagger(\mathbf{r}) \Psi_-(\mathbf{r}) \Psi_+(\mathbf{r})\end{aligned} \tag{1}$$

The intensity of the δ-interaction among the species is commonly expressed in terms of the s-wave scattering length a as $g = 4\pi\hbar^2 a/m$. This formalism allows broken symmetries like different external potentials and unequal populations or propagation velocities $\mathbf{U}_\sigma = \hbar \mathbf{q}_\sigma / m$ of the field operators, that for superfluids are generally expressed in terms of quasiparticle operators $b_{\alpha\sigma}^\dagger, b_{\alpha\sigma}$ through the Bogoliubov–de Gennes (BdG) transformation [26]

$$\Psi_\sigma = \sum_\alpha \left(\mathrm{u}_{\alpha\sigma} b_{\alpha\sigma}^\dagger - \mathrm{v}_{\alpha\sigma}^* b_{\alpha-\sigma} \right) \tag{2}$$

with the symmetries $\mathrm{u}_{\alpha-\sigma} = \mathrm{u}_{\alpha\sigma} \equiv \mathrm{u}_\alpha$, $\mathrm{v}_{\alpha-\sigma} = -\mathrm{v}_{\alpha\sigma} \equiv -\mathrm{v}_\alpha$. The one-body density, current and kinetic energy operators for each fermion species are

$$\hat{\rho}_\sigma(\mathbf{r}, \mathbf{r}') = \Psi_\sigma^\dagger(\mathbf{r}') \Psi_\sigma(\mathbf{r}) \tag{3}$$

$$\hat{\mathbf{j}}_\sigma(\mathbf{r}, \mathbf{r}') = \frac{\hbar}{2m\imath} (\nabla - \nabla') \hat{\rho}_\sigma(\mathbf{r}, \mathbf{r}') \tag{4}$$

$$\hat{\tau}_\sigma(\mathbf{r}, \mathbf{r}') = \frac{\hbar^2}{2m} \nabla \cdot \nabla' \hat{\rho}_\sigma(\mathbf{r}, \mathbf{r}') \tag{5}$$

The pair operator representing the anomalous density is

$$\hat{\kappa}(\mathbf{r},\mathbf{r}') = \Psi_+(\mathbf{r})\Psi_-(\mathbf{r}') \tag{6}$$

and the pair current and pair kinetic energy operators $\hat{\mathbf{j}}_\kappa$ and $\hat{\tau}_\kappa$ are defined as above with gradients operating upon $\hat{\kappa}(\mathbf{r},\mathbf{r}')$.

The dynamics is contained in the equation of motion (EOM)

$$\imath\hbar\frac{\partial\Psi_\sigma(\mathbf{r})}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + [V_\sigma(\mathbf{r}) - \mu_\sigma] + g\Psi^\dagger_{-\sigma}(\mathbf{r})\Psi_{-\sigma}(\mathbf{r})\right]\Psi_\sigma(\mathbf{r}) \tag{7}$$

and its Hermitian conjugate. These equations lead to the coupled EOM's for the spatial matrix elements of the above operators, that we omit here for the sake of briefness.

Is it convenient to introduce center-of-mass coordinates $(\mathbf{R},\mathbf{s})$ with $\mathbf{r} = \mathbf{R} + \mathbf{s}/2$ and $\mathbf{r}' = \mathbf{R} - \mathbf{s}/2$. The leading differential operators read

$$\hat{\mathbf{j}} = \frac{\hbar}{m\imath}\nabla_\mathbf{s} \tag{8}$$

$$\hat{\tau} = \frac{\hbar^2}{2m}\left(\frac{1}{4}\nabla^2_\mathbf{R} - \nabla^2_\mathbf{s}\right) \tag{9}$$

The diagonal terms of the coupled matrix EOM'S are then obtained taking the limit for s approaching zero. We need to keep in mind that the pairing tensor diverges in this limit. As proposed by previous authors [27], we introduce the gap matrix and the regular part κ_{reg} of the pair density

$$\Delta(\mathbf{R},\mathbf{s}) = -g\kappa_{reg}(\mathbf{R},\mathbf{s}) \tag{10}$$

$$\kappa(\mathbf{R},\mathbf{s}) = \frac{m}{4\pi\hbar^2}G_\mu(\mathbf{s})\Delta(\mathbf{R}) + \kappa_{reg}(\mathbf{R},\mathbf{s}) \tag{11}$$

with G_μ the one-body Green's function satisfying the equation

$$(\nabla^2_\mathbf{s} + k^2_\mu)G_\mu(\mathbf{s}) = -4\pi\delta(\mathbf{s}) \tag{12}$$

for $k^2_\mu = m(\mu_T - g\rho_T)/\hbar^2$, the subscript T indicating summation of the contributions from both species. In this way, we obtain the FD scheme in terms of the particle, momentum and kinetic energy densities of the fermion species, and of the order parameter and its first two moments (hereafter, κ stands for κ_{reg})

$$\frac{\partial\rho_\sigma}{\partial t} = -\nabla\cdot\mathbf{j}_\sigma \tag{13}$$

$$\frac{\partial\mathbf{j}_\sigma}{\partial t} = -\frac{\rho_\sigma}{m}\nabla\mu_\sigma + g\frac{\kappa\mathbf{j}^*_\kappa - \kappa^*\mathbf{j}_\kappa}{\imath\hbar} \tag{14}$$

