Free vibrations of anisotropic rectangular plates with holes and attached masses

C. A. Rossit†

Dpto. de Ingeniería, Instituto de Mecánica Aplicada, Universidad Nacional del Sur, 8000, Bahía Blanca, Argentina

P. M. Ciancio‡

Facultad de Ingeniería, Universidad Nacional del Centro de la Provincia de Buenos Aires, 7400, Olavarría, Argentina

(Received December 6, 2006, Accepted August 17, 2007)

Abstract. Anisotropic materials are increasingly required in modern technological applications. Certainly, civil, mechanical and naval engineers frequently deal with the situation of analyzing the dynamical behaviour of structural elements being composed of such materials. For example, panels of anisotropic materials must sometimes support electromechanical engines, and besides, holes are performed in them for operational reasons e.g., conduits, ducts or electrical connections. This study is concerned with the natural frequencies and normal modes of vibration of rectangular anisotropic plates supported by different combinations of the classical boundary conditions: clamped, simply – supported and free, and with additional complexities such holes of free boundaries and attached concentrated masses. A variational approach (the well known Ritz method) is used, where the displacement amplitude is approximated by a set of beam functions in each coordinate direction corresponding to the sides of the rectangular plate. Consequently each coordinate function satisfies the essential boundary conditions at the outer edge of the plate. The influence of the position and magnitude of both hole and mass, on the natural frequencies and modal shapes of vibration are studied for a generic anisotropic material. The classical Ritz method with beam functions as spatial approximation proved to be a suitable procedure to solve a problem of such analytical complexity.

Keywords: vibration of plates; anisotropic plates; concentrated mass; holes of free edge; Ritz method.

1. Introduction

The present study deals with the analysis of transverse vibrations of thin rectangular plates of anisotropic materials carrying concentrated masses rigidly attached and rectangular holes of free edges.

The proposed mechanical system is of great interest in many technological situations since it is quite common in a large variety of engineering fields: from plates supporting machinery with holes

† Professor, Corresponding author, E-mail: carossit@criba.edu.ar
‡ Graduate Student, E-mail: pciancio@fio.unicen.edu.ar

to printed circuit boards with electronic elements attached to them. A plate – like chassis or a printed circuit can be approximated as flat rectangular plates carrying concentrated masses with holes, subjected to vibration.

As it is known it does not appear possible to obtain an exact analytical solution for the mode shapes and natural frequencies of transverse vibration of such a complex structural system.

It is important to point out that the thorough treatise due to Lekhnitskii (1968) does not solve any problem of vibration of anisotropic plates. Nevertheless, there are several textbooks on anisotropic plates where the vibration problems are included (Reddy 1997, Whitney 1987).

The variational Ritz method (pointed out by Leissa 2005, Mikhlin 1964 as being incorrectly called the Rayleigh-Ritz method by some persons), is employed to perform the analysis.

The displacement amplitude is approximated by a set of beam functions in each principal coordinate direction as it has been done by pioneering works on the vibration of solid anisotropic plates (Ashton 1969, Ashton and Waddoups 1969, Ashton and Anderson 1969, Bert and Mayberry 1969, Mohan and Kingsbury 1971).

Unfortunately at least one of those "almost classical" works, the paper by Mohan and Kingsbury, published thirty five years ago commits a mathematical error since the eigenvalues are determinated by the Galerkin method. In view of the fact that the coordinate functions do not satisfy, generally, the natural boundary conditions the methodology is not admissible and the eigenvalues are not, in general, valid.

In the treatment of anisotropic vibrating rectangular plates with additional complexities, Avalos *et al.* (1991) considered doubly connected domains for the simply – supported case and Ciancio *et al.* (2006) studied the cantilever anisotropic plate with a rigidly attached mass.

The first five natural frequency coefficients are obtained for plates of different combinations of the classical boundary conditions with a centered orifice, and varying the position and magnitude of the concentrated mass. The considered structural systems are shown in Fig. 1.

The corresponding modal shapes are also studied.

Due to the quantity and variability of the parameters involved in the description of the dynamical behaviour of these kinds of structures, just a few representative cases will be considered to demonstrate the convenience of the procedure.

2. Approximate analytical solution

According to the classical thin anisotropic plate theory, (Lekhnitskii 1968), the energy functional corresponding to the vibrating described system is given by

$$J(W) = \frac{1}{2}\iint_{A_p}\left[D_{11}\left[\frac{\partial^2 W}{\partial \bar{x}^2}\right]^2 + 2D_{12}\frac{\partial^2 W}{\partial \bar{x}^2}\frac{\partial^2 W}{\partial \bar{y}^2} + D_{22}\left[\frac{\partial^2 W}{\partial \bar{y}^2}\right]^2 + 4D_{66}\left[\frac{\partial^2 W}{\partial \bar{x}\,\partial \bar{y}}\right]^2 + \right.$$
$$\left. + 4\left[D_{16}\frac{\partial^2 W}{\partial \bar{x}^2} + D_{26}\frac{\partial^2 W}{\partial \bar{y}^2}\right]\frac{\partial^2 W}{\partial \bar{x}\,\partial \bar{y}}\right]d\bar{x}\,d\bar{y} - \frac{1}{2}\rho h\omega^2\iint_{A_p} W^2 d\bar{x}\,d\bar{y} + \frac{1}{2}m\omega^2[W(\bar{x}_m, \bar{y}_m)]^2 \quad (1)$$

where $W = W(\bar{x}, \bar{y})$ is the deflection amplitude of the middle plane of the plate. D_{ij} are the well known flexural rigidities of the anisotropic plate.

A_p is the net area of the plate plan form: $A_p = A - A_h$ where A is the area of the whole rectangle: $a \times b$ and A_h is the area of the hole: $a_1 \times b_1$.

ρ, h are the density and the thickness of the plate, respectively, m is the magnitude of the concentrated mass, $W(\bar{x}_m, \bar{y}_m)$ is the plate displacement amplitude at the mass position $(\bar{x}_m, \bar{y}_m)$ and ω is the natural circular frequency of the system.

The rotatory inertia of the concentrated mass is neglected in the present analysis.

