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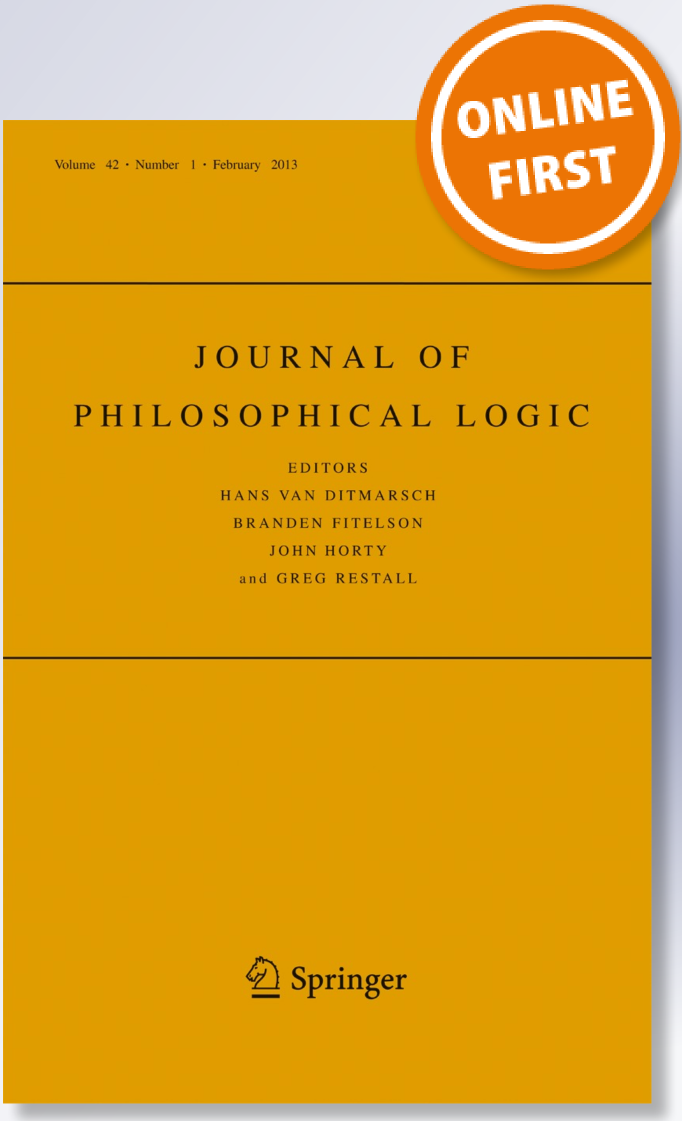

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Abstract The fact that many modal operators are part of an adjunction is probably folklore since the discovery of adjunctions. On the other hand, the natural idea of a minimal propositional calculus extended with a pair of adjoint operators seems to have been formulated only very recently. This recent research, mainly motivated by applications in computer science, concentrates on technical issues related to the calculi and not on the significance of adjunctions in modal logic. It then seems a worthy enterprise (both for these contemporary topical pursuits and also for historical interest) to trace the concept of adjunction back to the origins of the algebraic semantics of modal logic and to make explicit its ubiquity in this branch of mathematics.

Keywords Modal logic · Adjoint functors

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1 Introduction

In the introduction to [18], Goldblatt states that the symbol $\Box$ in propositional modal logic has "a long history of investigation in terms of modal interpretations of philosophical interest" (alethic, deontic, epistemic, temporal, etc.) and stresses that there are recent interpretations "of more mathematical concern". One of the motivations of our paper is the following thesis: left and right adjointness (in the usual categorical sense [27, 34]) determine useful mathematical *modalities* (Section 2.1 in [20]) capable of encompassing some classical situations as well as many of the newer applications. We collect mathematical and historical evidence supporting the claim that the thesis is relevant.

The fact that many modal operators are part of an adjunction is probably folklore since the discovery of adjunctions [27], but their explicit consideration in modal logic seems to be much more recent. The units and counits of two adjunctions are 'visible' in the usual presentations of tense logic but this fact is hardly ever emphasized. (The earliest exception we have found is [15]. See also the first example in [7].) Moreover, the natural idea of a minimal propositional calculus extended with a pair of adjoint operators seems to have been formulated only very recently in [44] and [10]. The work in [44] is motivated by epistemic applications and that in [10] by generalized fuzzy sets. But adjoint modal operators are far more common, and it might be useful to make their significance more explicit. We do so here by surveying the work of logicians, mathematicians and computer scientists who implicitly or explicitly have found adjunctions useful in their research on modal logic. This task will lead us back to Ore's [40] and Jónsson-Tarski's [26], but we cannot claim that our survey is exhaustive. Also, the emphasis will be put on the mathematical concepts and not on their historical development. This choice of emphasis has led us to introduce some original algebraic structures which may be of independent interest, but which are used here only as means to relate other people's work.

We will assume that the reader is familiar with some category theory. In particular, with *adjunctions* and *monads* as defined in Chapters IV and VI in [34]. We will also assume familiarity with basic concepts of modal logic; relying mostly on [3] and [20] for references to material on the subject.

As initial motivation we briefly discuss an important distinction between the way that adjointness appears in categorical logic and the way that it appears in the algebraic semantics of classical propositional modal logic. In their paper [36] on completeness results for intuitionistic and modal logics, Makkai and Reyes start by stating that categorical logic rests on Lawvere's insight that fundamental logical operations arise as adjoints to naturally given functors. The simplest examples of this phenomenon are conjunctions and disjunctions. Indeed, let P be a poset and consider the diagonal map $\Delta : P \to P \times P$. The poset P has finite meets if and only if Δ has a right adjoint (that is conveniently denoted by $\wedge : P \times P \to P$). Similarly, P has finite joins if and only if Δ has a left adjoint. Another important example (also highlighted in [36]) is exponentiation: a distributive lattice D is a Heyting algebra if and only if for every d in D, the monotone $d \wedge (\_) : D \to D$ has a right adjoint, sometimes denoted by $d \Rightarrow (\_)$. A fundamental observation due to Lawvere is that quantifiers also arise in this way (see, for example, [33] and [32]). If we let $\mathcal{C}$ be a category with finite limits, pulling back along a morphism $f : X \to Y$ induces a monotone morphism $f^* : \mathbf{Sub}Y \to \mathbf{Sub}X$ between posets of subobjects. If $\mathcal{C}$ is a topos, f^* has both a left adjoint $\exists_f : \mathbf{Sub}X \to \mathbf{Sub}Y$ and a right adjoint $\forall_f : \mathbf{Sub}X \to \mathbf{Sub}Y$. These adjoints are used to interpret the quantifiers of the first order *internal logic* of $\mathcal{C}$. (See [35] or [25].)

