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On Some Semi-Intuitionistic Logics

Abstract. Semi-intuitionistic logic is the logic counterpart to semi-Heyting algebras, which were defined by H. P. Sankappanavar as a generalization of Heyting algebras. We present a new, more streamlined set of axioms for semi-intuitionistic logic, which we prove translationally equivalent to the original one. We then study some formulas that define a semi-Heyting implication, and specialize this study to the case in which the formulas use only the lattice operators and the intuitionistic implication. We prove then that all the logics thus obtained are equivalent to intuitionistic logic, and give their Kripke semantics.

Keywords: Semi-intuitionistic logic, Semi-Heyting algebras, Intuitionistic logic, Heyting algebras.

1. Introduction

Semi-Heyting algebras were defined by H. P. Sankappanavar in [15] as a variety generalizing the one of Heyting algebras while retaining some important features, like the fact that they are all pseudocomplemented distributive lattices and their congruences are determined by filters. Semi-Heyting algebras are algebras $\mathbf{A} = \langle A, \vee, \wedge, \to, \top, \bot\rangle$ that satisfy the conditions:

$(SH1)$ $\langle A, \vee, \wedge, \top, \bot\rangle$ is a bounded lattice.
$(SH2)$ $x \wedge (x \to y) \approx x \wedge y$
$(SH3)$ $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$
$(SH4)$ $x \to x \approx \top$.

The resulting variety of Semi-Heyting algebras, denoted by SH, presents complex behavior, since one may define different implication operations over a single lattice to obtain different semi-Heyting algebras. For example, if the lattice is a chain with 5 elements, there are 10400 different ways of endowing it with a semi-Heyting implication [1]. One of the questions we set out to answer in this work is wether there are some subvarieties where the implication over a given lattice can be determined in a unique way.

A key feature of semi-Heyting algebras is that each one of them may be turned into a Heyting algebra by defining the implication $x \to_H y$ as

Presented by **Constantine Tsinakis**; *Received* September 19, 2013

Studia Logica (2015) 103: 303–344
DOI: 10.1007/s11225-014-9568-x

$x \to (x \wedge y)$. It is clear that Heyting algebras themselves are semi-Heyting algebras, so a lattice is the underlying lattice of a semi-Heyting algebra if and only if it is the underlying lattice of a Heyting algebra.

As part of the ongoing tradition of finding the relationship between algebraic structures and logics, one of us introduced in [4] semi-intuitionistic logic as the logic counterpart of semi-Heyting algebras, proving completeness. In section 2, we present a different set of axioms for semi-intuitionistic logic, which we prove in section 3 is translationally equivalent to the original one. The change we propose is in order to unify the algebraic and logic language, removing the need to distinguish between terms and formulas and also reduce the number of axioms. In section 4 we prove completeness again but this time using the fact that with the implication $\to_H$, semi-intuitionistic logic is an implicative logic in the sense of [13]. In section 5 we study some formulas that define a semi-Heyting implication, while in section 6 we specialize this study to the case in which the formulas use only the lattice operators and the intuitionistic implication. We prove then that all the logics thus obtained are equivalent to intuitionstic logic, give their Kripke semantics and prove completeness with respect to them. In the appendix we offer some of the longer proofs.

2. Semi-Intuitionistic Logic

Following [7], a *logical language* **L** is a set of connectives, each with a fixed arity $n \geq 0$. For a countably infinite set *Var* of propositional variables, the *formulas* of the logical language **L** are inductively defined as usual.

A *logic* in the language **L** is a pair $\mathcal{L} = \langle Fm, \vdash_{\mathcal{L}} \rangle$ where Fm is the set of formulas and $\vdash_{\mathcal{L}}$ is a substitution-invariant consequence relation on Fm. The set Fm may also be endowed with an algebraic structure, by considering the connectives of the language as operation symbols. The resulting algebra is often called the *algebra of formulas* and denoted by Fm. We will present finitary logics by means of their "Hilbert style" sets of axioms and inference rules.

We introduce now the semi-intuitionistic logic $\mathcal{SI}$, in the language $\{\wedge, \vee, \to, \bot, \top\}$, by means of the following set of axioms. Here $\alpha \to_H \beta$ is an abbreviation for $\alpha \to (\alpha \wedge \beta)$ that makes the axioms easier to read.

(S1) $\alpha \to_H (\alpha \vee \beta)$

(S2) $\beta \to_H (\alpha \vee \beta)$

(S3) $(\alpha \to_H \gamma) \to_H [(\beta \to_H \gamma) \to_H ((\alpha \vee \beta) \to_H \gamma)]$

(S4) $(\alpha \wedge \beta) \rightarrow_H \alpha$

(S5) $(\gamma \rightarrow_H \alpha) \rightarrow_H [(\gamma \rightarrow_H \beta) \rightarrow_H (\gamma \rightarrow_H (\alpha \wedge \beta))]$

(S6) $\top$

(S7) $\bot \rightarrow_H \alpha$

(S8) $((\alpha \wedge \beta) \rightarrow_H \gamma) \rightarrow_H (\alpha \rightarrow_H (\beta \rightarrow_H \gamma))$

(S9) $(\alpha \rightarrow_H (\beta \rightarrow_H \gamma)) \rightarrow_H ((\alpha \wedge \beta) \rightarrow_H \gamma)$

(S10) $(\alpha \rightarrow_H \beta) \rightarrow_H ((\beta \rightarrow_H \alpha) \rightarrow_H ((\alpha \rightarrow \gamma) \rightarrow_H (\beta \rightarrow \gamma)))$

(S11) $(\alpha \rightarrow_H \beta) \rightarrow_H ((\beta \rightarrow_H \alpha) \rightarrow_H ((\gamma \rightarrow \beta) \rightarrow_H (\gamma \rightarrow \alpha)))$

The inference rule is *semi-Modus Ponens* (SMP): For all $\Gamma \cup \{\phi, \gamma\} \subseteq Fm$, if $\Gamma \vdash_{\mathcal{SI}} \phi$ and $\Gamma \vdash_{\mathcal{SI}} \phi \rightarrow (\phi \wedge \gamma)$, then $\Gamma \vdash_{\mathcal{SI}} \gamma$. Notice that this is Modus Ponens for the implication $\rightarrow_H$.

Semi-intuitionistic logic was introduced in [4] using the language $\{\wedge, \vee, \rightarrow, \neg\}$ and a different set of axioms. We are going to denote that system $\mathcal{SI}'$. It was proved there that the logic $\mathcal{SI}'$ is complete with respect to the class of algebras SH. The logic $\mathcal{SI}'$ was defined using the axioms (S1) to (S11) replacing the axioms (S6) and (S7) by

(S6') $\alpha \rightarrow_H \alpha$

(S7') $(\alpha \wedge \neg\alpha) \rightarrow_H \beta$

and adding the axioms

(S12') $(\alpha \rightarrow_H \beta) \rightarrow_H [(\beta \rightarrow_H \gamma) \rightarrow_H (\alpha \rightarrow_H \gamma)]$

(S13') $(\alpha \wedge \beta) \rightarrow_H \beta$

(S14') $(\alpha \wedge (\beta \rightarrow \gamma)) \rightarrow_H (\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma)))$

(S15') $(\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma))) \rightarrow_H (\alpha \wedge (\beta \rightarrow \gamma))$

(S16') $(\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow_H (\alpha \wedge \beta)$

(S17') $(\alpha \wedge \beta) \rightarrow_H (\alpha \wedge (\alpha \rightarrow \beta))$

(S18') $(\alpha \rightarrow_H \neg\alpha) \rightarrow_H (\neg\alpha)$

with semi-Modus Ponens as the only inference rule.

We are going to verify that these two logics, $\mathcal{SI}$ and $\mathcal{SI}'$, are translationally equivalent (in the sense of Definition 3.3). As a consequence we will have that the logic $\mathcal{SI}$ is complete with respect to the variety SH.

Now we prove some of the results in the logics $\mathcal{SI}$ and $\mathcal{SI}'$ that will be needed. The proof of the following lemma is given in the Appendix.

Lemma 2.1. *The following statements hold in the logic $\mathcal{SI}$:*

(a) *If* $\Gamma \vdash_{\mathcal{SI}} \psi$ *then* $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow_H \psi$.

(b) $\vdash_{\mathcal{SI}} \alpha \rightarrow_H \alpha$

(c) *If* $\vdash_{\mathcal{SI}} \phi$ *then* $\vdash_{\mathcal{SI}} \alpha \rightarrow_H (\alpha \wedge \phi)$.

(d) $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H \beta$ (S13')

(e) $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H (\beta \wedge \alpha)$

(f) $\vdash_{\mathcal{SI}} \alpha \rightarrow \alpha$

(g) $\vdash_{\mathcal{SI}} \alpha \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \beta))$

(h) $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \gamma) \rightarrow_H \psi$ *if and only if* $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \wedge \gamma)) \rightarrow_H \psi$.

(i) $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)$

(j) *If* $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H \gamma$ *then* $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow_H \gamma$

(k) *If* $\vdash_{\mathcal{SI}} \alpha \rightarrow_H \beta$ *then* $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightarrow_H (\beta \wedge \gamma)$ *and* $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow_H (\gamma \wedge \beta)$.

(l) *If* $\vdash_{\mathcal{SI}} \alpha \rightarrow_H (\beta \wedge \gamma)$ *then* $\vdash_{\mathcal{SI}} \alpha \rightarrow_H \gamma$.

(m) *If* $\vdash_{\mathcal{SI}} \alpha \rightarrow_H \beta$ *and* $\vdash_{\mathcal{SI}} \beta \rightarrow_H \alpha$, *then* $\vdash_{\mathcal{SI}} (\alpha \rightarrow_H \gamma) \rightarrow_H (\beta \rightarrow_H \gamma)$ *and therefore* $\vdash_{\mathcal{SI}} (\beta \rightarrow_H \gamma) \rightarrow_H (\alpha \rightarrow_H \gamma)$.

(n) $\vdash_{\mathcal{SI}} (\alpha \rightarrow_H \beta) \rightarrow_H [(\beta \rightarrow_H \gamma) \rightarrow_H (\alpha \rightarrow_H \gamma)]$ (S12')

(o) $\vdash_{\mathcal{SI}} \alpha \rightarrow_H [\alpha \wedge ((\alpha \wedge \beta) \rightarrow_H \beta)]$

(p) $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \alpha) \rightarrow ((\alpha \wedge \beta) \wedge \beta)$

(q) $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \rightarrow \gamma)) \rightarrow_H (\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma)))$ (S14')

(r) $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma))) \rightarrow_H (\alpha \wedge (\beta \rightarrow \gamma))$ (S15')

(s) $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H (\alpha \wedge (\alpha \rightarrow \beta))$ (S17')

(t) $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow_H (\alpha \wedge \beta)$ (S16')

3. Equivalent Axiomatics

The logic systems $\mathcal{SI}$ and $\mathcal{SI}'$ use different languages, so to prove they are essentially the same we use the following definitions inspired by those found in [11].

Definition 3.1. Given two propositional languages $\mathbf{L}$ and $\mathbf{L}'$ using the same set *Var* of variables, a *translation* is a function $h : Fm_{\mathbf{L}} \rightarrow Fm_{\mathbf{L}'}$ satisfying:

1. If x_i is a propositional variable in $\mathbf{L}$, then $h(x_i) = x_i$;

2. If f is a k-place connective in $\mathbf{L}$, and $\{x_1, \ldots, x_k\} \subseteq Var$, to the formula $f(x_1, \ldots, x_k)$ we assign a formula β_f of $Fm_{\mathbf{L}'}$, where β_f contains only variables from $\{x_1, \ldots, x_k\}$. Then for any $\{\alpha_1, \ldots, \alpha_k\} \subseteq Fm_{\mathbf{L}}$,
$$h(f(\alpha_1, \ldots, \alpha_k)) = \beta_f(h(\alpha_1), \ldots, h(\alpha_k))$$

Definition 3.2. If $\mathcal{A}$ and $\mathcal{B}$ are logics in the languages $\mathbf{L}$ and $\mathbf{L}'$, respectively, a translation h from $\mathbf{L}$ into $\mathbf{L}'$ is *sound* if $h(\phi)$ is provable in $\mathcal{B}$ whenever ϕ is provable in $\mathcal{A}$. That is,
$$\text{If } \vdash_{\mathcal{A}} \phi \text{ then } \vdash_{\mathcal{B}} h(\phi).$$

Furthermore, we assume that in both logics there is a connective $\leftrightarrow$ such that the following axiom schema and inference rules governing this connective are valid:

(B1) $\vdash \phi \leftrightarrow \phi$;

(B2) $\phi \leftrightarrow \psi \vdash \psi \leftrightarrow \phi$;

(B3) $\phi \leftrightarrow \psi, \psi \leftrightarrow \gamma \vdash \phi \leftrightarrow \gamma$;

(B4) $\alpha_1 \leftrightarrow \beta_1, \ldots, \alpha_k \leftrightarrow \beta_k \vdash f(\alpha_1, \ldots, \alpha_k) \leftrightarrow f(\beta_1, \ldots, \beta_k)$, where f is any k-place connective in the system.

Definition 3.3. [11] We say that the logics $\mathcal{A}$ and $\mathcal{B}$ are *translationally equivalent* if there are translations h_1 and h_2 so that

1. Both h_1 and h_2 are sound;
2. For any formula ϕ in $Fm_{\mathbf{L}}$, $\vdash_{\mathcal{A}} \phi \leftrightarrow h_2(h_1(\phi))$;
3. For any formula ϕ in $Fm_{\mathbf{L}'}$, $\vdash_{\mathcal{B}} \phi \leftrightarrow h_1(h_2(\phi))$.

For the rest of this section, we fix the languages $\mathbf{L} = \{\wedge, \vee, \rightarrow, \bot, \top\}$ and $\mathbf{L}' = \{\wedge, \vee, \rightarrow, \neg\}$, and use a common set of variables Var. For $\alpha, \beta \in Fm_{\mathbf{L}}$ or $Fm_{\mathbf{L}'}$, let $\alpha \leftrightarrow_H \beta$ be the formula $(\alpha \rightarrow_H \beta) \wedge (\beta \rightarrow_H \alpha)$. We check that this connective satisfies the conditions (B1) to (B4).

The proofs of Lemmas 3.4-3.8 are given in the Appendix.

Lemma 3.4. *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *Then the following conditions are equivalent:*

(a) $\Gamma \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$,

(b) $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow_H \beta$ *and* $\Gamma \vdash_{\mathcal{SI}} \beta \rightarrow_H \alpha$,

Lemma 3.5. *Let* $\{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$.

a) $\vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \alpha$

b) $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}} \beta \leftrightarrow_H \alpha$

c) $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \gamma$

d) $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \leftrightarrow_H (\beta \wedge \delta)$

e) $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \vee \gamma) \leftrightarrow_H (\beta \vee \delta)$

f) $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \to \gamma) \leftrightarrow_H (\beta \to \delta)$

In the proofs of the last two lemmas we don't use the axioms (S6) and (S7), so we can use the same proofs for the logic $\mathcal{SI}'$.

As a consequence of Lemma 3.5 we have:

COROLLARY 3.6. $\vdash_{\mathcal{SI}} \bot \leftrightarrow_H \bot$ *and* $\vdash_{\mathcal{SI}} \top \leftrightarrow_H \top$.

LEMMA 3.7. *Let* $\{\alpha, \beta\} \subseteq Fm_{\mathbf{L}'}$. *Then* $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \neg\alpha \leftrightarrow_H \neg\beta$.

From the results above, conditions (B1) to (B4) hold in both logics $\mathcal{SI}$ and $\mathcal{SI}'$.

In order to prove a version of the deduction theorem we need the following lemma.

LEMMA 3.8. *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$.

a) $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \bot)) \to_H \beta$

b) *If* $\Gamma \vdash_{\mathcal{SI}} \alpha$ *and* $\Gamma \vdash_{\mathcal{SI}} \beta$ *then* $\Gamma \vdash_{\mathcal{SI}} \alpha \wedge \beta$.

c) $\vdash_{\mathcal{SI}} \alpha \to_H (\alpha \wedge (\bot \to \bot))$.

d) *If* $\Gamma \vdash_{\mathcal{SI}} \beta \leftrightarrow_H \bot$ *then* $\Gamma \vdash_{\mathcal{SI}} \beta \to_H (\alpha \wedge \bot)$ *and* $\Gamma \vdash_{\mathcal{SI}} (\alpha \wedge \bot) \to_H \beta$.

e) $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to_H (\alpha \to \bot))] \to_H \bot$ *and* $\vdash_{\mathcal{SI}} \bot \to_H [\alpha \wedge (\alpha \to_H (\alpha \to \bot))]$.

THEOREM 3.9. (Deduction theorem for $\mathcal{SI}$ and $\mathcal{SI}'$) *Let* $\Gamma \cup \{\phi, \psi\} \subseteq Fm_{\mathbf{L}}$. *Then*

$$\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \psi \textit{ iff } \Gamma \vdash_{\mathcal{SI}} \phi \to_H \psi.$$

The corresponding result also holds for $\mathcal{SI}'$.

PROOF. Assume that $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \psi$. We shall prove $\Gamma \vdash_{\mathcal{SI}} \phi \to_H \psi$ by induction on the proof for ψ. If ψ is an axiom of $\mathcal{SI}$ or a formula in Γ, then $\Gamma \vdash_{\mathcal{SI}} \psi$. By Lemma 2.1, part (a) we have $\Gamma \vdash_{\mathcal{SI}} \phi \to_H \psi$.

Since SMP is the only inference rule, we may assume that there is some formula α such that $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \alpha$ and $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \alpha \to_H \psi$. Then by inductive hypothesis we have,

1. $\Gamma \vdash_{\mathcal{SI}} \phi \to_H \alpha$ and
2. $\Gamma \vdash_{\mathcal{SI}} \phi \to_H (\alpha \to_H \psi)$.

3. $\Gamma \vdash_{\mathcal{SI}} \phi \rightarrow_H \phi$ by Lemma 2.1, part (b).
4. $\Gamma \vdash_{\mathcal{SI}} \phi \rightarrow_H (\phi \wedge \alpha)$ by (S5) and SMP applied to 1 and 3.
5. $\Gamma \vdash_{\mathcal{SI}} (\phi \wedge \alpha) \rightarrow_H \psi$ by (S9) and SMP applied to 2.
6. $\Gamma \vdash_{\mathcal{SI}} \phi \rightarrow_H \psi$ by (S12') and SMP applied to 4 and 5.

For the other implication, we assume that $\Gamma \vdash_{\mathcal{SI}} \phi \rightarrow_H \psi$. Then $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \phi \rightarrow_H \psi$. Since $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \phi$, $\Gamma \cup \{\phi\} \vdash_{\mathcal{SI}} \psi$ obtains.

The same proof can be used to prove the result for $\mathcal{SI}'$, since only axioms common to both are used. See also [4, Theorem 3.18]. ■

Some further results in $\mathcal{SI}'$ are needed.

Lemma 3.10. *$\mathcal{SI}'$ proves the following:*

a) $\vdash_{\mathcal{SI}'} \neg\alpha \leftrightarrow_H (\alpha \rightarrow_H (\neg\alpha))$
b) $\Gamma \vdash_{\mathcal{SI}'} \neg\beta \rightarrow_H (\beta \rightarrow (\alpha \wedge \neg\alpha))$
c) $\Gamma \vdash_{\mathcal{SI}'} (\beta \rightarrow (\alpha \wedge \neg\alpha)) \rightarrow_H \neg\beta$
d) $\Gamma \vdash_{\mathcal{SI}'} (\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha))) \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha))$

We now define translations that let us verify the equivalence.

Definition 3.11. Let h_1 be the translation from $Fm_{\mathbf{L}}$ to $Fm_{\mathbf{L}'}$ defined by:

- $h_1(\alpha \wedge \beta) = h_1(\alpha) \wedge h_1(\beta)$,
- $h_1(\alpha \vee \beta) = h_1(\alpha) \vee h_1(\beta)$,
- $h_1(\alpha \rightarrow \beta) = h_1(\alpha) \rightarrow h_1(\beta)$,
- $h_1(\top) = x \rightarrow x$ and $h_1(\bot) = x \wedge \neg x$, where x is a fixed variable.