$$\begin{aligned}\frac{\partial\tau_\sigma}{\partial t} &= -\nabla\cdot\mathbf{j}_\tau - \nabla(V_T + g\rho_T)\cdot\mathbf{j}_\sigma + g\frac{\kappa\tau^*_\kappa - \kappa^*\tau_\kappa}{\imath\hbar}\\ &\quad - g(\mathbf{j}^*_\kappa\cdot\nabla\kappa + \mathbf{j}_\kappa\cdot\nabla\kappa^*) - \frac{\hbar^2}{4m}\nabla^2(V_\sigma + g\rho_{-\sigma} - V_{-\sigma} - g\rho_\sigma)\end{aligned}$$

$$- g\frac{\hbar^2}{4m}\frac{\kappa^*\nabla^2\kappa - \kappa\nabla^2\kappa^*}{\imath\hbar} \tag{15}$$

$$\imath\hbar\frac{\partial\kappa}{\partial t} = \left(-\frac{\hbar^2}{4m}\nabla^2 + V_T - \mu_T\right)\kappa + \lim_{s\to 0}\left(-\frac{\hbar^2}{m}\nabla_{\mathbf{s}}^2\kappa\right) \tag{16}$$

$$\imath\hbar\frac{\partial \mathbf{j}_\kappa}{\partial t} = \left(-\frac{\hbar^2}{4m}\nabla^2 + V_T + g\rho_T - \mu_T\right)\mathbf{j}_\kappa + \lim_{s\to 0}\left(-\frac{\hbar^2}{m}\nabla_{\mathbf{s}}^2\mathbf{j}_\kappa\right) \tag{17}$$

$$+ \frac{\hbar}{2m\imath}\nabla(V_+ + g\rho_- - V_- - g\rho_+) - g\mathbf{j}_T\kappa$$
$$- g\frac{\hbar}{2m\imath}(\rho_- - \rho_+)\nabla\kappa \tag{18}$$

$$\imath\hbar\frac{\partial\tau_\kappa}{\partial t} = \left(-\frac{\hbar^2}{4m}\nabla^2 + V_T + g\rho_T - \mu_T\right)\tau_\kappa + \lim_{s\to 0}\left(-\frac{\hbar^2}{m}\nabla_{\mathbf{s}}^2\tau_\kappa\right)$$
$$- \imath\hbar\nabla(V_+ + g\rho_- - V_- - g\rho_+)\cdot\mathbf{j}_\kappa + \imath\hbar g\nabla\kappa\cdot(\mathbf{j}_- - \mathbf{j}_+)$$
$$- g\tau_T\kappa - \frac{\hbar^2}{4m}[\nabla^2(V_T + g\rho_T)\kappa - g\rho_T\nabla^2\kappa] \tag{19}$$

In (15), we have introduced the energy flow vector $\mathbf{j}_\tau = (\hbar/m\imath)\lim_{s\longrightarrow 0}\nabla\tau$. In fact, an infinite hierarchy of higher gradients of both the particle and the pair densities is created by the limiting process $\lim_{s\longrightarrow 0}$, starting with (4). This hierarchy is equivalent to an expansion of the matrix EOM's around $\mathbf{r} = \mathbf{r}'$; by doing so, we are respecting to a larger extent the matrix structure of quantum mechanics, somehow disregarded in the standard LDA. One can also consider that a gradient/momentum expansion can be viewed as an expansion in powers of $\hbar$, which certainly carries one beyond the classical limit; it is worthwhile to note that quantum hydrodynamics (see i.e., Refs. [3, 16]) incorporates just the particle density and the superfluid velocity into the theoretical frame. The set (13) to (19) can be closed by a truncation, replacing the given limits by some local functions in terms of a selected equation of state. This procedure is hereafter referred to as ESTF.

3 The Papenbrock-Bertsch Method

In this Section we present the various integrals for the homogeneous system, with pair interactions, employing the dimensional regularization method developed in Ref. [25], here denoted as PB. In the PB method, one writes the particle and pair densities, and related quantities like currents, kinetic energies and successive gradients as summations over single-particle orbitals $|\mathbf{k}\sigma\rangle$ employing the stationary amplitudes of the quasiparticle transformation (2), with $\mathrm{u}_{\mathbf{k}}(\mathbf{r}) = \mathrm{u}_{\mathbf{k}}e^{\imath\mathbf{k}\cdot\mathbf{r}}$ and $\mathrm{v}^*_{\mathbf{k}}(\mathbf{r}) = \mathrm{v}_{\mathbf{k}}e^{\imath\mathbf{k}\cdot\mathbf{r}}$ giving the particle velocity $\mathbf{U}_{\mathbf{k}}(\mathbf{r}) = \hbar\mathbf{k}/m$.

The BCS theory for asymmetric matter is well known in nuclear physics [28–30]; the occupation probability is $|\mathrm{v}_{\mathbf{k}}|^2 = 1 - |\mathrm{u}_{\mathbf{k}}|^2 = 1/2[1 - (\varepsilon + g\overline{\rho} - \overline{\mu})/E_{\mathbf{k}}]$, being $\varepsilon = \hbar^2k^2/2m$ the fermion kinetic energy, $\overline{\rho} = \rho_T/2$, $\overline{\mu} = \mu_T/2$, and $E_{\mathbf{k}} =$

$\sqrt{(\varepsilon + g\overline{\rho} - \overline{\mu})^2 + \Delta^2}$, with $\Delta = -g\kappa$. The quasiparticle energies take the form $E_{\mathbf{k}\sigma} = E_{\mathbf{k}} + \sigma\delta\mu$, with $\delta\mu = (\mu_+ - \mu_-)/2$ the excess chemical potential.