Intuitively, Makkai and Reyes restrict attention to modal operators that appear as adjoints to a 'more basic' functor. They define precisely the type of structures they want to study and prove deep and useful results about them. The examples we discuss in Section 4.1 are naturally part of this theory, but we mention it here to make explicit the following distinction with standard categorical logic: modal operators in general do not arise as adjoints to a 'more basic' functor. Consider a frame $\mathfrak{F} = (W, R)$ with set W of possible worlds and accessibility relation $R \subseteq W \times W$. Denote the boolean algebra structure on the power-set of W by $\mathcal{P}W$. Define the operator $m_R : \mathcal{P}W \to \mathcal{P}W$ by $m_R X = \{y \in W \mid (\exists x \in X)(yRx)\}$. The structure $(\mathcal{P}W, m_R)$ is denoted by $\mathfrak{F}^+$ and is called the *full complex algebra* of $\mathfrak{F}$. The *operator* m_R of $\mathfrak{F}^+$ does not have a left adjoint in general but it always has a right adjoint. This is well-known: $\mathcal{P}W$ is complete and m_R is *completely additive*, in the sense that it preserves arbitrary suprema (Exercise 5.2.7(a) in [3]). It follows that m_R has a right adjoint for general categorical reasons and we denote it by $h_R : \mathcal{P}W \to \mathcal{P}W$. The monotone function h_R does not have a further right adjoint in general, and what is more important to the present discussion is that neither m_R nor h_R is more 'naturally given' or canonical than the other. The operator m_R provides the interpretation of the usual 'possibility' symbol $\Diamond$ in $\mathfrak{F}^+$ (see Proposition 5.24 in [3]). The monotone function h_R has the following explicit definition $h_R X = \{y \in W \mid (\forall w \in W)(wRy \to w \in X)\}$ and gives semantics to the symbol H in the basic temporal language (Example 1.14 in [3]). Recall that the mnemonics for H is 'it always Has been the case'. Informally, we may say that looking once into the future is left adjoint to contemplating the whole past.

The adjunctions $m_R \dashv h_R : \mathcal{P}W \to \mathcal{P}W$ are used in [7] as the first example to motivate *Gaggle theory* whose aim, in turn, is to "provide a uniform semantical approach to a variety of logics" including modal, temporal and relevance logics based on the observation that many important principles in these involve Galois connections. The similarity with the motivation of the present paper should be clear. On the other hand, the intended applicability of Gaggle theory is much broader and its mathematical strategy is based on a generalization of the representation theorem due to Jónsson and Tarski. This combination of broader applicability and concentrated mathematical strategy makes Gaggle theory inefficient for the purposes of the present paper. But it should be stressed that many of the algebraic structures that we consider are instances of gaggles. (Incidentally, it must be said that we learned of Dunn's theory towards the end of the historical research presented here. This is somewhat surprising and we understand it as further evidence that the survey we present may be useful.)

More recent work relating adjunctions and modalities has abstracted the complex-algebra example above by choosing one operation and relying on the completeness of an underlying ordered structure to obtain an adjoint (see, for example, [2, 24, 46] and [43]). We propose that the adjunction is fundamental while completeness is accessory. In this sense, our paper is closer to [7, 10, 44].

In Section 2 we continue the above discussion on BAOs and adjoint operators by introducing the category of Boolean algebras equipped with an adjunction (BAAs). In Section 3 we generalize the idea of a BAA by relaxing the conditions on the underlying poset and mention examples in the literature that have profited from similar generalizations. Some emphasis is put on the fact that the categories of models we introduce are finitary algebraic categories; so they are susceptible to the same type of universal algebraic techniques that have been successfully applied to BAOs. The restriction of Goldblatt's duality for lattices-with-operators established in Proposition 3.8 improves the representation results proved in [44] and [10], and should be compared to the one provided by the general considerations of Gaggle theory [7]. Proposition 3.12 simplifies the axiomatization of Ewald's IK_t in [13] and relates it to Dunn's work on positive modal logic [9]. In Section 3.3 we recall relevant examples of monads and comonads that are also necessary for later sections. A further algebraic generalization is discussed in Section 4 where we consider modal operators that appear not *on* a lattice but *between* lattices. In Section 5 we discuss two examples of how categories more general than posets are used as models for modalities. In Section 6 we mention some examples that are not 'modes of adjointness'.

2 Boolean Algebras with Adjunctions

In this section we assume familiarity with the duality between the category **BAO** of Boolean algebras with operators and the category of descriptive frames (see [17] or Chapter 5 in [3]). Objects of **BAO** are usually considered to be pairs $(B, \Diamond)$ with B a Boolean algebra and $\Diamond : B \to B$ the underlying *operator* which is a function preserving finite suprema.

Define a *Boolean algebra with an adjunction* (BAA) as a triple $(B, \Diamond, H)$ where B is a Boolean algebra and $\Diamond, H : B \to B$ are monotone functions such that $\Diamond$ is left adjoint to H. For this reason we will usually write a BAA as $(B, \Diamond \dashv H)$. A *morphism* $f : (B, \Diamond, H) \to (B', \Diamond', H')$ is a Boolean algebra morphism $f : B \to B'$ preserving both $\Diamond$ and H. BAAs together with the morphisms between them form an algebraic category that we denote by **BAA**.

