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On some subvarieties of *IMTn*-algebras and the partitions of the m-cube

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Abstract. The aim of this paper is to study the variety $\mathbb{IMT}n$ and some of its subvarieties. After proving some general properties of $IMTn$-chains, we characterize $IMT4$-chains and we provide a complete description of some subvarieties of $\mathbb{IMT}4$. We establish a relationship between totally symmetric partitions of the m-cube and $IMT4$-chains, and as a consequence of this relationship, we give a procedure to construct all finite $IMT4$-chains.

1. Introduction and preliminaries

A *bounded, integral, residuated, lattice-ordered, commutative monoid,* or *bounded residuated lattice* for short, is an algebra $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that $\langle A, *, \top \rangle$ is a commutative monoid, $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice with greatest element $\top$ and least element $\bot$, and the residuation condition

$$a * b \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c \tag{1.1}$$

holds for any $a, b, c \in A$, where $\leq$ is the order given by the lattice structure.

On a residuated lattice $\mathcal{A}$, we consider the unary operation $\neg x := x \rightarrow \bot$ for all $x \in A$.

An *MTL-algebra* is a bounded residuated lattice satisfying the pre-linearity equation

$$(x \rightarrow y) \vee (y \rightarrow x) \approx \top, \tag{1.2}$$

and an *IMTL-algebra* is an *MTL*-algebra satisfying the involutive equation

$$\neg\neg x \approx x. \tag{1.3}$$

The variety of *IMTL*-algebras is represented by $\mathbb{IMTL}$.

The variety of *MTL*-algebras was introduced by F. Esteva and L. Godo in 2001 as the algebraic counterpart of the logic of all left-continuous t-norms and their residua [2]. This variety contains the variety of *BL*-algebras (introduced by Hájek in 1998), corresponding to the basic fuzzy logic, and the variety

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of *IMTL*-algebras, the algebraic counterpart of *IMTL* logic or involutive t-norm logic, which is the sentential logic with completeness relative to the class of involutive t-norms. An interesting problem is to determine and classify the axiomatic extensions of *BL* logic and *IMTL* logic, which is equivalent to determining the subvarieties of the equational class of *BL*-algebras and *IMTL*-algebras, respectively, and there is a considerable work done in the literature in that sense (see for instance [1, 3, 5, 6, 7, 11, 12]). This paper is a contribution to the study of some subvarieties of a particular subvariety of $\mathbb{IMTL}$, namely the variety $\mathbb{IMT}n$ of all *IMTL*-algebras that satisfy, for $n \in \omega$, the equation

$$\neg\, x^n \vee x \approx \top. \tag{1.4}$$

The study of the subvarieties of $\mathbb{IMT}n$ was initiated by J. Gispert and A. Torrens in [5], where they characterize and classify the subvarieties of the variety $\mathbb{IMT}3$, and that work is the starting point of the present paper.

This paper is organized as follows. In Section 2, we prove some general properties of the variety $\mathbb{IMT}n$. The complexity of the problem of characterizing the subvarieties of $\mathbb{IMT}n$ is reflected in Section 3, where we study the particular subvariety $\mathbb{IMT}4$. We give a description of $IMT4$-chains and we provide a complete description of several subvarieties of $\mathbb{IMT}4$. In Section 4, we establish a bijective correspondence between the finite chains in $\mathbb{IMT}4$ and the totally symmetric partitions of the m-cube. This correspondence allows us to determine the number of $IMT4$-structures that can be defined over a finite chain.

We assume that the reader has some familiarity with residuated lattices. We recommend [4] and [9] and the references given there.

In the next lemma, we list, for further reference, some well-known properties.

Lemma 1.1. *The following properties hold true in any residuated lattice $\mathcal{A}$, where a, b, c denote arbitrary elements of A:*

(1) $a \leq b$ *if and only if* $a \to b = \top$,
(2) *if* $a \leq b$, *then* $\neg\, b \leq \neg\, a$,
(3) $a * \neg\, a = \bot$,
(4) *if* $a \leq b$, *then* $a * c \leq b * c$,
(5) $a \to (b \to c) = (a * b) \to c$,
(6) $a * (a \to b) \leq b$.

The following lemma will be used throughout this paper.

Lemma 1.2. *Let $\mathcal{A} = \langle A, \wedge, \vee, *, \neg, \bot, \top\rangle$ be an algebra that satisfies the following properties:*

(1) $\langle A, \wedge, \vee, \bot, \top\rangle$ *is a bounded lattice,*
(2) $\langle A, *, \top\rangle$ *is a commutative monoid,*
(3) $\neg\neg\, a = a$ *for all* $a \in A$,
(4) *if* $a \leq b$, *then* $\neg\, b \leq \neg\, a$ *for all* $a, b \in A$,

(5) $a \leq \neg\, b$ *if and only if* $a * b = \perp$ *for all* $a, b \in A$.

Then $\mathcal{A} = \langle A, \wedge, \vee, *, \neg, \perp, \top\rangle$ *is a bounded residuated lattice, with* $x \to y := \neg\,(x * \neg\, y)$.

Proof. We define in A the implication $x \to y := \neg\,(x * \neg\, y)$. In order to prove that A is a residuated lattice, it is enough to check that

$$a * b \leq c \quad \text{if and only if} \quad a \leq b \to c.$$

First, we are going to see that $a \leq b$ if and only if $a \to b = \top$ for all $a, b \in A$. Let $a, b \in A$. If we assume that $a \leq b$, then by (3), we have that $a \leq \neg\,\neg\, b$. By (5), $a * \neg\, b \leq \perp$. Using (4), we have that $\neg\,(a * \neg\, b) \geq \neg \perp = \top$. Hence, $a \to b = \top$. The proof is similar for the converse.

To prove the desired condition, it is enough to observe that

$$\begin{aligned}(a * b) \to c &= \neg\,((a * b) * \neg\, c) = \neg\,(a * (b * \neg\, c)) = \neg\,(a * \neg\,\neg\,(b * \neg\, c)) \\ &= a \to (\neg\,(b * \neg\, c)) = a \to (b \to c).\end{aligned}$$

□

Observe that the operation $\to$ is uniquely determined by $\neg$ and $*$.

In this work, $\mathcal{C} = \langle C, \wedge, \vee, *, \to, \perp, \top\rangle$ will denote an *IMTL*-chain (or *IMTL*-algebra totally ordered by the lattice partial order). Observe that C contains at most one element satisfying the condition $\neg\, a = a$; this unique element will be represented by 0, whenever it exists. We consider the sets $C_+ = \{a \in C : a > \neg a\}$ and $C_- = \{a \in C : a \leq \neg\, a\}$ of positive and negative elements, respectively (see [11, Definition 3]). If $S = \{s \in C_+ : s \neq \top\}$ and $\neg\, S = \{s \in C_- : s \neq \perp, 0\}$, then either $C = \{\perp\} \cup \neg\, S \cup \{0\} \cup S \cup \{\top\}$ or $C = \{\perp\} \cup \neg\, S \cup S \cup \{\top\}$. By Lemma 1.1, for all $r, s \in S$, $\neg\, s * \neg r = \perp$, and we have that $s * \neg\, r = \perp$ if and only if $s \leq r$. If $0 \in C$, then $0^2 = \perp$ and $\neg\, s * 0 = \perp$. In particular, the product on an *IMTL*-chain $\mathcal{C}$ is completely determined by its values on $S \cup \{0\}$.

2. The variety $\mathbb{IMT}n$

In this section, we prove some general properties of the variety $\mathbb{IMT}n$. Recall that the variety $\mathbb{IMT}n$ is the subvariety of $\mathbb{IMTL}$ characterized by the identity $\neg\, x^n \vee x \approx \top$.

The class $\mathbb{IMT}n$ is a discriminator variety [8]. As in [5], if we take the term $\eta(x, y) = ((x \to y) * (y \to x))^n$, then the sentence

$$\forall x \forall y(((x \approx y) \Leftrightarrow \eta(x, y) = \top)\&((x \not\approx y) \Leftrightarrow \eta(x, y) = \perp))$$

holds in all *IMTn*-chains, and hence the interpretation of the term

$$t(x, y, z) = (\eta(x, y) \wedge z) \vee (\neg\eta(x, y) \wedge x)$$

gives a discriminator function in each *IMTn*-chain. Hence, subdirectly irreducible algebras in $\mathbb{IMT}n$ are all simple, and so each *IMTn*-algebra is semisimple. Moreover, since any *IMTL*-algebra is isomorphic to a subdirect product of *IMTL*-chains, then the variety $\mathbb{IMT}n$ is generated by *IMTn*-chains.

We can generalize [5, Lemma 2.4] as follows.

Lemma 2.1. *If $\mathcal{C}$ is an IMTn-chain with $|C| \geq 4$, then for all $a_1, a_2, \ldots, a_{n-1}$ in $C \setminus \{\top\}$,*

$$a_1 * a_2 * \cdots * a_{n-1} \neq \bot \quad \textit{implies} \quad a_1 * a_2 * \cdots * a_{n-1} = \min(C \setminus \{\bot\}).$$

The element $\min(C \setminus \{\bot\})$ will be represented by -1.

From Lemma 2.1, $\max S$ exists and we write $1 = \max S$. It is clear that $\neg 1 = -1$.

Let us write $\mathbb{IMT}n^-$ to denote the class $\mathbb{IMT}n \setminus \mathbb{IMT}(n-1)$. If $\mathcal{C} \in \mathbb{IMT}n^-$, then $1^n = \bot$ and $1^{n-1} = -1$. Indeed, the first identity is a consequence of (1.4), and since $1^{n-1} * 1 = 1^n = \bot$, we have that $1^{n-1} \leq \neg 1 = -1$. But $1^{n-1} \neq \bot$ by hypothesis, so $1^{n-1} = -1$.