Accordingly, one has the matrices

$$\rho_\sigma(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} [\mathrm{u}_{\mathbf{k}}(\mathbf{r})\mathrm{u}^*_{\mathbf{k}}(\mathbf{r}') f_{\mathbf{k}\sigma} + \mathrm{v}_{\mathbf{k}}(\mathbf{r}')\mathrm{v}^*_{\mathbf{k}}(\mathbf{r})(1 - f_{\mathbf{k}-\sigma})] \tag{20}$$

$$\mathbf{j}_\sigma(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} [\mathrm{u}_{\mathbf{k}}(\mathbf{r})\mathrm{u}^*_{\mathbf{k}}(\mathbf{r}') f_{\mathbf{k}\sigma} + \mathrm{v}_{\mathbf{k}}(\mathbf{r}')\mathrm{v}^*_{\mathbf{k}}(\mathbf{r})(1 - f_{\mathbf{k}-\sigma})]\mathbf{U}_{\mathbf{k}} \tag{21}$$

$$\tau_\sigma(\mathbf{r}, \mathbf{r}') = \frac{\hbar^2}{2m} \sum_{\mathbf{k}} \mathbf{k}^2 [\mathrm{u}_{\mathbf{k}}(\mathbf{r})\mathrm{u}^*_{\mathbf{k}}(\mathbf{r}') f_{\mathbf{k}\sigma} + \mathrm{v}_{\mathbf{k}}(\mathbf{r}')\mathrm{v}^*_{\mathbf{k}}(\mathbf{r})(1 - f_{\mathbf{k}-\sigma})] \tag{22}$$

$$\kappa(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} [\mathrm{u}_{\mathbf{k}}(\mathbf{r})\mathrm{v}^*_{\mathbf{k}}(\mathbf{r}')(1 - f_{\mathbf{k}\sigma}) - \mathrm{u}_{\mathbf{k}}(\mathbf{r}')\mathrm{v}^*_{\mathbf{k}}(\mathbf{r}) f_{\mathbf{k}-\sigma}] \tag{23}$$

and similarly for the pair current and kinetic energy, where $f_{\mathbf{k}\sigma} = 1/(1 + e^{E_{\mathbf{k}\sigma}/T})$ is the occupation number, at temperature T, of a quasiparticle with energy $E_{\mathbf{k}\sigma}$.

These summations are cast into integrals over the energy continuum with a density of states $\nu(\epsilon) = (2m^3)^{1/2}\epsilon^{1/2}/(2\pi^2)$ after the limit $\mathbf{r} \longrightarrow \mathbf{r}'$. In terms of the Fermi momentum $k_\mu = \sqrt{2m(\overline{\mu} - g\overline{\rho})/\hbar^2}$, the number of states per energy interval s

$$\nu(z)dz = \frac{k_\mu^3}{4\pi^2} z^{1/2} dz \tag{24}$$

with $z = \epsilon/(\overline{\mu} - g\overline{\rho})$, so that all single-particle and pair quantities can be expressed in terms of dimensionless energy integrals of the form

$$\int_0^\infty \frac{z^\alpha}{\sqrt{(z-1)^2 + x^2}} = -\frac{\pi}{\sin\alpha\pi}(1 + x^2)^{\alpha/2} P_\alpha\left(-\frac{1}{\sqrt{1 + x^2}}\right) \tag{25}$$

Here $P_\alpha(u)$ is an associated Legendre function, and $x = \Delta/(\overline{\mu} - g\overline{\rho})$. The PB regularization then turns all diverging integrals into analytical expressions by taking the appropriate limit of the parameter α.

These mathematical tricks are applied to every integral representing particle or pair properties in the BCS regime of an homogeneous, symmetric (i.e., $\rho_\pm = \rho_T/2$) superfluid at zero temperature, where the quasiparticle occupation numbers vanish. In particular, we are interested in the particle and kinetic energy densities

$$\begin{aligned}\rho &= -\frac{1}{2}\int_0^\infty \nu(z)dz \frac{z - 1}{\sqrt{(z-1)^2 + x^2}} \qquad (26)\\ &= -\frac{1}{2}\frac{k_\mu^3}{4\pi}(1 + x^2)^{1/4}\left[\sqrt{1 + x^2} P_{3/2}\left(-\frac{1}{\sqrt{1 + x^2}}\right) + P_{1/2}\left(-\frac{1}{\sqrt{1 + x^2}}\right)\right]\\ \tau &= -\frac{1}{2}(\overline{\mu} - g\overline{\rho})\int_0^\infty \nu(z)dz \frac{z(z - 1)}{\sqrt{(z-1)^2 + x^2}}\\ &= \frac{1}{2}\frac{k_\mu^3}{4\pi}(\overline{\mu} - g\overline{\rho})(1 + x^2)^{3/4}\left[\sqrt{1 + x^2} P_{5/2}\left(-\frac{1}{\sqrt{1 + x^2}}\right)\right.\end{aligned}$$

$$+ P_{3/2}\left(-\frac{1}{\sqrt{1+x^2}}\right)\Bigg] \tag{27}$$

In addition, the PB method computes the regular part of the pair density as

$$\kappa = \frac{\Delta}{2(\overline{\mu} - g\overline{\rho})}\int_0^\infty \frac{\nu(z)dz}{\sqrt{(z-1)^2 + x^2}} \tag{28}$$

For any nonvanishing value of Δ, this gives the gap equation

$$\frac{1}{k_\mu a} = (1+x^2)^{1/4} P_{1/2}\left(-\frac{1}{\sqrt{1+x^2}}\right) \tag{29}$$

with a the s-wave scattering length. We then get for the pair kinetic energy density

$$\tau_\kappa = \frac{\Delta}{2}\int_0^\infty \frac{z\nu(z)dz}{\sqrt{(z-1)^2 + x^2}} \equiv \overline{\mu}\kappa \tag{30}$$