As the length of the sides of the rectangular plate are a and b in the $\bar{x}$ and $\bar{y}$ directions respectively, the coordinates can be written in the dimensionless form

$$x = \bar{x}/a, y = \bar{y}/b$$

$$x_m = \bar{x}_m/a, y_m = \bar{y}_m/b \tag{2}$$

and the aspect ratio of the plate

$$\lambda = a/b$$

For simplicity, holes of the same aspect ratio of the plate are just considered: $a_1/b_1 = a/b = \lambda$.

The expression of the deflection of the plate is approximated in the form of a truncated series of beam functions $X_m(x)$ and $Y_n(y)$.

$$W(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y) \tag{3}$$

$X_m(x)$ and $Y_n(y)$ are the characteristic functions for the normal modes of vibration of beams with end conditions nominally similar to those of the opposite edges of the plate in each coordinate direction.

When the configuration of the plate leads to beams with both ends free, for example in the case of a cantilever plate, the first two characteristic functions correspond to rigid motions: translation and rotation.

Obviously $X_m(x)$ and $Y_n(y)$ do not satisfy the natural boundary conditions at outer and inner edges, as previously stated but this is legitimate when using the Ritz method (Nallim and Grossi 2003).

Substituting Eq. (3) into Eq. (1) and, requiring that $J(W)$ be a minimum with respect to the A_{mn}'s coefficients

$$\frac{\partial J[W]}{\partial A_{mn}} = 0 \qquad m = 1, 2, \ldots, M; \quad n = 1, 2, \ldots, N \tag{4}$$

one obtains a homogeneous linear system of equation in terms of the A_{mn}'s parameters.

From the non – triviality conditions, one can get natural frequency coefficients:

$\Omega_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$ as eigenvalues, and vibration modes as eigenvectors of the secular determinant.

The present study is concerned with the determination of the first five natural frequency coefficients Ω_1 to Ω_5 in the case of anisotropic rectangular plate, and their respective modal shapes.

3. Numerical results

The natural frequencies and modal shapes of the described plates are analyzed.

The plates are simply supported, clamped or free at their external edges.

The results of previous investigations show that the plate modal shapes and natural frequency

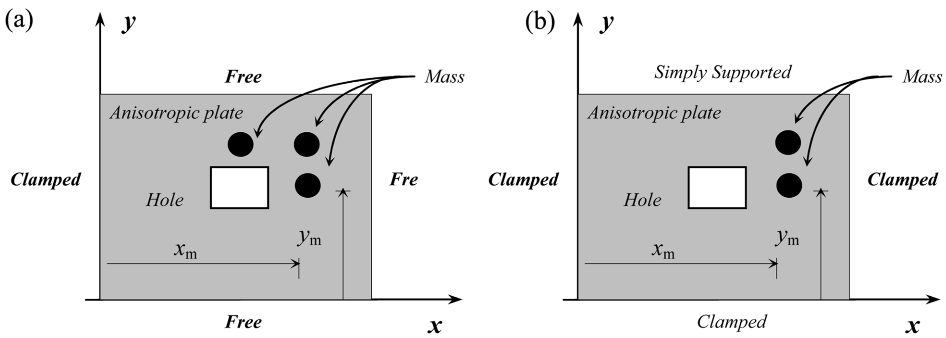

Fig. 1 Vibrating systems under consideration: (a) Cantilever anisotropic plate and (b) C-C-C-SS anisotropic plate

Table 1 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at (x_m = 0.5, y_m = 0.75)

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
	0	0	3.0663	4.8750	12.6397	18.8565	22.4569
		0	3.0331	4.8073	12.6503	18.6547	22.3895
	0.1	0.1	3.0289	4.5176	12.4955	18.1002	21.1687
		0.5	2.9987	3.7125	11.9093	15.7691	19.7586
2/3		1	2.8595	3.2408	11.4299	14.9039	19.5961
		0	2.6362	4.4472	12.3105	16.7700	21.7948
	0.2	0.1	2.6356	4.1703	12.1403	16.2680	20.7922
		0.5	2.6315	3.3914	11.4729	14.6065	18.8206
		1	2.6096	2.8444	10.9450	14.1058	18.4814
	0	0	2.8285	5.5269	18.9016	20.0922	27.5157
		0	2.7539	5.3895	18.9804	20.0111	27.4192
	0.1	0.1	2.7505	5.0583	18.4618	19.0017	26.1460
		0.5	2.7322	4.1367	15.4203	18.9893	24.9564
1		1	2.6911	3.4958	14.1884	18.9887	24.6828
		0	1.0822	4.5794	18.2573	19.4234	26.2727
	0.2	0.1	1.0781	4.3012	17.6624	18.5167	24.9961
		0.5	1.0614	3.5420	14.7672	18.3846	23.8515
		1	1.0401	3.0096	13.5522	18.3756	23.5981
	0	0	2.4493	6.1930	19.4740	24.7333	44.1457
		0	2.2369	5.8656	19.4341	24.9712	43.8940
	0.1	0.1	2.2357	5.4784	19.0262	22.7165	43.5189
		0.5	2.2296	4.4233	17.0592	20.7932	43.0314
3/2		1	2.2196	3.6832	15.9173	20.5385	42.8685
		0	2.3585	6.3847	19.3754	28.0044	44.7723
	0.2	0.1	2.3563	5.9087	18.5621	26.0624	44.0196
		0.5	2.3464	4.6692	16.6710	24.0524	43.1147
		1	2.3299	3.8486	15.8055	23.5755	42.8383

coefficients are strongly affected by the characteristic of anisotropic material and that such structures do not exhibit easily predictable behavior.

Therefore, and in view of the fact that the principal aim of the present work is to show the flexibility of the proposed procedure, just a generic arbitrary anisotropic material is considered ($D_{22} = D_{12} = D_{66} = D_{11}/2$, $D_{16} = D_{26} = D_{11}/3$), and two situations including the different boundary conditions are analyzed (Fig. 1).

In the first place, a cantilever anisotropic plate is studied.

Table 1 to Table 3 contain the first five frequency coefficients for a cantilever plate with a centered free edge hole and different locations and magnitudes of the concentrated mass. The variation of the aspect ratio of the plate and the magnitude of hole and mass are also taken into account.