The next lemma shows that in an *IMTn*-chain, the powers of 1 give us some information about the product operation. For example, for $\mathcal{C} \in \mathbb{IMT}n^-$, if n is even, then $1^{\frac{n}{2}}$ is in the lower half of the chain, whereas $1^{\frac{n}{2}-1}$ is in the upper half of the chain. A similar result holds for n odd and the elements $1^{\frac{n+1}{2}}$ and $1^{\frac{n-1}{2}}$. In addition, if $r, s \in S$ but they are smaller than $1^{\frac{n-1}{2}}$, then $s * r = -1$.

Lemma 2.2. *If $\mathcal{C} \in \mathbb{IMT}n^-$, then*

(1) *For n even, $1^{\frac{n}{2}} \leq \neg 1^{\frac{n}{2}}$ and $\neg 1^{\frac{n}{2}-1} < 1^{\frac{n}{2}-1}$.*
(2) *For n odd, $1^{\frac{n+1}{2}} < \neg 1^{\frac{n+1}{2}}$, $\neg 1^{\frac{n-1}{2}} < 1^{\frac{n-1}{2}}$, and if $r, s \in S, s \leq 1^{\frac{n-1}{2}}$, and $r \leq 1^{\frac{n-1}{2}}$, then $s * r = -1$.*
(3) *For $1 < m < n$, $1^{n-m} \leq \neg 1^m < 1^{n-m-1}$.*

Proof. (1): If n is even, $\bot = 1^n = 1^{\frac{n}{2}} * 1^{\frac{n}{2}}$, and so $1^{\frac{n}{2}} \leq \neg 1^{\frac{n}{2}}$. If we suppose that $\neg 1^{\frac{n}{2}-1} \not< 1^{\frac{n}{2}-1}$, then $1^{\frac{n}{2}-1} \leq \neg 1^{\frac{n}{2}-1}$, and by Lemma 1.1, we have $1^{n-2} = \bot$, and then $\mathcal{C} \in \mathbb{IMT}(n-2)$, a contradiction.

(2): If n is odd, $\bot = 1^n = 1^{\frac{n-1}{2}} * 1^{\frac{n+1}{2}}$, and then $1^{\frac{n-1}{2}} \leq \neg 1^{\frac{n+1}{2}}$. Since $1^{\frac{n+1}{2}} < 1^{\frac{n-1}{2}}$, it follows that $1^{\frac{n+1}{2}} < \neg 1^{\frac{n+1}{2}}$.

In order to prove that $\neg 1^{\frac{n-1}{2}} < 1^{\frac{n-1}{2}}$, suppose that $\neg 1^{\frac{n-1}{2}} \not< 1^{\frac{n-1}{2}}$. Then $1^{\frac{n-1}{2}} \leq \neg 1^{\frac{n-1}{2}}$. So $-1 = 1^{n-1} = 1^{\frac{n-1}{2}} * 1^{\frac{n-1}{2}} = \bot$, a contradiction.

From $s \leq 1^{\frac{n-1}{2}}$ and $r \leq 1^{\frac{n-1}{2}}$, we have $s * r \leq 1^{\frac{n-1}{2}} * 1^{\frac{n-1}{2}} = 1^{n-1} = -1$. As $r, s \in S$ and $s * r \neq \bot$, so we have that $s * r = -1$.

(3): The proof is similar. □

We shall write $a \prec b$ (or $b \succ a$) whenever $a < b$ and there is no element c in C with $a < c < b$. For any real number α, we write $[\alpha]$ to denote the greatest integer which is less than or equal to α.

Remark 2.3. Let $\mathcal{C} \in \mathbb{IMT}n^-$ such that $1^k \prec 1^{k-1} \prec \cdots \prec 1^2 \prec 1$, $k \leq [\frac{n-1}{2}]$; then $-1 = 1^{n-1} \prec 1^{n-2} \prec \cdots \prec 1^{n-k+1} \prec 1^{n-k}$. Indeed, if for some t with $1 \leq t < k$, we have $1^{n-t} < a < 1^{n-t-1}$, then $\neg 1^{n-t-1} < \neg a < \neg 1^{n-t}$, and since $1^{n-t-1} * 1^{t+1} = \bot$, we have that $1^{t+1} \leq \neg 1^{n-t-1}$, so $1^{t+1} < \neg a$, a contradiction.

The converse of the previous remark does not hold, as the following example shows.

Example 2.4. Let $\mathcal{C} = \langle C, \wedge, \vee, *, \neg, \perp, \top\rangle \in \mathbb{IMT}4^-$, such that $1 * \neg 1^2 = 1^3 = (\neg 1^2)^2 = \neg 1$, see Figure 1.

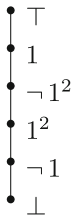

FIGURE 1

We have that $1^3 \prec 1^2$ $(1^{4-1} \prec 1^{4-2})$, but $1^2 \not\prec 1$.

Observe that for each $k \geq n+1$ for n odd, there exists a unique k-element chain in $\mathbb{IMT}n$ with $1^{\frac{n-1}{2}} \prec 1^{\frac{n-3}{2}} \prec \cdots \prec 1^2 \prec 1$, since by Lemma 2.2 (b), the product $s * r$ is uniquely determined for $s, r \leq 1^{\frac{n-1}{2}}$.

Proposition 2.5. *If $\mathcal{C} \in \mathbb{IMT}n$, $|C| \geq 4$, $a, b \in C$, and $a \prec b$, then $\neg a * b = -1$.*

Proof. Since $a < b$, we have that $\neg a * b \neq \perp$. Let us see that $\neg a * b \leq -1$. First observe that $b * 1 < b$. Indeed, we know that $b * 1 \leq b$. If we suppose that $b * 1 = b$, then $b * 1^2 = b * 1 = b$, and so $b * 1^n = b * 1^{n-1} = \cdots = b$, so $b = \perp$, which is not possible since $a \prec b$. Now, from $b * 1 < b$, since $a \prec b$, it follows that $b * 1 \leq a$. Then $\neg a * b * 1 \leq \neg a * a = \perp$, and this implies that $\neg a * b \leq -1$. □

Corollary 2.6. *Let $\mathcal{C} \in \mathbb{IMT}n$ with $|C| \geq 4$.*

(1) *If $0 \in C$ and $0 \prec s$, then $0 * s = -1$.*
(2) *If $\neg s \prec s$, then $s^2 = -1$.*

In [5, Lemma 2.7], J. Gispert and A. Torrens proved that the variety $\mathbb{IMT}3$ is locally finite. This result is no longer true for $n \geq 4$, as has been proved in [6, Theorem 4.31], where the authors construct an infinite 2-generated chain. In the following example, we provide an infinite 1-generated $IMT4$-chain.

Example 2.7. Consider the chain $B = \{b_i : i \in \omega\} \cup \{1\}$ ordered by

$$\cdots < b_6 < b_4 < b_2 < b_1 < b_3 < b_5 < \cdots < 1.$$

Let $-B = \{-b : b \in B\}$ and define an order in $D = \{\perp\} \cup -B \cup \{0\} \cup B \cup \{\top\}$ as follows: for $c \in D$, let $\perp \leq c \leq \top$, and for $b_i, b_j \in B$, let $0 < b_i$, $-b_i < 0$, $-1 < -b_i$, $-b_i < b_j$, and let $-b_i \leq -b_j$ if $b_j \leq b_i$. We define on D the operations $\neg$, $*$ and $\rightarrow$.

($\neg$): For all $b \in B$, let $\neg\top = \bot, \neg b = -b, \neg 0 = 0, \neg(-b) = b, \neg\bot = \top$.

($*$): For all $c, d \in D$, $b_i, b_j \in B$, and $e \in D$ with $e \neq \top$, let $c * d = d * c$, $c * \top = c$, $\bot * c = -b_i * -b_j = 0 * 0 = -1 * e = -b_i * 0 = \bot$, $0 * b_i = -1$, $1^2 = b_i * 1 = 0$, $-b_i * b_j = \begin{cases} -1 & \text{if } b_i < b_j, \\ \bot & \text{if } b_j \leq b_i; \end{cases}$

if $i \leq j$, $b_i * b_j = \begin{cases} -b_{j+1} & \text{if } (j \text{ odd and } i \neq j-1) \text{ or } (j \text{ even and } i = j), \\ -b_{j-1} & \text{if } (j \text{ even and } i \neq j) \text{ or } (j \text{ odd and } i = j-1). \end{cases}$

($\to$): For all $c, d \in D$, let $c \to d = \neg(c * \neg d)$.

It is not difficult to check that $\mathcal{D} = \langle D, \wedge, \vee, *, \to, \bot, \top\rangle \in \mathbb{IMT}4$. Observe that for all $n \in \omega$, we have that $b_{n+1} = \neg b_n^2$, and $\mathcal{D}$ is generated by the element b_1.

3. The variety $\mathbb{IMT}4$

The objective of this section is to characterize all the chains of the variety $\mathbb{IMT}4$. We also study some important subvarieties of $\mathbb{IMT}4$ and their lattice of subvarieties.

The following is a typical example of an $IMT4$-chain, in the sense that all chains in $\mathbb{IMT}4$ are of this form.

Example 3.1. Let $\mathbf{S} = \langle S, \ll, 1\rangle$ be a totally ordered set with greatest element 1 and let $\{\bot, \top, 0\} \cap S = \emptyset$.

Step 1. Consider $-S = \{-s : s \in S\}$ and $\Lambda(S) = \{\bot\} \cup -S \cup \{0\} \cup S \cup \{\top\}$.

We define a total order and a negation on $\Lambda(S)$. For all $a \in \Lambda(S)$, and all $r, s \in S$, let $\bot \leq a \leq \top$, $r \leq s$ if $r \ll s$, $0 \leq s$, $-s \leq 0$, $-s \leq r$, $-s \leq -r$ if $r \ll s$. For all $s \in S$, let $\neg\top = \bot$, $\neg s = -s$, $\neg 0 = 0$, $\neg(-s) = s$, $\neg\bot = \top$. Then $\mathbf{\Lambda}(\mathbf{S}) = \langle\Lambda(S), \wedge, \vee, \neg, \bot, \top\rangle$ is a chain with an involutive negation. Now we define a product for all $a, b \in \Lambda(S)$ and $r, s \in S$:

(a) $a * b = b * a$.
(b) $a * \top = a$, $\bot * a = \bot$.
(c) Any product of elements in $-S \cup \{0\}$ gives $\bot$, that is, $-r * -s = 0 * 0 = -s * 0 = \bot$.
(d) $0 * s = -1$, $s \in S$.
(e) The product of an element in $-S$ and an element in S is given by: $-r * s = \begin{cases} -1 & \text{if } r < s, \\ \bot & \text{if } s \leq r. \end{cases}$
(f) The product between elements s and r of S is: $s * r \in -S \cup \{0\}$, such that $(s * r) * t = \bot \Leftrightarrow s * (r * t) = \bot$ must hold for every $t \in S$.