All currents vanish in the homogeneous system, like any odd power of gradient operators acting on the densities. The procedure can be applied to integrals containing any power of ε^n, either for particles or for pairs, giving rise to the expression

$$\tau_\sigma^{(n)} = (-)^n \frac{k_\mu^3}{4\pi}(\overline{\mu} - g\overline{\rho})^n (1+x^2)^{n/2+1/4}\Bigg[\sqrt{1+x^2}\, P_{n+3/2}\left(-\frac{1}{\sqrt{1+x^2}}\right)$$
$$+ P_{n+1/2}\left(-\frac{1}{\sqrt{1+x^2}}\right)\Bigg] \tag{31}$$

for particles, and for pairs the recurrence formula

$$\tau_\kappa^{(n)} = -\frac{\Delta}{2}\tau_T^{(n-1)} + \overline{\mu}\tau_\kappa^{(n-1)}. \tag{32}$$

4 Homogeneous Superfluid

In this section we test the ESTF scheme for an homogeneous system of paired fermions, in the absence of external forces, and derive relations among the various magnitudes, denoted by superscripts (h). Assuming equilibrium ($\partial/\partial t = \nabla = 0$) with all currents vanishing, (13) to (19) show the following. First, (15) indicates that $\kappa^{(h)}$ and $\tau_\kappa^{(h)}$ should be real in equilibrium. Equation (16) then gives

$$\lim_{s\to 0}\left(-\frac{\hbar^2}{m}\nabla_{\mathbf{s}}^2\kappa\right) = \mu_T\kappa^{(h)} \tag{33}$$

consistently with (30), since in view of the relation (9), in an homogeneous system the limiting operator is twice the kinetic energy. In turn, (24) gives

$$\lim_{s\to 0}\left(-\frac{\hbar^2}{m}\nabla_{\mathbf{s}}^2\tau_\kappa\right) = (\mu_T - g\rho_T)\tau_\kappa^{(h)} + g\tau_T^{(h)}\kappa^{(h)} \tag{34}$$

as stems from (32) for $n=2$, noting that the limiting operator is just $2\,\tau_\kappa^{(2)}$.

With this in mind, our truncation criterion, aiming at applications to nonhomogeneous superfluids such as trapped gases, consists of (i) substituting the limiting operator in (16) by $2\,\tau_\kappa$; (ii) setting the limiting operator acting on the current in (18) equal to zero and (iii) replacing the limiting term in (19) locally by its expression (34) for an homogeneous system. In this way, the limiting term becomes a unique function of the local density.

Our next test for the validity of these criterion is the examination of fluctuations in an homogeneous system, in order to compare with the RPA predictions [7]. Assuming that all quantities $f(\mathbf{r},t)$ depart from equilibrium by amounts $\delta f(\mathbf{r},t) = \delta f^+ e^{\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \delta f^- e^{-\imath(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, the spatial amplitudes satisfy

$$\pm\hbar\omega\delta\rho^\pm = \pm\hbar\mathbf{k}\cdot\delta\mathbf{j}^\pm \tag{35}$$

$$\pm\hbar\omega\delta\mathbf{j}^\pm = \pm\hbar\mathbf{k}\frac{\rho}{m}\frac{\partial\mu}{\partial\rho}\delta\rho^\pm + 2\Delta(\delta\mathbf{j}_\kappa^\pm - \delta\mathbf{j}_\kappa^{*\mp}) \tag{36}$$

$$\pm\hbar\omega\delta\tau^\pm = \pm\mathbf{k}\cdot\delta\mathbf{j}_\tau^\pm + 2\Delta(\delta\tau_\kappa^+ - \delta\tau_\kappa^{-*}) + 2g\tilde{\tau}_\kappa(\delta\kappa^\pm - \delta\kappa^{*\mp}) \tag{37}$$

$$\pm\hbar\omega\delta\kappa^\pm = \left(\frac{\hbar^2k^2}{4m} - \mu\right)\delta\kappa^\pm + 2\delta\tau_\kappa^\pm \tag{38}$$

$$\pm\hbar\omega\delta\mathbf{j}_\kappa^\pm = \left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)\delta\mathbf{j}_\kappa^\pm + \Delta\delta\mathbf{j}^\pm \tag{39}$$

$$\begin{aligned}\pm\hbar\omega\delta\tau_\kappa^\pm &= \left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)\delta\tau_\kappa^\pm - g\tilde{\tau}\delta\kappa^\pm - g\kappa\delta\tau^\pm \\ &\quad + \left[g\tilde{\tau}_\kappa + \frac{\partial}{\partial\rho}\left[(\mu - g\rho)\tau_\kappa^{(h)} + g\tau^{(h)}\kappa^{(h)}\right]\right]\delta\rho^\pm\end{aligned} \tag{40}$$

with $\tilde{\tau} = \tau + \rho\hbar^2k^2/2m$ and $\tilde{\tau}_\kappa = \tau_\kappa + \kappa\hbar^2k^2/4m$. The equations stand for the total quantities of a symmetric mixture; for the sake of simplicity we remove subscripts T everywhere. Also implicit is the fact that the linearization procedure sets all nonfluctuating quantities at their real equilibrium values. Since the dynamics couples complex fluctuations to their conjugates, one has to consider also the EOM's for the latter, noting that δf^+ and δf^- are different in the general case.