Lemma 3.12. *The translation h_1 is sound.*

Proof. We want to check that for $\phi \in Fm_{\mathbf{L}}$, if $\vdash_{\mathcal{SI}} \phi$ then $\vdash_{\mathcal{SI}'} h_1(\phi)$. If ϕ is an axiom other than (S6) and (S7), this is trivial. The translation of (S6) is the axiom (S6') of $\mathcal{SI}'$. The translation of (S7), $h_1(\bot \rightarrow_H \alpha) = (x \wedge \neg x) \rightarrow_H h_1(\alpha)$, is an instance of axiom (S7') of $\mathcal{SI}'$.

Suppose now that for some $\beta \in Fm_{\mathbf{L}}$, $\vdash_{\mathcal{SI}} \beta$ and $\vdash_{\mathcal{SI}} \beta \rightarrow_H \phi$. Then, by inductive hypothesis, $\vdash_{\mathcal{SI}'} h_1(\beta)$ and $\vdash_{\mathcal{SI}'} h_1(\beta \rightarrow_H \phi)$. Hence $\vdash_{\mathcal{SI}'} h_1(\beta) \rightarrow_H h_1(\phi)$ and in consequence, by SMP, $\vdash_{\mathcal{SI}'} h_1(\phi)$. ■

Definition 3.13. Let $h_2 : Fm_{\mathbf{L}'} \rightarrow Fm_{\mathbf{L}}$ be the translation given by

- $h_2(\alpha \wedge \beta) = h_2(\alpha) \wedge h_2(\beta)$,
- $h_2(\alpha \vee \beta) = h_2(\alpha) \vee h_2(\beta)$,
- $h_2(\alpha \rightarrow \beta) = h_2(\alpha) \rightarrow h_2(\beta)$,

- $h_2(\neg\alpha) = h_2(\alpha) \to \perp$

Lemma 3.14. *The translation h_2 is sound.*

Proof. It is enough to check that the translations of axioms (S6'), (S7') and (S12') to (S18') are theorems of $\mathcal{SI}$.

The translation of (S6') is $h_2(\alpha) \to_H h_2(\alpha)$, which we proved in Lemma 2.1 (b).

The translation of (S7') is also a theorem:
$\vdash_{\mathcal{SI}} (h_2(\alpha) \wedge (h_2(\alpha) \to \perp)) \to_H h_2(\beta)$ by Lemma 3.8, part a).

The translations of (S12') to (S17') yield instances of those same axioms and were proved in Lemma 2.1.

Finally, we prove that the translation of (S18'), $(h_2(\alpha) \to_H (h_2(\alpha) \to \perp)) \to_H (h_2(\alpha) \to \perp)$, is also a theorem of $\mathcal{SI}$.

Let γ be the formula $\alpha \to_H (\alpha \to \perp)$.

1. $\vdash_{\mathcal{SI}} [\alpha \wedge \gamma] \to_H \perp$ and $\vdash_{\mathcal{SI}} \perp \to_H [\alpha \wedge \gamma]$ by Lemma 3.8 (e).
2. $\vdash_{\mathcal{SI}} [\alpha \wedge \gamma] \leftrightarrow_H \perp$ by Lemma 3.4 applied to 1.
3. $\vdash_{\mathcal{SI}} [\gamma \wedge \perp] \to_H [\alpha \wedge \gamma]$ and $\vdash_{\mathcal{SI}} [\alpha \wedge \gamma] \to_H [\gamma \wedge \perp]$ by Lemma 3.8 (d).
4. $\vdash_{\mathcal{SI}} [\alpha \wedge \gamma] \to_H [\gamma \wedge \alpha]$ and $\vdash_{\mathcal{SI}} [\gamma \wedge \alpha] \to_H [\alpha \wedge \gamma]$ by Lemma 2.1, (e).
5. $\vdash_{\mathcal{SI}} [\gamma \wedge \perp] \to_H [\gamma \wedge \alpha]$ and $\vdash_{\mathcal{SI}} [\gamma \wedge \alpha] \to_H [\gamma \wedge \perp]$ by Lemma 2.1(n) and SMP applied to 3 and 4.
6. $\vdash_{\mathcal{SI}} [[\gamma \wedge \perp] \to [\gamma \wedge \perp]] \to_H [[\gamma \wedge \alpha] \to [\gamma \wedge \perp]]$ by (S10) and SMP applied to the previous step.
7. $\vdash_{\mathcal{SI}} [\gamma \wedge [[\gamma \wedge \perp] \to [\gamma \wedge \perp]]] \to_H [\gamma \wedge [[\gamma \wedge \alpha] \to [\gamma \wedge \perp]]]$ by Lemma 2.1 (k) applied to step 6.
8. $\vdash_{\mathcal{SI}} [\gamma \wedge [[\gamma \wedge \alpha] \to [\gamma \wedge \perp]]] \to_H [\gamma \wedge [\alpha \to \perp]]$ by Lemma 2.1(r).
9. $\vdash_{\mathcal{SI}} [\gamma \wedge [\alpha \to \perp]] \to_H (\alpha \to \perp)$ by Lemma 2.1(d).
10. $\vdash_{\mathcal{SI}} [\gamma \wedge [[\gamma \wedge \perp] \to [\gamma \wedge \perp]]] \to_H (\alpha \to \perp)$ by Lemma 2.1(n) and SMP applied to 7, 8 and 9.
11. $\vdash_{\mathcal{SI}} [\gamma \wedge (\perp \to \perp)] \to_H [\gamma \wedge [[\gamma \wedge \perp] \to [\gamma \wedge \perp]]]$ by Lemma 2.1(q) .
12. $\vdash_{\mathcal{SI}} [\gamma \wedge (\perp \to \perp)] \to_H (\alpha \to \perp)$ by Lemma 2.1(n) and SMP applied to 10 and 11.
13. $\vdash_{\mathcal{SI}} \gamma \to_H [\gamma \wedge (\perp \to \perp)]$ by Lemma 2.1 (f) and (c).
14. $\vdash_{\mathcal{SI}} \gamma \to_H (\alpha \to \perp)$ by Lemma 2.1(n) and SMP applied to 12 and 13.
15. $\vdash_{\mathcal{SI}} (\alpha \to_H (\alpha \to \perp)) \to_H (\alpha \to \perp)$ replacing γ by its definition. ■

THEOREM 3.15. *The logics $\mathcal{SI}$ and $\mathcal{SI}'$ are translationally equivalent.*

PROOF. To prove this we shall consider the translations h_1 and h_2 introduced in Lemmas 3.12 and 3.14.

Let $\phi \in Fm_{\mathbf{L}}$. We prove now that

$$\vdash_{\mathcal{SI}} \phi \leftrightarrow_H h_2(h_1(\phi)) \tag{1}$$

by induction over the structure of ϕ. If $\phi = x$, with $x \in Var$, then $h_2(h_1(x)) = x$ and the result follows from 2.1 (b).

Suppose now that $\phi = \phi_1 \vee \phi_2$ with $\phi_1, \phi_2 \in Fm_{\mathbf{L}}$. By inductive hypothesis, we have that $\vdash_{\mathcal{SI}} \phi_1 \leftrightarrow_H h_2(h_1(\phi_1))$ and $\vdash_{\mathcal{SI}} \phi_2 \leftrightarrow_H h_2(h_1(\phi_2))$. Using Lemma 3.5, e), and theorem 3.9 this yields $\vdash_{\mathcal{SI}} (\phi_1 \vee \phi_2) \leftrightarrow_H (h_2(h_1(\phi_1)) \vee h_2(h_1(\phi_2)))$. By the definitions of h_1 and h_2, $h_2(h_1(\phi_1)) \vee h_2(h_1(\phi_2)) = h_2(h_1(\phi_1 \vee \phi_2))$, so we have the result for this case.

For $\phi = \phi_1 \to \phi_2$ and $\phi = \phi_1 \wedge \phi_2$ the proofs are similar. If $\phi = \bot$, then $h_2(h_1(\phi))$ is the formula $x \wedge (x \to \bot)$, for some $x \in Var$.

1. $\vdash_{\mathcal{SI}} \bot \to_H (x \wedge (x \to \bot))$ by (S7).
2. $\vdash_{\mathcal{SI}} (x \wedge (x \to \bot)) \to_H (x \wedge \bot)$ by Lemma 2.1(t).
3. $\vdash_{\mathcal{SI}} (x \wedge \bot) \to_H \bot$ by Lemma 2.1(d).
4. $\vdash_{\mathcal{SI}} (x \wedge (x \to \bot)) \to_H \bot$ by Lemma 2.1(n) and SMP applied to 2 and 3.
5. $\vdash_{\mathcal{SI}} \bot \leftrightarrow_H (x \wedge (x \to \bot))$ by Lemma 3.4 applied to 1 and 4.

Finally, for $\phi = \top$, $h_2(h_1(\phi))$ is $x \to x$ and we have:

1. Since $\top$ is the axiom (S6), by Lemma 2.1, a), $\vdash_{\mathcal{SI}} (x \to x) \to_H \top$.
2. $\vdash_{\mathcal{SI}} x \to x$ by Lemma 2.1, part (f).
3. $\vdash_{\mathcal{SI}} \top \to_H (x \to x)$ by the previous item and Lemma 2.1, (a).
4. $\vdash_{\mathcal{SI}} \top \leftrightarrow_H (x \to x)$ by Lemma 3.4 applied to items 1 and 3.

We check now that

$$\vdash_{\mathcal{SI}'} \psi \leftrightarrow_H h_1(h_2(\psi)) \tag{2}$$

for $\psi \in Fm_{\mathbf{L}'}$. The cases in which $\psi = x$, with $x \in Var$, $\psi = \psi_1 \vee \psi_2$, $\psi = \psi_1 \to \psi_2$ and $\psi = \psi_1 \wedge \psi_2$ with $\psi_1, \psi_2 \in Fm_{\mathbf{L}'}$ are verified as before. It only remains to check that

$$\vdash_{\mathcal{SI}'} \neg\alpha \leftrightarrow_H h_1(h_2(\neg\alpha)).$$

1. $\vdash_{\mathcal{SI}'} \neg\alpha \leftrightarrow_H (\alpha \to (x \wedge \neg x))$ by Lemma 3.10 (b).
2. $\vdash_{\mathcal{SI}'} \neg\alpha \to_H (\alpha \to (x \wedge \neg x))$ by Lemma 3.4 applied to item 1.

3. $\vdash_{\mathcal{SI}'} \alpha \leftrightarrow_H h_1(h_2(\alpha))$ by inductive hypothesis.
4. $\vdash_{\mathcal{SI}'} (x \wedge \neg x) \leftrightarrow_H (x \wedge \neg x)$ by Lemma 3.5 (a).
5. $\alpha \leftrightarrow_H h_1(h_2(\alpha)), (x \wedge \neg x) \leftrightarrow_H (x \wedge \neg x) \vdash_{\mathcal{SI}'} [\alpha \to (x \wedge \neg x)] \leftrightarrow_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by Lemma 3.5 f).
6. $\vdash_{\mathcal{SI}'} [\alpha \to (x \wedge \neg x)] \leftrightarrow_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by theorem 3.9 and SMP applied to 3 and 4.
7. $\vdash_{\mathcal{SI}'} [\alpha \to (x \wedge \neg x)] \to_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by Lemma 3.4.
8. $\vdash_{\mathcal{SI}'} (\neg\alpha) \to_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by Lemma 2.1(n) and SMP applied to 2 and 7.

   In a similar way we obtain,
9. $\vdash_{\mathcal{SI}'} [h_1(h_2(\alpha)) \to (x \wedge \neg x)] \to_H (\neg\alpha)$.
10. $\vdash_{\mathcal{SI}'} (\neg\alpha) \leftrightarrow_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by Lemma 3.4.

Finally, we note that $h_1(h_2(\neg\alpha))$ is the formula $h_1(h_2(\alpha)) \to (x \wedge \neg x)$. ■

From now on, we use interchangeably $x \to x$ and $\top$, $\neg x$ and $x \to \bot$, and $\bot$ and $\neg(x \to x)$.

4. Algebraic Semantics

The set of formulas $Fm_{\mathbf{L}}$ can be algebrized in the usual way. The homomorphisms from the formula algebra $\mathsf{Fm}_{\mathbf{L}}$ into an $\mathbf{L}$-algebra $\mathbf{A}$ are called *interpretations*. The set of all such interpretations is denoted by $Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$. If $h \in Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$ then the *interpretation of a formula* α under h is its image $h\alpha \in A$. For formulas in $Fm_{\mathbf{L}'}$, we first translate them via h_2 and then interpret them as $\mathbf{L}$-formulas.

Equations are pairs of formulas that will be written in the form $\alpha \approx \beta$. An equation $\alpha \approx \beta$ *is satisfied by the interpretation* h in $\mathbf{A}$ if $h\alpha = h\beta$. We denote this by $\models_{\mathbf{A}} \alpha \approx \beta[\![h]\!]$. An algebra $\mathbf{A}$ *satisfies the equation* $\alpha \approx \beta$ if all the interpretations in $\mathbf{A}$ satisfy it; in symbols,

$$\models_{\mathbf{A}} \alpha \approx \beta \text{ if and only if } \models_{\mathbf{A}} \alpha \approx \beta[\![h]\!] \text{ for all } h \in Hom(\mathsf{Fm}, \mathbf{A}).$$

A class of algebras K *satisfies the equation* $\alpha \approx \beta$ when all the algebras in K satisfy it; i.e.

$$\models_{\mathsf{K}} \alpha \approx \beta \text{ if and only if } \models_{\mathbf{A}} \alpha \approx \beta \text{ for all } \mathbf{A} \in \mathsf{K}.$$

Since the class of algebras we are working with is that of semi-Heyting algebras SH, we can introduce a pre-order relation on the set of formulas:

given $\alpha, \beta \in Fm_{\mathbf{L}}$ (or $Fm_{\mathbf{L}'}$), we write $\alpha \leq \beta$ iff $\models_{\mathsf{SH}} \alpha \approx \alpha \wedge \beta$. Furthermore, we will denote $\models_{\mathsf{SH}}$ simply by $\models$, and introduce the following notation; for $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$ and $\mathbf{A} \in \mathsf{SH}$:

$$\Gamma \models_{\mathbf{A}} \alpha \approx \beta \text{ if and only if}$$
$$\text{for all } \gamma \in \Gamma \text{ if } \models_{\mathbf{A}} \gamma \approx \top, \text{ then } \models_{\mathbf{A}} \alpha \approx \beta.$$

And

$$\Gamma \models \alpha \approx \beta \text{ if and only if } \Gamma \models_{\mathbf{A}} \alpha \approx \beta \text{ for all } \mathbf{A} \in \mathsf{SH}.$$

Using theorems 3.17 and 3.18 from [4] it is easy to verify the following theorem.

THEOREM 4.1. *For* $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}'}$ *the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{SI}'} \phi$

2. $\Gamma \models \phi \approx \top$

By theorems 3.9, 3.15, and 4.1, we obtain the next result:

THEOREM 4.2. *(Completeness) For* $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ *the following are equivalent:*

1. $\Gamma \vdash_{\mathcal{SI}} \phi$

2. $\Gamma \models \phi \approx \top$

It is worth noting that the inference rule SMP implies the traditional Modus Ponens (MP) for the connective $\rightarrow$.

LEMMA 4.3. *If* $\Gamma \vdash_{\mathcal{SI}} \alpha$ *and* $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow \beta$ *then* $\Gamma \vdash_{\mathcal{SI}} \beta$.

PROOF.

1. $\Gamma \vdash_{\mathcal{SI}} \alpha$ by hypothesis.
2. $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow \beta$ by hypothesis.
3. $\Gamma \vdash_{\mathcal{SI}} \alpha \wedge (\alpha \rightarrow \beta)$ by Lemma 3.8, part b).
4. $\Gamma \vdash_{\mathcal{SI}} \alpha \wedge \beta$ by SMP applied to 3 and Lemma 2.1(t).
5. $\Gamma \vdash_{\mathcal{SI}} \beta$ by SMP in 4 and Lemma 2.1(d). ■

On the other hand, the following example shows that Modus Ponens does not imply SMP. Consider the logic $\mathcal{B}$ defined by axioms (S1) to (S11), with MP as its only inference rule. We next present an algebraic model for the logic $\mathcal{B}$ that is not a semi-Heyting algebra.

Consider the algebra $\mathbf{A}$ with universe $\{\bot, \top\}$ and the operations $\wedge, \vee, \rightarrow$ defined by:

$\wedge$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\top$	$\top$

$\vee$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$

$\rightarrow$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$

This algebra $\mathbf{A}$ satisfies $x \rightarrow_H y = x \rightarrow (x \wedge y) \approx \top$ for any election of x and y, and all the axioms except (S6) are of that form. It is also easy to check that MP is satisfied in $\mathbf{A}$. Therefore, $\mathbf{A}$ is a model for the logic $\mathcal{B}$. It is also clear that $\mathbf{A}$ is not a semi-Heyting algebra.

Definition 4.4. [13] Let $\mathcal{L}$ be a logic in a language with a binary connective $\rightarrow$, either primitive or defined by a term in exactly two variables. Then $\mathcal{L}$ is called an implicative logic with respect to the binary connective $\rightarrow$ if the following conditions are satisfied:

(IL1) $\vdash_{\mathcal{L}} \alpha \rightarrow \alpha$.

(IL2) $\alpha \rightarrow \beta, \beta \rightarrow \gamma \vdash_{\mathcal{L}} \alpha \rightarrow \gamma$.

(IL3) For each connective f in the language of arity $n > 0$,
$$\left\{\begin{matrix} \alpha_1 \rightarrow \beta_1, \ldots, \alpha_n \rightarrow \beta_n \\ \beta_1 \rightarrow \alpha_1, \ldots, \beta_n \rightarrow \alpha_n \end{matrix}\right\} \vdash_{\mathcal{L}} f(\alpha_1, \ldots, \alpha_n) \rightarrow f(\beta_1, \ldots, \beta_n).$$

(IL4) $\alpha, \alpha \rightarrow \beta \vdash_{\mathcal{L}} \beta$.

(IL5) $\alpha \vdash_{\mathcal{L}} \beta \rightarrow \alpha$.

Definition 4.5. [13, Definition 6, page 181] Let $\mathcal{L}$ be an implicative logic on the language $\mathbf{L}$. An $\mathcal{L}$-algebra is an algebra $\mathbf{A}$ of similarity type $\mathbf{L}$ that has an element $\top$ with the following properties:

(LALG1) For all $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ and all $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$, if $\Gamma \vdash_{\mathcal{L}} \phi$ and $h\Gamma \subseteq \{\top\}$ then $h\phi = \top$

(LALG2) For all $a, b \in A$, if $a \rightarrow b = \top$ and $b \rightarrow a = \top$ then $a = b$.

The class of $\mathcal{L}$-algebras is denoted by $\mathsf{Alg}^*\mathcal{L}$.

Lemma 4.6. *$\mathcal{SI}$ is an implicative logic with respect to the binary connective $\rightarrow_H$.*

Proof. Observe that

$$\vdash_{\mathcal{SI}} \alpha \rightarrow_H \alpha$$

by Lemma 2.1, (b). Condition (IL2) follows from Lemma 2.1(n) by SMP. Condition (IL3) follows from Lemma 3.5. Condition (IL4) is SMP, and (IL5) follows from Lemma 2.1, part (a). ■

We note that the logic $\mathcal{SI}$ is not an implicative logic with respect to the operator $\to$. For this consider the two-element commutative semi-Heyting algebra $\bar{\mathbf{2}}$. If $x \vdash_{\mathcal{SI}} y \to x$ then, by the deduction theorem 3.9, it follows that $\vdash_{\mathcal{SI}} x \to_{H} (y \to x)$. Using completeness (theorem 4.2), $\bar{\mathbf{2}} \models x \to_{H} (y \to x) \approx \top$, which does not hold for $x = \top$ and $y = \bot$.