Table 2 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at ($x_m = 0.75$, $y_m = 0.5$)

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/3	0	0	3.0663	4.8750	12.6397	18.8565	22.4569
	0.1	0	3.0331	4.8073	12.6503	18.6547	22.3895
		0.1	2.8516	4.6959	11.4026	18.6263	22.3887
		0.5	2.3075	4.4910	9.6593	18.5904	22.3828
		1	1.9075	4.4055	9.0745	18.5784	32.1185
	0.2	0	2.6362	4.4472	12.3105	16.7700	21.7948
		0.1	2.5278	4.2678	10.8575	16.6717	21.7074
		0.5	2.1416	3.8984	8.9721	16.5860	21.3488
		1	1.8069	3.7386	8.3851	16.5639	21.1958
1	0	0	2.8285	5.5269	18.9016	20.0922	27.5157
	0.1	0	2.7539	5.3895	18.9804	20.0111	27.4192
		0.1	2.6012	5.2302	17.7760	19.7613	26.7249
		0.5	2.1488	4.9191	15.4208	19.6812	25.9909
		1	1.8049	4.7761	14.4980	19.6689	25.7901
	0.2	0	1.0822	4.5794	18.2573	19.4234	26.2727
		0.1	1.0590	4.2847	17.0637	18.8364	25.5745
		0.5	0.9764	3.5964	14.4855	18.7110	24.9838
		1	0.8928	3.1995	13.5450	18.6960	24.8417
3/2	0	0	2.4493	6.1930	19.4740	24.7333	44.1457
	0.1	0	2.2369	5.8656	19.4341	24.9712	43.8940
		0.1	2.1264	5.6379	19.2929	24.2895	41.4174
		0.5	1.7942	5.1715	18.8761	22.9645	37.3624
	0.2	1	1.5308	4.9407	18.5845	22.4117	36.0353
		0	2.3585	6.3847	19.3754	28.0044	44.7723
		0.1	2.2352	6.1529	19.2357	26.6257	42.4859
		0.5	1.8726	5.6765	18.8161	24.1201	39.356
		1	1.5913	5.4396	18.5145	23.1757	38.4687

All the values are determined taking $M = N = 10$ in Eq. (3).

Fig. 2 to Fig. 8 show the modal shapes for the cantilever anisotropic plate with the free edge hole and the concentrated mass.

Then an anisotropic rectangular plate with three outer edges clamped and the remaining simply supported, is analyzed.

Table 4 and Table 5 show values for similar features of this situation, as those considered for the cantilever plate.

As well Fig. 9 to Fig. 13 show the modal shapes of some particular cases of this plate.

In order to evaluate the accuracy of the expounded procedure, in Table 6 a comparison is made with the results obtained by Cupial (1997) for a highly anisotropic simply supported plate, by means of the Ritz method using orthogonal polynomials.

Table 3 Frequency coefficients values for a cantilever (CFFF) anisotropic, doubly connected plate with a concentrated mass attached at ($x_m = 0.75$, $y_m = 0.75$)

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/3	0	0	3.0663	4.8750	12.6397	18.8565	22.4569
	0.1	0	3.0331	4.8073	12.6503	18.6547	22.3895
		0.1	2.9938	4.0409	12.2414	17.9541	18.2756
		0.5	2.3474	3.4122	12.0826	13.8717	17.7570
		1	1.9228	2.9447	11.2692	11.9367	16.7336
	0.2	0	2.6362	4.4472	12.3105	16.7700	21.7948
		0.1	2.6349	3.7253	12.2740	15.8876	21.6851
		0.5	2.4753	2.6610	12.2110	14.9249	21.5809
		1	1.9157	2.6436	12.1840	14.6491	21.5528
1	0	0	2.8285	5.5269	18.9016	20.0922	27.5157
	0.1	0	2.7539	5.3895	18.9804	20.0111	27.4192
		0.1	2.7501	4.2228	15.1750	17.6949	23.1222
		0.5	2.1519	3.6119	14.9921	17.4857	22.7534
		1	1.7593	3.1616	14.7835	17.2516	19.9787
	0.2	0	1.0822	4.5794	18.2573	19.4234	26.2727
		0.1	1.0795	3.8509	18.0962	19.3740	25.9693
		0.5	1.0677	2.6277	17.8462	19.3196	25.4568
		1	1.0497	2.0603	17.7509	19.3034	25.2551
3/2	0	0	2.4493	6.1930	19.4740	24.7333	44.1457
	0.1	0	2.2369	5.8656	19.4341	24.9712	43.8940
		0.1	2.2069	5.0441	19.4287	24.8912	43.1459
		0.5	2.0641	3.6974	19.4211	24.7707	41.6088
		1	1.8642	3.1739	19.4183	24.7244	40.9251
	0.2	0	2.3585	6.3847	19.3754	28.0044	44.7723
		0.1	2.31768	5.4787	19.3575	27.8792	43.9428
		0.5	2.1358	4.0422	19.3333	27.6864	42.1260
		1	1.9065	3.5002	19.3247	27.6113	41.3487

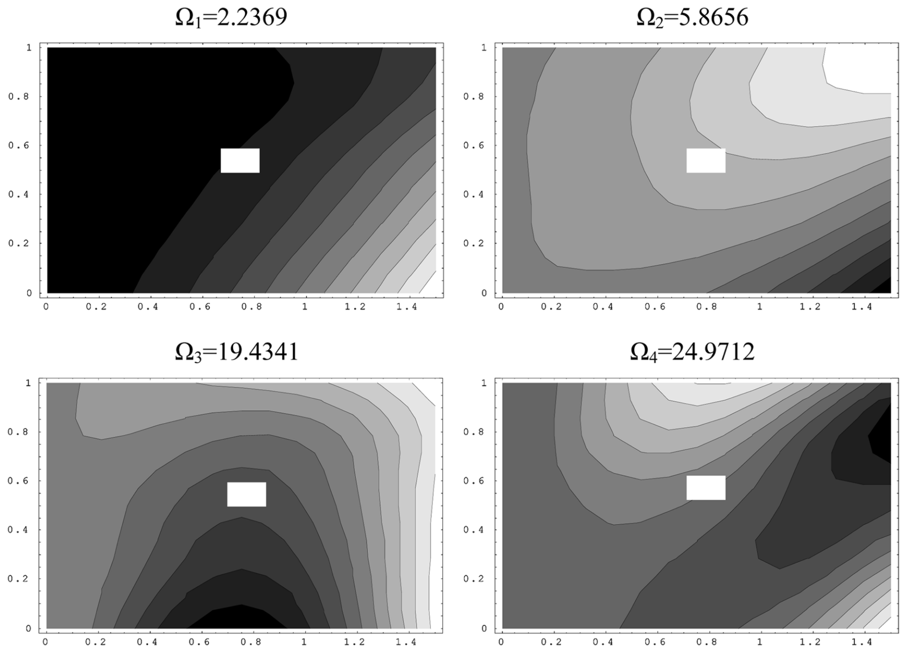

Fig. 2 First four modal shapes of vibration of a cantilever anisotropic plate of $\lambda = 3/2$, hole of $a_1/a = 0.1$ and without attached mass