We note that (35), (36) and (39) are decoupled from the rest. This permits us to examine this particular system, together with the respective equations for the amplitudes corresponding to the same phase factor. It is useful to switch variables to even $(\delta f^+ + \delta f^{-*})$ and odd $(\delta f^+ - \delta f^{-*})$ amplitudes and write

$$\hbar\omega\delta\rho_{\text{odd}} = \hbar\mathbf{k}\cdot\delta\mathbf{j}_{\text{odd}} \tag{41}$$

$$\hbar\omega\delta\mathbf{j}_{\text{odd}} = c_s^2\hbar\mathbf{k}\delta\rho_{\text{odd}} \tag{42}$$

$$\hbar\omega\delta\rho_{\text{even}} = \hbar\mathbf{k}\cdot\delta\mathbf{j}_{\text{even}} \tag{43}$$

$$\hbar\omega\delta\mathbf{j}_{\text{even}} = c_s^2\hbar\mathbf{k}\cdot\delta\rho_{\text{even}} + 4\Delta\delta\mathbf{j}_{\kappa,\text{odd}} \tag{44}$$

$$\hbar\omega\delta\mathbf{j}_{\kappa,\mathrm{even}} = \left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)\delta\mathbf{j}_{\kappa,\mathrm{odd}} + \Delta\delta\mathbf{j}_{\mathrm{odd}} \tag{45}$$

$$\hbar\omega\delta\mathbf{j}_{\kappa,\mathrm{odd}} = \left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)\delta\mathbf{j}_{\kappa,\mathrm{even}} + \Delta\delta\mathbf{j}_{\mathrm{even}} \tag{46}$$

The first two members of this array represent a real mode, the sound-like one with $\omega_1 = c_s k$, being $c_s^2 = \rho/m(\partial\mu/\partial\rho)$ the usual sound velocity squared. The remaining four fluctuations are then enslaved by the particle and current density oscillation. Furthermore, if the sound mode is not excited, one gets the solutions

$$\begin{aligned}(\hbar\omega)^2 = {} & \frac{1}{2}\left[\left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)^2 + 4\Delta^2 + (\hbar\omega_1)^2\right] \\ & \pm \frac{1}{2}\sqrt{\left[\left(\frac{\hbar^2k^2}{4m} + g\rho - \mu\right)^2 + 4\Delta^2 + (\hbar\omega_1)^2\right]^2 - (\hbar\omega_1)^4}\end{aligned} \tag{47}$$

At zero momentum, the positive root yields 2 $E_{\mathbf{k}=0}$ as predicted by the RPA [7]. The perturbations of the pair density and the kinetic energies are enslaved by these fluctuations according to (37), (39) and (40) and their complex conjugates.

In addition to these particle density modes, the equations for the pair density and kinetic energies exhibit additional solutions corresponding to vanishing $\delta\rho$. Taking into account that in transport theory, one has for the heat flux the relation $\mathbf{j}_\tau = -K\nabla\tau$, with K a thermal conductivity, the particle kinetic energy undergoes a diffusive behavior, leading to a vanishing $\delta\tau$ at equilibrium. We then see that if $\delta\rho = \delta\tau = 0$, a real gapped mode due to pairing fluctuations exists, determined by (38) and (40). One is left with a simple eigenvalue equation for ω at $k = 0$

$$\hbar\omega = g\overline{\rho} - \mu \pm \sqrt{(g\overline{\rho})^2 - 2g\tau} \tag{48}$$

Taking into account the relation $\tau = 3/5[\rho(\overline{\mu} - g\overline{\rho}) - |\Delta|^2/g]$ that can be extracted from (26) and (27), the above gapped eigenvalues read

$$\hbar\omega = g\overline{\rho} - \mu \pm \sqrt{\frac{17}{5}(g\overline{\rho})^2 - \frac{6}{5}g\rho\overline{\mu} + \frac{6}{5}|\Delta|^2} \tag{49}$$

We now observe that the traditional formulation of the RPA for the superconducting electron gas as presented by Anderson [7] follows form the derivation and analysis of EOM's for fluctuation operators $\rho^{\mathbf{q}}_{\mathbf{k}\sigma} = c^\dagger_{\mathbf{k}+\mathbf{q},\sigma}c_{\mathbf{k}\sigma}$ and $\kappa^{\mathbf{q}}_{\mathbf{k}\sigma} = c_{\mathbf{k}+}c_{-\mathbf{k}-\mathbf{q},-}$, with $c^\dagger, c$ particle creation and annihilation operators in momentum–spin representation. The procedure establishes EOM's for these fluctuations, driven by a standard BCS Hamiltonian; the case $\mathbf{q} = 0$ corresponding to the particle and pair densities naturally decouples from the system, and the stable solutions coincide with the BCS ones. These EOM's for the above fluctuations and their Hermitian conjugates possess two kind of equilibrium solutions. If terms containing fluctuations of the Hamiltonian are disregarded, the excitation spectrum is particle-like and reproduces the BCS quasiparticle spectrum. If the fluctuations in the Hamiltonian are kept, under assumptions valid in the weak coupling limit it is shown that there appear two formal solutions representing collective excitations: one branch is the well known Anderson-Bogoliubov longitudinal mode that coincides with collisionless sound in a neutral Fermi gas, the other branch lies in the quasiparticle continuum, near or above the superfluid gap.