Since $\mathcal{SI}$ is an implicative logic with respect to the binary connective $\to_{H}$, we have the next result using [13, Theorem 7.1, p. 222].

Theorem 4.7. *The logic $\mathcal{SI}$ is complete with respect to the class $\mathsf{Alg}^*\mathcal{SI}$. In other words, for all $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$,*

$$\Gamma \vdash_{\mathcal{SI}} \phi \text{ if and only if } h\Gamma \subseteq \{\top\} \text{ implies } h\phi = \top,$$

for all $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ and all $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SI}$.

As a consequence, by theorem 4.2, we have

Corollary 4.8. $\mathsf{Alg}^*\mathcal{SI} = \mathsf{SH}$.

5. Formulas that Define Semi-Heyting Implications

In this section we characterize semi-intuitionistic formulas in two variables that define semi-Heyting implications.

Lemma 5.1. *Let $\epsilon(x, y) \in Fm_{\mathbf{L}}$ be a formula in two variables. Then*

$$\models x \wedge \epsilon(y, z) \approx x \wedge \epsilon(x \wedge y, x \wedge z).$$

Proof. By an easy induction on the construction of the formula $\epsilon(x, y)$. The key step is the one for the implication, where (SH3) is used. Let $a, b, c \in A$ with $\mathbf{A} \in \mathsf{SH}$. We assume that for $i = 1, 2$, $x \wedge \epsilon_i(y, z) \approx x \wedge \epsilon_i(x \wedge y, x \wedge z)$.

If $\epsilon(a, b) = \epsilon_1(a, b) \to \epsilon_2(a, b)$ then $a \wedge \epsilon(b, c) = a \wedge (\epsilon_1(b, c) \to \epsilon_2(b, c)) = a \wedge ((a \wedge \epsilon_1(b, c)) \to (a \wedge \epsilon_2(b, c))) = a \wedge ((a \wedge \epsilon_1(a \wedge b, a \wedge c)) \to (a \wedge \epsilon_2(a \wedge b, a \wedge c))) = a \wedge (\epsilon_1(a \wedge b, a \wedge c) \to \epsilon_2(a \wedge b, a \wedge c)) = a \wedge \epsilon(a \wedge b, a \wedge c)$. ■

Looking at the definition of semi-Heyting algebras and using Lemma 5.1, we can easily see which formulas define operations that yield new semi-Heyting algebras:

Lemma 5.2. *Let $\langle A, \vee, \wedge, \to, \bot, \top \rangle$ be a semi-Heyting algebra. Then a formula in two variables $\epsilon(x, y) \in Fm_{\mathbf{L}}$ satisfies the following conditions:*

a) $\models \epsilon(x, x) \approx \top$,

b) $\models x \wedge \epsilon(x, y) \approx x \wedge y$,

if and only if $\langle A, \vee, \wedge, \epsilon, \bot, \top \rangle \in \mathsf{SH}$.

We denote with IF the set of all the formulas in two variables that define a semi-Heyting implication.

The following lemma states that we can always define a Heyting algebra structure over any semi-Heyting algebra. Moreover, among all the semi-Heyting implication operations that can be defined in a given distributive lattice, the Heyting implication is the greatest one.

Lemma 5.3. [2, Lemma 4.1] *Let* $\langle A, \vee, \wedge, \rightarrow, \bot, \top\rangle$ *be a semi-Heyting algebra. If we define* $a \rightarrow_H b = a \rightarrow (a \wedge b)$ *for every* $a, b \in A$, *then* $\langle A, \vee, \wedge, \rightarrow_H, \bot, \top\rangle$ *is a Heyting algebra. Furthermore,* $a \rightarrow b \leq a \rightarrow_H b$ *for every* $a, b \in A$.

Proof. We prove first that $\rightarrow_H$ is a semi-Heyting implication. Let $a, b, c \in A$ with $\mathbf{A} \in \mathsf{SH}$. Conditions a) and b) from Lemma 5.2 are verified as follows: $a \rightarrow_H a = a \rightarrow (a \wedge a) = \top$, so we have a). For condition b), $a \wedge (a \rightarrow_H b) = a \wedge (a \rightarrow (a \wedge b)) = a \wedge a \wedge b = a \wedge b$.

Finally, to prove that $\rightarrow_H$ is a Heyting algebra implication, it is enough to prove that for all $a, b \in A$, $(a \wedge b) \rightarrow_H a = (a \wedge b) \rightarrow ((a \wedge b) \wedge a) = \top$, which follows from a) and elementary lattice properties.

For the second statement, we calculate $(a \rightarrow b) \wedge (a \rightarrow_H b) = (a \rightarrow b) \wedge [a \rightarrow (a \wedge b)] = (a \rightarrow b) \wedge [(a \wedge (a \rightarrow b)) \rightarrow (a \wedge b \wedge (a \rightarrow b))] = (a \rightarrow b) \wedge ((a \wedge b) \rightarrow (a \wedge b)) = (a \rightarrow b) \wedge \top = a \rightarrow b$. Thus $a \rightarrow b \leq a \rightarrow_H b$. ∎

Pseudocomplemented lattices or *p-lattices* have been studied by many authors (see, for example [3], [8], [9], [13]). Semi-Heyting algebras are pseudocomplemented [14] (with $\neg x = x \rightarrow \bot$) as well.

Since there may be more than one semi-Heyting implication over a given lattice, we denote with the pair $\langle A, \rightarrow\rangle$ the algebras in the variety SH to make explicit which implication we are considering. Next we show that regardless the implication chosen, they all agree on the pseudocomplement.

Corollary 5.4. *Let* $\mathbf{A} = \langle A, \rightarrow\rangle \in \mathsf{SH}$. *Then* $\models_{\mathbf{A}} x \rightarrow \bot \approx x \rightarrow_H \bot$, *where* $\rightarrow_H$ *is defined as in Lemma 5.3.*

Proof. Let $a \in A$. From Lemma 5.3 we have that $a \rightarrow \bot = a \rightarrow (a \wedge \bot) = a \rightarrow_H \bot$. ∎

In Lemma 5.3 we saw that from any semi-Heyting implication $\rightarrow$ we can obtain the Heyting implication. The next result shows that to recover the original implication, it is enough to know its values for $x < y$.

Lemma 5.5. *Let* $\mathbf{A} = \langle A, \rightarrow\rangle \in \mathsf{SH}$. *Then*

$$\models_{\mathbf{A}} x \rightarrow y \approx (x \rightarrow_H y) \wedge ((x \wedge y) \rightarrow y).$$

Proof. Let $a, b \in A$. Then we calculate $(a \to (a \wedge b)) \wedge ((a \wedge b) \to b) = (a \to (a \wedge b)) \wedge [(a \wedge (a \to (a \wedge b))) \to b] = (a \to (a \wedge b)) \wedge [(a \wedge (a \to (a \wedge b))) \to (b \wedge ((b \wedge a) \to (b \wedge a \wedge b)))] = (a \to (a \wedge b)) \wedge [(a \wedge (a \to (a \wedge b))) \to (b \wedge (a \to (a \wedge b)))] = (a \to (a \wedge b)) \wedge (a \to b) = (a \to b) \wedge (a \to (a \wedge b)) = (a \to b) \wedge [(a \wedge (a \to b)) \to (a \wedge b \wedge (a \to b))] = (a \to b) \wedge ((a \wedge b) \to (a \wedge b)) = (a \to b) \wedge \top = a \to b$. ■

In [14] the variety of commutative semi-Heyting algebras was defined. These are the semi-Heyting algebras satisfying the identity $x \to y \approx y \to x$.

The next result generalizes [2, Lemma 4.3].

Lemma 5.6. *Let $\mathbf{A} = \langle A, \to \rangle$ be a semi-Heyting algebra. If we define on A the implication*

$$x \leftrightarrow_H y = (x \to_H y) \wedge (y \to_H x),$$

then:

(a) *$\langle A, \leftrightarrow_H \rangle$ is a commutative semi-Heyting algebra.*

(b) $\models_{\mathbf{A}} x \leftrightarrow_H y \leq x \to y$

(c) *$\leftrightarrow_H$ is the only commutative semi-Heyting operation that can be defined over A.*

Proof.

(a) Let us check that $\leftrightarrow_H$ is a semi-Heyting implication. In first place, $a \leftrightarrow_H a = (a \to_H a) \wedge (a \to_H a) = \top \wedge \top = \top$. Now we calculate $a \wedge (a \leftrightarrow_H b) = a \wedge (a \to_H b) \wedge (b \to_H a) = a \wedge b \wedge (b \to_H a) = a \wedge b \wedge a = a \wedge b$ so $\langle \mathbf{A}, \leftrightarrow_H \rangle$ is a semi-Heyting algebra. It is straightforward to check that the implication is commutative.

(b) We prove that $a \leftrightarrow_H b \leq a \to b$ on $\mathbf{A}$. Indeed $(a \to b) \wedge (a \leftrightarrow_H b) = (a \to b) \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = [(a \wedge (a \to (a \wedge b))) \to (b \wedge (a \to (a \wedge b)))] \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = ((a \wedge b) \to b) \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = [(a \wedge b \wedge (b \to (a \wedge b))) \to (b \wedge (b \to (a \wedge b)))] \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = ((a \wedge b) \to (a \wedge b)) \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = \top \wedge (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = (a \to (a \wedge b)) \wedge (b \to (a \wedge b)) = a \leftrightarrow_H b$. Thus, $a \leftrightarrow_H b \leq a \to b$.

(c) Let $\to$ and $\to'$ be two commutative semi-Heyting implications defined over $\mathbf{A}$. Then $(a \to b) \wedge (a \to' b) = (a \to b) \wedge [(a \wedge (a \to b)) \to' (b \wedge (a \to b))] = (a \to b) \wedge [(a \wedge b) \to' (b \wedge (b \to a))] = (a \to b) \wedge [(a \wedge b) \to' (b \wedge a)] = a \to b$. Therefore, $(a \to b) \leq (a \to' b)$. In a similar fashion, one proves $(a \to' b) \leq (a \to b)$. ■

The next result determines a necessary and sufficient condition for semi-Heyting terms to be semi-Heyting implications.

Theorem 5.7. *A formula* $\epsilon(x, y)$ *defines a semi-Heyting implication if and only if*

$$x \leftrightarrow_H y \leq \epsilon(x, y) \leq x \rightarrow_H y.$$

Proof. If $\epsilon(x, y)$ defines a semi-Heyting implication, it follows immediately from Lemmas 5.3 and 5.6 that $x \leftrightarrow_H y \leq \epsilon(x, y) \leq x \rightarrow_H y$.

Assume now that $x \leftrightarrow_H y \leq \epsilon(x, y) \leq x \rightarrow_H y$. Then $\models x \leftrightarrow_H x \leq \epsilon(x, x)$ so $\models x \leftrightarrow_H x \approx \top$.

Since $x \wedge y = x \wedge (x \leftrightarrow_H y) \leq x \wedge \epsilon(x, y) \leq x \wedge (x \rightarrow_H y) = x \wedge y$, we obtain $x \wedge \epsilon(x, y) = x \wedge y$. By Lemma 5.1 it follows that $\epsilon(x, y)$ defines a semi-Heyting implication. ■

As an example, let us consider the totally ordered lattice

$$\mathbf{C} : \perp < a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots < \top$$

with a countable number of elements.

We define on $\mathbf{C}$ an operation $\rightarrow$ by:

$$x \rightarrow y = \begin{cases} y & \text{if} \quad x > y, \\ a_{k+1} & \text{if} \quad x = \perp,\ y = a_k \text{ with } k \geq 1, \\ \top & \text{otherwise.} \end{cases}$$

Using [1, Lemmas 2.2 and 2.3] we can check that $\langle C, \rightarrow\rangle$ is a semi-Heyting chain.

We define $x \xrightarrow{n} y$ recursively:

$$x \xrightarrow{1} y = x \rightarrow y \quad \text{and} \quad x \xrightarrow{n+1} y = x \rightarrow (x \xrightarrow{n} y).$$

Observe that in $\mathbf{C}$, $\perp \xrightarrow{n} a_1 = a_{n+1}$. Now we consider the formulas:

$$\epsilon_n(x, y) := (x \leftrightarrow_H y) \vee (\neg x \wedge (x \xrightarrow{n} y)).$$

It follows immediately that $x \leftrightarrow_H y \leq \epsilon_n(x, y)$. Since $\neg x \leq x \rightarrow_H y$ for all x, we also have that $\epsilon_n(x, y) \leq x \rightarrow_H y$, so $\epsilon_n \in IF$.

In the algebra $\mathbf{C}$, taking $x = \perp$ and $y = a_1$, the formula $\epsilon_n(x, y)$ yields $\epsilon_n(\perp, a_1) = a_{n+1}$. Consequently all the formulas in $\{\epsilon_n(x, y)\}_{n\in\mathbb{N}}$ give, in general, different semi-Heyting implications on $\mathbf{C}$.

In Heyting algebra, however, all these formulas collapse.

Lemma 5.8. *If* $\mathbf{A} = \langle A; \rightarrow_H\rangle$ *is a Heyting algebra then* $\epsilon_n^{\mathbf{A}}(a, b) = \epsilon_m^{\mathbf{A}}(a, b)$ *for all* $a, b \in A$ *and* $n, m \in \mathbb{N}$.

Proof. We check first that $a \rightarrow_H b \leq a \xrightarrow{n}_H b$ for all $n \in \mathbb{N}$ by induction over n. For $n = 1$ it is immediate. For the inductive step: $a \wedge (a \rightarrow_H b) \leq$

$a \wedge (a \xrightarrow{k}_H b) \leq a \xrightarrow{k}_H b$. So $a \rightarrow_H b \leq a \rightarrow_H (a \xrightarrow{k}_H b) = a \xrightarrow{k+1}_H b$ since $\mathbf{A}$ is a Heyting algebra.

It is easy to verify that $a \wedge (a \xrightarrow{n}_H b) = a \wedge b$. Then $a \wedge (a \xrightarrow{n}_H b) \leq b$, so $a \xrightarrow{n}_H b \leq a \rightarrow_H b$. Then we have $\epsilon_n^{\mathbf{A}}(a,b) = (a \leftrightarrow_H b) \vee (\neg a \wedge (a \rightarrow_H b)) = \epsilon_m^{\mathbf{A}}(a,b)$. ■

6. Intuitionistic Formulas that Define Semi-Heyting Implications

We have seen in Lemma 5.3 that in every semi-Heyting algebra, we have the intuitionistic implication $\rightarrow_H$. This implication, unlike semi-intuitionistic implications in general, is uniquely determined for each underlying lattice. Therefore, when we consider formulas in the language $\{\wedge, \vee, \bot, \top, \rightarrow_H\}$, they have a unique interpretation in a lattice once the variables have been assigned. We call these formulas *H-formulas*. These formulas can of course also be written in the language $\mathbf{L}$.

Definition 6.1. Let IF_H be the set of H-formulas in two variables that define a semi-intuitionistic implication, that is, that fulfill the conditions of Lemma 5.2.

In this section our first goal is to prove that each formula in IF_H determines a subvariety of SH that is term equivalent to the one of Heyting algebras, and also a semi-intuitionistic propositional calculus translationally equivalent to the intuitionistic one.

First we will present an infinite family of formulas of this kind and prove that they yield different implications.

Let $\mathcal{F}_{\mathcal{H}}(x)$ be the free Heyting algebra on one generator x (see for example [10], [3]). There exists in this algebra an infinite sequence of H-formulas on the variable x, $\{M_i(x)\}_{i\geq 1}$ pairwise non-equivalent and such that $x \leq M_i(x)$. For each $i \in I$ we define the formula

$$\mu_i(x,y) = (x \leftrightarrow_H y) \vee (y \wedge M_i(x)).$$

Lemma 6.2. *The family of H-formulas $\{\mu_i(x,y)\}_{i\in I}$ define non-equivalent semi-intuitionistic implications.*

Proof. It is clear that $x \leftrightarrow_H y \leq \mu_i(x,y)$. Since $x \wedge y \wedge M_i(x) = x \wedge y \leq y$, $y \wedge M_i(x) \leq x \rightarrow_H y$, so $\mu_i(x,y) \leq x \rightarrow_H y$. Therfore, by Theorem 5.7, $\mu_i(x,y) \in IF_H$.

If $i \neq j$ then the formulas $\mu_i(x,y)$ and $\mu_j(x,y)$ are not equivalent. To see this, it is enough to consider in the semi-Heyting algebra $\mathcal{F}_{\mathcal{H}}(x)$,

$$\mu_i(x, \top) = x \vee M_i(x) = M_i(x)$$

and since $M_i(x) \neq M_j(x)$ we have that $\mu_i(x, \top) \neq \mu_j(x, \top)$. ■

For $\epsilon \in IF_H$ we introduce the following subvariety of the variety of semi-Heyting algebras.

$$\mathsf{SH}_\epsilon = \{\mathbf{A} \in \mathsf{SH} : \models_{\mathbf{A}} x \to y \approx \epsilon\}.$$

In all of these varieties, the implication operation is uniquely determined, since ϵ is an H-formula.

Corollary 6.3. *The varieties associated to the formulas $\mu_i(x, y)$ and $\mu_j(x, y)$, SH_{μ_i} and SH_{μ_j}, are incomparable.*

We can construct another sequence of non-equivalent H-formulas, and different also from the formulas μ_i. Consider a sequence of formulas $\{N_j(x)\}_{j\geq 1}$ such that $N_j(x) \geq x \vee \neg x$ in the algebra $\mathcal{F}_{\mathcal{H}}(x)$. Now let

$$\nu_j(x, y) = (x \leftrightarrow_H y) \vee (y \wedge N_j(x)) \vee (\neg x \wedge \neg\neg y).$$

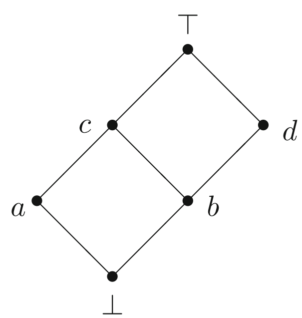
Lemma 6.4. *The H-formulas $\nu_j(x, y)$ are in IF_H. If $i \neq j$ then $\nu_i \neq \nu_j$. Furthermore, the formulas ν_j are not equivalent to μ_i for any i.*

Proof. In the semi-Heyting algebra $\mathcal{F}_{\mathcal{H}}(x)$ we calculate

$$\nu_i(x, \top) = x \vee N_i(x) \vee \neg x = N_i(x)$$

so, since $N_i(x) \neq N_j(x)$ in $\mathcal{F}_{\mathcal{H}}(x)$,we have that $\nu_i(x, \top) \neq \nu_j(x, \top)$ if $i \neq j$. Therefore $\{\nu_i\}_{i\geq 1}$ is an infinite sequence of non-equivalent formulas.

Finally, consider the Heyting algebra $\mathbf{A}$ in the diagram below:



We have that $\nu_i^{\mathbf{A}}(\bot, b) = \neg b \vee (b \wedge N_i(\bot)) \vee (\neg\bot \wedge \neg\neg b) = \neg b \vee (b \wedge N_i(\bot)) \vee \neg\neg b = a \vee (b \wedge N_i(\bot)) \vee d = \top$. On the other hand, $\mu_j^{\mathbf{A}}(\bot, b) = \neg b \vee (b \wedge M_j(\bot))$. Since $M_j(x)$ is an H-formula then $M_j(\bot) \in \{\bot, \top\}$. Therefore

$$\mu_j^{\mathbf{A}}(\bot, b) = \begin{cases} \neg b \vee b \text{ if } M_j(\bot) = \top \\ \neg b \quad\quad \text{if } M_j(\bot) = \bot \end{cases} = \begin{cases} a \vee b \text{ if } M_j(\bot) = \top \\ a \quad\quad \text{if } M_j(\bot) = \bot \end{cases}$$
$$= \begin{cases} c \text{ if } M_j(\bot) = \top \\ a \text{ if } M_j(\bot) = \bot \end{cases}$$

So $\nu_i^{\mathbf{A}}(\bot, b) \neq \mu_j^{\mathbf{A}}(\bot, b)$. ■

THEOREM 6.5. *The varieties* SH_ϵ *are term-equivalent to the variety* H *of Heyting algebras, for all* $\epsilon \in IF_H$.