It is easy to see, by Lemma 1.2, that $\mathbf{\Lambda}(\mathbf{S})^* = \langle\Lambda(S), \wedge, \vee, \neg, *, \bot, \top\rangle$ is a chain of $\mathbb{IMT}4$.

Step 2. Observe that in $\mathbf{\Lambda}(\mathbf{S})^*$, if $1^2 < 0$, then $s * r < 0$ for all $s, r \in S$. In this case, if we consider $\Omega(S) = \Lambda(S) \setminus \{0\}$, then $\Omega(S)$ is a subuniverse of $\Lambda(S)$, and we will write $\mathbf{\Omega}(\mathbf{S})^* = \langle \Omega(S), \wedge, \vee, \neg, *, \bot, \top\rangle$ to denote the corresponding subalgebra of $\mathbf{\Lambda}(\mathbf{S})^*$.

If in the previous example, instead of the condition "$s * r \in -S \cup \{0\}$ such that $(s*r)*t = \bot \Leftrightarrow s*(r*t) = \bot$" we put "$s*r = -1$", we have the unique product such that $\mathbf{\Lambda}(\mathbf{S})^*$ is an *IMT*3-chain.

Consider $\mho = \langle\{\frac{1}{k} : k \in \omega, k \geq 1\}, \leq\rangle$, and for $m \in \omega$ with $m \geq 1$, $\mathbf{m} = \langle\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}, \leq\rangle$ where $\leq$ is the usual order. If $S = \emptyset$, $\mathbf{\Omega}(\emptyset)^*$ will be denoted by $\mathbf{\Omega}(\mathbf{0})^*$ and is isomorphic to the two element Boolean algebra. Also, $\mathbf{\Lambda}(\emptyset)^* = \mathbf{\Lambda}(\mathbf{0})^*$ is isomorphic to the three-element MV-chain, and $\mathbf{\Omega}(\mathbf{1})^*$ is isomorphic to the four-element IMT3-chain. If $|C| > 4$, there exists more than one product operation that can be defined to obtain a structure of IMT4-chain on C.

Let us see now that all the chains in $\mathbb{IMT}4$ are as in the previous example.

Let $\mathcal{C}$ be a chain in $\mathbb{IMT}4$. Recall that $1^4 = \bot$ and $1^3 = \bot$ if and only if $\mathcal{C} \in \mathbb{IMT}3$, and so if $s \in S = \{s \in C_+ : s \neq \top\}$, then $s^3 \in \{\bot, -1\}$. In addition, $1^2 \leq \neg 1^2$, and for all $r, s \in S$, $s*r \leq \neg(s*r)$.

Lemma 3.2. *Let $\mathcal{C} \in \mathbb{IMT}4$ with $|C| > 4$. Let $r, s \in S$.*

(1) *If $r < s$, then $\neg r * s = -1$.*
(2) *If $0 \in C$, then $0 * s = -1$.*

Proof. (1): Since $1^2 \leq \neg 1^2$, we have that $1^2 \leq r$. Then $\neg r * 1^2 \leq \neg r * r = \bot$. Therefore, $\neg r * 1 \leq -1$ and consequently $\neg r * 1 = -1$ since $1 \not\leq r$. As well as $s \leq 1$, then $\neg r * s \leq \neg r * 1 = -1$. So $\neg r * s = -1$ because $s \not\leq r$.

(2): From $1^2 \leq \neg 0$, we have $1^2 * 0 = \bot$. Thus, $1 * 0 = -1$. Since $s \in S$, we have that $0 * s = -1$. □

Theorem 3.3. *If $\mathcal{C} \in \mathbb{IMT}4$, then there exists a totally ordered set S such that $\mathcal{C}$ is isomorphic to either $\mathbf{\Lambda}(\mathbf{S})^*$ or $\mathbf{\Omega}(\mathbf{S})^*$.*

Observe that in order to construct a structure of an $\mathbb{IMT}4$-algebra over a totally ordered set, it is enough to define an operation $*$ over the set S that satisfies $s*r \in -S \cup \{0\}$ with $(s*r)*t = \bot \Leftrightarrow s*(r*t) = \bot$ (see Example 3.1).

In the next subsections, we shall study three subvarieties of $\mathbb{IMT}4$ and their corresponding lattices of subvarieties.

3.1. The subvariety $\mathbb{MIMT}4$.

We consider now the subvariety $\mathbb{MIMT}4$ of $\mathbb{IMT}4$ characterized within $\mathbb{IMT}4$ by the equation

$$\neg x^2 \vee (\neg x^2 \to x^2) \approx \top. \tag{3.1}$$

Lemma 3.4. *Let $\mathcal{C} \in \mathbb{IMT}4$ such that $|C| > 4$. The following conditions are equivalent:*

(1) *$\mathcal{C} \in \mathbb{MIMT}4$.*
(2) *$0 \in C$ and $s * r = 0$ for all $s, r \in S$.*

Proof. (2) implies (1): It is clear that if $0 \in C$ and $s * r = 0$ for all $s, r \in S$, then $\mathcal{C}$ satisfies (3.1).

(1) implies (2): Suppose that $\neg a^2 \vee (\neg a^2 \to a^2) = \top$ for every $a \in C$. Since $|C| \geq 5$, there exists $d \notin \{\bot, -1, 1, \top\}$ such that $\neg d \leq d$. If $\neg d^2 = \top$, then $d \leq \neg d$, and thus $d = 0$. If $\neg d^2 \to d^2 = \top$, then $d^2 = 0$. So $0 \in C$. Now let $s, r \in S$ such that $s \leq r$. Then $s^2 \leq s * r$. Since $\neg s^2 \neq \top$, we have that $\neg s^2 \to s^2 = \top$, and consequently $0 = \neg s^2 = s^2 = s * r$. □

Observe that there is just one structure of an $IMTn$-algebra for chains $\mathcal{C}$ with $|C| \leq 4$, namely the 2-element Boolean algebra $\mathbf{\Omega}(\mathbf{0})^*$, the three-element MV-chain $\mathbf{\Lambda}(\mathbf{0})^*$, and the four-element $IMT3$-chain $\mathbf{\Omega}(\mathbf{1})^*$. We have that $\mathbf{\Omega}(\mathbf{0})^*$ and $\mathbf{\Lambda}(\mathbf{0})^*$ belong to $\mathbb{MIMT}4$, whereas $\mathbf{\Omega}(\mathbf{1})^*$ does not.

If $\mathcal{C}$ is a chain with $|C| > 4$, there is just one product $*_0$ that can be defined on C such that $\mathcal{C} \in \mathbb{MIMT}4$; $\mathcal{C}$ is of the form $\mathbf{\Lambda}(\mathbf{S})^{*_0}$, with $s *_0 r = 0$ for all $s, r \in S$.

Observe that $\mathbb{MIMT}4$ is a locally finite variety.

For all $k > 0$, we consider the term $S_k(x_0, \ldots, x_k) = \bigvee_{i<k} (x_i \to x_{i+1})$. The following result follows from [5, Lemma 3.11].

Lemma 3.5. *For each $\mathcal{C} \in \mathbb{IMTL}$, $|C| \leq k$ if and only if $S_k(a_0, \ldots, a_k) = \top$ for all $a_0, \ldots, a_k \in C$.*

It is easy to see that for any $m \in \omega$, $\mathbf{\Lambda}(\mathbf{m})^{*_0}$ is a subalgebra of $\mathbf{\Lambda}(\mho)^{*_0}$. From this and the above, the next result follows immediately, where $\mathbb{V}(K)$ denotes the variety generated by a class K of algebras and $\mathcal{T}$ is the trivial variety.

Theorem 3.6. *The variety $\mathbb{MIMT}4$ is the variety generated by $\mathbf{\Lambda}(\mho)^{*_0}$, and the proper nontrivial subvarieties of $\mathbb{MIMT}4$ are $\mathbb{V}(\mathbf{\Omega}(\mathbf{0})^*)$ and $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_0})$ for $m \in \omega$. They are axiomatized within $\mathbb{MIMT}4$ by the equation $\neg x \vee x \approx \top$ and by $S_{2m+3}(x_0, \ldots, x_{2m+3}) \approx \top$ for $m \in \omega$, respectively. The lattice of subvarieties of $\mathbb{MIMT}4$ is the following $(\omega+1)$-type chain:*

$$\mathcal{T} \subset \mathbb{V}\left(\mathbf{\Omega}(\mathbf{0})^*\right) \subset \mathbb{V}\left(\mathbf{\Lambda}(\mathbf{0})^*\right) \subset \mathbb{V}\left(\mathbf{\Lambda}(\mathbf{1})^{*_0}\right) \subset \cdots \subset \mathbb{MIMT}4 = \mathbb{V}(\mathbf{\Lambda}(\mho)^{*_0}).$$

Remark 3.7. It is known ([5]) that there is just one structure of an $IMT3$-algebra definable on each chain $\Lambda(S)$. Let $*_3$ denote the product operation such that $\mathbf{\Lambda}(\mathbf{S})^{*_3} \in \mathbb{IMT}3$, and where $*$ is any product such that $\mathbf{\Lambda}(\mathbf{S})^* \in \mathbb{IMT}4$. Recall that $*_0$ is the unique product such that $\mathbf{\Lambda}(\mathbf{S})^{*_0} \in \mathbb{MIMT}4$. Then for all $a, b \in \Lambda(S)$, we have $a *_3 b \leq a * b \leq a *_0 b$.