Since left adjoints preserve suprema, the assignment $(B, \Diamond, H) \mapsto (B, \Diamond)$ extends to a faithful functor **BAA** $\to$ **BAO**. As adjoints are unique, this functor is injective on objects so **BAA** is a (non-full) subcategory of **BAO**. This functor has a left adjoint for general reasons. Intuitively, this left adjoint **BAO** $\to$ **BAA** takes a BAO $(B, \Diamond)$ and freely adds a right adjoint to $\Diamond$. (Although not directly applicable to the issue at hand, it seems relevant to mention that the problem of freely adding right adjoints to morphisms in a category has a very concrete solution [6].)

The algebraic category **BAA** can be presented by extending the usual presentation of **BAO** with a unary symbol H plus axioms stating that H is a semilattice morphism $(B, \wedge, \top) \to (B, \wedge, \top)$ and axioms stating that $x \leq H\Diamond x$ (the unit of the adjunction) and that $\Diamond H x \leq x$ (the counit).

Example 2.1 (Full Complex Algebras) Consider a frame $\mathfrak{F} = (W, R)$ and its associated full complex algebra $\mathfrak{F}^+ = (\mathcal{P}W, m_R)$ in **BAO**. The discussion in the introduction implies that $\mathfrak{F}^+$ extends to an object $(\mathcal{P}W, m_R \dashv h_R)$ in **BAA**.

Example 2.2 (Semi-symmetric Modal Boolean Algebras) These are introduced in p. 348 of [8] and they are exactly BAAs, so Dunn should be credited for this concept. He does not consider explicitly the category **BAA** but it is clear from his definition that the theory of semi-symmetric modal Boolean algebras has an equational presentation therefore inducing an algebraic category. Dunn also introduces loc. cit. *symmetric modal Boolean algebras* but this is misleading from a categorical perspective because the naturally determined categories of algebras are isomorphic over **Set**. Put differently, the distinction between symmetric and semi-symmetric modal Boolean algebras should be seen as distinguishing two different presentations of the same algebraic category. (See discussion after Example 2.5 below.)

Example 2.3 (Galois Algebras) In the introduction to [46], a *Galois algebra* is defined as a complete Boolean algebra B equipped with a function that preserves all suprema; let us call this function $\Diamond$. Galois algebras are not organized into a category in [46] but we can let **cBAA** $\to$ **BAA** be the full subcategory determined by those objects in **BAA** whose underlying boolean algebras are complete. That is, the full subcategory of **BAA** determined by Galois algebras as defined above. The completeness hypothesis is justified in [46] in two ways. First, because it implies that the function $\Diamond$ has a right adjoint. Second, because it allows to apply the Knaster-Tarski fixpoint Theorem. The *existence* of an adjoint to $\Diamond$ seems to be the main use of completeness in [46]. In fact, the whole paper advocates the use of *finitary* algebraic properties in calculations. So for many of the purposes in [46] the objects in **BAA** are good enough. On the other hand, von Karger argues that "In all practical applications, the Boolean algebra is complete" and that "most finitely disjunctive operators distribute over infinite disjunctions as well". While this may be true, having a finitary algebraic category implies that one has all the machinery of universal algebra available for its study. This machinery is an important part of the algebraic model theory of modal logic. For example, Birkhoff's Theorem is an essential ingredient in the proof of the Goldblatt-Thomason Theorem (see 5.54 in [3]). While **BAA** is a finitely presentable algebraic category, it is not clear to us if the forgetful functor **cBAA** $\to$ **Set** has a left adjoint. (Recall that the free complete Boolean algebra on $\mathbb{N}$ does not exist.)

Example 2.4 (Modal-like Operators in Boolean Lattices) The idea to consider adjunctions on general Boolean algebras as modal operators is quite explicit in [24]. Rough set approximation operators, temporal logic and linguistic modifiers determined by L-sets are mentioned as motivations. The most concrete part of [24], though, concentrates on complete atomic Boolean algebras.

Example 2.5 (Classical Modal Algebras) Complete boolean algebras equipped with a function that preserves all suprema are called *Classical modal algebras* in [43]. This is, of course, the same concept as the Galois algebras discussed in Example 2.3. In contrast with that example, completeness of classical modal algebras is assumed for "simplicity of presentation" and it is mentioned that "An alternative would be to put aside the completeness criteria and instead ask for existence of adjoints for each join (meet) preserving operator". See footnote in p. 394 of [43].

Before presenting another example it is convenient to discuss an equivalent definition of **BAO** which is closer to the usual formulation of propositional normal modal logics. We introduced objects of **BAO** as pairs $(B, \Diamond)$ with B a Boolean algebra and $\Diamond$ a $\vee$-semilattice morphism. This definition leads to a very simple equational presentation. But for some purposes it is useful to consider an object of **BAO** as a triple $(B, \Diamond, \Box)$ with $(B, \Diamond)$ as before and $\Box : (B, \wedge, \top) \to (B, \wedge, \top)$ a $\wedge$-semilattice morphism satisfying $\Diamond = \neg\Box\neg$ and $\Box = \neg\Diamond\neg$. This definition is highly redundant but it is closer to the standard presentation of classical propositional modal logic which involves a symbol $\Box$ satisfying $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$ (essentially preservation of finite infima). There is really no reason to consider $\Diamond$ as more fundamental than $\Box$ in a BAO; any morphism of BAOs as originally defined preserves $\Box$.

Similarly, we can consider BAAs as structures of the form $(B, \Diamond \dashv H, P \dashv \Box)$ with B a Boolean algebra and $\Diamond, H, P, \Box : B \to B$ monotone functions such that $\Diamond \dashv H$, $P \dashv \Box$ and $\Box = \neg\Diamond\neg$ and $P = \neg H \neg$. Notice that the units of the adjunctions imply that for every element q in B, $q \leq H\Diamond q$ and $q \leq \Box P q$. These are the algebraic counterpart of the *converse axioms* in the definition of *normal temporal logic* (see Definition 4.33 in [3]). The symbol P is for 'it was true at some Past time'. Altogether, the category **BAA** provides an algebraic semantics for normal temporal logic, just as **BAO** provides an algebraic semantics for normal modal logic (Theorem 5.27 loc. cit.).