PROOF. The equivalence is given by the equation $x \rightharpoonup_H y = \epsilon(x, x \wedge y)$ and the fact that ϵ is an H-formula. ■

DEFINITION 6.6. Let ϵ be a formula in IF_H. The logic $\mathcal{SI}_\epsilon$ is the axiomatic extension of $\mathcal{SI}$ defined by adding the axiom schema:

(S_ϵ) $(\alpha \to \beta) \leftrightarrow_H \epsilon$

As an example, let $\mathcal{SIC}$ denote the logic $\mathcal{SI}_{\alpha \leftrightarrow_H \beta}$, that is, the axiomatic extension of $\mathcal{SI}$ characterized by the axiom schema:

(C) $(\alpha \to \beta) \leftrightarrow_H (\alpha \leftrightarrow_H \beta)$

The logic $\mathcal{SIC}$ is interesting, for it provides a new interpretation of the implicative connective. One has that, for instance, $\bot \to \top = \bot$.

From Lemma 4.6, we have the following:

LEMMA 6.7. *Let* ϵ *be a formula in* IF_H. *The logic* $\mathcal{SI}_\epsilon$ *is implicative with respect to the binary connective* $\rightharpoonup_H$.

LEMMA 6.8. *Let* ϵ *be a formula in* IF_H. *Then* $\mathsf{Alg}^*\mathcal{SI}_\epsilon = \mathsf{SH}_\epsilon$.

PROOF. Let us take $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SI}_\epsilon$. By Theorem 4.7, $\mathsf{Alg}^*\mathcal{SI}_\epsilon \subseteq \mathsf{Alg}^*\mathcal{SI} = \mathsf{SH}$. If we take $\Gamma = \emptyset$ and $\phi = (x \to y) \leftrightarrow_H \epsilon$ in condition (LALG1) of Definition 4.5, we have that $\mathbf{A} \in \mathsf{SH}_\epsilon$.

Consider now $\mathbf{A} \in \mathsf{SH}_\epsilon$. We want to check that $\mathbf{A}$ is a $\mathcal{SI}_\epsilon$-algebra. Suppose that $\Gamma \vdash_{\mathcal{SI}_\epsilon} \phi$ and $h\Gamma \subseteq \{\top\}$ with $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$, and $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$. If $\phi = (x \to y) \leftrightarrow_H \epsilon$, since $\mathbf{A} \in \mathsf{SH}_\epsilon$, $h\phi = \top$. Thus $\mathsf{SH}_\epsilon \subseteq \mathsf{Alg}^*\mathcal{SI}_\epsilon$. ■

Since $\mathcal{SI}_\epsilon$ is an implicative logic with respect to the binary connective $\rightharpoonup_H$, by Lemma 6.7 and Lemma 6.8, we have the following result.

THEOREM 6.9. *The logic* $\mathcal{SI}_\epsilon$ *is complete with respect to the class* $\mathsf{Alg}^*\mathcal{SI}_\epsilon = \mathsf{SH}_\epsilon$*:*
For all $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$, $\Gamma \vdash_{\mathcal{SI}_\epsilon} \phi$ *if and only if* $h\Gamma \subseteq \{\top\}$ *implies* $h\phi = \top$ *for all* $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ *and all* $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SI}_\epsilon$.

6.1. Translations

In [4] it is proved that the intuitionistic logic $\mathcal{I}$ is an axiomatic extension of the logic $\mathcal{SI}$. Here we will give translations proving that if $\epsilon \in IF_H$, then $\mathcal{SI}_\epsilon$ is translationally equivalent to $\mathcal{I}$.

We use the language $\mathbf{L} = \{\wedge, \vee, \bot, \top, \rightarrow\}$ and we write $\alpha \rightarrow_H \beta$ as an abbreviation of $\alpha \rightarrow (\alpha \wedge \beta)$, while $\alpha \rightarrow_\epsilon \beta$ stands for $\epsilon(\alpha, \beta)$. Notice that if h is a semi-Heyting algebra homomorphism, we have of course that $h(x \rightarrow y) = h(x) \rightarrow h(y)$ but also $h(x \rightarrow_H y) = h(x) \rightarrow_H h(y)$ and $h(x \rightarrow_\epsilon y) = h(x) \rightarrow_\epsilon h(y)$.

In a Heyting algebra, the equation $x \rightarrow y \approx x \rightarrow_H y$ is valid, while in the variety SH_ϵ, $x \rightarrow y \approx x \rightarrow_\epsilon y$ is valid.

Lemma 6.10. *Let* $\cdot^\circ : Fm \rightarrow Fm$ *be the translation:*

- $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$
- $(\alpha \vee \beta)^\circ = \alpha^\circ \vee \beta^\circ$
- $(\alpha \rightarrow \beta)^\circ = \alpha^\circ \rightarrow_H \beta^\circ$.

Then this translation is sound.

Proof. Consider a formula $\alpha \in Fm$ such that $\vdash_{\mathcal{I}} \alpha$. We need to prove that α° is a theorem of $\mathcal{SI}_\epsilon$. For this, we take an arbitrary algebra $\langle A, \rightarrow\rangle \in \mathsf{SH}_\epsilon$, and $h \in Hom(\mathsf{Fm}, \langle A, \rightarrow\rangle)$. We define on A the implication $x \rightarrow_H y = x \rightarrow (x \wedge y)$. It will be enough to prove that $h(\alpha^\circ) = \top$. By Lemma 5.3, $\langle A, \rightarrow_H\rangle$ is a Heyting algebra. Now let $h' \in Hom(\mathsf{Fm}, \langle A, \rightarrow_H\rangle)$ be the only homomorphism such that $h'(x) = h(x)$ for every variable x.

We prove by induction over the formula α that $h'(\alpha) = h(\alpha^\circ)$. If $\alpha = \alpha_1 \rightarrow \alpha_2$, then $h'(\alpha) = h'(\alpha_1 \rightarrow \alpha_2) = h'(\alpha_1) \rightarrow_H h'(\alpha_2)$. By inductive hypothesis, this is $h(\alpha_1^\circ) \rightarrow_H h(\alpha_2^\circ) = h(\alpha_1^\circ \rightarrow_H \alpha_2^\circ) = h((\alpha_1 \rightarrow \alpha_2)^\circ)$. The other inductive steps are trivial. Since $\vdash_{\mathcal{I}} \alpha$, by completeness, $h'(\alpha) = \top$, so $h(\alpha^\circ) = \top$. ■

The formula ϵ provides also a translation of $\mathcal{SI}_\epsilon$ in $\mathcal{I}$.

Lemma 6.11. *Let* $\epsilon(x, y) \in IF_H$. *The translation* $\cdot^* : Fm \rightarrow Fm$*:*

- $(\alpha \wedge \beta)^* = \alpha^* \wedge \beta^*$
- $(\alpha \vee \beta)^* = \alpha^* \vee \beta^*$
- $(\alpha \rightarrow \beta)^* = \alpha^* \rightarrow_\epsilon \beta^*$

is sound.

Note that the translation $\cdot^*$ *depends on the formula* $\epsilon(x, y)$.

Proof. Let $\alpha \in Fm$ such that $\vdash_{\mathcal{SI}_\epsilon} \alpha$. To prove that $\vdash_{\mathcal{I}} \alpha^*$ we take a Heyting algebra $\langle A, \to \rangle$ and $h \in Hom(\mathsf{Fm}, \langle A, \to \rangle)$. It will be enough to prove that $h(\alpha^*) = \top$. On A we define the implication $x \to_\epsilon y = \epsilon(x, y)$. Then we have that $\langle A, \to_\epsilon \rangle \in \mathsf{SH}_\epsilon$. Now consider the homomorphism $h' \in Hom(\mathsf{Fm}, \langle A, \to_\epsilon \rangle)$ such that $h'(x) = h(x)$ for all variables x.

We prove inductively that for all $\alpha \in Fm$, $h(\alpha^*) = h'(\alpha)$. If $\alpha = \alpha_1 \to \alpha_2$ with $\alpha_1, \alpha_2 \in Fm$ then $h'(\alpha) = h'(\alpha_1 \to \alpha_2) = h'(\alpha_1) \to_\epsilon h'(\alpha_2) = h(\alpha_1^*) \to_\epsilon h(\alpha_2^*) = h(\alpha_1^* \to_\epsilon \alpha_2^*) = h((\alpha_1 \to \alpha_2)^*)$.

The other inductive steps are trivial. Since $\vdash_{\mathcal{SI}_\epsilon} \alpha$, by completeness, $h'(\alpha) = \top$, so $h(\alpha^*) = \top$. ■

Since both $\mathcal{I}$ and $\mathcal{SI}_\epsilon$ are axiomatic extensions of $\mathcal{SI}$, Lemma 3.5 and Corollary 3.6 hold for these logics, so conditions (B1) to (B4) are satisfied as well. The next three lemmas conclude the proof of the translational equivalence between $\mathcal{I}$ and $\mathcal{SI}_\epsilon$.

Lemma 6.12. *For every formula $\alpha \in Fm$,*

$$\vdash_{\mathcal{I}} \alpha \leftrightarrow (\alpha^\circ)^*.$$

Proof. Let $\mathbf{A} = \langle A, \to \rangle$ be a Heyting algebra and $h \in Hom(Fm, \langle A, \to \rangle)$. By completeness, it will be enough to show that $h(\alpha) = h((\alpha^\circ)^*)$ for all $\alpha \in Fm$. We proceed by induction on α, and as before, the only non-trivial case is the one of the implication.

If $\alpha = \alpha_1 \to \alpha_2$, $h((\alpha^\circ)^*) = h(((\alpha_1 \to \alpha_2)^\circ)^*) = h((\alpha_1^\circ \to (\alpha_1^\circ \wedge \alpha_2^\circ))^*) = h(\alpha_1^{\circ *} \to_\epsilon (\alpha_1^{\circ *} \wedge \alpha_2^{\circ^*})) = h(\alpha_1^{\circ^*}) \to_\epsilon (h(\alpha_1^{\circ *}) \wedge h(\alpha_2^{\circ^*}))$. By inductive hypothesis then, $h((\alpha^\circ)^*) = h(\alpha_1) \to_\epsilon (h(\alpha_1) \wedge h(\alpha_2))$. Since $\to_\epsilon$ is a semi-Heyting implication, by Lemma 5.3, $h(\alpha_1) \to_\epsilon (h(\alpha_1) \wedge h(\alpha_2)) = h(\alpha_1) \to_H h(\alpha_2)$. Given that h is a homomorphism, and $\mathbf{A}$ is a Heyting algebra, $h(\alpha_1) \to_H h(\alpha_2) = h(\alpha_1 \to \alpha_2) = h(\alpha)$. ■

Lemma 6.13. *Let $\alpha, \beta \in Fm$ and $\epsilon(x, y) \in IF_H$. Then*

$$(\alpha \to_\epsilon \beta)^\circ = \alpha^\circ \to_\epsilon \beta^\circ.$$

Proof. We show this by induction on the formula ϵ.

Assume that $\alpha \to_\epsilon \beta = \gamma(\alpha, \beta) \wedge \delta(\alpha, \beta)$. Then $(\alpha \to_\epsilon \beta)^\circ = (\gamma \wedge \delta)^\circ = \gamma^\circ \wedge \delta^\circ$. By inductive hypothesis this is $\gamma(\alpha^\circ, \beta^\circ) \wedge \delta(\alpha^\circ, \beta^\circ) = \alpha^\circ \to_\epsilon \beta^\circ$.

If $\alpha \to_\epsilon \beta = \gamma(\alpha, \beta) \to_H \delta(\alpha, \beta)$. Then $(\alpha \to_\epsilon \beta)^\circ = (\gamma \to_H \delta)^\circ = (\gamma \to (\gamma \wedge \delta))^\circ = \gamma^\circ \to_H (\gamma^\circ \wedge \delta^\circ) = \gamma^\circ \to (\gamma^\circ \wedge \delta^\circ)$. By inductive hypothesis this is $\gamma(\alpha^\circ, \beta^\circ) \to (\gamma(\alpha^\circ, \beta^\circ) \wedge \delta(\alpha^\circ, \beta^\circ)) = \gamma(\alpha^\circ, \beta^\circ) \to_H \delta(\alpha^\circ, \beta^\circ) = \alpha^\circ \to_\epsilon \beta^\circ$.

The rest of the induction steps are similar. ■

Lemma 6.14. *For every formula $\alpha \in Fm$,*

$$\vdash_{\mathcal{SI}_\epsilon} \alpha \leftrightarrow (\alpha^*)^\circ.$$

Proof. Let $\langle \mathbf{A}, \to \rangle \in \mathsf{SH}_\epsilon$ and $h \in Hom(\mathsf{Fm}, \mathbf{A})$. By completeness, it will be enough to prove that $h(\alpha) = h((\alpha^*)^\circ)$. We proceed by induction on α, where the step for the implication is as follows.

If $\alpha = \alpha_1 \to \alpha_2$, then $h((\alpha^*)^\circ) = h(((\alpha_1 \to \alpha_2)^*)^\circ) = h((\alpha_1^* \to_\epsilon \alpha_2^*)^\circ)$. By Lemma 6.13, this equals $h(\alpha_1^{*\circ} \to_\epsilon \alpha_2^{*\circ}) = h(\alpha_1^{*\circ}) \to_\epsilon h(\alpha_2^{*\circ})$. By inductive hypothesis this is $h(\alpha_1) \to_\epsilon h(\alpha_2) = h(\alpha_1 \to_\epsilon \alpha_2) = h(\alpha_1 \to \alpha_2) = h(\alpha)$. ■

Thus we have proved the following:

Theorem 6.15. *For any $\epsilon \in IF_H$, the logic $\mathcal{SI}_\epsilon$ is translationally equivalent to the intuitionistic logic $\mathcal{I}$.*

In particular, each one of the examples at the beginning of this section gives a logic which is equivalent to $\mathcal{I}$.

Our next goal is to prove that for any axiomatic extension $\mathcal{A}$ of $\mathcal{SI}$ that is translationally equivalent to $\mathcal{I}$, there exists a formula δ such that $\mathcal{A}$ is equivalent to $\mathcal{SI}_\delta$ in the sense that they prove the same theorems.

Let $\mathcal{A}$ be a logic that is an axiomatic extension of $\mathcal{SI}$ and is translationally equivalent to $\mathcal{I}$. There exist then translations $h_1, h_2 : Fm \to Fm$ such that for all $\phi \in Fm$ the following conditions hold:

(a) If $\vdash_{\mathcal{I}} \phi$ then $\vdash_{\mathcal{A}} h_1\phi$

(b) If $\vdash_{\mathcal{A}} \phi$ then $\vdash_{\mathcal{I}} h_2\phi$

(c) $\vdash_{\mathcal{I}} \phi \leftrightarrow_H h_2(h_1(\phi))$

(d) $\vdash_{\mathcal{A}} \phi \leftrightarrow_H h_1(h_2(\phi))$.

We consider the formula $\delta = (h_2(x \to y))^\circ$. Here we use first the translation h_2 that gives an intuitionistic translation of the implication in $\mathcal{A}$, and then apply the translation $\cdot^\circ$, so that the resulting formula can be written using only the implication $\to_H$.

Lemma 6.16. *For any $\alpha \in Fm$, $\vdash_{\mathcal{A}} \alpha$ if and only if $\vdash_{\mathcal{SI}_\delta} (h_2(\alpha))^\circ$.*

Proof. Assume that $\vdash_{\mathcal{A}} \alpha$. Then by condition (b) above, $\vdash_{\mathcal{I}} h_2(\alpha)$. By Lemma 6.10, $\vdash_{\mathcal{SI}_\delta} (h_2(\alpha))^\circ$. Then the translation $*$ corresponding to the logic $\mathcal{SI}_\delta$ introduced in Lemma 6.11 is such that $\vdash_{\mathcal{I}} ((h_2(\alpha))^\circ)^*$. Therefore, by Lemma 6.12, we have that $\vdash_{\mathcal{I}} h_2(\alpha)$. From condition (a) it follows that $\vdash_{\mathcal{A}} h_1(h_2(\alpha))$. Finally, $\vdash_{\mathcal{A}} \alpha$ is valid by (d). ■

Lemma 6.17. *For every $\alpha \in Fm$, $\vdash_{\mathcal{A}} \alpha \to \beta$ if and only if $\vdash_{\mathcal{SI}_\delta} \alpha \to \beta$.*

Proof. By the definition of δ, we have that $\vdash_{\mathcal{SI}_\delta} (\alpha \to \beta) \leftrightarrow_H ((h_2(\alpha \to \beta))^\circ)$. Using Lemma 3.4, this means that $\vdash_{\mathcal{SI}_\delta} \alpha \to \beta$ if and only if $\vdash_{\mathcal{SI}_\delta}$

$(h_2(\alpha \to \beta))^{\circ}$. Therefore, by Lemma 6.16, $\vdash_{\mathcal{SI}_\delta} \alpha \to \beta$ if and only if $\vdash_{\mathcal{A}}$ $\alpha \to \beta$. ■

THEOREM 6.18. *The logics $\mathcal{A}$ and $\mathcal{SI}_\delta$ have the same theorems.*

PROOF. We will prove that $\vdash_{\mathcal{A}} \alpha$ if and only if $\vdash_{\mathcal{SI}_\delta} \alpha$. By Lemma 2.1,a), and 3.9, $\vdash_{\mathcal{A}} \alpha$ if and only if $\vdash_{\mathcal{A}} (\beta \to \beta) \to_H \alpha$. By Lemma 6.17, $\vdash_{\mathcal{A}} (\beta \to \beta) \to_H \alpha$ if and only if $\vdash_{\mathcal{SI}_\delta} (\beta \to \beta) \to_H \alpha$. Using again Lemma 2.1,a), $\vdash_{\mathcal{SI}_\delta} (\beta \to \beta) \to_H \alpha$ if and only if$\vdash_{\mathcal{SI}_\delta} \alpha$. ■

6.2. Kripke Models and Priestley Representations

Given a formula $\epsilon \in IF_H$, it is easy to find the Kripke models for the logic $\mathcal{SI}_\epsilon$. Just as in the case of intuitionistic logic (see [6]), we show how to construct an algebra in SH_ϵ from a Kripke model, and conversely, build a Kripke model from an algebra in SH_ϵ in such a way that validity of formulas in the algebraic model coincides with validity in the Kripke model.

We first give a Kripke-style semantics for each of the logics $\mathcal{SI}_\epsilon$ (which, we know, are translationally equivalent to the intuitionistic calculus $\mathcal{I}$).

DEFINITION 6.19. Let $\epsilon \in IF_H$. $\langle W, R, \Vdash_\epsilon\rangle$ is a *Kripke model* for the logic $\mathcal{SI}_\epsilon$ (or ϵ-Kripke model) if W is a non-empty set, $R \subseteq W \times W$ is a reflexive and transitive binary relation and $\Vdash_\epsilon \subseteq W \times Fm$ is a relation such that for all $v, w \in W$ the following conditions hold:

(1) For all $\alpha \in Var$, if $w \Vdash_\epsilon \alpha$ and wRv, then $v \Vdash_\epsilon \alpha$.

(2) $w \Vdash_\epsilon \alpha \wedge \beta$ iff $w \Vdash_\epsilon \alpha$ and $w \Vdash_\epsilon \beta$.

(3) $w \Vdash_\epsilon \alpha \vee \beta$ iff $w \Vdash_\epsilon \alpha$ or $w \Vdash_\epsilon \beta$.

(4) $w \Vdash_\epsilon \top$ for all $w \in W$ and $w \Vdash_\epsilon \bot$ for no $w \in W$.