In the ESTF frame, one has to keep in mind that the existence of the Fourier components $f_{\mathbf{k}}$ of the six quantities involved make room to the existence of off-diagonal fields $f(\mathbf{r},\mathbf{r}') \equiv f(\mathbf{s})$. It is possible to verify explicitly that the off-diagonal EOM's (9) and (11) for $\rho_\sigma(\mathbf{s})$ and $\delta\kappa(\mathbf{s})$ give rise to the BCS equilibrium solutions. Disturbances of the equilibrium configuration that involve spatial perturbations contain the sound-like oscillation and the BCS quasiparticle excitations. The ESTF formalism show that these modes are compatible with a static pair density, since the pair condensate only participates with the pair current, which is driven by potential flow, as seen in (46).

The appearance of the gapped modes, namely the pairing vibrations, is due to the incorporation of the particle kinetic energy. Equation (48) shows that the gap dependence of the latter in equilibrium is in charge of these high energy excitations, and explains the failure of the standard TF + BCS approach in accounting for these massive modes. The current ESTF description overcomes the limitation in earlier approaches [21–23] where the hierarchy was truncated at a lower level.

5 The Confined Superfluid

In most cold atoms experiments the superfluid is confined by an approximately parabolic trap. In this case the coupled equations (13)–(19) cannot be solved analytically but are amenable to a numerical treatment. The inhomogeneous equilibrium states are found by imposing the vanishing of the time derivatives. The resulting system of equations for vanishing stationary currents can be split into a coupled system for κ and τ_κ

$$0 = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_T - \mu\right)\kappa + 2\tau_\kappa \tag{50}$$

$$\begin{aligned} 0 &= \left(-\frac{\hbar^2}{4m}\nabla^2 + V_T + g\rho - \mu\right)\tau_\kappa - g\tau_\kappa - \frac{\hbar^2}{4m}[\nabla^2(V_T + g\rho)\kappa - g\rho\nabla^2\kappa] \\ &\quad -\left[(\mu^{(h)}[\rho] - g\rho)\tau_K^{(h)}[\rho] + g\tau^{(h)}[\rho]\kappa^{(h)}[\rho]\right] \end{aligned} \tag{51}$$

where τ has been approximated in a LDA fashion as $\tau^{(h)}[\rho]$ with $\tau^{(h)}$ given by (27). The density profile and chemical potential are obtained from the EOS $\mu^{(h)}[\rho] = \mu - V_T(\mathbf{r})$ by requiring the density to be normalized to N.

In Fig. 1 we show the results for the inhomogeneous gaps and τ_κ at different levels of approximations. In particular we compare a pure LDA approximation for the homogeneous gap in the PB approach, with the STF and ESTF approximations. As seen in the gap profiles, the oscillations around the trap center are strongly reduced in the ESTF as compared to the STF, while the largest differences are found around the point of maximum gradient in the gap. Consistently with the approximation scheme

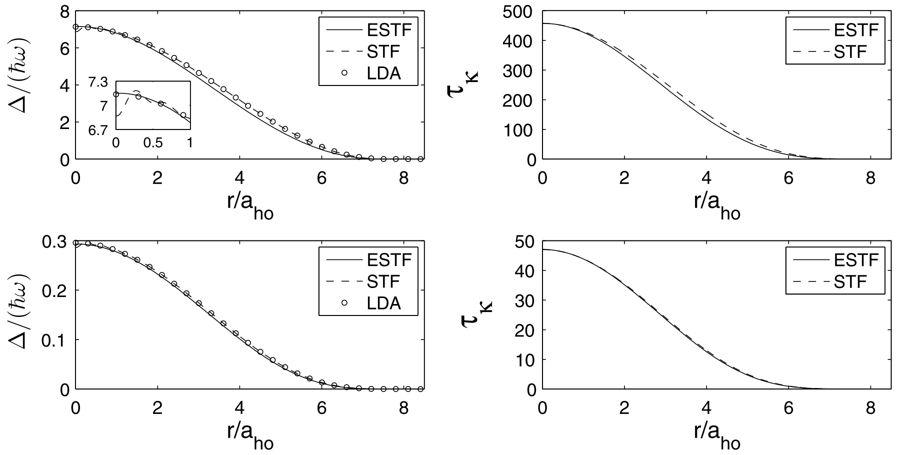

Fig. 1 Comparison of the gap profile Δ (*left column*) and τ_κ (in arbitrary units, *right column*) as calculated in the LDA, STF and ESTF (this article) approximations. The *top* and *bottom rows* correspond to $a = -114$ and -50 nm respectively

the corrections are smaller for the lower interaction strength. Most interesting to note is the validity of our previous approximation for τ_κ, since even for the highest interaction strength the plain LDA compares very well to the ESTF values.

The collective fluctuations in the confined superfluid can be also split as for the homogeneous fluid; in this case, the fluctuations of scalar quantities such as particle, gap and energy density profiles can be written as $\delta g(\mathbf{r}) = g(r)Y_{lm}(\hat{r})$ with Y_{lm} a spherical harmonic function, while currents are expanded in terms of the vector spherical harmonic functions [32] Y^L_{lm} as $\delta\mathbf{j} = f_1(r)\mathbf{Y}^{l+1}_{lm}(\hat{r}) + f_0(r)\mathbf{Y}^{l}_{lm}(\hat{r}) + f_{-1}(r)\mathbf{Y}^{l-1}_{lm}(\hat{r})$. The 3D coupled eigenvalue equations can then be cast into a system of 1D coupled equations for the radial amplitudes of a given multipolarity l analogous to (41)–(46), with the wavevector $\mathbf{k}$ replaced by the $-i\nabla$ operator and the chemical potential μ and sound velocity c_s replaced by their local values $\mu - V_T(\mathbf{r})$ and $c_s(r)$.