Example 2.6 (Modal Kleene Algebras) We review some of the material in [39] where extensions of Kleene algebras are used to give a sound and complete semantics for propositional Hoare logic. A *Kleene algebra* is a structure $(K, +, 0, \cdot, 1, (\_)^*)$ such that $(K, +, 0, \cdot, 1)$ is a semi-ring, $(K, +, 0)$ is a commutative idempotent monoid and $(\_)^* : K \to K$ is a unary operation satisfying certain conditions which we do not need explicitly stated here. Such a structure is equipped with a partial order defined by $a \leq b$ if and only if $a + b = b$. A *Kleene algebra with tests* (KAT) is a two-sorted structure (K, B) where K is a Kleene algebra and $B \subseteq K$ is a Boolean algebra such that $0_K = 0_B$ and $1_K = 1_B$. A *Kleene algebra with domain* (KAD) is a structure (K, B, δ) where (K, B) is a KAT and $\delta : K \to B$ is a function such that for all a in K and p in B, $a \leq (\delta a) \cdot a$ and $\delta(p \cdot a) \leq p$. For such a structure, define a family (indexed by $a \in K$) of 'forward diamond' operators $\Diamond_a : B \to B$ by $\Diamond_a p = \delta(a \cdot p)$. A *weak converse* for a Kleene algebra K is an involutive operation $(\_)^\circ : K \to K$ which preserves $+$ and satisfies $(a \cdot b)^\circ = b^\circ \cdot a^\circ$, $(a^*)^\circ = (a^\circ)^*$ and $p^\circ \leq p$. If (K, B, δ) is a KAD with a weak converse then define the *codomain* operation $\rho : K \to B$ by $\rho a = \delta(a^\circ)$ and $\blacklozenge_a : B \to B$ by $\blacklozenge_a p = \rho(p \cdot a)$. If we also define $\Box_a = \neg\Diamond_a\neg$ and $\blacksquare_a = \neg\blacklozenge_a\neg$ then it follows that $\Diamond_a \dashv \blacksquare_a$ and $\blacklozenge_a \dashv \Box_a$. In other words, for every KAD (K, B, δ) equipped with a weak converse, each $a \in K$ induces a BAA $(B, \Diamond_a \dashv \blacksquare_a, \blacklozenge_a \dashv \Box_a)$. To give a taste of the application related to Hoare logic let us just say that partial correctness assertions of the form $\{\phi\}\alpha\{\psi\}$

can be interpreted in a Kleene algebra with domain and codomain in such a way that ϕ and ψ are interpreted as p and q in B respectively and α is interpreted as an element a in K. Then, $\{\phi\}\alpha\{\psi\}$ holds if and only if $\blacklozenge_a p \leq q$ in B, or equivalently $p \leq \Box_a q$.

It is clear that the operation $(B, \Diamond \dashv H, P \dashv \Box) \mapsto (B, P \dashv \Box, \Diamond \dashv H)$ produces an object of **BAA**. This is, of course, related to the operation that takes a frame (W, R) and produces the frame $(W, \breve{R})$ where $\breve{R}$ is the converse of R. The fixpoints of this operation have the following simple (and surely folklore) characterization.

Proposition 2.7 *Let $(\mathcal{P}W, \Diamond \dashv H, P \dashv \Box)$ be the 'extended' full complex algebra of a frame (W, R). Then the following are equivalent:*

(1) $\Diamond \dashv \Box$,
(2) *R is symmetric,*
(3) $H = \Box$,
(4) $\Diamond = P$.

Proof Using the formulas $\Box = \neg\Diamond\neg$ and $P = \neg H \neg$ we can calculate the following more explicit definitions for $\Box$ and P. $\Box X = \{y \in W \mid (\forall w \in W)(yRw \Rightarrow w \in X)\}$ and $PX = \{y \in W \mid (\exists x \in X)(xRy)\}$.

Assume first that $\Diamond \dashv \Box$. The unit of the adjunction states that $p \leq \Box\Diamond p$ for every p in $\mathcal{P}W$. So the usual axiom (B) for symmetry holds. If R is symmetric then clearly $H = \Box$ (compare the explicit definitions of H and $\Box$). The fact that the last two items and the first item are equivalent follows from uniqueness of adjoints. $\Box$

2.1 Galois Connections and Conjugate Pairs

It is natural that the adjunctions relating temporal operators did not appear in the 1958 paper by Prior because Kan's paper [27] on adjunctions was published the same year (and according to Section 4.4 in [20], Prior's work was presented in August 1954 during the New Zealand Philosophy Congress). We do not know who can be credited with the observation that $\Diamond \dashv H$ and $P \dashv \Box$. The fact is implicit in the 1988 work by Ghilardi and Meloni. Although the main interest of [15] is first order modal (and tense) logic, the relation between forward and backwards modalities is clearly expressed in their axiom (ADJ). More explicit are the Examples in Section 3 of [7] and it is relevant to mention that in the abstract of that paper the author says that the ideas there were first presented in a seminar in 1979 (at the university of Victoria).

It is interesting to remark that, as pointed out in [24], Galois connections induced by relations are studied by Ore in his 1944 paper [40]. It is well-known that Galois connections are adjoint pairs (see IV.5 in [34]). On the other hand, we don't know if Ore's work had any influence on Jónsson and Tarski who discovered left-adjoints between Boolean algebras. This fact is known among some modal logicians, but since it is not easy to find in print, we include here a brief discussion. Let A be a Boolean algebra.

Definition 2.8 (Definition 1.11 in [26]) Let f and g be functions on A to A. We say that g is a *conjugate* of f if, for any $x, y \in A$, we have

$$(fx) \wedge y = 0 \text{ if, and only if, } (gy) \wedge x = 0.$$

If, in particular, a function f is conjugate of itself, then we call f *self-conjugate*.

It is obvious (Theorem 1.12 in [26]) that if g is a conjugate of f then f is a conjugate of g and so it is justified to say that "f and g are conjugate".

Lemma 2.9 *If f and g are conjugate then they are monotone.*

Proof It is enough to prove that $x \leq y$ implies $fx \leq fy$. First notice that, since $(fy) \wedge \neg(fy) = 0$, $y \wedge g(\neg(fy)) = 0$ and so, $g(\neg(fy)) \leq \neg y$, for any y. Now, if $x \leq y$ then $0 \geq x \wedge \neg y \geq x \wedge g(\neg(fy))$. It follows that $(fx) \wedge \neg(fy) = 0$ and so $fx \leq fy$. $\square$

Theorem 1.13 in [26] provides an explicit formula for a conjugate, Theorem 1.14 gives a characterization of functions with a conjugate and Theorem 1.15 gives a characterization of conjugate pairs. All these results follow from the following.