3.2. The subvariety $\mathbb{IIMT}4$.

Recall that $\mathcal{C}$ denotes an IMTL-chain; let $T = \{s \in C : \neg s \leq s, s \neq \top\}$, that is, $T = S \cup \{0\}$ if $0 \in C$, and $T = S$, otherwise. We are going to consider the subvariety of $\mathbb{IMT}4$ generated by the chains $\mathcal{C}$ in which T has least element i and $s * r = \neg i$ for all $r, s \in T \setminus \{i\}$.

Lemma 3.8. *Let $\mathcal{C} \in \mathbb{IMT}4$ such that $|C| \geq 4$. If T has least element i, then $s * i = -1$ for all $s \in S$.*

Proof. From $1^4 = \bot$, we have $1^2 \leq \neg 1^2$, so $\neg 1^2 \in T$. Then $i \leq \neg 1^2$, thus $\bot = 1^2 * i = 1 * 1 * i$. Hence, as $1 * i \neq \bot$, we have that $1 * i = -1$. But $s \leq 1$ for all $s \in S$, and $\neg i < s$, so we get $s * i = -1$. $\square$

Lemma 3.9. *Let $\mathcal{C} \in \mathbb{IMT}4$ with $|C| \geq 5$. The following conditions are equivalent:*

(1) *T has least element i and $s * r = \neg i$ for all $r, s \in T \setminus \{i\}$.*
(2) *$\mathcal{C}$ satisfies the identity*

$$x \vee y \vee z \vee \neg x^2 \vee \neg y^2 \vee [(z \to \neg z) \wedge \neg \eta(z, \neg z)] \vee t_1(x, y, z) \vee t_2(x, y, z) \approx \top \quad (3.2)$$

$$\text{with} \quad t_1(x, y, z) = [(x \to z) \vee (y \to z)] \wedge [(x * y) \to \neg z],$$
$$t_2(x, y, z) = [\neg z \to (x * y)] \wedge [\neg \eta(x * y, \neg z) \vee (z \to (x \wedge y))].$$

Proof. (1) implies (2): Let $a, b, c \in C$. If $\top \in \{a, b, c\}$ and $a \leq \neg a$, $b \leq \neg b$, or $c < \neg c$, it is easy to check (3.2). So we choose $a, b \in S$ and $c \in T$. If we consider $a = i$, by Lemma 3.8, $a * b = i * b = -1$. Note that in this case, $\neg c \geq -1 = a * b$ and $a = i \leq c$. Hence, $t_1(i, b, c) = \top$. For $b = i$, we can prove, in a similar way, that $t_1(a, i, c) = \top$.

Therefore, we can take $a, b \in T \setminus \{i\}$ and $c \in T$. By hypothesis, $a * b = \neg i$. Then $\neg c \leq \neg i = a * b$. If $c = i$, then $c \leq a \wedge b$, and if $c \neq i$, then $\neg c < \neg i = a * b$. Consequently, $t_2(a, b, c) = \top$.

(2) implies (1): Consider $\mathcal{C} \in \mathbb{IMT}4$ that satisfies (3.2). Suppose that $\mathcal{C} \in \mathbb{IMT}3$ and consider $x = y = 1$ and $z = d \in T$ with $d \neq 1$. Since $|C| \geq 5$, we have that $\neg 1^2 \neq \top$. Using that $d \in T$, $(d \to \neg d) \wedge \neg \eta(d, \neg d) \neq \top$. Hence, $t_1(1, 1, d) = \top$ or $t_2(1, 1, d) = \top$. Since $d \neq 1$, so $t_1(1, 1, d) \neq \top$. Therefore, $t_2(1, 1, d) = \top$. Thus, $\neg d \leq -1$, a contradiction. Then $\mathcal{C} \notin \mathbb{IMT}3$. Let $a \in T$ and take $x = y = 1$ and $z = a$. Since $a, 1 \neq \top$, so $1^2 \neq \bot$, $a \geq \neg a$, and $t_1(1, 1, a) \vee t_2(1, 1, a) = \top$. Assume that $t_1(1, 1, a) = \top$. Hence, $1^2 \leq \neg a \leq -1$, and thus $1^3 = \bot$, a contradiction since $\mathcal{C} \notin \mathbb{IMT}3$. Then $t_2(1, 1, a) = \top$. Observe that $\neg a \leq 1^2$. This is equivalent to $a \geq \neg 1^2$, and therefore $\neg 1^2$ is the least element of T.

Let $a, b \in T \setminus \{\neg 1^2\}$. Then we can take $x = a$, $y = b$, and $c = \neg 1^2$. Therefore, $t_2(a, b, \neg 1^2) = \top$. Hence, $1^2 \leq a * b$, and consequently $a * b = 1^2$. $\square$

Let $\mathbb{IIMT}4$ be the subvariety of $\mathbb{IMT}4$ characterized by (3.2) and let $*_i$ be the unique product that can be defined on a chain $\mathcal{C}$ in such a way that $\mathcal{C} \in \mathbb{IIMT}4$.

Observe that if C has a middle element 0, then $s *_i r = 0$ for all $r, s \in S$, and consequently $*_i = *_0$ ($\mathbf{\Lambda}(\mathbf{S})^{*_i} = \mathbf{\Lambda}(\mathbf{S})^{*_0}$). In particular, $\mathbb{MIMT}4 \subseteq \mathbb{IIMT}4$. If C has no 0, then $\mathcal{C}$ is of the form $\mathbf{\Omega}(\mathbf{S})^{*_i}$, where S has least element i and $r *_i s = -i$ for all $r, s \in S \setminus \{i\}$, and $s *_i i = -1$ for all $s \in S$. Observe also that $\mathbb{IIMT}4$ does not contain $\mathbb{IMT}3$ as a subvariety.

Let $\mho_i = \langle \{i\} \cup \{\frac{1}{k} : k \in \omega, k \geq 1\}, \leq \rangle$ be ordered by $i < \frac{1}{k}$ for all $k \in \omega$ with $\leq$ the usual order on $\{\frac{1}{k} : k \in \omega\}$.

The variety $\mathbb{IIMT}4$ is locally finite. If $\mathcal{C}$ is a finite chain in $\mathbb{IIMT}4$, then there exists an $m \in \omega$ such that $\mathcal{C}$ is isomorphic to $\mathbf{\Lambda}(\mathbf{m})^{*_i}$ or $\mathbf{\Omega}(\mathbf{m})^{*_i}$. If $\mathcal{C}$ is finitely generated, then $\mathcal{C}$ is finite and is isomorphic to a subalgebra of $\mathbf{\Lambda}(\mho)^{*_i}$ or $\mathbf{\Omega}(\mho_i)^{*_i}$. It is also clear that $\mathbb{IIMT}4 = \mathbb{V}(\mathbf{\Lambda}(\mho)^{*_i}, \mathbf{\Omega}(\mho_i)^{*_i})$.

It is easy to see that for a chain $\mathcal{C} \in \mathbb{IMT}4$, $0 \notin C$ if and only if the identity $\neg\eta(x, \neg x) \approx \top$ holds in $\mathcal{C}$.

So we have the following result which is similar to [5, Theorem 3.13]. We write S_r to abreviate $S_r(x_0, \ldots, x_r)$.

Theorem 3.10. *Every proper nontrivial subvariety of* $\mathbb{IIMT}4$ *is of one the following types with the identities that characterize them within* $\mathbb{IIMT}4$*:*

(1) $\mathbb{MIIMT}4$ *and its subvarieties.*
(2) $\mathbb{V}(\mathbf{\Omega}(\mho_i)^{*_i})$, *characterized by* $\neg\eta(x, \neg x) \approx \top$.
(3) $\mathbb{V}(\mathbf{\Lambda}(\mho)^{*_i}, \mathbf{\Omega}(\mathbf{k})^{*_i})$ *for some* $k \in \omega$, *characterized by*

$$(\neg x^2 \vee (\neg x^2 \to x^2)) \vee (S_{2k+2} \wedge (\neg\eta(x, \neg x))) \approx \top.$$

(4) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_i}, \mathbf{\Omega}(\mho_i)^{*_i})$ *for some* $m \in \omega$, *characterized by*

$$(S_{2m+3} \wedge (\neg x^2 \vee (\neg x^2 \to x^2))) \vee \neg\eta(x, \neg x) \approx \top.$$

(5) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_i}, \mathbf{\Omega}(\mathbf{k})^{*_i})$ *for some* $k, m \in \omega$, *characterized by*

$$(S_{2m+3} \wedge (\neg x^2 \vee (\neg x^2 \to x^2))) \vee (S_{2k+2} \wedge (\neg\eta(x, \neg x))) \approx \top.$$

In Figure 2 we can see the lattice of subvarieties of $\mathbb{IIMT}4$, where we have omitted the superscript $*_i$ and where $\mathbb{V}_{mk} = \mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_i}, \mathbf{\Omega}(\mathbf{k})^{*_i})$.

3.3. The subvariety $\mathbb{CIMT}4$**.** Our objective in this subsection is to study the subvariety of $\mathbb{IMT}4$ in which the product on the chains behaves as the one in the $\mathbb{IMT}3$-chains, except for the element 1, in which $1^2 \neq -1$. Recall that in an $\mathbb{IMT}3$-chain, $s * r = -1$ for all $r, s \in S$ (see the observation following Example 3.1).