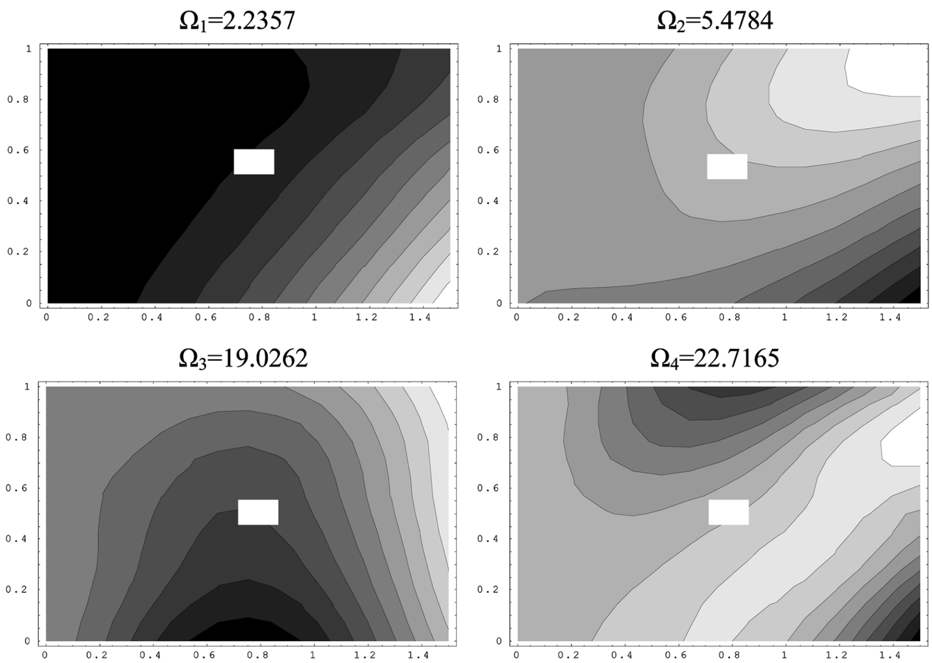

Fig. 3 First four modal shapes of vibration of a cantilever anisotropic plate of $\lambda = 3/2$, hole of $a_1/a = 0.1$ and a concentrated mass ($M = 0.1$) attached at ($x_m = 0.5$, $y_m = 0.75$)

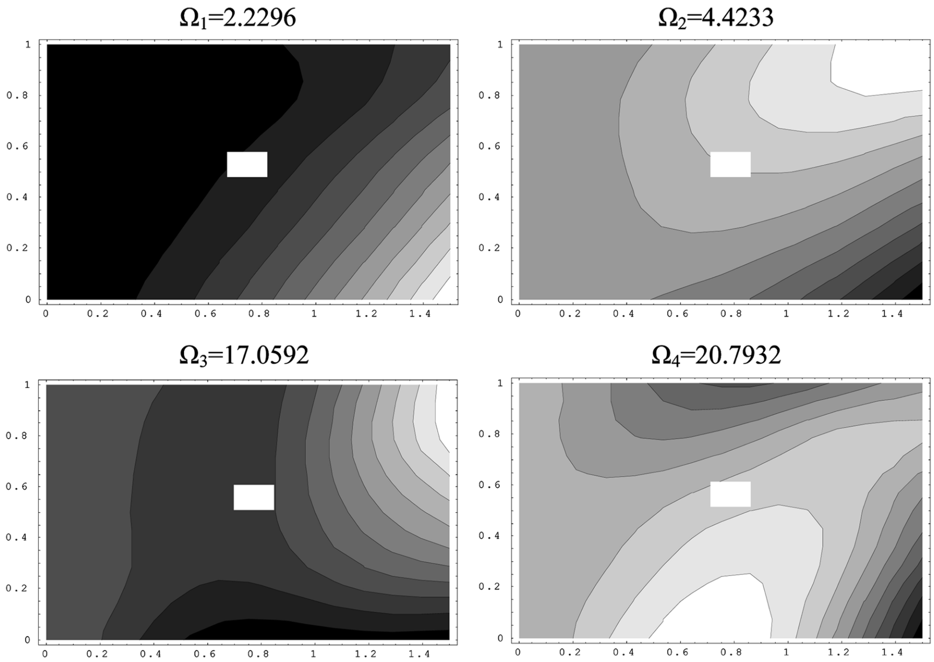

Fig. 4 First four modal shapes of vibration of a cantilever anisotropic plate of $\lambda = 3/2$, hole of $a_1/a = 0.1$ and a concentrated mass ($M = 0.5$) attached at ($x_m = 0.5$, $y_m = 0.75$)