Proposition 2.10 *Let $f, g : A \to A$ be monotone functions. Then g is a conjugate of f if and only if $f \dashv \neg g \neg$.*

Proof Assume that g is a conjugate of f and calculate:

$$fx \leq y \Leftrightarrow (fx) \wedge \neg y = 0 \Leftrightarrow x \wedge g(\neg y) = 0 \Leftrightarrow x \leq \neg(g(\neg y))$$

for any $x, y \in A$. We leave the converse to the reader. $\square$

As we mentioned above, Proposition 2.10 is not easy to find in the literature. For example, the concept of conjugate does not appear in [3] and only self-conjugacy is briefly mentioned in Section 3.3 of [20]. The only published statements relating conjugates and adjoints that we know of are in [24] and in Venema's Chapter 6 in [4] (see p. 385, paragraph after Proposition 129).

Corollary 2.11 *In every BAA $(B, \Diamond \dashv H, P \dashv \Box)$, $\Diamond$ and P are conjugate.*

Theorem 2.12 in [26] shows that if f and g are conjugate then their *perfect extensions* f^+ and g^+ are also conjugate. We can restate this in a way closer to the spirit of this paper: the operation $(\_)^+$ preserves left adjoints. (The corresponding result with respect to *completions* is proved in [16]. It is also proved there that conjugated functions are better behaved w.r.t. completions than completely additive maps. See discussion after Lemma 18 loc. cit. Again, this can be formulated as a statement about left adjoints.) The discussion on cylindric and relation algebras is done in Section 3 of [26] and relies on self-conjugate functions.

Corollary 2.12 *For every BAA* $(B, \Diamond \dashv H, P \dashv \Box)$ *the following are equivalent:*

(1) $\Diamond$ *is self-conjugate.*
(2) P *is self-conjugate.*
(3) $\Diamond \dashv \Box$.

Proof Follows from Corollary 2.11 and Proposition 2.7. □

The footnote in p. 903 of [26] states that conjugate pairs were first discussed in Tarski's 1927 paper *Sur quelques propriétés caractéristiques des images d'ensembles* published in the *Annales de la Société Polonaise de Mathématique*, but we have not been able to obtain a copy of this paper.

It is also relevant to remark that in p. 924 of [26] the definition of conjugate pair of functions on a Boolean algebra B is generalized to functions $B \to B'$ between two Boolean algebras.

Altogether, although it had not been discovered at the time of the publication of [26], the notion of adjunction played an important and explicit role in the paper.

2.2 Comments on Duality

The simplicity of BAOs and the irresistible intuitive appeal of Kripke structures are best seen together in a categorical duality (see [17] or Chapter 5 in [3]). It is well-known that the operation $\mathfrak{F} \mapsto \mathfrak{F}^+$ that maps a frame to its corresponding full complex algebra extends to a full embedding $\mathbf{Fr} \to \mathbf{BAO}^{\mathrm{op}}$ where $\mathbf{Fr}$ is the category of frames and *bounded* morphisms (see Section 3.3 in [3]). The fact that $\mathfrak{F}^+$ extends to an object in **BAA** does not imply that the functor just mentioned lands in **BAA**. In fact, it does not. Instead, one has to consider the category $\mathbf{Fr}_t$ of frames and *temporal* bounded morphisms (as defined just before Proposition 2.14 in [3]) between them. The operation $(\_)^+$ then extends to a functor $\mathbf{Fr}_t \to \mathbf{BAA}^{\mathrm{op}}$.

It is also well-known that $(\_)^+ : \mathbf{Fr} \to \mathbf{BAO}^{\mathrm{op}}$ is not a duality. Intuitively, because there are more BAOs than full complex algebras. In order to get a duality, the definition of frame is enriched to that of a *descriptive* frame. A category **dFr** of descriptive frames and bounded morphisms between them is obtained and the functor $\mathbf{Fr} \to \mathbf{BAO}^{\mathrm{op}}$ is extended to a duality $\mathbf{dFr} \to \mathbf{BAO}^{\mathrm{op}}$. In Section 3.1 we establish a duality between **BAA** and a category of *relational* Stone spaces.

Another option is to consider the full image of the functor $\mathbf{BAA} \to \mathbf{BAO}$. Denote the resulting full subcategory by $\mathbf{BAA}_b \to \mathbf{BAO}$. This category has the same objects of **BAA** but it has more morphisms. Indeed, all the boolean algebra morphisms that preserve $\Diamond$ and $\Box$. The category $\mathbf{BAA}_b$ is probably not an algebraic category but the functor $\mathbf{Fr} \to \mathbf{BAO}^{\mathrm{op}}$ factors through the embedding $(\mathbf{BAA}_b)^{\mathrm{op}} \to \mathbf{BAO}^{\mathrm{op}}$ and it may induce an interesting duality with a category of frames.

3 Lattices Equipped with an Adjunction

Denote the algebraic category of lattices by **Lat**. Define a *lattice with adjunction* (Lata) as a triple $(A, \mathsf{L}, \mathsf{R})$ with A a lattice and $\mathsf{L}, \mathsf{R} : A \to A$ monotone functions such that L is left adjoint to R. It is therefore natural to write a Lata as $(A, \mathsf{L} \dashv \mathsf{R})$. A morphism $f : (A, \mathsf{L} \dashv \mathsf{R}) \to (A', \mathsf{L}' \dashv \mathsf{R}')$ of lattices with adjunction is a morphism $f : A \to A'$ of lattices such that $f\mathsf{L} = \mathsf{L}'f$ and $f\mathsf{R} = \mathsf{R}'f$. Lattices with adjunction and morphisms between them form a category that we denote by **Lata**. It is an algebraic category because it can be presented by extending the standard presentation of **Lat** with two unary operators L and R together with axioms $\mathsf{L}\bot = \bot$, $\mathsf{L}(x \vee y) = (\mathsf{L}x) \vee (\mathsf{L}y)$, $\mathsf{R}\top = \top$, $\mathsf{R}(x \wedge y) = (\mathsf{R}x) \wedge (\mathsf{R}y)$ and the axioms saying that there is a unit $x \leq \mathsf{RL}x$ and a counit $\mathsf{LR}x \leq x$; just as we presented **BAA** by extending a presentation of the category of Boolean algebras.