Lemma 3.11. *Let* $\mathcal{C} \in \mathbb{IMT}4$ *with* $|C| \geq 4$. *If* $-1 \prec 1^2$, *then for all* $r, s \in S$, $s * r = 1^2$ *if* $s = r = 1$ *and* $s * r = -1$ *otherwise.*

Proof. Suppose that $r < 1$. Then $r \leq \neg 1^2$, and so

$$s * r \leq 1 * \neg 1^2 = 1 * (1 \to \neg 1) \leq \neg 1.$$

Since $s * r \neq \perp$, it follows that $s * r = -1$. $\square$

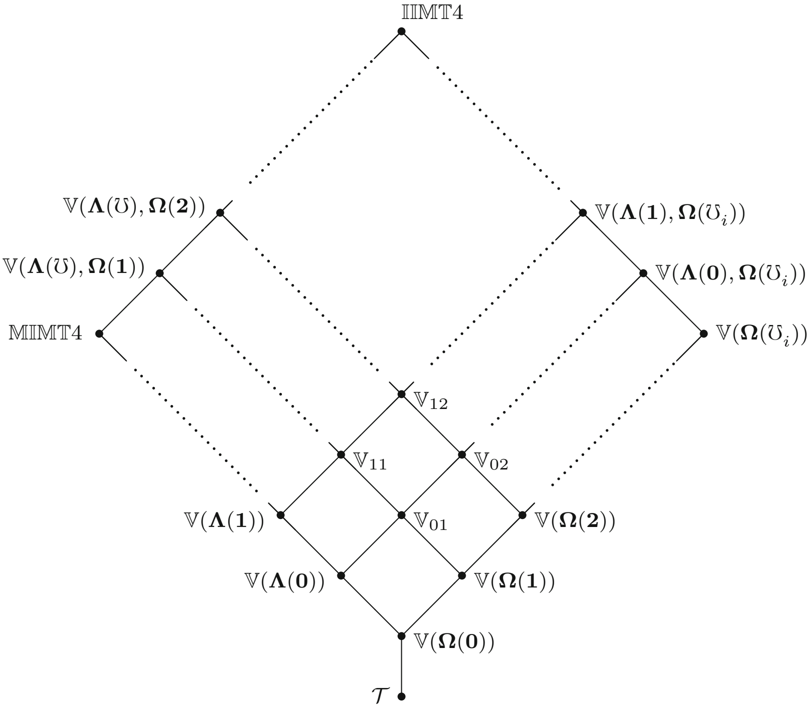

FIGURE 2. Lattice of subvarieties of $\mathbb{IIMT}4$

Lemma 3.12. *Let $\mathcal{C} \in \mathbb{IMT}4$, such that $|C| > 4$. Then, $-1 \prec 1^2$ if and only if $\mathcal{C}$ satisfies the identity*

$$t_1(x,y) \vee t_2(x) \vee t_3(x,y) \vee t_4(x,y) \approx \top, \tag{3.3}$$

where

$$\begin{aligned}
t_1(x,y) &= (x \to \neg\, x) \vee (y \to \neg\, y) \vee x \vee y,\\
t_2(x) &= \neg\, x^3 \wedge \eta((\neg\, x^2)^3, x^2),\\
t_3(x,y) &= [(y * \neg\, x^3) \to (\neg\, x^3)^2] \wedge [\neg\, \eta(y * \neg\, x^3, (\neg\, x^3)^2)]\\
&\qquad \wedge [\neg\, \eta(y, \neg\, x^3)] \wedge [\neg\, \eta(x^3, \bot)] \wedge [\eta(y * \neg\, x^3, x^3)],\\
t_4(x,y) &= [\eta(y, \neg\, x^3)] \wedge [x^3 \to (y * \neg\, x^3)] \wedge [\neg\, \eta(x^3, y * \neg\, x^3)] \wedge [\neg\, \eta(x^3, \bot)].
\end{aligned}$$

Proof. Suppose that $-1 \prec 1^2$. Let $a, b \in C$.

If $a = \top$ or $b = \top$ or $a \leq \neg\, a$ or $b \leq \neg\, b$, then $t_1(a,b) = \top$. Hence, we can assume that $a, b \neq \top$, $\neg\, a < a$, and $\neg\, b < b$.

If $a \neq 1$, by Lemma 3.11, we have that $a^2 = -1$. Hence, $a^3 = \bot$. Consequently, $(\neg\, a^2)^3 = 1^3 = -1 = a^2$ and $t_2(a) = \top$.

Assume that $a = 1$. If $b = 1$, then we have that $\neg\, a^3 = 1 = b$. Since $b * \neg\, a^3 = 1^2 \succ -1 = a^3$, so $a^3 \to (b * \neg\, a^3) = \top$ and $\eta(a^3, b * \neg\, a^3) = \bot$. It is clear that $\eta(a^3, \bot) = \bot$ because $1^3 = -1$. Then $t_4(a,b) = \top$. Now we consider the case in which $b < 1$. By Lemma 3.11, we can prove that $b * \neg a^3 = b * 1 = -1 < 1^2 = (\neg a^3)^2$. Hence, $(b * \neg a^3) \to ((\neg a^3)^2) = \top$ and $\eta(b * \neg a^3, (\neg a^3)^2) = \bot$. So, $\eta(a^3, \bot) = \bot$, $\neg a^3 = 1 > b$, and consequently, $\eta(b, \neg a^3) = \bot$. Finally, $b * \neg a^3 = b * 1 = -1 = a^3$. Hence, $t_3(a, b) = \top$.

Let $\mathcal{C} \in \mathbb{IMT}4$ and suppose that $\mathcal{C} \in \mathbb{IMT}4$ satisfies (3.3). If we consider the element $1 \in C$, then $t_1(1,1) \vee t_2(1) \vee t_3(1,1) \vee t_4(1,1) = \top$. It is clear that $t_1(1,1) \neq \top$. Now we will prove that it is not possible to have $t_2(1) = \top$. If $t_2(1) = \top$, then $\neg 1^3 = \top$ and $(\neg 1^2)^3 = 1^2$. Moreover, since $1^3 = \bot$, so $1^2 \leq -1$. If $1^2 = \bot$, then $\bot = 1^2 = (\neg 1^2)^3 = \top^3 = \top$, and if $1^2 = -1$, then $-1 = 1^2 = (\neg 1^2)^3 = 1^3 = \bot$. Then $t_2(1) \neq \top$. This shows that $t_3(1,1) = \top$ or $t_4(1,1) = \top$. Then we have that $\neg \eta(1^3, \bot) = \top$. So, $1^3 = -1$. Hence, $1 * \neg 1^3 = 1^2 = (\neg 1^3)^2$. Consequently, $\eta(1 * \neg 1^3, (\neg 1^3)^2) = \top$. So, $t_3(1,1) \neq \top$. Therefore, the only possible case is to have the condition $t_4(1,1) = \top$, and thus we have that $1^3 \to (1 * \neg 1^3) = \top$ and $\neg \eta(1^3, 1 * \neg 1^3) = \top$. Hence, we have $-1 = 1^3 < 1 * \neg 1^3 = 1^2$. Now we consider the elements $a = 1, b \in C$ such that $b \prec 1$ in (3.3). As before, we have that $t_3(1, b) = \top$ or $t_4(1, b) = \top$. Since $\neg a^3 = 1 \neq b$, we have that $\eta(b, \neg a^3) = \bot$. Hence, $t_3(1, b) = \top$. Moreover, since $\eta(b * \neg a^3, a^3) = \top$, so $b * 1 = b * \neg 1^3 = b * \neg a^3 = a^3 = 1^3 = -1$. Then $b * 1^2 = (-1) * 1 = \bot$. By residuation, $1^2 \leq \neg b$. Therefore, $-1 \prec 1^2$. $\square$

Let $\mathbb{CIMT}4$ denote the subvariety of $\mathbb{IMT}4$ characterized by the equation (3.3). Observe that the Boolean algebra $\mathbf{\Omega}(\mathbf{0})^*$ and the 3-element MV-chain $\mathbf{\Lambda}(\mathbf{0})^*$ belong to $\mathbb{CIMT}4$, whereas the 4-element chain $\mathbf{\Omega}(\mathbf{1})^*$ does not.

If $*_c$ denotes the product of the Lemma 3.12 in a chain $\mathcal{C}$, it is clear that the chains in the variety $\mathbb{CIMT}4$ are of the form $\mathbf{\Lambda}(\mathbf{S})^{*_c}$ or $\mathbf{\Omega}(\mathbf{S})^{*_c}$, where $S = \{\dots, a, 1\}$ is a totally ordered set with greatest element 1, $a \prec 1$, and

$$s *_c r = \begin{cases} \neg a & \text{if } s = r = 1, \\ \neg 1 & \text{otherwise,} \end{cases}$$

for all $r, s \in S$.

Observe that the only chain in $\mathbb{CIMT}4$ in which the condition $\neg 1^2 = 1^2$ holds is the 5-element chain $\mathbf{\Lambda}(\mathbf{1})^{*_c}$.

$\mathbb{CIMT}4$ is locally finite. For all $m \in \omega$, $\mathbb{V}(\mathbf{\Lambda}(\mathbf{0})^{*_c}) \subseteq \mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_c})$, and $\mathbf{\Lambda}(\mathbf{1})^{*_c}$ is not a proper subalgebra of any *CIMT*4-chain. Moreover, for $2 \leq k \leq m$, we have that $\mathbb{V}(\mathbf{\Omega}(\mathbf{k})^{*_c}) \subseteq \mathbb{V}(\mathbf{\Omega}(\mathbf{m})^{*_c})$, $\mathbb{V}(\mathbf{\Omega}(\mathbf{k})^{*_c}) \subseteq \mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_c})$ and that $\mathbb{V}(\mathbf{\Lambda}(\mathbf{k})^{*_c}) \subseteq \mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*_c})$. It is also clear that $\mathbb{CIMT}4 = \mathbb{V}(\mathbf{\Lambda}(\mho)^{*_c}, \mathbf{\Lambda}(\mathbf{1})^{*_c})$ and that $\mathbb{V}(\mathbf{\Omega}(\mho)^{*_c}) \subseteq \mathbb{V}(\mathbf{\Lambda}(\mho)^{*_c})$.

In order to characterize by equations all the subvarieties of $\mathbb{CIMT}4$ we state the following lemma.