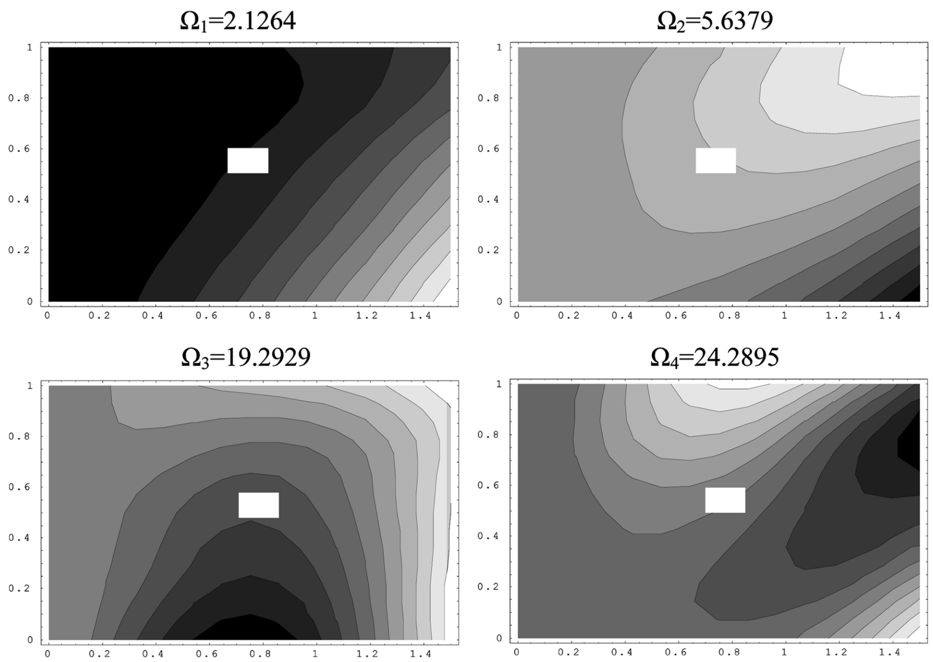

Fig. 5 First four modal shapes of vibration of a cantilever anisotropic plate of $\lambda = 3/2$, hole of $a_1/a = 0.1$ and a concentrated mass ($M = 0.1$) attached at ($x_m = 0.75$, $y_m = 0.5$)

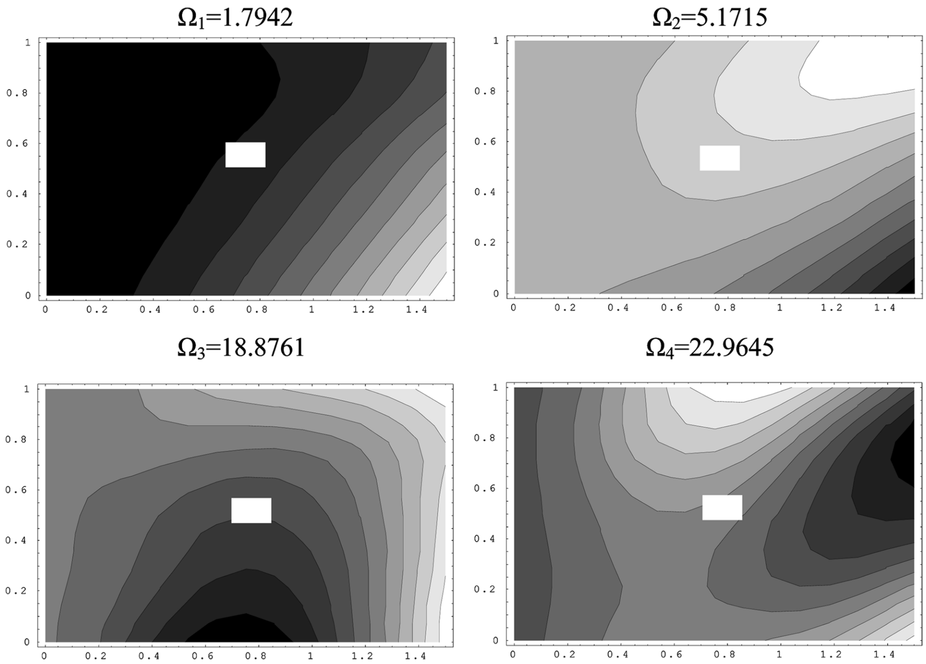

Fig. 6 First four modal shapes of vibration of a cantilever anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.5) attached at (x_m = 0.75, y_m = 0.5)

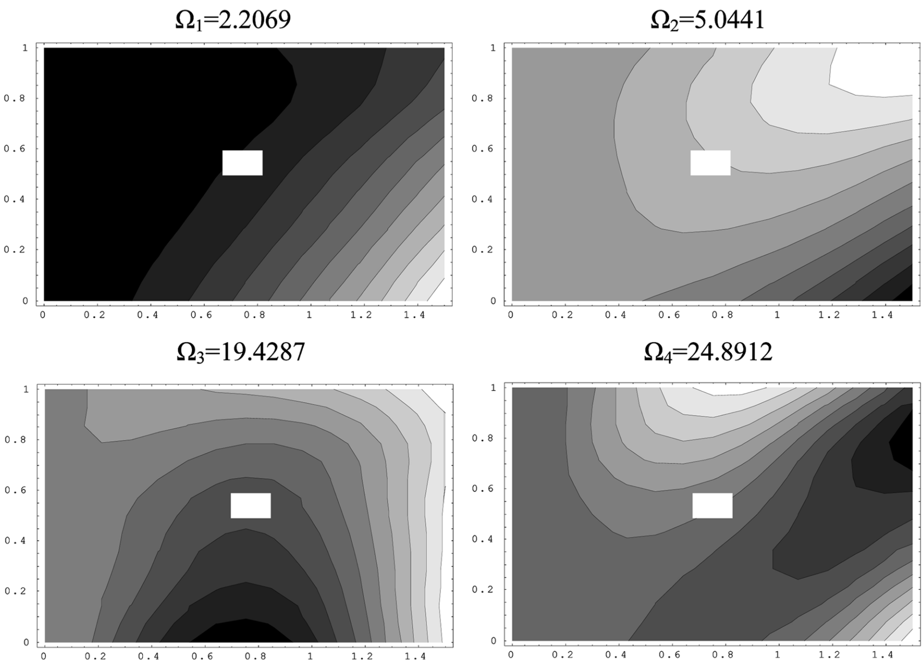

Fig. 7 First four modal shapes of vibration of a cantilever anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.1) attached at (x_m = 0.75, y_m = 0.75)