We invite the reader to think of L as a 'modality of existential character' and R as 'modality of universal character', together in an adjointness situation like $\Diamond \dashv H$ or $P \dashv \Box$. But bare in mind that while 'existential/universal character' are informal ideas, left/right adjointness are precise concepts. (See also the discussion below Example 4.4.)

Concerning notation, we believe that using symbols $\Diamond$ and $\Box$ instead of L and R respectively is not a good idea. The presentation of **Lata** is not meant to abstract the situation isolated in Proposition 2.7 but, as we mentioned above, to abstract situations such as $\Diamond \dashv H$ or $P \dashv \Box$. An alternative is to write $\Diamond \dashv \blacksquare$ or $\blacklozenge \dashv \Box$. We have chosen the symbols clearly suggesting the Left and Right adjointness role of the operators.

Of course, there are two obvious forgetful functors **BAA** $\to$ **Lata** mapping the object $(B, \Diamond \dashv H, P \dashv \Box)$ to $(B, \Diamond \dashv H)$ and $(B, P \dashv \Box)$ respectively. We now discuss other examples.

Example 3.1 If H is a Heyting algebra then every $a \in H$ determines an adjunction $a \wedge (\_) \dashv a \Rightarrow (\_)$. These operations are not usually considered as modal operators, but if H is a Boolean algebra then $(H, a \wedge (\_))$ is an unquestionable BAO.

More generally, if L is a lattice equipped with a symmetric monoidal closed structure $(L, \cdot, \multimap)$ then, for any $a \in L$, $(L, a \cdot (\_) \dashv a \multimap (\_))$ is a Lata.

Example 3.2 (Positive Logic with Adjoint Modalities) Let **dLata** $\to$ **Lata** be the full subcategory determined by those objects whose underlying lattice is distributive. A complete and cut-free sequent calculus to derive validities in distributive Latas is introduced in [44]. As an application of the work in [14], a representation theorem (for perfect distributive Latas) in terms of ordered Kripke structures is proved. Also in [44], the authors propose an epistemic reading of the operators $\mathsf{L} \dashv \mathsf{R}$: fixing an agent, interpret $\mathsf{L}m$ as the agent's 'uncertainty' about m and $\mathsf{R}m$ as the agent's 'belief' about m. This epistemic interpretation is also explored in [2, 43] where the aim is to study information update due to information exchanges between agents. In this context, there is a complete lattice of 'epistemic propositions' and each agent has an associated adjunction. As in Example 2.3, completeness of the underlying lattices is mainly used to produce an adjoint for a complete operator.

Example 3.3 (Temporal Heyting Algebras) The *modalized Heyting calculus* is introduced in [12] as an extension of the Heyting propositional calculus with a unary operator $\Box$ and three axioms. The models of this calculus are called *frontal Heyting algebras* and are defined as pairs $(H, \Box)$ with H a Heyting algebra and $\Box : H \to H$ a function satisfying $\Box(p \wedge q) = \Box p \wedge \Box q$, $p \leq \Box p$ and $\Box p \leq q \vee (q \Rightarrow p)$. (See Definition 2 loc. cit.) The first part of Section 2 in [12] discusses the interest of mHC mentioning, in particular, its connections with propositional quantification in intuitionistic logic, topology, categorical logic and intuitionistic temporal logic. Esakia defines a *temporal* Heyting algebra as a frontal algebra $(H, \Box)$ such that $\Box$ has a left adjoint (that he denotes by $\Diamond$, but which is motivated as an operator looking once into the past).

Example 3.4 (Intuitionistic Propositional Logic with Galois Connections) As a natural generalization of the work mentioned in Example 2.4, Heyting algebras equipped with an adjunction are studied in [10]. A Hilbert-style calculus is introduced together with its semantics in such algebras and also in partially ordered Kripke structures. Different problems relating these structures are addressed, including a representation result.

3.1 Duality for Distributive Lattices

Recall that we denote by **dLata** the category of distributive Latas. We establish a duality for **dLata** that improves the representation results in [44] and [10]. For convenience, all lattices in this section are assumed to be distributive.

In the case of full complex algebras we discussed adjunctions $\mathsf{L} \dashv \mathsf{R}$ where L is a modal operator of existential character looking in one direction and its right adjoint R is an operator of universal character looking in the opposite direction. In this section we give a precise result supporting the intuition that all distributive lattices equipped with an adjunction are of this form. The result is a duality for **dLata** obtained by suitably restricting that in Theorem 2.3.3 in [19]. We recall some of the relevant material, but the reader is assumed to be familiar with Section 2 of Goldblatt's paper.

A *lattice with operators* is a triple (A, f, g) where A is a lattice, $f : A \to A$ is a $\vee$-semilattice morphism and $g : A \to A$ is a $\wedge$-semilattice morphism. A morphism $\phi : (A, f, g) \to (A', f', g')$ is a map $A \to A'$ between the underlying lattices such that $\phi(fx) = f'(\phi x)$ and $\phi(gx) = g'(\phi x)$. Following the notation in [19], we denote by **DLO** the category of lattices with operators and morphisms between them.

Recall that a *Priestley space* is a pair $(X, \leq)$ where X is a compact topological space, $\leq$ is a partial order on the underlying set of X and the space X is totally order-separated with respect to $\leq$. A *relational* Priestley space is a 4-tuple $(X, \leq, R, Q)$ where $(X, \leq)$ is a Priestley space and R and Q are binary relations on X satisfying a number of conditions made explicit at the end of p. 184 in [19]. Let **RPS** be the category of relational Priestley spaces with *continuous bounded* morphisms between them (Section 2.3 loc. cit.). For convenience, let us state the following particular case of Theorem 2.3.3 in [19].