The collective modes can be separated as before in density modes (with induced pairing) and pure pairing fluctuations. Here we anticipate results for the density modes of our extended approach in a spherical trap. A more thorough analysis including the pairing fluctuations will be given elsewhere [31]. The frequency spectrum of density modes can be separated in two decoupled blocks: one corresponding to the usual collisionless sound spectrum for the odd components and the other frequency block containing the pair currents. For vanishingly small gap, i.e., in the weak-coupling BCS limit, both energy blocks are degenerate and the pair current is negligible. In the general case, even the density fluctuations in the usual sound deviate from the standard hydrodynamic result due to the coupling to the pairing gap. In Fig. 2 we compare the low-energy density fluctuations $\delta\rho$ with l equal to 0 and 2 for two values of the scattering length, $a = -114$ and -50 nm. We note that the real part of the density fluctuation, i.e. the physical one, is proportional to the even density fluctuation only, so that $\delta\rho = \delta\rho_{\text{even}}$. In addition to the shorter spatial extent of the fluctuation for the strongest interaction, the figure shows that the differences

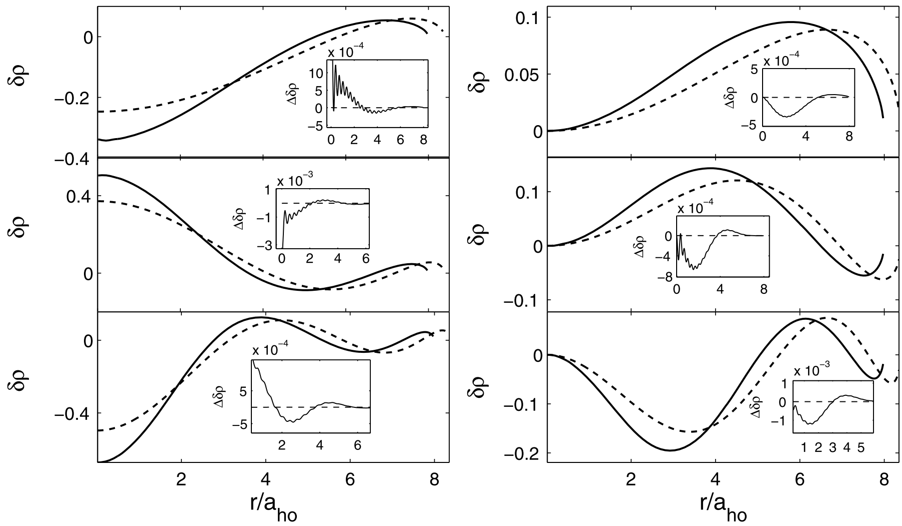

Fig. 2 Low-energy density fluctuations (in arbitrary units) with $l = 0$ (*left column*) and 2 (*right column*) as functions of r (in units of $a_{\rm ho}$). The *solid* and *dashed lines* correspond to mixtures with $a = -114$ nm and $a = -50$ nm, respectively. The *insets* show the difference between the STF and ESTF results for the fluctuation

between the STF and ESTF are noticeable for the largest coupling and near the trap center.

For the parameters considered the ESTF produces qualitatively the same density fluctuations as the standard TF or STF approaches; the newest feature is that it gives us access to the spatial profile of the pair current fluctuation. In Fig. 3 we show the radial amplitude of the pair current for $l = 0$.

As shown by (14) the particle dynamics is directly coupled to the imaginary part of the pair current and thus to the odd fluctuation $\delta\mathbf{j}_{\kappa,\rm odd}$, while the even fluctuation $\delta\mathbf{j}_{\kappa,\rm even}$ enters at a second order in Δ through its coupling with the odd part. We find that the amplitude of the odd component may be an order of magnitude larger than the even component, as a result of the direct coupling. Equations (44)–(46) indicate that the equilibrium gap provides the intensity of the current-current coupling; indeed, the short wavelength oscillations in both panels of Fig. 3 resemble those at the center of the trap in Fig. 1 and stem from the Laplacian terms in (45)–(46). These rapid spatial oscillations become damped as the gap decreases.

To the best of our knowledge, the meaning of the pair current has not been put forward in the literature. A microscopic analysis based on the action of the current operator on the pair density (cf. (31)) can relate this current to a local internal angular momentum of the paired fermions, with averages taken with respect to the pair density [19]. Such a quantity vanishes in local equilibrium, thus its fluctuation—in the present case, induced by a sound-like density perturbation—might be traced to a local microscopic vorticity capable of coupling to a macroscopic rotational velocity field. In other words the coupled pairs could behave as scattering centers for vortex lines. This possibility opens interesting perspectives from the experimental side, since the nucleation and stability of vortices in trapped superfluids could perhaps ex-

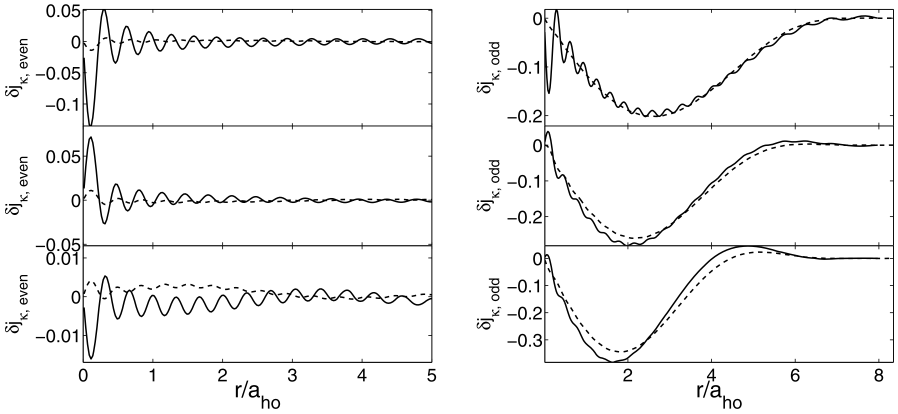

Fig. 3 Pair current fluctuations (in arbitrary units) for the monopolar density fluctuations depicted in Fig. 2. The *left* and *right columns* correspond to $\delta\mathbf{j}_{\kappa,\mathrm{even}}$ and $\delta\mathbf{j}_{\kappa,\mathrm{odd}}$ respectively

hibit some traces of such a coupling. We intend to explore these questions further in a nextcoming investigation.