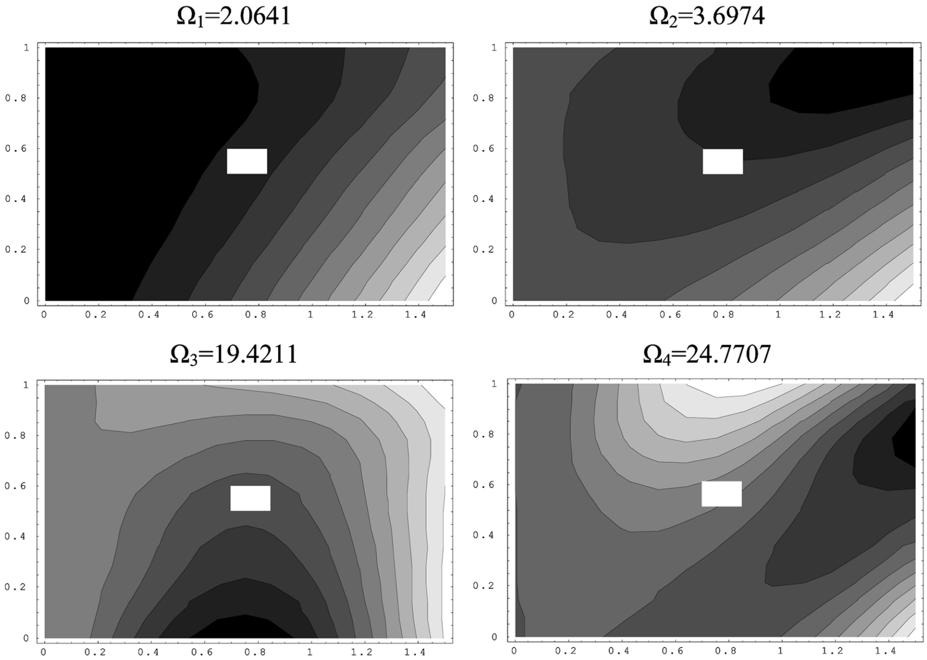

Fig. 8 First four modal shapes of vibration of a cantilever anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.5) attached at (x_m = 0.75, y_m = 0.75)

Table 4 Frequency coefficients values for a CCCS anisotropic, doubly connected plate with a concentrated mass attached at (x_m = 0.75, y_m = 0.75)

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/3	0	0	25.7724	35.2778	49.9407	65.525	70.1787
	0.1	0	24.8513	32.0440	45.7753	62.6881	65.6947
		0.1	23.1764	29.9772	43.9352	50.9795	62.8010
		0.5	15.9805	27.7657	39.1383	46.0149	62.7838
		1	12.0293	27.4321	39.9398	45.9639	62.7820
	0.2	0	26.8278	35.2345	52.3965	66.5107	72.0796
		0.1	24.3510	33.4069	51.6343	52.6782	71.3595
		0.5	16.2107	31.2550	43.2288	52.4258	70.8826
		1	12.1436	30.8584	42.1582	52.4228	70.8141
1	0	0	30.5162	51.7154	72.4358	81.633	102.175
	0.1	0	28.0672	43.7729	68.6633	70.8784	100.769
		0.1	26.0828	42.3615	56.8117	68.7228	92.5347
		0.5	19.0186	37.4857	47.9728	68.7166	88.0908
		1	14.6184	36.0325	47.1737	68.7159	87.4502
	0.2	0	31.5729	51.1067	73.8302	81.3427	101.600
		0.1	28.6852	49.1300	60.2238	79.7175	96.5330
		0.5	19.8401	42.6949	54.1184	79.3450	93.9024
		1	15.0443	41.1919	53.6819	79.2907	93.4821

Table 4 Contined

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
3/2	0	0	42.1101	78.4318	99.5379	124.335	150.353
	0.1	0	35.6861	56.8301	93.4766	102.985	148.993
		0.1	32.5645	56.1075	78.3357	97.6760	131.609
		0.5	23.5599	53.0553	62.9248	96.9603	125.301
		1	18.2394	51.3574	61.0942	96.8765	124.417
	0.2	0	42.9540	76.2962	101.445	120.163	150.184
		0.1	38.6180	72.2201	82.2312	116.612	137.545
		0.5	26.5340	61.0356	79.8809	115.118	133.292
		1	20.1489	58.7002	79.4949	114.878	132.702

Table 5 Frequency coefficients values for a CCCS anisotropic, doubly connected plate with a concentrated mass attached at ($x_m = 0.75$, $y_m = 0.5$)

$\lambda = a/b$	a_1/a	$M = m/m_p$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/3	0	0	25.7724	35.2778	49.9407	65.525	70.1787
	0.1	0	24.8513	32.0440	45.7753	62.6881	65.6947
		0.1	22.1185	30.9576	43.9352	51.2905	64.6980
		0.5	14.6141	29.6753	39.1383	47.7237	64.6016
		1	10.9771	29.3880	38.2088	47.8887	64.5890
	0.2	0	26.8278	35.2345	52.3965	66.5107	72.0796
		0.1	24.0822	34.3506	46.2095	57.4780	69.9709
		0.5	15.9229	32.6360	39.5466	55.5072	69.6295
		1	11.9210	32.1790	38.7735	55.3244	69.5856
1	0	0	30.5162	51.7154	72.4358	81.633	102.175
	0.1	0	28.0672	43.7729	68.6633	70.8784	100.769
		0.1	25.3999	40.4442	61.1848	69.4842	88.8229
		0.5	17.5011	36.1917	54.6164	69.3986	83.9458
		1	13.2953	35.3238	53.5916	69.3885	83.2865
	0.2	0	31.5729	51.1067	73.8302	81.3427	101.600
		0.1	28.3847	47.2353	64.4369	81.0546	89.3167
		0.5	19.2178	41.5076	58.6972	80.8308	85.5898
		1	14.5130	40.3806	57.9792	80.7803	85.1403
3/2	0	0	42.1101	78.4318	99.5379	124.335	150.353
	0.1	0	35.6861	56.8301	93.4766	102.985	148.993
		0.1	32.4520	50.8013	83.3812	101.480	136.766
		0.5	22.4512	44.8990	76.3090	101.026	129.671
		1	17.0762	43.8567	75.0657	100.954	128.459
	0.2	0	42.9540	76.2962	101.445	120.163	150.184
		0.1	38.6887	64.2182	99.8588	111.626	139.620
		0.5	26.0067	54.8097	97.8596	107.335	135.749
		1	19.5929	53.3858	97.3649	106.754	135.185