Corollary 3.5 *There are functors*

$$\begin{tikzcd} \mathbf{RPS} \arrow[r, bend left, "\Phi"] & \mathbf{DLO}^{\mathrm{op}} \arrow[l, bend left, "\Theta"] \end{tikzcd}$$

showing that **DLO** *is dual to* **RPS***.*

The functor Φ takes a relational Priestley space $(X, \leq, R, Q)$ and produces the lattice with operators $(\mathrm{cl}(X, \leq), \exists_R, \forall_Q)$ where

(1) $\mathrm{cl}(X, \leq)$ is the lattice of complex cones of the Priestley space $(X, \leq)$,
(2) $\exists_R$ is the $\vee$-semilattice morphism defined by

$$\exists_R U = \{y \in X \mid (\exists x \in U)(xRy)\}$$

and
(3) $\forall_Q$ is the $\wedge$-semilattice morphism defined by

$$\forall_Q U = \{y \in X \mid (\forall x \in X)(xQy \Rightarrow x \in U)\}.$$

Clearly, **dLata** appears as the variety **dLata** $\to$ **DLO** of those (A, f, g) such that $f \dashv g$. The purpose of this section is to identify what relational Priestley spaces are mapped (via Φ) to the subcategory $\mathbf{dLata}^{\mathrm{op}} \to \mathbf{DLO}^{\mathrm{op}}$.

Lemma 3.6 *Let $(X, \leq, R, Q)$ be in* **RPS***. If R equals the converse $\breve{Q}$ of Q then $\exists_R \dashv \forall_Q$.*

Proof By hypothesis,

$$\exists_R U = \{y \in X \mid (\exists x \in U)(xRy)\} = \{y \in X \mid (\exists x \in U)(yQx)\}$$

which is left adjoint to $\forall_Q U = \{y \in X \mid (\forall x \in X)(xQy \Rightarrow x \in U)\}$, just as in the case $m \dashv h$ among full complex algebras (see Section 1). $\square$

To prove that this result can be reversed, it is useful to recall the definition of $\Theta : \mathbf{DLO}^{\mathrm{op}} \to \mathbf{RPS}$. If A is a lattice, we denote its associated Priestley space of prime filters by A_+. If (A, f, g) is in **DLO** then $\Theta(A, f, g) = (A_+, R_f, Q_g)$ where

(1) $R_f(G, F)$ iff $fG \subseteq F$ and
(2) $Q_g(G, F)$ iff $g^{-1}F \subseteq G$

for all $G, F \in A_+$. (See p. 186 in [19].)

Lemma 3.7 *Let (A, f, g) be in* **DLO***. If $f \dashv g$ then $R_f = (Q_g)\breve{}$.*

Proof Of course, $(Q_g)\breve{}(G, F)$ iff $g^{-1}G \subseteq F$. Therefore it is enough to show that

$$g^{-1}G \subseteq F \quad \Leftrightarrow \quad fG \subseteq F$$

under the hypothesis that $f \dashv g$.

Assume first that $g^{-1}G \subseteq F$ and let $u \in G$. Then $u \leq g(fu)$ by adjointness. As G is upper closed, $g(fu) \in G$. But then $fu \in F$ by assumption. So $fG \subseteq F$.

Now assume that $fG \subseteq F$ and let $gu \in G$. Then $f(gu) \in F$ by assumption. As $f(gu) \leq u$ by adjunction, $u \in F$ because F is upper closed. Hence, $g^{-1}G \subseteq F$. $\square$

We can therefore conclude the following duality result.

Proposition 3.8 *The equivalence* $\Phi : \mathbf{RPS} \to \mathbf{DLO}^{\mathrm{op}}$ *restricts to an equivalence as below*

$$\begin{tikzcd} \mathbf{aRPS} \arrow[r, "\cong"] \arrow[d] & \mathbf{dLata}^{\mathrm{op}} \arrow[d] \\ \mathbf{RPS} \arrow[r, "\Phi"'] & \mathbf{DLO}^{\mathrm{op}} \end{tikzcd}$$

where $\mathbf{aRPS} \to \mathbf{RPS}$ *is the full subcategory of those relational Priestley spaces* $(X, \leq, R, Q)$ *such that* R *is the converse of* Q.

Proposition 3.8 extends the representation result given in [44]. Also, it is trivial to restrict this duality to one for Heyting algebras equipped with an adjunction. Just consider the (non-full) subcategory of **aRPS** determined by those objects that have an underlying *Heyting space* and by *Heyting morphisms* between them. The resulting duality simplifies and extends the representation result proved in [10]. Further restricting to Stone spaces produces the duality for **BAA**s promised in Section 2.2. Compare also with Theorem 3.6 in [26]. (See [5] for a proof that the category of Heyting algebras is dual to the category of Heyting spaces. The paper also describes some of the history of this 1974 result which is attributed to Adams and also to Esakia.)

What is unclear to us is the relation with the representation result which appears as a corollary of the main result of Gaggle theory. Let us briefly discuss this. The general strategy in [7] is to generalize the results in [26]. The reason to look for such generalization is best described by a quotation: "The problem with the Jónsson-Tarski result is that [...] the context is more restrictive that one would like. For example, the underlying structure must be a Boolean algebra, and the 'operators' must distribute over Boolean disjunction in each of their places". (See second paragraph in Section 2 of [7].) In particular, the objects that Gaggle theory considers in place of Boolean algebras with operators are defined so as to include, as instances, monoidal closed lattices $(L, \cdot, \multimap)$ as in the comment after Example 3.1. Notice that the 'operator' $\multimap$ is covariant in one argument but contravariant in the other. Informally, a *gaggle* is a 'lattice with operators' of the sort suggested above together with a (non-canonical) choice of operator (called the *head*) such that every other operator satisfies an *abstract law of residuation* with respect to the head. (See Section 7 loc. cit.) The abstract law of residuation is a generalization of the notion of Galois connection and, for example, any monoidal closed lattice $(L, \cdot, \multimap)$ is a gaggle when the monoidal operation $\cdot$ is considered as the head of the family of 'operators' $\{\cdot, \multimap\}$. Also, any $(D, \mathsf{L} \dashv \mathsf{R})$ in **dLata** is a gaggle when either L or R is considered as the head of $\{\mathsf{L}, \mathsf{R}\}$. The main result loc. cit. is a representation theorem for gaggles which, as a corollary, induces a representation result for distributive Latas. This should be compared to that induced by our duality of Proposition 3.8.