Lemma 3.13. *Let $\mathcal{C} \in \mathbb{CIMT}4$ such that $|C| > 4$. Then $|C| > 5$ if and only if $\mathcal{C}$ satisfies the identity $\neg\eta(\neg x^2, x^2) \approx \top$.*

Proof. Suppose that $|C| > 5$ and let $a \in C$, $a \neq \top$. Since $\mathcal{C} \in \mathbb{CIMT}4$, $-1 \prec 1^2$, and by Lemma 3.11, either $-1 \prec a^2$ or $a^2 \leq -1$. So $a^2 \neq 0$ for all $a \in C$, that is $\neg a^2 \neq a^2$. So, $\eta(\neg a^2, a^2) = \bot$, and then $\neg\eta(\neg a^2, a^2) = \top$.

Suppose now that $\mathcal{C}$ satisfies the identity $\neg\eta(\neg x^2, x^2) \approx \top$. If we take $x = 1$ in this identity, we get $(\neg 1^2 \to 1^2) * (1^2 \to \neg 1^2) = \bot$, so $\neg 1^2 \neq 1^2$, that is, $\neg 1^2 \neq 0$. As in addition, $-1 \prec 1^2$, we have that $|C| > 5$. □

In the following theorem, we denote the term $S_r(x_0, \ldots, x_r)$ by S_r.

Theorem 3.14. *Every proper nontrivial subvariety of* $\mathbb{CIMT}4$ *is of one of the following types, given with the identities that characterize them within* $\mathbb{CIMT}4$*:*

(1) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{l})^{*c})$ *for* $0 \leq l \leq 1$*, characterized by* $S_{2l+3} \approx \top$.
(2) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*c})$ *for* $m \in \omega$ *with* $m \geq 2$*, characterized by* $S_{2m+3} \wedge \neg\eta(\neg x^2, x^2) \approx \top$.
(3) $\mathbb{V}(\mathbf{\Lambda}(\mho)^{*c})$*, characterized by* $\neg\eta(\neg x^2, x^2) \approx \top$.
(4) $\mathbb{V}(\mathbf{\Omega}(\mho)^{*c})$*, characterized by* $\neg\eta(x, \neg x) \approx \top$.
(5) $\mathbb{V}(\mathbf{\Omega}(\mathbf{k})^{*c})$ *for* $k \in \omega$ *with* $k \neq 1$*, characterized by* $S_{2k+2} \wedge \neg\eta(x, \neg x) \approx \top$.
(6) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{1})^{*c}, \mathbf{\Lambda}(\mathbf{m})^{*c})$ *for* $m \in \omega$ *with* $m \geq 2$*, characterized by* $S_{2l+3} \vee (S_{2m+3} \wedge \neg\eta(\neg x^2, x^2)) \approx \top$.
(7) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{l})^{*c}, \mathbf{\Omega}(\mho)^{*c})$ *for* $0 \leq l \leq 1$*, characterized by* $S_{2l+3} \vee \neg\eta(x, \neg x) \approx \top$.
(8) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{l})^{*c}, \mathbf{\Omega}(\mathbf{k})^{*c})$ *for* $0 \leq l \leq 1$ *and* $k \in \omega$*,* $k \neq 1$*, characterized by* $S_{2l+3} \vee (S_{2k+2} \wedge \neg\eta(x, \neg x)) \approx \top$.
(9) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*c}, \mathbf{\Omega}(\mho)^{*c})$ *for* $m \in \omega$ *with* $m \geq 2$*, characterized by* $(S_{2m+3} \wedge \neg\eta(\neg x^2, x^2)) \vee \neg\eta(x, \neg x) \approx \top$.
(10) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{m})^{*c}, \mathbf{\Omega}(\mathbf{k})^{*c})$ *for* $m \in \omega$ *and* $m \geq 2$ *and for* $k \in \omega$ *with* $m < k$ *and* $k \geq 3$*, characterized by* $(S_{2m+3} \wedge \neg\eta(\neg x^2, x^2)) \vee (S_{2k+2} \wedge \neg\eta(x, \neg x)) \approx \top$.
(11) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{1})^{*c}, \mathbf{\Lambda}(\mathbf{m})^{*c}, \mathbf{\Omega}(\mho)^{*c})$ *for* $m \in \omega$ *with* $m \geq 2$*, characterized by* $S_5 \vee (S_{2m+3} \wedge \neg\eta(\neg x^2, x^2)) \vee \neg\eta(x, \neg x) \approx \top$.
(12) $\mathbb{V}(\mathbf{\Lambda}(\mathbf{1})^{*c}, \mathbf{\Lambda}(\mathbf{m})^{*c}, \mathbf{\Omega}(\mathbf{k})^{*c})$ *for* $m \in \omega$ *and* $m \geq 2$ *and for* $k \in \omega$ *with* $m < k$ *and* $k \geq 3$*, characterized by* $S_5 \vee (S_{2m+3} \wedge \neg\eta(\neg x^2, x^2)) \vee (S_{2k+2} \wedge \neg\eta(x, \neg x)) \approx \top$.

Figure 3 depicts the lattice of subvarieties of $\mathbb{CIMT}4$.

3.4. Other subvarieties of $\mathbb{IMT}4$.

For each $k \in \omega$ with $k \geq 2$, consider the term $S_k^2(x_0, \ldots, x_k) = \bigvee_{i<k} \left(x_i^2 \to x_{i+1}^2\right)$ and the subvariety $\mathbb{IMT}4^k$ characterized within $\mathbb{IMT}4$ by the identity $S_k^2(x_0, \ldots, x_k) \approx \top$. If a chain $\mathcal{C} \in \mathbb{IMT}4$ satisfies the identity $S_k^2(x_0, \ldots, x_k) \approx \top$, then $\{x \in C : x = y^2, y \in C\}$ has at most k elements.

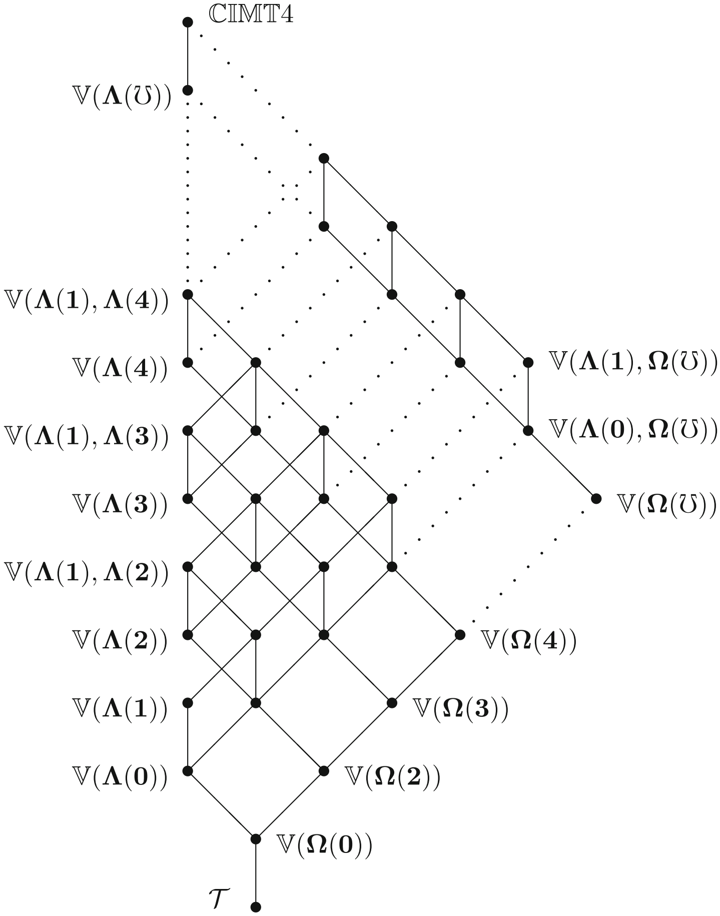

FIGURE 3. Lattice of subvarieties of $\mathbb{CIMT}4$

We have the following chain of subvarieties of $\mathbb{IMT}4$:

$$\mathbb{IMT}4^2 \subset \mathbb{IMT}4^3 \subset \cdots \subset \mathbb{IMT}4^k \subset \cdots \subset \mathbb{IMT}4.$$

Observe that $\mathbb{IMT}4^2 = \mathbb{V}(\mathbf{\Omega}(\mathbf{0}))$. But $\mathbb{IMT}4^3 = \mathbb{MIMT}4 \vee \mathbb{IMT}3$. Indeed, if $\mathcal{C} \in \mathbb{IMT}4^3$ and $\mathcal{C} \notin \mathbb{MIMT}4$, then $|C| \geq 4$ ($\mathbf{\Omega}(\mathbf{0})$ and $\mathbf{\Lambda}(\mathbf{0})$ belong to $\mathbb{MIMT}4$) and $1^2 < \neg 1^2$. Since $1 * \neg 1^2 \leq 1 * 1 = 1^2 = \neg(\neg 1^2)$, it follows that $1 * (\neg 1^2)^2 = \perp$. Thus, $1^2 = (\neg 1^2)^2 \leq -1$, and than $1^2 = -1$. So $\mathcal{C} \in \mathbb{IMT}3$.

We also have $\mathbb{IIMT}4 \subset \mathbb{IMT}4^4$ and $\mathbb{CIMT}4 \subset \mathbb{IMT}4^4$.

Finally, consider the subvarieties V_1 and V_2 characterized within $\mathbb{IMT}4$ by the identities $\neg\,\eta(x, \neg\,x) \approx \top$ and $\neg\,\eta(x^2, \neg x^2) \approx \top$, respectively.