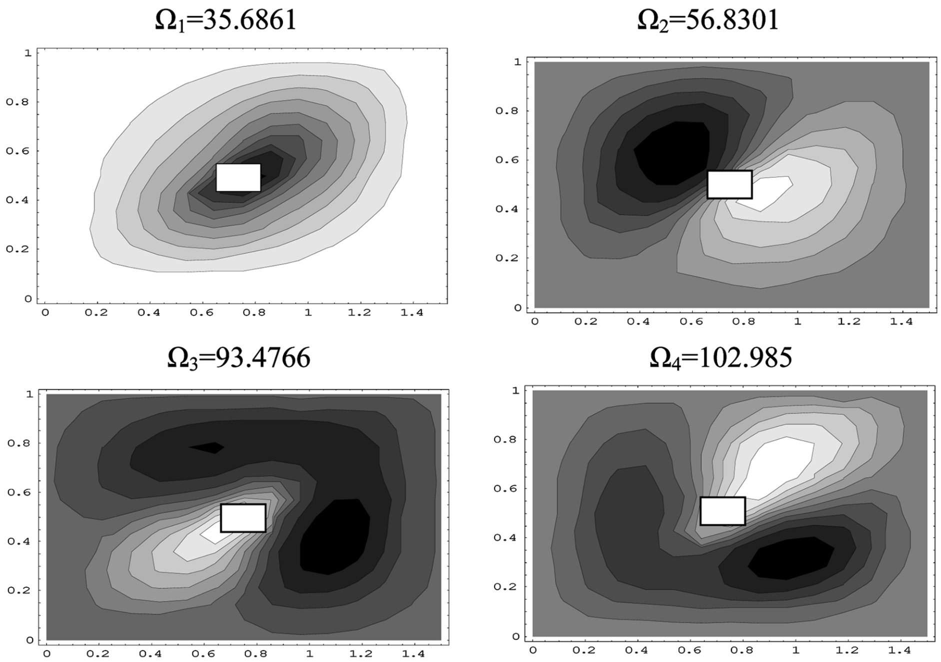

Fig. 9 First four modal shapes of vibration of a CCCS anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and without attached mass

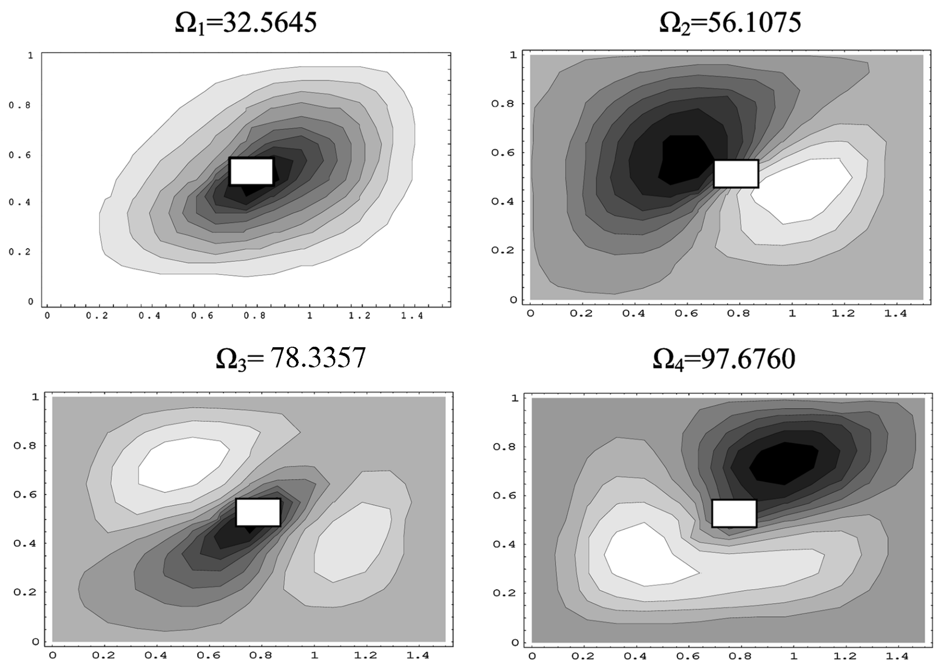

Fig. 10 First four modal shapes of vibration of a CCCS anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.1) attached at (x_m = 0.75, y_m = 0.75)

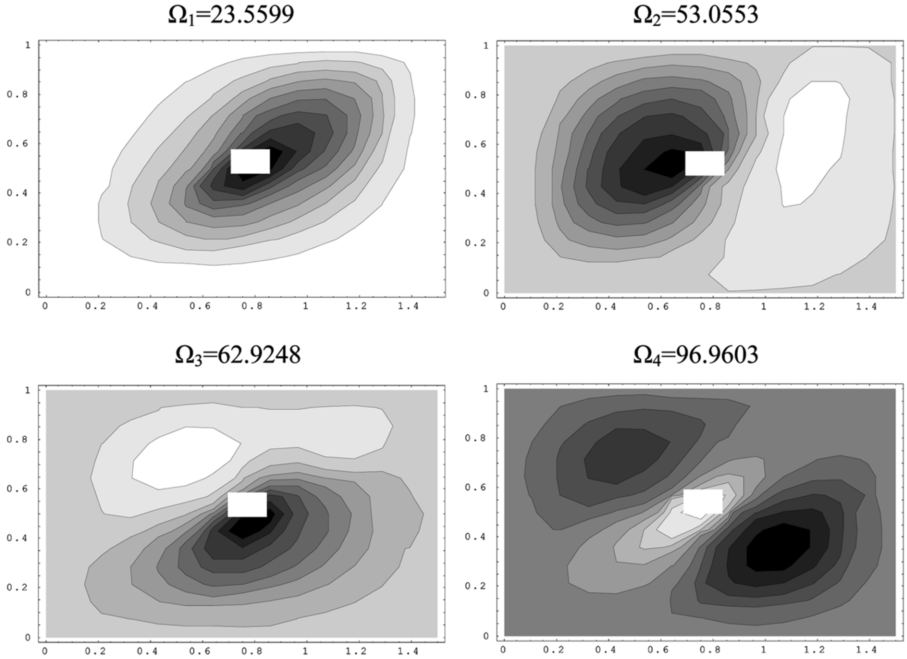

Fig. 11 First four modal shapes of vibration of a CCCS anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.5) attached at (x_m = 0.75, y_m = 0.75)