(5) $w \Vdash_\epsilon \alpha \to \beta$ iff $\mathbf{P}_\epsilon(w)$,

where $\mathbf{P}_\epsilon(w)$ is a predicate associated to the formula $\epsilon(\alpha, \beta)$ defined as follows:

$$\mathbf{P}_\epsilon(w) \text{ iff } \begin{cases} w \Vdash_\epsilon \alpha & \text{if } \epsilon(\alpha,\beta) = \alpha \\ w \Vdash_\epsilon \beta & \text{if } \epsilon(\alpha,\beta) = \beta \\ w \Vdash_\epsilon \alpha \text{ and } w \nVdash_\epsilon \alpha & \text{if } \epsilon(\alpha,\beta) = \bot \\ w \Vdash_\epsilon \alpha & \text{if } \epsilon(\alpha,\beta) = \top \\ \mathbf{P}_{\epsilon_1}(w) \text{ and } \mathbf{P}_{\epsilon_2}(w) & \text{if } \epsilon(\alpha,\beta) = \epsilon_1(\alpha,\beta) \wedge \epsilon_2(\alpha,\beta) \\ \mathbf{P}_{\epsilon_1}(w) \text{ or } \mathbf{P}_{\epsilon_2}(w) & \text{if } \epsilon(\alpha,\beta) = \epsilon_1(\alpha,\beta) \vee \epsilon_2(\alpha,\beta) \\ \text{for all } v \in W \text{ such that } wRv, & \\ \text{if } \mathbf{P}_{\epsilon_1}(v) \text{ then } \mathbf{P}_{\epsilon_2}(v) & \text{if } \epsilon(\alpha,\beta) = \epsilon_1(\alpha,\beta) \to_H \epsilon_2(\alpha,\beta) \end{cases}$$

with $\epsilon_1(\alpha, \beta)$ and $\epsilon_2(\alpha, \beta)$ H-formulas. Notice that although $\epsilon \in IF_H$, its subformulas are not necessarily in IF_H, so we need to define $\mathbf{P}_\gamma(w)$ recursively for all H-formulas γ.

If $\epsilon(\alpha, \beta)$ is simply $\alpha \to_H \beta$, the semantics above coincides with the Kripke semantics for intuitionistic logic. We denote the satisfaction relation for this case by $\Vdash_H$.

As an example, for the formula $(\alpha \to_H \beta) \wedge (\beta \to_H \alpha)$, the predicate $\mathbf{P}_{(\alpha \to_H \beta)\wedge(\beta \to_H \alpha)}(w)$ is "if $w \Vdash_H \alpha$ then $w \Vdash_H \beta$ and if $w \Vdash_H \beta$ then $w \Vdash_H \alpha$". That is, " $w \Vdash_H \alpha$ iff $w \Vdash_H \beta$".

Definition 6.20. A formula α is *ϵ-Kripke-valid in the model $\langle W, R, \Vdash_\epsilon\rangle$* if $w \Vdash_\epsilon \alpha$ for each $w \in W$. We denote this by $\langle W, R\rangle \Vdash_\epsilon \alpha$. The formula is *$\epsilon$-Kripke-valid* if it is valid in every model.

Given $\epsilon \in IF_H$, and an ϵ-Kripke model $\langle W, R, \Vdash_\epsilon\rangle$, a set $X \subseteq W$ is *increasing* if for all $x \in X$ and $v \in W$, if xRv then $v \in X$. Let A be the set of increasing subsets of W ordered by inclusion. On A we define $X \to_H Y$ as the biggest increasing set contained in $(W \setminus X) \cup Y$. It is well known that this operation is the relative pseudocomplement or intuitionistic implication on the lattice A ([6]). Now we can define on A the operation $X \to Y = \epsilon^{\mathbf{A}}(X, Y)$ for all $X, Y \in A$.

Lemma 6.21. *$\mathbf{A} = \langle A, \cup, \cap, \to, W, \emptyset\rangle$ is an algebra in SH_ϵ.*

Proof. It is clear that $\langle A, \cap, \cup, \emptyset, W\rangle$ is a bounded lattice. By the definition of the implication on A, $X \to Y = \epsilon^{\mathbf{A}}(X, Y)$ so $\mathbf{A} \in \mathsf{SH}_\epsilon$. ■

Lemma 6.22. *For each $\alpha \in Fm$, the set $m(\alpha) = \{w \in W : \ w \Vdash_\epsilon \alpha\}$ is an increasing subset of W.*

Proof. Let $w \in m(\alpha)$ and $v \in W$ be such that wRv. So $w \Vdash_\epsilon \alpha$. We want to prove that $v \Vdash_\epsilon \alpha$. We proceed by induction on α.

If $\alpha \in Var$, by (1) in Definition 6.19 it follows that $v \Vdash_\epsilon \alpha$.

Let $\alpha_1, \alpha_2 \in Fm$ be such that $\alpha = \alpha_1 \wedge \alpha_2$. Since $w \Vdash_\epsilon \alpha$, from (2) in Definition 6.19, we have that $w \Vdash_\epsilon \alpha_1$ and $w \Vdash_\epsilon \alpha_2$. By the inductive hypothesis on the formulas α_1 and α_2, $v \Vdash_\epsilon \alpha_1$ and $v \Vdash_\epsilon \alpha_2$. Thus $v \Vdash_\epsilon \alpha$.

The case in which $\alpha = \alpha_1 \vee \alpha_2$ is similar.

Assume now that $\alpha = \alpha_1 \to \alpha_2$. From $w \Vdash_\epsilon \alpha_1 \to \alpha_2$, we know that $\mathbf{P}_\epsilon(w)$ holds. Now we need to prove by induction on the formula ϵ that $\mathbf{P}_\epsilon(v)$ holds.

- If $\epsilon(\alpha_1, \alpha_2) = \alpha_1$ then, from (5), $w \Vdash_\epsilon \alpha_1$. Then by the inductive hypothesis on α_1, $v \Vdash_\epsilon \alpha_1$. So $\mathbf{P}_\epsilon(v)$.

- If $\epsilon = \bot$ and $\mathbf{P}_\epsilon(w)$, we get a contradiction.
- Let $\epsilon_1(\alpha_1, \alpha_2), \epsilon_2(\alpha, \alpha_2)$ be H-formulas such that $\epsilon(\alpha_1, \alpha_2) = \epsilon_1(\alpha_1, \alpha_2) \wedge \epsilon_2(\alpha_1, \alpha_2)$. Since $\mathbf{P}_\epsilon(w)$ holds, it follows that $\mathbf{P}_{\epsilon_1}(w)$ and $\mathbf{P}_{\epsilon_2}(w)$. By inductive hypothesis, $\mathbf{P}_{\epsilon_1}(v)$ and $\mathbf{P}_{\epsilon_2}(v)$, so $\mathbf{P}_\epsilon(v)$.
- If $\epsilon(\alpha_1, \alpha_2) = \epsilon_1(\alpha_1, \alpha_2) \rightharpoonup_H \epsilon_2(\alpha_1, \alpha_2)$ with $\epsilon_1(\alpha_1, \alpha_2), \epsilon_2(\alpha_1, \alpha_2)$ H-formulas, what we need to show is that for each $z \in W$ such that vRz and $\mathbf{P}_{\epsilon_1}(z)$, $\mathbf{P}_{\epsilon_2}(z)$ obtains. Since $\mathbf{P}_\epsilon(w)$, then for all $z' \in W$ such that wRz', if $\mathbf{P}_{\epsilon_1}(z')$ then $\mathbf{P}_{\epsilon_2}(z')$. Given that wRv and vRz, it follows that wRz and $\mathbf{P}_{\epsilon_2}(z)$.

The remaining cases are similar to those above. ■

The following lemma is proved in the Appendix.

Lemma 6.23. *The function $m : \mathsf{Fm} \to \mathbf{A}$ defined by $m(\alpha) = \{w \in W : w \Vdash_\epsilon \alpha\}$ is a homomorphism.*

It is clear from the construction above that a formula α is Kripke-valid in a Kripke model if and only if $m(\alpha) = \top^{\mathbf{A}} = W$ in its corresponding algebra.

Now we take an algebra $\mathbf{A} \in \mathsf{SH}_\epsilon$ and a homomorphism $m : \mathsf{Fm} \to \mathbf{A}$ and find an ϵ-Kripke model $\langle W, R, \Vdash_\epsilon \rangle$ such that $\langle W, R \rangle \Vdash_\epsilon \alpha$ iff $m(\alpha) = \top$.

To do this, take W the set of prime filters of $\mathbf{A}$, and let R be the inclusion relation. We say that $F \Vdash_\epsilon \alpha$ if and only if $m(\alpha) \in F$ for all prime filters F of $\mathbf{A}$.

Lemma 6.24. *For each $F \in W$, $\mathbf{P}_\epsilon(F)$ holds if and only if $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$.*

Proof. Proceeding by induction on ϵ, we have:

- If $\epsilon(\alpha, \beta) = \alpha$, and $\mathbf{P}_\epsilon(F)$ then $F \Vdash_\epsilon \alpha$, so $m(\alpha) \in F$. Then $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$. The other inclusion is similar.
- If $\epsilon = \bot$, $\mathbf{P}_\epsilon(F)$ is never true, the same as the proposition $\bot \in F$, since F is a prime filter.
- In the case $\epsilon = \epsilon_1 \wedge \epsilon_2$ with $\epsilon_1(\alpha, \beta), \epsilon_2(\alpha, \beta)$ H-formulas, if $\mathbf{P}_\epsilon(F)$ holds then $\mathbf{P}_{\epsilon_1}(F)$ and $\mathbf{P}_{\epsilon_2}(F)$. By inductive hypothesis, $\epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$ and $\epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$. Since F is a filter, $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$. The converse is similar.
- Assume now that $\epsilon = \epsilon_1 \rightharpoonup_H \epsilon_2$ with $\epsilon_1(\alpha, \beta), \epsilon_2(\alpha, \beta)$ H-formulas.

  Suppose $\mathbf{P}_\epsilon(F)$ holds. Then for each $F_1 \in W$ such that $F \subseteq F_1$, if $\mathbf{P}_{\epsilon_1}(F_1)$ then $\mathbf{P}_{\epsilon_2}(F_1)$. If $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) = \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \rightharpoonup_H \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) \notin F$, by [6, Lemma 6.2] we have that the filter $\bar{F}$ generated by F and $\epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$ does not include the element $\epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$. Then by [6, Lemma 6.4], there exists a prime filter F_1 such that $\bar{F} \subseteq F_1$ and $\epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) \notin F_1$. Then $F \subseteq F_1$ and by the inductive hypothesis on $\epsilon_2(x, y)$, $\mathbf{P}_{\epsilon_2}(F_1)$ does not hold. Since $\epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \in F_1$, $\mathbf{P}_{\epsilon_1}(F_1)$ holds, a contradiction. Then we conclude that $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$.

Now assume $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) = \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \to_H \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$. Let $F_1 \in W$ be such that $F \subseteq F_1$ and assume that $P_{\epsilon_1}(F_1)$. We need to prove that $\mathbf{P}_{\epsilon_2}(F_1)$ holds. Since $P_{\epsilon_1}(F_1)$ holds, then $\epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \in F_1$. Since $F \subseteq F_1$, we also have that $\epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \to_H \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) \in F_1$. It follows that $\epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))) \in F_1$ and by inductive hypothesis, $\mathbf{P}_{\epsilon_2}(F_1)$. ■

Theorem 6.25. *$\langle W, R, \Vdash_\epsilon \rangle$ is an ϵ-Kripke model such that $\langle W, R \rangle \Vdash_\epsilon \alpha$ iff $m(\alpha) = \top$.*

Proof. Conditions (1) through (4) from Definition 6.19 follow immediately and are proved as in [6].

To prove that condition (5) holds, we need to show that $F \Vdash_\epsilon \alpha \to \beta$ if and only if $\mathbf{P}_\epsilon(F)$. From the definition for $\Vdash_\epsilon$, it follows that $F \Vdash_\epsilon \alpha \to \beta$ if and only if $m(\alpha \to \beta) \in F$. Since m is a homomorphism, $F \Vdash_\epsilon \alpha \to \beta$ if and only if $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \in F$. By Lemma 6.24, we can conclude that $F \Vdash_\epsilon \alpha \to \beta$ if and only if $\mathbf{P}_\epsilon(F)$.

The conclusion that $\langle W, R \rangle \Vdash_\epsilon \alpha$ if and only if $m(\alpha) = \top$ is immediate. ■

From Lemma 6.21 and Theorem 6.25 we conclude the next result.

Theorem 6.26. *The equation $\alpha \approx \top$ is satisfied in every algebra of SH_ϵ if and only if α is Kripke-valid in every ϵ-Kripke model.*

With respect to the Priestley representation of algebras in the variety SH_ϵ, it is simply the Priestley representation of the Heyting algebra (see [12], [5]) obtained defining $\to_H$ over them. The semi-Heyting implication operation can then be recovered by letting $x \to y = \epsilon(x, y)$ in the Heyting algebra obtained from its Priestley representation.

Acknowledgements. We gratefully acknowledge the constructive comments and corrections offered by the referee. This work was partially supported by CONICET (Consejo Nacional de Investigaciones Científicas y Técnicas).

Appendix: Proofs

Proof. of Lemma 2.1

(a)
1. $\Gamma \vdash_{\mathcal{SI}} (\psi \wedge \alpha) \to_H \psi$ by axiom (S4).
2. $\Gamma \vdash_{\mathcal{SI}} [(\psi \wedge \alpha) \to_H \psi] \to_H [\psi \to_H (\alpha \to_H \psi)]$ by (S8).
3. $\Gamma \vdash_{\mathcal{SI}} \psi \to_H (\alpha \to_H \psi)$ by SMP applied to 1 and 2.
4. $\Gamma \vdash_{\mathcal{SI}} \psi$ by hypothesis.
5. $\Gamma \vdash_{\mathcal{SI}} \alpha \to_H \psi$ by SMP applied to 3 and 4.

(b) Let ψ the formula $(\alpha \wedge \beta) \to_H \alpha$.
1. $\vdash_{\mathcal{SI}} (\alpha \wedge \psi) \to_H \alpha$ by axiom (S4).
2. $\vdash_{\mathcal{SI}} [(\alpha \wedge \psi) \to_H \alpha] \to_H [((\alpha \wedge \psi) \to_H \alpha) \to_H ((\alpha \wedge \psi) \to_H (\alpha \wedge \alpha))]$ by axiom (S5).
3. $\vdash_{\mathcal{SI}} ((\alpha \wedge \psi) \to_H \alpha) \to_H ((\alpha \wedge \psi) \to_H (\alpha \wedge \alpha))$ by SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} (\alpha \wedge \psi) \to_H (\alpha \wedge \alpha)$ by SMP applied to 1 and 3.
5. $\vdash_{\mathcal{SI}} (\alpha \wedge \alpha) \to_H \alpha$ by axiom (S4).
6. $\vdash_{\mathcal{SI}} \psi$ by axiom (S4).
7. $\vdash_{\mathcal{SI}} (\alpha \wedge \alpha) \to_H \psi$ by part (a) and 6.
8. $\vdash_{\mathcal{SI}} [(\alpha \wedge \alpha) \to_H \alpha] \to_H [((\alpha \wedge \alpha) \to_H \psi) \to_H ((\alpha \wedge \alpha) \to_H (\alpha \wedge \psi))]$ by axiom (S5).
9. $\vdash_{\mathcal{SI}} ((\alpha \wedge \alpha) \to_H \psi) \to_H ((\alpha \wedge \alpha) \to_H (\alpha \wedge \psi))$ by SMP applied to 5 and 8.
10. $\vdash_{\mathcal{SI}} (\alpha \wedge \alpha) \to_H (\alpha \wedge \psi)$ by SMP applied to 7 and 9.
11. $\vdash_{\mathcal{SI}} ((\alpha \wedge \alpha) \to_H (\alpha \wedge \psi)) \to_H [((\alpha \wedge \psi) \to_H (\alpha \wedge \alpha)) \to_H ((\alpha \to (\alpha \wedge \psi)) \to_H (\alpha \to (\alpha \wedge \alpha)))]$ by axiom (S11).
12. $\vdash_{\mathcal{SI}} ((\alpha \wedge \psi) \to_H (\alpha \wedge \alpha)) \to_H ((\alpha \to (\alpha \wedge \psi)) \to_H (\alpha \to (\alpha \wedge \alpha)))$ by SMP applied to 10 and 11.
13. $\vdash_{\mathcal{SI}} (\alpha \to (\alpha \wedge \psi)) \to_H (\alpha \to (\alpha \wedge \alpha))$ by SMP applied to 4 and 12.
14. $\vdash_{\mathcal{SI}} (\alpha \to_H \psi) \to_H (\alpha \to_H \alpha)$ by the definition of $\to_H$.
15. $\vdash_{\mathcal{SI}} \alpha \to_H \psi$ by axiom (S4) and by part (a).
16. $\vdash_{\mathcal{SI}} \alpha \to_H \alpha$ by SMP applied to 14 and 15.

(c)
1. $\vdash_{\mathcal{SI}} \phi$ by hypothesis.
2. $\vdash_{\mathcal{SI}} \alpha \to_H \alpha$ by part (b).
3. $\vdash_{\mathcal{SI}} \alpha \to_H \phi$ by (a).

   Therefore,
4. $\vdash_{\mathcal{SI}} (\alpha \to_H \alpha) \to_H [(\alpha \to_H \phi) \to_H (\alpha \to_H (\alpha \wedge \phi))]$ by axiom (S5).
5. $\vdash_{\mathcal{SI}} \alpha \to_H (\alpha \wedge \phi)$ SMP applied to 2, 3 and 4.

(d)
1. $\vdash_{\mathcal{SI}} \beta \to_H \beta$ by part (b).
2. $\vdash_{\mathcal{SI}} \alpha \to_H (\beta \to_H \beta)$ by part (a).
3. $\vdash_{\mathcal{SI}} (\alpha \to_H (\beta \to_H \beta)) \to_H ((\alpha \wedge \beta) \to_H \beta)$ by (S9).
4. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \beta$ by SMP applied to 2 and 3.

(e)
1. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \beta$ by part (d).
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \alpha$ by (S4).

3. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightharpoonup_H \beta) \rightharpoonup_H [((\alpha \wedge \beta) \rightharpoonup_H \alpha) \rightharpoonup_H ((\alpha \wedge \beta) \rightharpoonup_H (\beta \wedge \alpha))]$ by axiom (S5).
4. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightharpoonup_H \alpha) \rightharpoonup_H ((\alpha \wedge \beta) \rightharpoonup_H (\beta \wedge \alpha))$ by SMP applied to 1 and 3.
5. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H (\beta \wedge \alpha)$ by SMP applied to 2 and 4.

(f)
1. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H (\alpha \wedge \alpha)) \rightharpoonup_H [((\alpha \wedge \alpha) \rightharpoonup_H \alpha) \rightharpoonup_H ((\alpha \to (\alpha \wedge \alpha)) \rightharpoonup_H (\alpha \to \alpha))]$ by (S11).
2. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \alpha) \rightharpoonup_H [(\alpha \rightharpoonup_H \alpha) \rightharpoonup_H (\alpha \rightharpoonup_H (\alpha \wedge \alpha))]$ by (S5).
3. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H \alpha$ by part (b).
4. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\alpha \wedge \alpha)$ SMP applied to 3 and 2.
5. $\vdash_{\mathcal{SI}} ((\alpha \wedge \alpha) \rightharpoonup_H \alpha) \rightharpoonup_H ((\alpha \to (\alpha \wedge \alpha)) \rightharpoonup_H (\alpha \to \alpha))$ by SMP applied to 4 and 1.
6. $\vdash_{\mathcal{SI}} (\alpha \wedge \alpha) \rightharpoonup_H \alpha$ by (S4).
7. $\vdash_{\mathcal{SI}} (\alpha \to (\alpha \wedge \alpha)) \rightharpoonup_H (\alpha \to \alpha)$ by SMP applied to 5 and 6.
8. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \alpha)) \rightharpoonup_H (\alpha \to \alpha)$ the definition of $\rightharpoonup_H$.
9. $\vdash_{\mathcal{SI}} \alpha \to \alpha$ by SMP applied to 3 and 8.