V_1 and V_2 are not locally finite, since we can construct in a similar way as in the example 2.7 an infinite 1-generated chain that belongs to both V_1 and V_2. Indeed, consider the set $S = \{i\} \cup \{b_i : i \in \omega\} \cup \{1\}$ ordered by

$$i < \cdots < b_6 < b_4 < b_2 < b_1 < b_3 < b_5 < \cdots < 1,$$

and the chain $\mathbf{\Omega}(\mathbf{S})^{*}$ where $*$ is defined over S by: $1^2 = b_i * 1 = -i$ for all $b_i \in S$, $i * b = -1$ for all $b \in S$, and for all $b_i, b_j \in S$ such that $i \leq j$,

$$b_i * b_j = \begin{cases} -b_{j+1} & \text{if } (j \text{ odd and } i \neq j-1) \text{ or } (j \text{ even } i = j), \\ -b_{j-1} & \text{if } (j \text{ even and } i \neq j) \text{ or } (j \text{ odd and } i = j-1). \end{cases}$$

It is clear that $\mathbf{\Omega}(\mathbf{S})^{*}$ belongs to $\mathbb{IMT}4$, satisfies the identities $\neg\,\eta(x^2, \neg\, x^2) \approx \top$ and $\neg\,\eta(x, \neg\, x) \approx \top$, and is generated by the element b_1.

4. Finite chains in $\mathbb{IMT}4$ and partitions of the m-cube

The purpose of the next results is to investigate the structure of the finite chains in the variety $\mathbb{IMT}4$. In the first place, for $\mathbf{m} = \langle\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}, \leq\rangle$, we shall find a method for constructing all possible products on $\mathbf{\Lambda}(\mathbf{m})$ and then we shall determine how many *IMT*4-structures can be defined on the finite chain $\mathbf{\Lambda}(\mathbf{m})$.

For $m \geq 1$, we say that the finite set $\Pi \subseteq \{1, 2, \ldots, m\}^3$ is a *totally symmetric plane partition in the m-cube* (TSPP) ([13]) if it satisfies the following conditions:

(P1) If $\langle i, j, k\rangle \in \Pi$ and $i' \leq i, j' \leq j, k' \leq k$, then $\langle i', j', k'\rangle \in \Pi$.
(P2) If $\langle i, j, k\rangle \in \Pi$, then $\langle j, i, k\rangle \in \Pi$.
(P3) If $\langle i, j, k\rangle \in \Pi$, then $\langle j, k, i\rangle \in \Pi$.

We will denote by π_{ij} the cardinal of the set $\{k : \langle i, j, k\rangle \in \Pi\}$.

We are going to give a relationship between the totally symmetric plane partitions in the m-cube and the products definable over a finite chain $\mathbf{\Lambda}(\mathbf{m})$.

To begin with, let us see that starting from a TSPP we can get a totally ordered $\mathbb{IMT}4$-algebra with an odd number of elements.

Consider a totally symmetric plane partition Π in the m-cube. Recall that a product operation on $\mathbf{\Lambda}(\mathbf{m})$ is uniquely determined if we define it over the "upper half" of the chain, that is, over the set $\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}$. Thus, we define the following operation $*_{\Pi}$:

$$\frac{i}{m} *_{\Pi} \frac{j}{m} = -\frac{\pi_{ij}}{m},$$

where $0 = -\frac{0}{m}$. And we define the operation in the rest of the chain $\mathbf{\Lambda}(\mathbf{m})$ as in example 3.1.

Lemma 4.1. $\mathbf{\Lambda}(\mathbf{m})^{*_{\Pi}} \in \mathbb{IMT}4$*.*

Proof. First observe that $*_{\Pi}$ is commutative, so it is enough to prove that $*_{\Pi}$ is associative, that is, let us see that for $\frac{i}{m}, \frac{j}{m}, \frac{k}{m} \in \{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}$, $\left(\frac{i}{m} *_{\Pi} \frac{j}{m}\right) *_{\Pi} \frac{k}{m} = \perp$ if and only if $\frac{i}{m} *_{\Pi} \left(\frac{j}{m} *_{\Pi} \frac{k}{m}\right) = \perp$. Suppose that $\left(\frac{i}{m} *_{\Pi} \frac{j}{m}\right) *_{\Pi} \frac{k}{m} = \perp$. Hence, $-\frac{\pi_{ij}}{m} *_{\Pi} \frac{k}{m} = \perp$ and, consequently, $\frac{\pi_{ij}}{m} \geq \frac{k}{m}$, that is, $\pi_{ij} \geq k$. Since Π is decreasing, we have that $\langle i, j, k\rangle \in \Pi$. Using (P3), $\langle j, k, i\rangle \in \Pi$. Therefore, $\pi_{jk} \geq i$. Consequently, $\frac{j}{m} *_{\Pi} \frac{k}{m} \leq -\frac{i}{m}$ and $\frac{i}{m} *_{\Pi} \left(\frac{j}{m} *_{\Pi} \frac{k}{m}\right) = \perp$. The converse is similar. $\square$

Let us see the previous construction in an example. For $m = 2$, the totally symmetric plane partitions in the 2-cube $\{1, 2\}^3$ are

- $\Pi_1 = \emptyset$,
- $\Pi_2 = \{\langle 1, 1, 1\rangle\}$,
- $\Pi_3 = \{\langle 1, 1, 1\rangle, \langle 1, 1, 2\rangle, \langle 1, 2, 1\rangle, \langle 2, 1, 1\rangle\}$,
- $\Pi_4 = \{\langle 1, 1, 1\rangle, \langle 1, 1, 2\rangle, \langle 1, 2, 1\rangle, \langle 1, 2, 2\rangle, \langle 2, 1, 1\rangle, \langle 2, 1, 2\rangle, \langle 2, 2, 1\rangle\}$,
- $\Pi_5 = \Pi_4 \cup \{\langle 2, 2, 2\rangle\}$.

From these TSPP's we can define the following products on $\Lambda(2)$ to obtain different structures of $\mathbb{IMT}4$-algebras $\mathbf{\Lambda(2)}$.

$$\Lambda(2) : \perp < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < \top$$

Let T_i denote the table operation associated to the TSPP Π_i (recall that it is enough to define the operation for $\{\frac{1}{2}, 1\}$).

T_1

$*$	1	$\frac{1}{2}$
1	0	0
$\frac{1}{2}$	0	0

T_2

$*$	1	$\frac{1}{2}$
1	0	0
$\frac{1}{2}$	0	$-\frac{1}{2}$

T_3

$*$	1	$\frac{1}{2}$
1	0	$-\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{2}$	-1

T_4

$*$	1	$\frac{1}{2}$
1	$-\frac{1}{2}$	-1
$\frac{1}{2}$	-1	-1

T_5

$*$	1	$\frac{1}{2}$
1	-1	-1
$\frac{1}{2}$	-1	-1

FIGURE 4. The operations associated to the TSPP Π_i

Now we study the inverse problem, that is, we want to find a procedure to build a TSPP from a finite $\mathbb{IMT}4$-chain with an odd number of elements.

Lemma 4.2. *Let* $\mathbf{\Lambda(m)}^* \in \mathbb{IMT}4$*. If we define*

$$\Pi^* = \left\{\langle i, j, k\rangle : \left(\frac{i}{m} * \frac{j}{m}\right) * \frac{k}{m} = \perp\right\},$$

then Π^* *is a* TSPP *in the* m*-cube.*

Proof. Assume that $\langle i, j, k\rangle \in \Pi^*$ and that $i' \leq i, j' \leq j$ and $k \leq k'$ with $1 \leq i, j, k \leq m$. By properties of the operation $*$, we have that

$$\left(\frac{i'}{m} * \frac{j'}{m}\right) * \frac{k'}{m} \leq \left(\frac{i}{m} * \frac{j}{m}\right) * \frac{k}{m} = \perp.$$

Then we have that $\langle i', j', k'\rangle \in \Pi^*$, proving condition (P1). Conditions (P2) and (P3) follow from the commutativity and associativity of $*$. $\square$

For example, if we start from $\mathbf{\Lambda(2)}^*$, where $*$ is the product given by the table operation T_3, it is easy to see that

$$\Pi^* = \Pi_3 = \{\langle 1, 1, 1\rangle, \langle 1, 1, 2\rangle, \langle 1, 2, 1\rangle, \langle 2, 1, 1\rangle\}.$$

In Lemmas 4.1 and 4.2, we have shown a procedure to construct a TSPP from an $\mathbb{IMT}4$ chain with an odd number of elements, and conversely, we can construct such a chain from a TSPP. We are going to prove that this correspondence is bijective.

Lemma 4.3. *Let* Π *be a* TSPP *in the m-cube. Then* $\Pi = \Pi^{*_\Pi}$.

Proof. Consider $\langle i, j, k\rangle \in \Pi$. By definition of $*_\Pi$, we have that $\frac{i}{m} *_\Pi \frac{j}{m} = -\frac{\pi_{ij}}{m}$. From $k \le \pi_{ij}$ it follows that $\frac{i}{m} *_\Pi \frac{j}{m} \le -\frac{k}{m}$. Then $\left(\frac{i}{m} *_\Pi \frac{j}{m}\right) *_\Pi \frac{k}{m} = \bot$, and thus, $\langle i, j, k\rangle \in \Pi^{*_\Pi}$.

Assume now that $\langle i, j, k\rangle \in \Pi^{*_\Pi}$. Hence, $-\frac{\pi_{ij}}{m} \le -\frac{k}{m}$, and consequently, $k \le \pi_{ij}$. Observe that $\pi_{ij} \ne 0$ since $1 \le k$. Then using that $\langle i, j, \pi_{ij}\rangle \in \Pi$ and $k \le \pi_{ij}$, we have that $\langle i, j, k\rangle \in \Pi$ by (P1). □

Lemma 4.4. *Let* $\mathbf{\Lambda(m)}^* \in \mathbb{IMT}4$. *Then* $\mathbf{\Lambda(m)}^* \cong \mathbf{\Lambda(m)}^{*_{\Pi^*}}$.

Proof. For $\frac{i}{m}, \frac{j}{m} \in \{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} = 1\}$ we verify that $\frac{i}{m} * \frac{j}{m} = \frac{i}{m} *_{\Pi^*} \frac{j}{m}$. By definition of $*_{\Pi^*}$, we have $\frac{i}{m} *_{\Pi^*} \frac{j}{m} = -\frac{\pi^*_{ij}}{m}$. We consider the following cases.