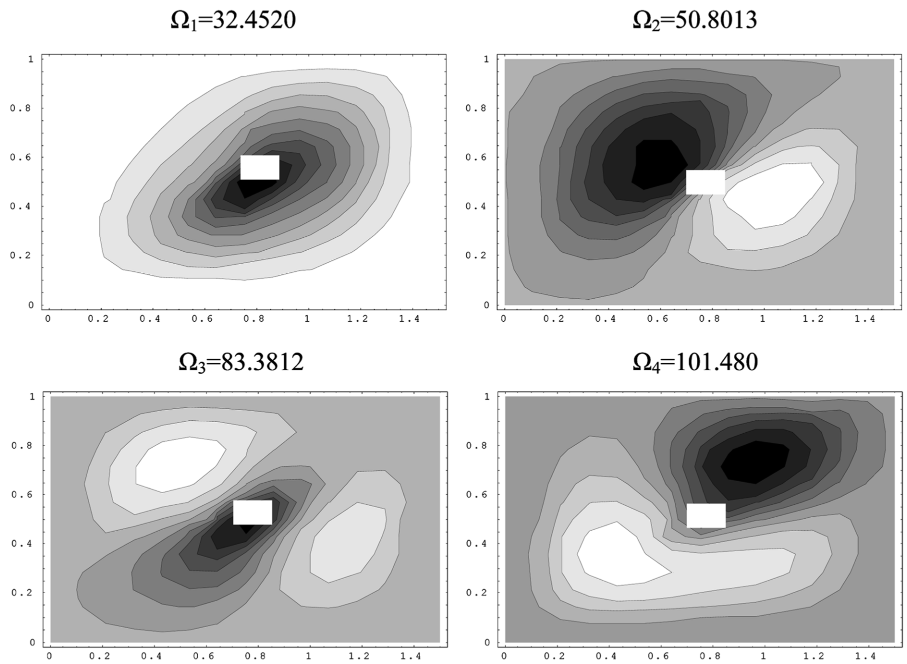

Fig. 12 First four modal shapes of vibration of a CCCS anisotropic plate of λ = 3/2, hole of a_1/a = 0.1 and a concentrated mass (M = 0.1) attached at (x_m = 0.75, y_m = 0.5)

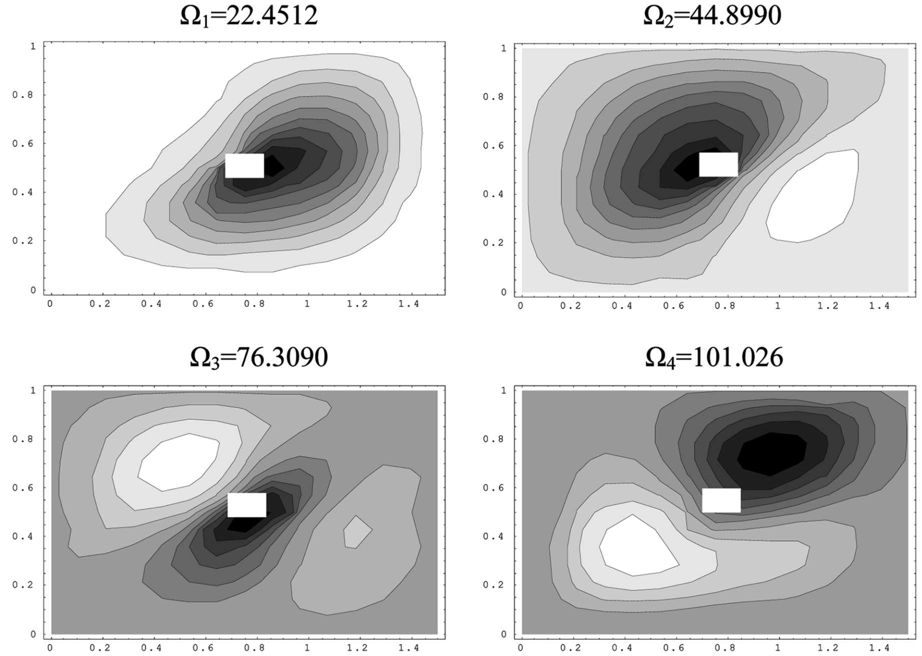

Fig. 13 First four modal shapes of vibration of a CCCS anisotropic plate of $\lambda = 3/2$, hole of $a_1/a = 0.1$ and a concentrated mass ($M = 0.5$) attached at $x_m = 0.75$, $y_m = 0.5$

Table 6 The non-dimensional frequency $\omega a^2[12(1-\nu_{12}\nu_{21})\rho/E_1h^3]^{1/2}$ of a single-layer (30°) simply supported square plate

Mode	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
Cupial, 1997	11.233	20.282	32.995	34.866	47.133	48.283
Present approach	11.372	20.485	33.234	35.085	47.592	48.659

The analysis is done for the composite material properties: $E_1 = 138$ [GPa], $E_2 = 8.96$ [GPa], $G_{12} = 7.1$ [GPa], and $\nu_{12} = 0.30$.

This case is chosen because, as Cupial (1997) himself and Whitney (1972) stated, the convergence of the Ritz method using beam functions may be slow for the free vibration frequencies of highly anisotropic plates with simply supported edges.

Present computing facilities and a convenient and straightforward algorithm (Felix *et al.* 2004) make possible to increase the number in terms in Eq. (3) without difficulty.

For this particular case $M = N = 30$ is taken and the obtained values show good accuracy from an engineering viewpoint.

4. Conclusions

As a general conclusion, one may say that the Ritz method, using beam function provides an accurate and convenient procedure to attack a difficult elastodynamics problem: the vibration of thin rectangular plates with structural and mechanical complexities like for the present situation where anisotropic material characteristics, doubly connected domain and attached masses are present.

The obtained values are the outcome of an algorithm, relatively simple to implement, (Felix *et al.* 2004) which allows studying those situations where the plates possess additional complexities, with the only assistance of a P.C.

Acknowledgements

The present study has been sponsored by CONICET Research and Development Program, ANCPYT and by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur.

The authors are deeply indebted to Professor Emeritus Patricio A. A. Laura for his valuable suggestions. The authors are also indebted to the anonymous reviewers for their constructive criticism.

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