(g)
1. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H (\alpha \wedge \beta)$ by part (b).
2. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightharpoonup_H (\alpha \wedge \beta)) \rightharpoonup_H (\alpha \rightharpoonup_H (\beta \rightharpoonup_H (\alpha \wedge \beta)))$ by axiom (S8).
3. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\beta \rightharpoonup_H (\alpha \wedge \beta))$ by SMP applied to 1 and 2.

(h)
1. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \gamma) \rightharpoonup_H \psi$ by hypothesis.
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H (\gamma \rightharpoonup_H \psi)$ by axiom (S8) and SMP applied to 1.
3. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))$ by axiom (S8) and SMP applied to 2.
4. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))] \rightharpoonup_H \alpha$ by (S4).
5. $\vdash_{\mathcal{SI}} (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi)) \rightharpoonup_H ((\beta \wedge \gamma) \rightharpoonup_H \psi)$ by axiom (S9).
6. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H [(\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi)) \rightharpoonup_H ((\beta \wedge \gamma) \rightharpoonup_H \psi)]$ by part (a).
7. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))] \rightharpoonup_H ((\beta \wedge \gamma) \rightharpoonup_H \psi)$ by axiom (S9) and SMP applied to 6.
8. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))] \rightharpoonup_H [\alpha \wedge ((\beta \wedge \gamma) \rightharpoonup_H \psi)]$ by axiom (S5) and SMP applied to 4 and 7.
9. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\beta \wedge \gamma) \rightharpoonup_H \psi)] \rightharpoonup_H \alpha$ by (S4).
10. $\vdash_{\mathcal{SI}} ((\beta \wedge \gamma) \rightharpoonup_H \psi) \rightharpoonup_H (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))$ by axiom (S8).
11. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H [((\beta \wedge \gamma) \rightharpoonup_H \psi) \rightharpoonup_H (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))]$ by part (a).
12. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\beta \wedge \gamma) \rightharpoonup_H \psi)] \rightharpoonup_H (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))$ by axiom (S9) and SMP applied to 11.
13. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\beta \wedge \gamma) \rightharpoonup_H \psi)] \rightharpoonup_H [\alpha \wedge (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))]$ by axiom (S5) and SMP applied to 9 and 12.
14. $\vdash_{\mathcal{SI}} [\alpha \to (\alpha \wedge (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi)))] \rightharpoonup_H [\alpha \to (\alpha \wedge ((\beta \wedge \gamma) \rightharpoonup_H \psi))]$ by axiom (S11) and SMP applied to 8 and 13.
15. $\vdash_{\mathcal{SI}} [\alpha \rightharpoonup_H (\beta \rightharpoonup_H (\gamma \rightharpoonup_H \psi))] \rightharpoonup_H [\alpha \rightharpoonup_H ((\beta \wedge \gamma) \rightharpoonup_H \psi)]$ by the definition of $\rightharpoonup_H$ and 14.
16. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H ((\beta \wedge \gamma) \rightharpoonup_H \psi)$ by SMP applied to 3 and 15.
17. $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \wedge \gamma)) \rightharpoonup_H \psi$ by axiom (S9) and SMP applied to 16.
The other implication can be verified in a similar way.

(i)
1. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow_H (\alpha \wedge \beta)$ by part (e).
2. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow_H (\beta \wedge \alpha)$ by part (b).
3. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow_H [(\alpha \wedge \beta) \wedge (\beta \wedge \alpha)]$ by axiom (S5) and SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge (\beta \wedge \alpha)] \rightarrow_H (\beta \wedge \alpha)$ by part (d).
5. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \rightarrow ((\alpha \wedge \beta) \wedge (\beta \wedge \alpha))] \rightarrow_H [(\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)]$ by axiom (S11) and SMP applied to 3 and 4.
6. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightarrow_H (\beta \wedge \alpha)) \rightarrow_H [(\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)]$ by the definition of $\rightarrow_H$.
7. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H (\beta \wedge \alpha)$ by part (e).
8. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow (\beta \wedge \alpha)$ by SMP applied to 6 and 7.

(j)
1. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H (\beta \wedge \alpha)$ by (e)
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H \gamma$ by hypothesis.
3. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightarrow_H ((\beta \wedge \alpha) \wedge \gamma)$ by axiom (S5) and SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} [\beta \wedge (\alpha \wedge \gamma)] \rightarrow_H \beta$ by axiom (S4).
5. $\vdash_{\mathcal{SI}} ((\beta \wedge \alpha) \wedge \gamma) \rightarrow_H \beta$ by part (h) applied to 4.
6. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightarrow_H \alpha$ by axiom (S4).
7. $\vdash_{\mathcal{SI}} \beta \rightarrow_H [(\alpha \wedge \gamma) \rightarrow_H \alpha]$ by part (a).
8. $\vdash_{\mathcal{SI}} (\beta \wedge (\alpha \wedge \gamma)) \rightarrow_H \alpha$ by axiom (S9) and SMP applied to 7.
9. $\vdash_{\mathcal{SI}} ((\beta \wedge \alpha) \wedge \gamma) \rightarrow_H \alpha$ by part (h) applied to 8.
10. $\vdash_{\mathcal{SI}} ((\beta \wedge \alpha) \wedge \gamma) \rightarrow_H (\alpha \wedge \beta)$ by axiom (S5) and SMP applied to 5 and 9.
11. $\vdash_{\mathcal{SI}} [(\beta \wedge \alpha) \rightarrow (\alpha \wedge \beta)] \rightarrow_H [(\beta \wedge \alpha) \rightarrow ((\beta \wedge \alpha) \wedge \gamma)]$ by axiom (S11) and SMP applied to 3 and 10.
12. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow (\alpha \wedge \beta)$ by part (i).
13. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow ((\beta \wedge \alpha) \wedge \gamma)$ by SMP applied to 11 and 12.
14. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightarrow_H \gamma$ by definicin de $\rightarrow_H$.

(k)
1. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow_H (\alpha \wedge \gamma)$ by part (e).
2. $\vdash_{\mathcal{SI}} \alpha \rightarrow_H \beta$ by hypothesis.
3. $\vdash_{\mathcal{SI}} \gamma \rightarrow_H (\alpha \rightarrow_H \beta)$ by part (a).
4. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow_H \beta$ by axiom (S9) and SMP applied to 3.
5. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow_H ((\alpha \wedge \gamma) \wedge \beta)$ by axiom (S5) and SMP applied to 1 and 4.
6. $\vdash_{\mathcal{SI}} (\gamma \wedge \beta) \rightarrow_H \gamma$ by axiom (S4).
7. $\vdash_{\mathcal{SI}} \alpha \rightarrow_H [(\gamma \wedge \beta) \rightarrow_H \gamma]$ by part (a).
8. $\vdash_{\mathcal{SI}} (\alpha \wedge (\gamma \wedge \beta)) \rightarrow_H \gamma$ by axiom (S9) and SMP applied to 7.
9. $\vdash_{\mathcal{SI}} ((\alpha \wedge \gamma) \wedge \beta) \rightarrow_H \gamma$ by part (h).
10. $\vdash_{\mathcal{SI}} (\alpha \wedge (\gamma \wedge \beta)) \rightarrow_H \alpha$ by axiom (S4).
11. $\vdash_{\mathcal{SI}} ((\alpha \wedge \gamma) \wedge \beta) \rightarrow_H \alpha$ by part (h).
12. $\vdash_{\mathcal{SI}} ((\alpha \wedge \gamma) \wedge \beta) \rightarrow_H (\gamma \wedge \alpha)$ by axiom (S5) and SMP applied to 9 and 11.
13. $\vdash_{\mathcal{SI}} [(\alpha \wedge \gamma) \rightarrow (\gamma \wedge \alpha)] \rightarrow_H [(\alpha \wedge \gamma) \rightarrow ((\alpha \wedge \gamma) \wedge \beta)]$ by axiom (S11) and SMP applied to 5 and 12.
14. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightarrow (\gamma \wedge \alpha)$ by part (i).
15. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightarrow ((\alpha \wedge \gamma) \wedge \beta)$ by SMP applied to 13 and 14.

16. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H \beta$ by the definition of $\rightharpoonup_H$.
17. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H \gamma$ by part (d).
18. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H (\beta \wedge \gamma)$ by axiom (S5) and SMP applied to 16 and 17.
    Also,
19. $\vdash_{\mathcal{SI}} (\gamma \wedge (\alpha \wedge \beta)) \rightharpoonup_H \gamma$ by axiom (S4).
20. $\vdash_{\mathcal{SI}} ((\gamma \wedge \alpha) \wedge \beta) \rightharpoonup_H \gamma$ by part (h) applied to 19.
21. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H \alpha$ by axiom (S4).
22. $\vdash_{\mathcal{SI}} \gamma \rightharpoonup_H [(\alpha \wedge \beta) \rightharpoonup_H \alpha]$ by part (a).
23. $\vdash_{\mathcal{SI}} (\gamma \wedge (\alpha \wedge \beta)) \rightharpoonup_H \alpha$ by axiom (S9) and SMP applied to 22.
24. $\vdash_{\mathcal{SI}} ((\gamma \wedge \alpha) \wedge \beta) \rightharpoonup_H \alpha$ by (h).
25. $\vdash_{\mathcal{SI}} ((\gamma \wedge \alpha) \wedge \beta) \rightharpoonup_H (\alpha \wedge \gamma)$ by axiom (S5) and SMP applied to 23 and 20.
26. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H \beta$ by part (j) applied to 4.
27. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H (\gamma \wedge \alpha)$ by (e).
28. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H ((\gamma \wedge \alpha) \wedge \beta)$ by axiom (S5) and SMP applied to 27 and 26.
29. $\vdash_{\mathcal{SI}} [(\gamma \wedge \alpha) \rightarrow (\alpha \wedge \gamma)] \rightharpoonup_H [(\gamma \wedge \alpha) \rightarrow [(\gamma \wedge \alpha) \wedge \beta]]$ by axiom (S11) and SMP applied to 25 and 28.
30. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow (\alpha \wedge \gamma)$ by part (i).
31. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightarrow [(\gamma \wedge \alpha) \wedge \beta]$ by SMP applied to 29 and 30.
32. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightharpoonup_H \beta$ by the definition of $\rightharpoonup_H$.
33. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightharpoonup_H \gamma$ by el axioma (S4).
34. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightharpoonup_H (\gamma \wedge \beta)$ by axiom (S5) and SMP applied to 32 and 33.

(l)
1. $\vdash_{\mathcal{SI}} (\beta \wedge \gamma) \rightharpoonup_H \gamma$ by part (d).
2. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H [(\beta \wedge \gamma) \rightharpoonup_H \gamma]$ by part (a).
3. $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \wedge \gamma)) \rightharpoonup_H \gamma$ by axiom (S9) and SMP applied to 2.
4. $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \wedge \gamma)) \rightharpoonup_H \alpha$ by axiom (S4).
5. $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \wedge \gamma)) \rightharpoonup_H (\alpha \wedge \gamma)$ by axiom (S5) and SMP applied to 4 and 3.
6. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\beta \wedge \gamma)$ by hypothesis.
7. $\vdash_{\mathcal{SI}} \gamma \rightharpoonup_H (\alpha \rightharpoonup_H (\beta \wedge \gamma))$ by part (a).
8. $\vdash_{\mathcal{SI}} (\gamma \wedge \alpha) \rightharpoonup_H (\beta \wedge \gamma)$ by axiom (S9) and SMP applied to 7.
9. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H (\beta \wedge \gamma)$ by part (j).
10. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H \alpha$ by axiom (S4).
11. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \rightharpoonup_H (\alpha \wedge (\beta \wedge \gamma))$ by axiom (S5) and SMP applied to 10 and 9.
12. $\vdash_{\mathcal{SI}} (\alpha \rightarrow (\alpha \wedge (\beta \wedge \gamma))) \rightharpoonup_H (\alpha \rightarrow (\alpha \wedge \gamma))$ by axiom (S11) and SMP applied to 5 and 11.
13. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H (\beta \wedge \gamma)) \rightharpoonup_H (\alpha \rightharpoonup_H \gamma)$ by the definition of $\rightharpoonup_H$.
14. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H \gamma$ by SMP applied to 6.

(m)
1. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \gamma) \rightharpoonup_H (\alpha \rightharpoonup_H \gamma)$ by part (b).
2. $\vdash_{\mathcal{SI}} ((\alpha \rightharpoonup_H \gamma) \wedge \alpha) \rightharpoonup_H \gamma$ by axiom (S9) and SMP applied to 1.
3. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \rightharpoonup_H \gamma)] \rightharpoonup_H \gamma$ by part (j).
4. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \rightharpoonup_H \gamma)] \rightharpoonup_H (\alpha \rightharpoonup_H \gamma)$ by part (d).
5. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H \beta$ by hypothesis.
6. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \gamma) \rightharpoonup_H (\alpha \rightharpoonup_H \beta)$ by part (a).

7. $\vdash_{\mathcal{SI}} [(\alpha \to_H \gamma) \wedge \alpha] \to_H \beta$ by axiom (S9) and SMP applied to 6.
8. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to_H \gamma)] \to_H \beta$ by part (j) applied to 7.
9. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to_H \gamma)] \to_H [\beta \wedge (\alpha \to_H \gamma)]$ by axiom (S5) and SMP applied to 8 and 4.
10. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to_H \gamma)] \to_H [[\beta \wedge (\alpha \to_H \gamma)] \wedge \gamma]$ by axiom (S5) and SMP applied to 9 and 3.
11. $\vdash_{\mathcal{SI}} [\beta \wedge (\alpha \to_H \gamma)] \to_H [[\alpha \wedge (\alpha \to_H \gamma)] \to_H [[\beta \wedge (\alpha \to_H \gamma)] \wedge \gamma]]$ by part (a).
12. $\vdash_{\mathcal{SI}} [[\beta \wedge (\alpha \to_H \gamma)] \wedge [\alpha \wedge (\alpha \to_H \gamma)]] \to_H [[\beta \wedge (\alpha \to_H \gamma)] \wedge \gamma]]$ by axiom (S9) and SMP applied to 11.
13. $\vdash_{\mathcal{SI}} \beta \to_H \alpha$ by hypothesis.
14. $\vdash_{\mathcal{SI}} (\beta \wedge (\alpha \to_H \gamma)) \to_H (\alpha \wedge (\alpha \to_H \gamma))$ by part (k).
15. $\vdash_{\mathcal{SI}} (\gamma \wedge (\beta \wedge (\alpha \to_H \gamma))) \to_H (\gamma \wedge (\alpha \wedge (\alpha \to_H \gamma)))$ by part (k).
16. $\vdash_{\mathcal{SI}} (\gamma \wedge (\beta \wedge (\alpha \to_H \gamma))) \to_H (\alpha \wedge (\alpha \to_H \gamma))$ by part (l).
17. $\vdash_{\mathcal{SI}} ((\beta \wedge (\alpha \to_H \gamma)) \wedge \gamma) \to_H (\alpha \wedge (\alpha \to_H \gamma))$ by part (j).
18. $\vdash_{\mathcal{SI}} ((\beta \wedge (\alpha \to_H \gamma)) \wedge \gamma) \to_H (\beta \wedge (\alpha \to_H \gamma))$ by axiom (S4).
19. $\vdash_{\mathcal{SI}} ((\beta \wedge (\alpha \to_H \gamma)) \wedge \gamma) \to_H [(\beta \wedge (\alpha \to_H \gamma)) \wedge (\alpha \wedge (\alpha \to_H \gamma))]$ by axiom (S5) and SMP applied to 17 and 18.
20. $\vdash_{\mathcal{SI}} [(\beta \wedge (\alpha \to_H \gamma)) \to [(\beta \wedge (\alpha \to_H \gamma)) \wedge (\alpha \wedge (\alpha \to_H \gamma))]] \to_H [(\beta \wedge (\alpha \to_H \gamma)) \to ((\beta \wedge (\alpha \to_H \gamma)) \wedge \gamma)]$ by axiom (S11) and SMP applied to 12 and 19.
21. $\vdash_{\mathcal{SI}} [(\beta \wedge (\alpha \to_H \gamma)) \to_H (\alpha \wedge (\alpha \to_H \gamma))] \to_H [(\beta \wedge (\alpha \to_H \gamma)) \to_H \gamma]$ by the definition of $\to_H$.
22. $\vdash_{\mathcal{SI}} (\beta \wedge (\alpha \to_H \gamma)) \to_H \gamma$ by SMP applied to 14 and 21.
23. $\vdash_{\mathcal{SI}} ((\alpha \to_H \gamma) \wedge \beta) \to_H \gamma$ by part (j).
24. $\vdash_{\mathcal{SI}} (\alpha \to_H \gamma) \to_H (\beta \to_H \gamma)$ by axiom (S8) and SMP applied to 23.

The other half is similar.

(n)

1. $\vdash_{\mathcal{SI}} (\beta \to_H \gamma) \to_H (\beta \to_H \gamma)$ by part (b).
2. $\vdash_{\mathcal{SI}} ((\beta \to_H \gamma) \wedge \beta) \to_H \gamma$ by axiom (S9) and SMP applied to 1.
3. $\vdash_{\mathcal{SI}} (\beta \wedge (\beta \to_H \gamma)) \to_H \gamma$ by part (j) applied to 2.
4. $\vdash_{\mathcal{SI}} \alpha \to_H [(\beta \wedge (\beta \to_H \gamma)) \to_H \gamma]$ by part (a).
5. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge (\beta \to_H \gamma))] \to_H \gamma$ by axiom (S9) and SMP applied to 4.
6. $\vdash_{\mathcal{SI}} (\beta \wedge (\beta \to_H \gamma)) \to_H \beta$ by axiom (S4).
7. $\vdash_{\mathcal{SI}} \alpha \to_H [(\beta \wedge (\beta \to_H \gamma)) \to_H \beta]$ by part (a).
8. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge (\beta \to_H \gamma))] \to_H \beta$ by axiom (S9) and SMP applied to 7.
9. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge (\beta \to_H \gamma))] \to_H \alpha$ by axiom (S4).
10. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge (\beta \to_H \gamma))] \to_H (\alpha \wedge \beta)$ by axiom (S5) and SMP applied to 9 and 8.
11. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge (\beta \to_H \gamma))] \to_H [(\alpha \wedge \beta) \wedge \gamma]$ by axiom (S5) and SMP applied to 10 and 5.
12. $\vdash_{\mathcal{SI}} [(\beta \wedge (\alpha \wedge \beta)) \wedge \gamma] \to_H \gamma$ by part (d).
13. $\vdash_{\mathcal{SI}} [\beta \wedge ((\alpha \wedge \beta) \wedge \gamma)] \to_H \gamma$ by part (h) applied to 12.
14. $\vdash_{\mathcal{SI}} [((\alpha \wedge \beta) \wedge \gamma) \wedge \beta] \to_H \gamma$ by part (j).
15. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \gamma) \to_H (\beta \to_H \gamma)$ by axiom (S8) and SMP applied to 14.

16. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H \beta$ by part (d).
17. $\vdash_{\mathcal{SI}} \gamma \rightharpoonup_H [(\alpha \wedge \beta) \rightharpoonup_H \beta]$ by part (a).
18. $\vdash_{\mathcal{SI}} [\gamma \wedge (\alpha \wedge \beta)] \rightharpoonup_H \beta$ by axiom (S9) and SMP applied to 17.
19. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H \beta$ by part (j).
20. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \wedge \gamma)] \rightharpoonup_H \alpha$ by axiom (S4).
21. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H \alpha$ by part (h) applied to 20.
22. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H [\beta \wedge (\beta \rightharpoonup_H \gamma)]$ by axiom (S5) and SMP applied to 19 and 15.
23. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H [\alpha \wedge [\beta \wedge (\beta \rightharpoonup_H \gamma)]]$ by axiom (S5) and SMP applied to 21 and 22.
24. $\vdash_{\mathcal{SI}} [[(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H \gamma] \rightharpoonup_H [[\alpha \wedge [\beta \wedge (\beta \rightharpoonup_H \gamma)]] \rightharpoonup_H \gamma]$ by part (m) in view if 11 and 23.
25. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge \gamma] \rightharpoonup_H \gamma$ by part (d).
26. $\vdash_{\mathcal{SI}} [\alpha \wedge [\beta \wedge (\beta \rightharpoonup_H \gamma)]] \rightharpoonup_H \gamma$ by SMP applied to 24 and 25.
27. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge (\beta \rightharpoonup_H \gamma)) \rightharpoonup_H \gamma$ by part (h) applied to 26.
28. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H [(\beta \rightharpoonup_H \gamma) \rightharpoonup_H \gamma]$ by axiom (S8) and SMP applied to 27.
29. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \beta) \rightharpoonup_H (\alpha \rightharpoonup_H \beta)$ by part (b).
30. $\vdash_{\mathcal{SI}} ((\alpha \rightharpoonup_H \beta) \wedge \alpha) \rightharpoonup_H \beta$ by axiom (S9) and SMP applied to 29.
31. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightharpoonup_H \beta)) \rightharpoonup_H \beta$ by part (j).
32. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightharpoonup_H \beta)) \rightharpoonup_H \alpha$ by axiom (S4).
33. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightharpoonup_H \beta)) \rightharpoonup_H (\alpha \wedge \beta)$ by axiom (S5) and SMP applied to 32 and 31.
34. $\vdash_{\mathcal{SI}} (\beta \wedge \alpha) \rightharpoonup_H \beta$ by axiom (S4).
35. $\vdash_{\mathcal{SI}} \beta \rightharpoonup_H (\alpha \rightharpoonup_H \beta)$ by axiom (S8) and SMP applied to 34.
36. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\beta \rightharpoonup_H (\alpha \rightharpoonup_H \beta))$ by part (a).
37. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H (\alpha \rightharpoonup_H \beta))$ by axiom (S9) and SMP applied to 36.
38. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H \alpha$ by axiom (S4).
39. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H (\alpha \wedge (\alpha \rightharpoonup_H \beta))$ by axiom (S5) and SMP applied to 38 and 37.
40. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \rightharpoonup_H [(\beta \rightharpoonup_H \gamma) \rightharpoonup_H \gamma]] \rightharpoonup_H [(\alpha \wedge (\alpha \rightharpoonup_H \beta)) \rightharpoonup_H [(\beta \rightharpoonup_H \gamma) \rightharpoonup_H \gamma]]$ by part (m) and items 33 and 39.
41. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightharpoonup_H \beta)) \rightharpoonup_H [(\beta \rightharpoonup_H \gamma) \rightharpoonup_H \gamma]$ by SMP applied to 28 and 40.
42. $\vdash_{\mathcal{SI}} [(\alpha \wedge (\alpha \rightharpoonup_H \beta)) \wedge (\beta \rightharpoonup_H \gamma)] \rightharpoonup_H \gamma$ by axiom (S9) and SMP applied to 41.
43. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\alpha \rightharpoonup_H \beta) \wedge (\beta \rightharpoonup_H \gamma))] \rightharpoonup_H \gamma$ by part (h).
44. $\vdash_{\mathcal{SI}} [((\alpha \rightharpoonup_H \beta) \wedge (\beta \rightharpoonup_H \gamma)) \wedge \alpha] \rightharpoonup_H \gamma$ by part (j).
45. $\vdash_{\mathcal{SI}} ((\alpha \rightharpoonup_H \beta) \wedge (\beta \rightharpoonup_H \gamma)) \rightharpoonup_H (\alpha \rightharpoonup_H \gamma)$ by axiom (S8) and SMP applied to 44.
46. $\vdash_{\mathcal{SI}} (\alpha \rightharpoonup_H \beta) \rightharpoonup_H [(\beta \rightharpoonup_H \gamma) \rightharpoonup_H (\alpha \rightharpoonup_H \gamma)]$ by axiom (S8) and SMP applied to 45.

(o) 1. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \wedge \beta)) \rightharpoonup_H (\alpha \wedge \beta)$ by part (d).
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoonup_H \beta$ by part (d).
3. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \wedge \beta)) \rightharpoonup_H \beta$ by part (n) and SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H [(\alpha \wedge \beta) \rightharpoonup_H \beta]$ by axiom (S8) and SMP applied to 3.

5. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H \alpha$ by part (b).
6. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H [\alpha \wedge [(\alpha \wedge \beta) \rightharpoondown_H \beta]]$ by axiom (S5) and SMP applied to 4 and 5.

(p)
1. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to (\alpha \wedge \beta)$ by part (f).
2. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \alpha) \rightharpoondown_H (\alpha \wedge \beta)$ by (S4).
3. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \beta) \rightharpoondown_H (\alpha \wedge \beta)$ by (S4).
4. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H \alpha$ by (S4).
5. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H (\alpha \wedge \beta)$ by part (b).
6. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H ((\alpha \wedge \beta) \wedge \alpha)$ by axiom (S5) and SMP applied to 4 and 5.
7. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H \beta$ by part (d).
8. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H (\alpha \wedge \beta)$ by part (b).
9. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \rightharpoondown_H ((\alpha \wedge \beta) \wedge \beta)$ by axiom (S5) and SMP applied to 7 and 8.
10. $\vdash_{\mathcal{SI}} (((\alpha \wedge \beta) \wedge \beta) \rightharpoondown_H (\alpha \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \rightharpoondown_H ((\alpha \wedge \beta) \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \to (\alpha \wedge \beta)) \rightharpoondown_H ((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta))))$ by (S11).
11. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightharpoondown_H ((\alpha \wedge \beta) \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \to (\alpha \wedge \beta)) \rightharpoondown_H ((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)))$ by SMP applied to 3 and 10.
12. $\vdash_{\mathcal{SI}} (((\alpha \wedge \beta) \to (\alpha \wedge \beta)) \rightharpoondown_H ((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)))$ by SMP applied to 9 and 11.
13. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \rightharpoondown_H ((\alpha \wedge \beta) \wedge \alpha)) \rightharpoondown_H ((((\alpha \wedge \beta) \wedge \alpha) \rightharpoondown_H (\alpha \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta))))$ by axiom (S10).
14. $\vdash_{\mathcal{SI}} (((\alpha \wedge \beta) \wedge \alpha) \rightharpoondown_H (\alpha \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)))$ by SMP applied to 6 and 13.
15. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta))$ by SMP applied to 2 and 14.
16. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \to (\alpha \wedge \beta)) \rightharpoondown_H (((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta))$ by part (n) and SMP applied to 12 and 15.
17. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)$ by SMP applied to 1 and 16.

(q)
1. $\vdash_{\mathcal{SI}} (\beta \rightharpoondown_H (\alpha \wedge \beta)) \rightharpoondown_H [((\alpha \wedge \beta) \rightharpoondown_H \beta) \rightharpoondown_H [(\beta \to \gamma) \rightharpoondown_H ((\alpha \wedge \beta) \to \gamma)]]$ by (S10).
2. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H (\beta \rightharpoondown_H (\alpha \wedge \beta))$ by part (g).
3. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H [((\alpha \wedge \beta) \rightharpoondown_H \beta) \rightharpoondown_H [(\beta \to \gamma) \rightharpoondown_H ((\alpha \wedge \beta) \to \gamma)]]$ by part (n) and SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\alpha \wedge \beta) \rightharpoondown_H \beta)] \rightharpoondown_H [(\beta \to \gamma) \rightharpoondown_H ((\alpha \wedge \beta) \to \gamma)]$ by axiom (S9) and SMP applied to 3.
5. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H [\alpha \wedge ((\alpha \wedge \beta) \rightharpoondown_H \beta)]$ by part (o).
6. $\vdash_{\mathcal{SI}} \alpha \rightharpoondown_H [(\beta \to \gamma) \rightharpoondown_H ((\alpha \wedge \beta) \to \gamma)]$ by part (n) and SMP applied to 4 and 5.
7. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \to \gamma)] \rightharpoondown_H ((\alpha \wedge \beta) \to \gamma)$ by axiom (S9) and SMP applied to 6.
8. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \to \gamma)] \rightharpoondown_H \alpha$ by (S4).

9. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \to \gamma)] \to_H [\alpha \wedge ((\alpha \wedge \beta) \to \gamma)]$ by axiom (S5) and SMP applied to 7 and 8.
10. $\vdash_{\mathcal{SI}} ((\alpha \wedge \gamma) \to_H \gamma) \to_H ((\gamma \to_H (\alpha \wedge \gamma)) \to_H (((\alpha \wedge \beta) \to \gamma) \to_H ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))))$ by axiom (S11).
11. $\vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H \gamma$ by part (d).
12. $\vdash_{\mathcal{SI}} (\gamma \to_H (\alpha \wedge \gamma)) \to_H (((\alpha \wedge \beta) \to \gamma) \to_H ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)))$ by SMP applied to 10 and 11.
13. $\vdash_{\mathcal{SI}} \alpha \to_H (\gamma \to_H (\alpha \wedge \gamma))$ by part (g).
14. $\vdash_{\mathcal{SI}} \alpha \to_H (((\alpha \wedge \beta) \to \gamma) \to_H ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)))$ by part (n) and SMP applied to 12 and 13.
15. $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \to \gamma)) \to_H ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))$ by axiom (S9) and SMP applied to 14.
16. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \to \gamma)] \to_H ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))$ by part (n) and SMP applied to 9 and 15.
17. $\vdash_{\mathcal{SI}} [\alpha \wedge (\beta \to \gamma)] \to_H [\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))]$ by axiom (S5) and SMP applied to 8 and 16.

(r)
1. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \to_H \beta) \to_H [(\beta \to_H (\alpha \wedge \beta)) \to_H [((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \to_H (\beta \to (\alpha \wedge \gamma))]]$ by axiom (S10).
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \beta$ by part (d).
3. $\vdash_{\mathcal{SI}} (\beta \to_H (\alpha \wedge \beta)) \to_H [((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \to_H (\beta \to (\alpha \wedge \gamma))]$ by SMP applied to 1 and 2.
4. $\vdash_{\mathcal{SI}} \alpha \to_H (\beta \to_H (\alpha \wedge \beta))$ by part (g).
5. $\vdash_{\mathcal{SI}} \alpha \to_H [((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \to_H (\beta \to (\alpha \wedge \gamma))]$ by part (n) and SMP applied to 3 and 4.
6. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))] \to_H (\beta \to (\alpha \wedge \gamma))$ by (S9) and SMP applied to 5.
7. $\vdash_{\mathcal{SI}} (\gamma \to_H (\alpha \wedge \gamma)) \to_H [((\alpha \wedge \gamma) \to_H \gamma) \to_H [(\beta \to (\alpha \wedge \gamma)) \to_H (\beta \to \gamma)]]$ by (S11).
8. $\vdash_{\mathcal{SI}} \alpha \to_H (\gamma \to_H (\alpha \wedge \gamma))$ by part (g).
9. $\vdash_{\mathcal{SI}} \alpha \to_H [((\alpha \wedge \gamma) \to_H \gamma) \to_H [(\beta \to (\alpha \wedge \gamma)) \to_H (\beta \to \gamma)]]$ by part (n) and SMP applied to 7 and 8.
10. $\vdash_{\mathcal{SI}} [\alpha \wedge ((\alpha \wedge \gamma) \to_H \gamma)] \to_H [(\beta \to (\alpha \wedge \gamma)) \to_H (\beta \to \gamma)]]$ by (S9) and SMP applied to 9.
11. $\vdash_{\mathcal{SI}} \alpha \to_H [\alpha \wedge ((\alpha \wedge \gamma) \to_H \gamma)]$ by part (o).
12. $\vdash_{\mathcal{SI}} \alpha \to_H [(\beta \to (\alpha \wedge \gamma)) \to_H (\beta \to \gamma)]]$ by part (n) and SMP applied to 10 and 11.
13. $\vdash_{\mathcal{SI}} (\alpha \wedge (\beta \to (\alpha \wedge \gamma))) \to_H (\beta \to \gamma)$ by (S9) and SMP applied to 12.
14. $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \to_H \alpha$ by (S4).
15. $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \to_H (\alpha \wedge (\beta \to (\alpha \wedge \gamma)))$ by axiom (S5) and SMP applied to 6 and 14.
16. $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \to_H (\beta \to \gamma)$ by part (n) and SMP applied to 13 and 15.
17. $\vdash_{\mathcal{SI}} (\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \to_H (\alpha \wedge (\beta \to \gamma))$ by axiom (S5) and SMP applied to 14 and 16.

(s) 1. $\vdash_{\mathcal{SI}} ((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)$ by part (p).
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H (\alpha \wedge \beta)$ by (b).
3. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H [((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)]]$ by (a) and 1.
4. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H [(\alpha \wedge \beta) \wedge [((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)]]$ by (S5) and SMP applied to 2 and 3.
5. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge [((\alpha \wedge \beta) \wedge \alpha) \to ((\alpha \wedge \beta) \wedge \beta)]] \to_H [(\alpha \wedge \beta) \wedge (\alpha \to \beta)]$ by part (r).
6. $\vdash_{\mathcal{SI}} [(\alpha \wedge \beta) \wedge (\alpha \to \beta)] \to_H (\alpha \to \beta)$ by part (d).
7. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H (\alpha \to \beta)$ by part (n) and SMP applied to 4, 5 and 6.
8. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H \alpha$ by axiom (S4).
9. $\vdash_{\mathcal{SI}} (\alpha \wedge \beta) \to_H (\alpha \wedge (\alpha \to \beta))$ by axiom (S5) and SMP applied to 7 and 8.

(t) 1. $\vdash_{\mathcal{SI}} (\alpha \to_H \top) \to_H [(\top \to_H \alpha) \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]]$ by (S10).
2. $\vdash_{\mathcal{SI}} ((\alpha \to_H \top) \wedge (\top \to_H \alpha)) \to_H ((\alpha \to \beta) \to_H (\top \to \beta))$ by (S9) and SMP applied to 1.
3. $\vdash_{\mathcal{SI}} ((\top \to_H \alpha) \wedge (\alpha \to_H \top)) \to_H ((\alpha \to \beta) \to_H (\top \to \beta))$ using part (j) applied to 2.
4. $\vdash_{\mathcal{SI}} (\top \to_H \alpha) \to_H [(\alpha \to_H \top) \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]]$ by (S8) and SMP applied to 3.
5. $\vdash_{\mathcal{SI}} (\alpha \wedge \top) \to_H \alpha$ by (S4).
6. $\vdash_{\mathcal{SI}} \alpha \to_H (\top \to_H \alpha)$ by axiom (S8) and SMP applied to 5.
7. $\vdash_{\mathcal{SI}} \alpha \to_H [(\alpha \to_H \top) \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]]$ by part (n) and SMP applied to 4 and 6.
8. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to_H \top)) \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]$ by axiom (S8) and SMP applied to 7.
9. $\vdash_{\mathcal{SI}} ((\alpha \to_H \top) \wedge \alpha) \to_H (\alpha \wedge (\alpha \to_H \top))$ by part (e).
10. $\vdash_{\mathcal{SI}} ((\alpha \to_H \top) \wedge \alpha) \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]$ by part (n) and SMP applied to 8 and 9.
11. $\vdash_{\mathcal{SI}} (\alpha \to_H \top) \to_H [\alpha \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]]$ by axiom (S8) and SMP applied to 10.
12. $\vdash_{\mathcal{SI}} \top$ by (S6).
13. $\vdash_{\mathcal{SI}} \alpha \to_H \top$ by part (a).
14. $\vdash_{\mathcal{SI}} \alpha \to_H [(\alpha \to \beta) \to_H (\top \to \beta)]$ by SMP applied to 11 and 13.
15. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H (\top \to \beta)$ by axiom (S9) and SMP applied to 14.
16. $\vdash_{\mathcal{SI}} ((\top \wedge \beta) \to_H \beta) \to_H [(\beta \to_H (\top \wedge \beta)) \to_H [(\top \to \beta) \to_H (\top \to (\top \wedge \beta))]]$ by (S11).
17. $\vdash_{\mathcal{SI}} (\top \wedge \beta) \to_H \beta$ by part (d).
18. $\vdash_{\mathcal{SI}} (\beta \to_H (\top \wedge \beta)) \to_H [(\top \to \beta) \to_H (\top \to (\top \wedge \beta))]$ by SMP applied to 16 and 17.
19. $\vdash_{\mathcal{SI}} \beta \to_H (\beta \wedge \top)$ by part (c) and 12.
20. $\vdash_{\mathcal{SI}} (\beta \wedge \top) \to_H (\top \wedge \beta)$ by part (e).
21. $\vdash_{\mathcal{SI}} \beta \to_H (\top \wedge \beta)$ by part (n) and SMP applied to 19 and 20.
22. $\vdash_{\mathcal{SI}} (\top \to \beta) \to_H (\top \to (\top \wedge \beta))$ by SMP applied to 18 and 21.
23. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H (\top \to (\top \wedge \beta))$ by part (n) and SMP applied to 15 and 22.

24. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H (\top \to_H \beta)$ by the definition of $\to_H$.
25. $\vdash_{\mathcal{SI}} [(\alpha \wedge (\alpha \to \beta)) \wedge \top] \to_H \beta$ by axiom (S9) and SMP applied to 24.
26. $\vdash_{\mathcal{SI}} [\top \wedge (\alpha \wedge (\alpha \to \beta))] \to_H [(\alpha \wedge (\alpha \to \beta)) \wedge \top]$ by part (e).
27. $\vdash_{\mathcal{SI}} [\top \wedge (\alpha \wedge (\alpha \to \beta))] \to_H \beta$ by part (n) and SMP applied to 25 and 26.
28. $\vdash_{\mathcal{SI}} \top \to_H [(\alpha \wedge (\alpha \to \beta)) \to_H \beta]$ by axiom (S8) and SMP applied to 27.
29. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H \beta$ by SMP applied to 12 and 28.
30. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H \alpha$ by axiom (S4).
31. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \beta)) \to_H (\alpha \wedge \beta)$ by axiom (S5) and SMP applied to 29 and 30. ■

Proof. of Lemma 3.4.
(a) implies (b):

1. $\Gamma \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \wedge (\beta \to_H \alpha)$ by hypothesis.
2. $\Gamma \vdash_{\mathcal{SI}} [(\alpha \to_H \beta) \wedge (\beta \to_H \alpha)] \to_H (\alpha \to_H \beta)$ by (S4).
3. $\Gamma \vdash_{\mathcal{SI}} \alpha \to_H \beta$ by SMP.
4. Similarly, $\Gamma \vdash_{\mathcal{SI}} \beta \to_H \alpha$, using Lemma 2.1(d).

(b) implies (a):

1. $\Gamma \vdash_{\mathcal{SI}} \alpha \to_H \beta$ by hypothesis.
2. $\Gamma \vdash_{\mathcal{SI}} \beta \to_H \alpha$ by hypothesis.
3. $\Gamma \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H (\alpha \to_H \beta)$ by Lemma 2.1, part (b).
4. $\Gamma \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H (\beta \to_H \alpha)$ by 2 and Lemma 2.1, part (a).
5. $\Gamma \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H [(\alpha \to_H \beta) \wedge (\beta \to_H \alpha)]$ by (S5) and SMP applied to 3 and 4.
6. $\Gamma \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$ by SMP applied to 1 and 5 and the definition of $\leftrightarrow_H$. ■

Proof. of Lemma 3.5

a) 1. $\vdash_{\mathcal{SI}} \alpha \to_H \alpha$ by Lemma 2.1, part (b).
   2. $\vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \alpha$ by Lemma 3.4.

b) 1. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$.
   2. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}} \alpha \to_H \beta$ and $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}} \beta \to_H \alpha$ by Lemma 3.4.
   3. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}} \beta \leftrightarrow_H \alpha$ by Lemma 3.4.

c) 1. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$
   2. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \alpha \to_H \beta$ by Lemma 3.4.
   3. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \beta \to_H \alpha$ by Lemma 3.4.
   4. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \beta \leftrightarrow_H \gamma$.
   5. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \beta \to_H \gamma$ by Lemma 3.4.
   6. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \gamma \to_H \beta$ by Lemma 3.4.
   7. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H ((\beta \to_H \gamma) \to_H (\alpha \to_H \gamma))$ by Lemma 2.1(n).

8. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \alpha \to_H \gamma$ by SMP.
   In a similar manner,
9. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \gamma \to_H \alpha$.
10. $\alpha \leftrightarrow_H \beta, \beta \leftrightarrow_H \gamma \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \gamma$ by Lemma 3.4.

d)
1. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$.
2. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \to_H \beta$ by Lemma 3.4.
3. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H \alpha$ by (S4).
4. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H \beta$ by Lemma 2.1(n) and SMP from 2 and 3.
5. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \leftrightarrow_H \delta$.
6. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \to_H \delta$ by Lemma 3.4.
7. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H \gamma$ by Lemma 2.1(d).
8. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H \delta$ by Lemma 2.1(n) and SMP from 6 and 7.
9. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} ((\alpha \wedge \gamma) \to_H \beta) \to_H [((\alpha \wedge \gamma) \to_H \delta) \to_H ((\alpha \wedge \gamma) \to_H (\beta \wedge \delta))]$ by (S5).
10. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \to_H (\beta \wedge \delta)$ SMP from 4, 8 and 9.
    In a similar way,
11. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\beta \wedge \delta) \to_H (\alpha \wedge \gamma)$.
12. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \wedge \gamma) \leftrightarrow_H (\beta \wedge \delta)$ by Lemma 3.4.

e)
1. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$ by identity.
2. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \to_H \beta$ by Lemma 3.4.
3. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \beta \to_H (\beta \vee \delta)$ by (S1).
4. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \to_H (\beta \vee \delta)$ by Lemma 2.1(n) and SMP applied to 2 and 3.
5. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \leftrightarrow_H \delta$ by identity.
6. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \to_H \delta$ by Lemma 3.4.
7. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \delta \to_H (\beta \vee \delta)$ by (S2).
8. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \to_H (\beta \vee \delta)$ by Lemma 2.1(n) and SMP applied to 6 and 7.
9. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \to_H (\beta \vee \delta)) \to_H [(\gamma \to_H (\beta \vee \delta)) \to_H ((\alpha \vee \gamma) \to_H (\beta \vee \delta))]]$ by (S3).
10. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \vee \gamma) \to_H (\beta \vee \delta)$ SMP applied to 4, 8 and 9.
    Analogously,
11. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\beta \vee \delta) \to_H (\alpha \vee \gamma)$.
12. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \vee \gamma) \leftrightarrow_H (\beta \vee \delta)$ by Lemma 3.4.

f)
1. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \leftrightarrow_H \beta$ by identity.
2. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \alpha \to_H \beta$ and $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \beta \to_H \alpha$, by Lemma 3.4.
3. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \to_H \beta) \to_H [(\beta \to_H \alpha) \to_H ((\alpha \to \gamma) \to_H (\beta \to \gamma))]$ by (S10).
4. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \to \gamma) \to_H (\beta \to \gamma)$ by SMP applied to 2 and 3.
5. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \leftrightarrow_H \delta$ by identity.

6. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \gamma \rightarrow_H \delta$ and $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} \delta \rightarrow_H \gamma$ by Lemma 3.4.
7. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\delta \rightarrow_H \gamma) \rightarrow_H [(\gamma \rightarrow_H \delta) \rightarrow_H ((\beta \rightarrow \gamma) \rightarrow_H (\beta \rightarrow \delta))]$ by (S11).
8. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\beta \rightarrow \gamma) \rightarrow_H (\beta \rightarrow \delta)$ by SMP applied to 6 and 7.
9. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \rightarrow \gamma) \rightarrow_H (\beta \rightarrow \delta)$ by (S12') and SMP applied to 4 and 8.
   Analogously,
10. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\beta \rightarrow \delta) \rightarrow_H (\alpha \rightarrow \gamma)$.
11. $\alpha \leftrightarrow_H \beta, \gamma \leftrightarrow_H \delta \vdash_{\mathcal{SI}} (\alpha \rightarrow \gamma) \leftrightarrow_H (\beta \rightarrow \delta)$ by Lemma 3.4. ■

Proof. of Lemma 3.7.

1. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \alpha \leftrightarrow_H \beta$ by identity.
2. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \alpha \rightarrow_H \beta$ by Lemma 3.4.
3. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\neg\beta \wedge \alpha) \rightarrow_H \neg\beta$ by axiom (S4).
4. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\neg\beta \wedge \alpha) \rightarrow_H \alpha$ by axiom (S13').
5. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\neg\beta \wedge \alpha) \rightarrow_H \beta$ by axiom (S12') and SMP applied to 4 and 2.
6. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\neg\beta \wedge \alpha) \rightarrow_H (\beta \wedge \neg\beta)$ by axiom (S5) and SMP applied to 5 and 3.
7. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\beta \wedge \neg\beta) \rightarrow_H \neg\alpha$ by axiom (S7').
8. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\neg\beta \wedge \alpha) \rightarrow_H \neg\alpha$ by axiom (S12') and SMP applied to 6 and 7.
9. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \neg\beta \rightarrow_H (\alpha \rightarrow_H \neg\alpha)$ by axiom (S8) and SMP applied to 8.
10. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} (\alpha \rightarrow_H \neg\alpha) \rightarrow_H \neg\alpha$ by axiom (S18').
11. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \neg\beta \rightarrow_H \neg\alpha$ by axiom (S12') and SMP applied to 9 and 10.
    Symmetrically, using that $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \beta \rightarrow_H \alpha$, we have that
12. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \neg\alpha \rightarrow_H \neg\beta$.
13. $\alpha \leftrightarrow_H \beta \vdash_{\mathcal{SI}'} \neg\alpha \leftrightarrow_H \neg\beta$ by Lemma 3.4. ■

Proof. of Lemma 3.8.

a)
1. $\vdash_{\mathcal{SI}} \bot \rightarrow_H \beta$ by (S7).
2. $\vdash_{\mathcal{SI}} (\alpha \wedge \bot) \rightarrow_H \bot$ by Lemma 2.1(d).
3. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightarrow \bot)) \rightarrow_H (\alpha \wedge \bot)$ by Lemma 2.1(t).
4. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \rightarrow \bot)) \rightarrow_H \beta$ by Lemma 2.1(n) and SMP applied to the previous items.

b)
1. $\Gamma \vdash_{\mathcal{SI}} (\alpha \rightarrow_H \alpha) \rightarrow_H [(\alpha \rightarrow_H \beta) \rightarrow_H (\alpha \rightarrow_H (\alpha \wedge \beta))]$ by (S5).
2. $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow_H \alpha$ by Lemma 2.1, part (b).
3. $\Gamma \vdash_{\mathcal{SI}} (\alpha \rightarrow_H \beta) \rightarrow_H (\alpha \rightarrow_H (\alpha \wedge \beta))$ by SMP applied to 1 and 2.
4. $\Gamma \vdash_{\mathcal{SI}} \beta$ by hypothesis.
5. $\Gamma \vdash_{\mathcal{SI}} \alpha \rightarrow_H \beta$ by Lemma 2.1, part (a).

6. $\Gamma \vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\alpha \wedge \beta)$ by SMP applied to 3 and 5.
7. $\Gamma \vdash_{\mathcal{SI}} \alpha$ by hypothesis.
8. $\Gamma \vdash_{\mathcal{SI}} \alpha \wedge \beta$ by SMP applied to 6 and 7.

c)
1. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H \alpha$ by Lemma 2.1, part (b).
2. $\vdash_{\mathcal{SI}} \bot \to \bot$ by Lemma 2.1, part (f).
3. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\bot \to \bot)$ by Lemma 2.1, part (a).
4. $\vdash_{\mathcal{SI}} \alpha \rightharpoonup_H (\alpha \wedge (\bot \to \bot))$ by (S5) and SMP applied to 1 and 3.

d)
1. $\Gamma \vdash_{\mathcal{SI}} \beta \leftrightharpoons_H \bot$ by hypothesis.
2. $\Gamma \vdash_{\mathcal{SI}} \beta \rightharpoonup_H \bot$ by Lemma 3.4.
3. $\Gamma \vdash_{\mathcal{SI}} \bot \rightharpoonup_H \alpha$ (S7).
4. $\Gamma \vdash_{\mathcal{SI}} \beta \rightharpoonup_H \alpha$ using Lemma 2.1(n) and SMP applied to 2 and 3.
5. $\Gamma \vdash_{\mathcal{SI}} (\beta \rightharpoonup_H \alpha) \rightharpoonup_H [(\beta \rightharpoonup_H \bot) \rightharpoonup_H (\beta \rightharpoonup_H (\alpha \wedge \bot))]$ (S5).
6. $\Gamma \vdash_{\mathcal{SI}} \beta \rightharpoonup_H (\alpha \wedge \bot)$ SMP applied to 4, 2 and 5.
7. $\Gamma \vdash_{\mathcal{SI}} (\alpha \wedge \bot) \rightharpoonup_H \bot$ by Lemma 2.1(d).
8. $\Gamma \vdash_{\mathcal{SI}} \bot \rightharpoonup_H \beta$ (S7).
9. $\Gamma \vdash_{\mathcal{SI}} (\alpha \wedge \bot) \rightharpoonup_H \beta$ using Lemma 2.1(n) and SMP applied to 7 and 8.

e)
1. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to (\alpha \wedge (\alpha \to \bot)))] \rightharpoonup_H [\alpha \wedge (\alpha \wedge (\alpha \to \bot))]$ by Lemma 2.1(t).
2. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \wedge (\alpha \to \bot))] \rightharpoonup_H (\alpha \wedge (\alpha \to \bot))$ by Lemma 2.1(d).
3. $\vdash_{\mathcal{SI}} (\alpha \wedge (\alpha \to \bot)) \rightharpoonup_H (\alpha \wedge \bot)$ by Lemma 2.1(t).
4. $\vdash_{\mathcal{SI}} (\alpha \wedge \bot) \rightharpoonup_H \bot$ using Lemma 2.1(d).
5. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \to (\alpha \wedge (\alpha \to \bot)))] \rightharpoonup_H \bot$ by Lemma 2.1(n) and SMP.
6. $\vdash_{\mathcal{SI}} [\alpha \wedge (\alpha \rightharpoonup_H (\alpha \to \bot))] \rightharpoonup_H \bot$ by the definition of $\rightharpoonup_H$.
   The other result is direct from (S7). ■

Proof. of Lemma 3.10.

a)
1. $\neg\alpha \vdash_{\mathcal{SI}'} \neg\alpha$
2. $\neg\alpha \vdash_{\mathcal{SI}'} (\alpha \rightharpoonup_H \neg\alpha)$ by Lemma 2.1, part (a).
3. $\vdash_{\mathcal{SI}'} \neg\alpha \rightharpoonup_H (\alpha \rightharpoonup_H \neg\alpha)$ by theorem 3.9.
4. $\vdash_{\mathcal{SI}'} (\alpha \rightharpoonup_H \neg\alpha) \rightharpoonup_H \neg\alpha$ by (S18').
5. $\vdash_{\mathcal{SI}'} (\alpha \rightharpoonup_H \neg\alpha) \leftrightharpoons_H \neg\alpha$ by 3, 4 and Lemma 3.4.

b)
1. $\Gamma \vdash_{\mathcal{SI}'} (\alpha \wedge \neg\alpha) \rightharpoonup_H \beta$ by (S7').
2. $\Gamma \vdash_{\mathcal{SI}'} (\alpha \wedge \neg\alpha) \rightharpoonup_H \neg\beta$ by (S7').
3. $\Gamma \vdash_{\mathcal{SI}'} (\alpha \wedge \neg\alpha) \rightharpoonup_H (\beta \wedge \neg\beta)$ by (S5) and SMP applied to 1 and 2.
4. $\Gamma \vdash_{\mathcal{SI}'} (\beta \wedge \neg\beta) \rightharpoonup_H (\alpha \wedge \neg\alpha)$ by a similar argument.
5. $\Gamma \vdash_{\mathcal{SI}'} (\beta \wedge \neg\beta) \leftrightharpoons_H (\alpha \wedge \neg\alpha)$ by Lemma 3.4.
6. $\Gamma \vdash_{\mathcal{SI}'} r_1(\beta \wedge \neg\beta) \rightharpoonup_H r_1(\alpha \wedge \neg\alpha)$ with $r_1(x) = \neg\beta \rightharpoonup_H (\beta \to x)$ by Lemma 3.4 and conditions (B1) to (B4).
7. $\Gamma, \neg\beta, \beta \vdash_{\mathcal{SI}'} \neg\beta$.
8. $\Gamma, \neg\beta \vdash_{\mathcal{SI}'} \beta \rightharpoonup_H \neg\beta$ by Theorem 3.9 applied to 7.
9. $\Gamma, \neg\beta \vdash_{\mathcal{SI}'} \beta \to (\beta \wedge \neg\beta)$ by definition of $\rightharpoonup_H$ in 8.
10. $\Gamma \vdash_{\mathcal{SI}'} r_1(\beta \wedge \neg\beta)$ by Theorem 3.9 applied to 9.
11. $\Gamma \vdash_{\mathcal{SI}'} r_1(\alpha \wedge \neg\alpha)$ by SMP applied to 6 and 10.
12. $\Gamma \vdash_{\mathcal{SI}'} \neg\beta \rightharpoonup_H (\beta \to (\alpha \wedge \neg\alpha))$ using the definition of r_1 in 11.

c)
1. $\Gamma \vdash_{\mathcal{SI}'} (\beta \wedge \neg\beta) \leftrightarrow_H (\alpha \wedge \neg\alpha)$ as in part b).
2. $\Gamma \vdash_{\mathcal{SI}'} r_2(\beta \wedge \neg\beta) \rightarrow_H r_2(\alpha \wedge \neg\alpha)$ with $r_2(x) = (\beta \to x) \rightarrow_H \neg\beta$ by Lemma 3.4 and conditions (B1) to (B4).
3. $\Gamma \vdash_{\mathcal{SI}'} (\beta \rightarrow_H \neg\beta) \rightarrow_H \neg\beta$ by axiom (S18').
4. $\Gamma \vdash_{\mathcal{SI}'} (\beta \to (\beta \wedge \neg\beta)) \rightarrow_H \neg\beta$ by the definition of $\rightarrow_H$ in 3.
5. $\Gamma \vdash_{\mathcal{SI}'} r_2(\beta \wedge \neg\beta)$ using the definition of r_2 in 4.
6. $\Gamma \vdash_{\mathcal{SI}'} r_2(\alpha \wedge \neg\alpha)$ by SMP applied to 2 and 5.
7. $\Gamma \vdash_{\mathcal{SI}'} (\beta \to (\alpha \wedge \neg\alpha)) \rightarrow_H \neg\beta$ using the definition of r_2 in 6.

d)
1. $\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha)) \vdash_{\mathcal{SI}'} \beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha))$.
2. $\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha)), \beta \vdash_{\mathcal{SI}'} \beta \rightarrow_H (\alpha \wedge \neg\alpha)$ by Theorem 3.9 applied to 1.
3. $\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha)), \beta \vdash_{\mathcal{SI}'} \beta$.
4. $\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha)), \beta \vdash_{\mathcal{SI}'} \alpha \wedge \neg\alpha$ using SMP in 2 and 3.
5. $\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha)) \vdash_{\mathcal{SI}'} \beta \rightarrow_H (\alpha \wedge \neg\alpha)$ by Theorem 3.9 applied to 4.
6. $\Gamma \vdash_{\mathcal{SI}'} (\beta \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha))) \rightarrow_H (\beta \rightarrow_H (\alpha \wedge \neg\alpha))$ by Theorem 3.9 applied to 5. ∎

Proof. of Lemma 6.23.

Let $\alpha, \beta \in Fm$, and let us check that $m(\alpha \wedge \beta) = m(\alpha) \cap m(\beta)$. If $w \in m(\alpha \wedge \beta)$, then $w \Vdash_\epsilon \alpha \wedge \beta$, so $w \Vdash_\epsilon \alpha$ and $w \Vdash_\epsilon \beta$. Therefore, $w \in m(\alpha)$ and $w \in m(\beta)$, so $m(\alpha \wedge \beta) \subseteq m(\alpha) \cap m(\beta)$. The other direction has a similar proof.

The proof of $m(\alpha \vee \beta) = m(\alpha) \cup m(\beta)$ is also straightforward. It is clear that $m(\top) = W$ and $m(\bot) = \emptyset$.

Lastly, we prove that $m(\alpha \to \beta) = m(\alpha) \to m(\beta)$. Observe that $m(\alpha \to \beta) = \{w \in W : \ w \Vdash_\epsilon \alpha \to \beta\} = \{w \in W : \ \mathbf{P}_\epsilon(w)\}$ and $m(\alpha) \to m(\beta) = \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$. We prove that

$$\{w \in W : \ \mathbf{P}_\epsilon(w)\} = \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$$

by induction over ϵ. We exhibit only some of the cases.

- If $\epsilon(\alpha, \beta) = \alpha$, then $\{w \in W : \ \mathbf{P}_\epsilon(w)\} = \{w \in W : \ w \Vdash_\epsilon \alpha\} = m(\alpha) = \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$.
- If $\epsilon = \bot$ then $\{w \in W : \ \mathbf{P}_\epsilon(w)\} = \emptyset = \bot^{\mathbf{A}}$.
- When $\epsilon = \epsilon_1 \wedge \epsilon_2$ for H-formulas ϵ_1 and ϵ_2, $\{w \in W : \ \mathbf{P}_\epsilon(w)\} = \{w \in W : \ \mathbf{P}_{\epsilon_1}(w) \text{ and } \mathbf{P}_{\epsilon_2}(w)\} = \{w \in W : \ \mathbf{P}_{\epsilon_1}(w)\} \cap \{w \in W : \ \mathbf{P}_{\epsilon_2}(w)\} = \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \cap \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta)) = \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$.
- If $\epsilon = \epsilon_1 \rightarrow_H \epsilon_2$ for H-formulas $\epsilon_1(\alpha, \beta)$ and $\epsilon_2(\alpha, \beta)$, then $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) = \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta)) \rightarrow_H \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$ is the biggest increasing subset contained in $\big(W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))\big) \cup \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$.

  To check that $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \subseteq \{w \in W : \ \mathbf{P}_\epsilon(w)\}$, let $w \in \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$. By condition (5) of Definition 6.19, it is enough to consider $v \in W$ such that wRv and $\mathbf{P}_{\epsilon_1}(v)$ and then prove that $\mathbf{P}_{\epsilon_2}(v)$. Since $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$ is increasing, $v \in \epsilon^{\mathbf{A}}(m(\alpha), m(\beta)) \subseteq \big(W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))\big) \cup \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$ so

$v \in W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$ or $v \in \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$. By inductive hypothesis on ϵ_1 it follows that $\{w \in W : \mathbf{P}_{\epsilon_1}(w)\} = \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$. Therefore, since $\mathbf{P}_{\epsilon_1}(v)$, $v \in \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$. As a consequence, $v \in \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$, so $\mathbf{P}_{\epsilon_2}(v)$ because by inductive hypothesis we also have that $\{w \in W : \mathbf{P}_{\epsilon_2}(w)\} = \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$.

Going in the other direction, let $w \in W$ be such that $\mathbf{P}_{\epsilon}(w)$. We prove first that the increasing set $M = \{v \in W : wRv\} \subseteq \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$ so $w \in \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$, given that R is a reflexive relation. Let $v \in M$. If $v \notin W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$ then $v \in \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))$. By the inductive hypothesis, this means that $\mathbf{P}_{\epsilon_1}(v)$. Since $\mathbf{P}_{\epsilon}(w)$ holds, then $\mathbf{P}_{\epsilon_2}(v)$ because wRv. Therefore $v \in \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$, so $M \subseteq \big(W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))\big) \cup \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$. Since $\epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$ is the biggest of the increasing sets contained in $\big(W \setminus \epsilon_1^{\mathbf{A}}(m(\alpha), m(\beta))\big) \cup \epsilon_2^{\mathbf{A}}(m(\alpha), m(\beta))$, $M \subseteq \epsilon^{\mathbf{A}}(m(\alpha), m(\beta))$. ∎

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