$\pi^*_{ij} = 0$: Then there is no k with $1 \le k \le m$ such that $\langle i, j, k\rangle \in \Pi^*$. Thus, $\left(\frac{i}{m} * \frac{j}{m}\right) * \frac{k}{m} = -1$ for all $1 \le k \le m$. Hence, $\frac{i}{m} * \frac{j}{m} > -\frac{k}{m}$ for all $1 \le k \le m$. Then $\frac{i}{m} * \frac{j}{m} = 0 = \frac{i}{m} *_{\Pi^*} \frac{j}{m}$.

$\pi^*_{ij} > 0$: Then $\langle i, j, \pi^*_{ij}\rangle \in \Pi^*$. Consequently, $\left(\frac{i}{m} * \frac{j}{m}\right) * \frac{\pi^*_{ij}}{m} = \bot$. Hence, we have that $\frac{i}{m} * \frac{j}{m} \le -\frac{\pi^*_{ij}}{m}$. Suppose now that there exists $1 \le d \le m$ such that $\frac{i}{m} * \frac{j}{m} = -\frac{d}{m}$ with $d > \pi^*_{ij}$. Then $\langle i, j, d\rangle \in \Pi^*$, a contradiction. Hence, $\frac{i}{m} * \frac{j}{m} = -\frac{\pi^*_{ij}}{m} = \frac{i}{m} *_{\Pi^*} \frac{j}{m}$. □

From the previous results, we have the following theorem.

Theorem 4.5. *There exists a bijective correspondence between the* TSPP's *in an m-cube and the* $\mathbb{IMT}4$ *chains with* $2m + 3$ *elements.*

With *TS*(m) denoting the number of totally symmetric plane partitions in the m-cube, J. R. Stembridge proved in [13, Corollary 5.2] that for $m \ge 1$,

$$TS(m) = \prod_{1\le i\le j\le k\le m} \frac{i+j+k-1}{i+j+k-2}.$$

Then by Theorem 4.5, we have the following result.

Corollary 4.6. *In* $\mathbb{IMT}4$*, there are*

$$\prod_{1\le i\le j\le k\le m} \frac{i+j+k-1}{i+j+k-2}$$

non-isomorphic chains with $n = 2m + 3$ *elements, for* $m \ge 1$.

For $n = 5, 7, 9, 11, 13, 15, 17, 19$, this formula gives 2, 5, 16, 66, 352, 2431, 21760, and 252586, algebras respectively.

In the following lemmas, we will see that the formula of the previous corollary also applies to chains with an even number of elements. In fact, we will prove that there exists a bijective correspondence between the chains $\mathbf{\Lambda(m)}^*$ and $\mathbf{\Omega(m+1)}^*$ with $m \geq 1$.

Some of the proofs of the next lemmas are long but computational, and will be omitted.

Lemma 4.7. *Let* $\mathbf{\Lambda(m)}^* \in \mathbb{IMT}4$ *with* $m \geq 1$*. If* $\star$ *is defined on* $\mathbf{m+1}$ *by*

$$\frac{i}{m+1} \star \frac{j}{m+1} = \begin{cases} -\frac{k+1}{m+1} & \textit{if } 2 \leq i, j \leq m+1, \frac{i-1}{m} * \frac{j-1}{m} = -\frac{k}{m} \neq 0, \\ -\frac{1}{m+1} & \textit{if } 2 \leq i, j \leq m+1, \frac{i-1}{m} * \frac{j-1}{m} = 0, \\ -1 & \textit{if } i = 1 \textit{ or } j = 1, \end{cases}$$

then $\mathbf{\Omega(m+1)}^\star \in \mathbb{IMT}4$*.*

Lemma 4.8. *Let* $\mathbf{\Omega(m+1)}^* \in \mathbb{IMT}4$ *with* $m \geq 1$*. If* $\times$ *is defined on* $\mathbf{m}$ *by*

$$\frac{i}{m} \times \frac{j}{m} = \begin{cases} -\frac{k-1}{m} & \textit{if } \frac{i+1}{m+1} * \frac{j+1}{m+1} = -\frac{k}{m+1}, 2 \leq k \leq m+1, \\ 0 & \textit{if } \frac{i+1}{m+1} * \frac{j+1}{m+1} = -\frac{1}{m+1}, \end{cases}$$

then $\mathbf{\Lambda(m)}^\times \in \mathbb{IMT}4$*.*

Lemma 4.9. *If* $\mathbf{\Lambda(m)}^* \in \mathbb{IMT}4$ *with* $m \geq 1$*, then* $\mathbf{\Lambda(m)}^* \cong \mathbf{\Lambda(m)}^\times$*.*

Proof. We have to prove that $\frac{i}{m} * \frac{j}{m} = \frac{i}{m} \times \frac{j}{m}$ for all $\frac{i}{m}, \frac{j}{m} \in \mathbf{m}$. If $\frac{i}{m} * \frac{j}{m} = -\frac{k}{m} \neq 0$, then $\frac{i+1}{m+1} \star \frac{j+1}{m+1} = -\frac{k+1}{m+1}$, and $\frac{i}{m} \times \frac{j}{m} = -\frac{(k+1)-1}{m}$. And if $\frac{i}{m} * \frac{j}{m} = 0$, then $\frac{i+1}{m+1} \star \frac{j+1}{m+1} = -\frac{1}{m+1}$, and by definition, $\frac{i}{m} \times \frac{j}{m} = 0$. □

Lemma 4.10. *If* $\mathbf{\Omega(m+1)}^* \in \mathbb{IMT}4$ *with* $m \geq 1$*, then we have* $\mathbf{\Omega(m+1)}^* \cong \mathbf{\Omega(m+1)}^\star$*.*

Proof. Let us see that $\frac{i}{m+1} * \frac{j}{m+1} = \frac{i}{m+1} \star \frac{j}{m+1}$ for all $\frac{i}{m+1}, \frac{j}{m+1} \in \mathbf{m+1}$.

By Lemma 3.8, if $i = 1$ or $j = 1$, then $\frac{i}{m+1} * \frac{j}{m+1} = -1$, and by definition, $\frac{i}{m+1} \star \frac{j}{m+1} = -1$. If $2 \leq i, j \leq m+1$ and $\frac{i}{m+1} * \frac{j}{m+1} = -\frac{k}{m+1}$ for $k \geq 2$, then $\frac{i-1}{m} \times \frac{j-1}{m} = -\frac{k-1}{m}$ and $\frac{i}{m+1} \star \frac{j}{m+1} = -\frac{k}{m+1}$. Finally, if $2 \leq i, j \leq m+1$ and $\frac{i}{m+1} * \frac{j}{m+1} = -\frac{1}{m+1}$, then $\frac{i-1}{m} \times \frac{j}{m} = 0$ and $\frac{i}{m+1} \star \frac{j}{m+1} = -\frac{1}{m+1}$. □

From Lemma 4.9 and 4.10, we obtain the following theorem.

Theorem 4.11. *There exists a bijective correspondence between the sets* $\mathcal{O} = \{\mathbf{\Lambda(m)}^* : \mathbf{\Lambda(m)}^* \in \mathbb{IMT}4\}$ *and* $\mathcal{E} = \{\mathbf{\Omega(m+1)}^* : \mathbf{\Omega(m+1)}^* \in \mathbb{IMT}4\}$ *for* $m \geq 1$*.*

Now consider the set

$$\mathcal{O}' = \{\mathbf{\Lambda(m+1)}^* : \mathbf{\Lambda(m+1)}^* \in \mathbb{IMT}4 \text{ and } 1^2 < 0\}.$$

We are going to provide a procedure to obtain all the chains $\mathbf{\Lambda(m+1)}^*$ with $1^2 < 0$ from the chains $\mathbf{\Lambda(m)}^*$. In addition, we will establish a bijection between the sets $\mathcal{O}$ and $\mathcal{O}'$.

Lemma 4.12. *Let* $\mathbf{\Lambda}(\mathbf{m})^* \in \mathbb{IMT}4$ *with* $m \geq 1$. *If* $\times$ *is the binary operation on* $\mathbf{m+1}$ *defined by*

$$\frac{i}{m+1} \times \frac{j}{m+1} = \begin{cases} -\frac{1}{m+1} & \text{if } 2 \leq i,j \leq m+1, \frac{i-1}{m} * \frac{j-1}{m} = 0, \\ -\frac{k+1}{m+1} & \text{if } 2 \leq i,j \leq m+1, \frac{i-1}{m} * \frac{j-1}{m} = -\frac{k}{m} \neq 0, \\ -1 & \text{if } i = 1 \text{ or } j = 1, \end{cases}$$

then $\mathbf{\Lambda}(\mathbf{m+1})^\times \in \mathbb{IMT}4$.

Lemma 4.13. *Let* $\mathbf{\Lambda}(\mathbf{m+1})^* \in \mathbb{IMT}4$ *with* $m \geq 1$ *and* $1^2 < 0$. *If* $\otimes$ *is the binary operation defined on* $\mathbf{m}$ *by*

$$\frac{i}{m} \otimes \frac{j}{m} = \begin{cases} 0 & \text{if } \frac{i+1}{m+1} * \frac{j+1}{m+1} = -\frac{1}{m+1}, \\ -\frac{k-1}{m} & \text{if } \frac{i+1}{m+1} * \frac{j+1}{m+1} = -\frac{k}{m+1} \neq -\frac{1}{m+1}, \end{cases}$$

then $\mathbf{\Lambda}(\mathbf{m})^\otimes \in \mathbb{IMT}4$.

Then we have the following theorem.

Theorem 4.14. *There exists a bijection between the sets* $\mathcal{O}$ *and* $\mathcal{O}'$.

Observe that there is a bijection between the set of $IMT4$-chains $\mathbf{\Lambda}(\mathbf{m+1})^*$ with the property that $1 * \frac{i}{m+1} = 0$ for all $\frac{i}{m+1} \in \mathbf{m+1}$ and the set of TSPP's of the m-cube. Thus, we have the same number of $IMT4$-chains with $2m+3$ elements as chains with $2m+5$ elements such that $1 * \frac{i}{m+1} = 0$ for all $\frac{i}{m+1} \in \mathbf{m+1}$.

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