It is important to mention that the fluiddynamical description is not related in an obvious manner to either standard quantum hydrodynamics—where the leading quantities are the particle density and the gradient of the phase of the condensate wave function—or to the Gross-Pitaevskii EOM (see i.e., Refs. [3] and [16]). In particular, the pair current $\mathbf{j}_\kappa$ is not the superfluid current; while the latter derives from the gradient of the superfluid order parameter—say κ—, one can easily realize (see e.g., (20) to 23) that the former is constructed with the gradients of the particle wave functions entering the microscopic structure of the pair density.

Finally, we would like to comment on the lowest dipolar mode, given than a branch of the spectrum is determined solely by the equations for the confined superfluid analogous to (41) and (42), there is a mode at $\omega = \omega_0$ as in the LDA and STF approaches corresponding to a generalized Kohn mode [33]. However, at variance of the LDA and STF approximations due to the coupling to the pair current fluctuation the density fluctuation is not exactly proportional to a rigid translation of the particle density and deviates from the derivative of the equilibrium density. Nonetheless for moderate values of the gap profile as considered here these deviations can be safely neglected as shown in the inset of Fig. 4.

6 Summary

In this paper we have derived in detail the FD formulation for a fermionic superfluid, complementing the mass, momentum and energy conservation laws of standard hydrodynamics with EOM's for the pair density, pair current density and pair kinetic energy of the system with two species. The present treatment overcomes the most serious limitation of the simpler STF approach where the moment hierarchy of the generalized density matrix of superfluids was truncated at a lower level [21], namely, the

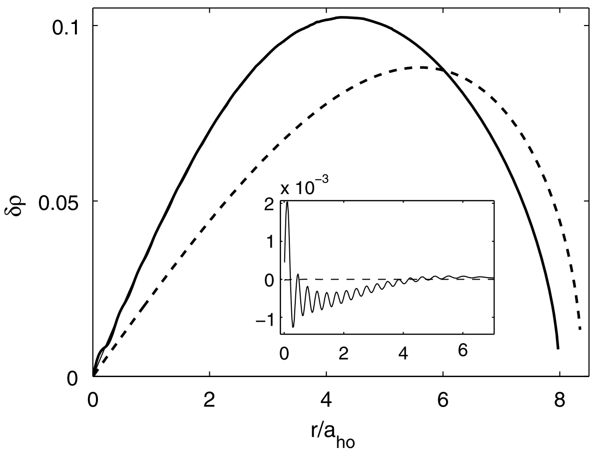

Fig. 4 Comparison of the dipolar density mode at $\omega = \omega_0$ for $a = -114$ (*solid lines*) and -50 (*dashed lines*) nm. The inset shows the difference between the ESTF and STF approximations for each value of a

nonappearance of a gapped density fluctuation originating in the internal oscillations of pairs. We have shown that in a homogeneous fermion superfluid, the particle kinetic energy—disregarded in our previous calculations [21–23]—estimated through a local EOS according to the proposal of Ref. [25] is the agent of the incorporation of the gap in the structure of the mode frequency at zero momentum. The spectrum of a homogeneous superfluid so derived is then in a one-to-one correspondence with the original RPA one [7–9].

The advantage of our so-called ESTF treatment is the formulation in terms of macroscopic fields for all conserved quantities and their fluctuations, rather than on single particle wave functions. In addition to facilitating numerical estimates, a macroscopic view can be helpful for comprehension of many observable features of trapped cold gases. This has been tested for the case of a sample of Li atoms in a parabolic trap. The equilibrium configurations for density and gap profiles calculated in the ESTF frame are very similar to those in the previous STF one, assessing the validity of the latter, much simpler on computational grounds. More important differences arise in the deviations from equilibrium in the trap. The classification of spectral branches is the same as for a homogeneous superfluid, so that collisionless sound is a leading density fluctuation, capable of inducing perturbations in the particle and pair current densities—apart from kinetic energies and pair fluctuations, not contemplated in the present study.

To summarize, we believe that our approach is preferable to previous one commented here, like standard LDA, since it permits, for example, to identify the particle kinetic energy as an agent in the emergence of the pairing vibration, to associate the first moment of the anomalous density with the internal motion of the pair, and to treat small amplitude density and pairing vibrations on an equal footing. We note that the pair current density is a quantity not yet investigated, whose fluctuations may show up in relation to nucleation and stabilization of macroscopic vorticity in trapped superfluids. The concept of an intrinsic pair angular momentum, and its coupling to the vorticity field is well-known in the hydrodynamic theory of superfluid 3He [19], however in such a context this quantity is related to the textures of an anisotropic superfluid. We believe that this topic is worth being pursued, since it may help to develop important details of the structure and dynamics of paired fermions in quantum liquids and trapped gases.

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