Finally, all these representability results suggest a potentially interesting definability theory. Intuitively, the classical theory studies what classes of relational structures can be defined with modalities of existential and universal character facing in one direction. In the temporal case, we have access to both characters and both directions. The ‘definability theory’ suggested by the results above should study what classes of relational Priestley spaces can be defined using adjoint modal operators $\mathsf{L} \dashv \mathsf{R}$. Simple examples showing that the idea is nontrivial may be found already at the level of bidirectional frames for classical propositional tense logic. The next result presents one of the simplest cases.

Proposition 3.9 *For any frame* (W, R)*, the following are equivalent:*

(1) $(W, R) \models p \Rightarrow \Diamond Hp$.
(2) *For all* $u \in W$*, there exists* $v \in W$ *such that* uRv *and for every* $u' \in W$*,* $u'Rv$ *implies* $u = u'$.

Also, the resulting class of frames is not definable in terms of $\Diamond$ *and* $\Box$.

Proof Assume first that the second item holds. Take an arbitrary valuation V on (W, R), and a state $u \in W$ such that $(W, R, V), u \models p$. We need to show that Hp holds at some state ‘on the right’. For this, take the element v given by the second condition. It satisfies that uRv and we claim that $(W, R, V), v \models Hp$. Indeed, if $u'Rv$ then $u = u'$ and so, $(W, R, V), u' \models p$.

For the converse, assume that $(W, R) \models p \Rightarrow \Diamond Hp$ and let $u \in W$. Let V be the valuation determined by $Vp = \{u\}$ and $Vq = \emptyset$ for every $q \neq p$. It follows that $(W, R, V), u \models \Diamond Hp$. So there is a $v \in W$ such that uRv and $(W, R, V), v \models Hp$. In turn this means that for every $u' \in W$, $u'Rv$ implies $(W, R, V), u' \models p$. That is, $u' \in Vp$ and hence $u = u'$.

To prove the last part of the proposition we use the Goldblatt-Thomason Characterization (Theorem 3.19 in [3]). We show that the class of frames defined by the first order conditions of the second item of the statement is not closed under bounded morphic images. Let $(\mathbb{N}, R)$ be the frame given by $nR(n+1)$. It can be pictured by $0 \to 1 \to 2 \to \ldots$. The frame $2\mathbb{N} = (\mathbb{N}, R) + (\mathbb{N}, R)$ can be pictured as two ‘parallel’ copies of $\mathbb{N}$ as on the left below

$$
\begin{tikzcd}
0 \arrow[r, "R"] & 1 \arrow[r, "R"] & 2 \arrow[r, "R"] & \cdots \\
\bar{0} \arrow[r, "R"] & \bar{1} \arrow[r, "R"] & \bar{2} \arrow[r, "R"] & \cdots
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[dr, "R'"] & & & \\
& 1 \arrow[r, "R'"] & 2 \arrow[r, "R'"] & \cdots \\
0' \arrow[ur, "R'"'] & & &
\end{tikzcd}
$$

On the other hand, consider the set $\mathbb{N}' = \mathbb{N} + \{0'\}$ and the frame $(\mathbb{N}', R')$ given by $nR'(n+1)$ for every $n \in \mathbb{N}$ and $0'R'1$. It can be pictured as on the right above. Now let $f : 2\mathbb{N} \to \mathbb{N}'$ be the function such that $fn = n$ for every $n \in \mathbb{N}$, $f\bar{0} = 0'$ and, for each $m \geq 1$, $f\bar{m} = m$. This function is clearly a surjective bounded morphism in the sense of Definition 3.13 in [3]. While its domain of shape '$\rightrightarrows$' clearly validates $p \Rightarrow \Diamond Hp$, its codomain of shape '$\rightarrowtail$' clearly does not. $\square$

3.2 Intuitionistic Tense Logic

If we consider the two adjunctions $\Diamond \dashv H$ and $P \dashv \square$ as fundamental in a BAA, then it is natural to consider structures of the form $(A, \Diamond, H, P, \square)$ where both $(A, \Diamond, H)$ and $(A, P, \square)$ are in **dLata**. But of course, there should be a minimum of interaction between the two Lata structures.

In [9] Dunn defines a minimal normal modal logic with operators $\square$ and $\Diamond$ but without negation or implication. One of the main results shows that it is complete with respect to the usual interpretation in Kripke models. As the author says, the only non-obvious postulates are

$$\Diamond p \wedge \square q \vdash \Diamond(p \wedge q) \qquad \square(p \vee q) \vdash \square p \vee \Diamond q$$

See the abstract loc. cit.

The two previous paragraphs suggest the following.

Definition 3.10 A *tense Lata* is a structure $(A, \Diamond \dashv H, P \dashv \square)$ such that both $(A, \Diamond, H)$ and $(A, P, \square)$ are in **dLata** and moreover the conditions

$$\Diamond p \wedge \square q \leq \Diamond(p \wedge q) \qquad Pp \wedge Hq \leq P(p \wedge q)$$

and

$$\square(p \vee q) \leq \square p \vee \Diamond q \qquad H(p \vee q) \leq Hp \vee Pq$$

hold.

The conditions defining tense Latas can be equationally presented and so an algebraic category of very natural structures is induced. Moreover, there is an obvious forgetful functor from **BAA** to the category of tense Latas. Now, in order to compare these structures with already existing material on intuitionistic temporal logic it is relevant to introduce a weaker concept.

Definition 3.11 An *Ewald Lata* is a structure $(A, \Diamond \dashv H, P \dashv \square)$ as above such that

$$\Diamond p \wedge \square q \leq \Diamond(p \wedge q) \qquad Pp \wedge Hq \leq P(p \wedge q)$$

hold.

In other words, an Ewald Lata is like a tense Lata but satisfying only the first row of axioms in Definition 3.10. It is fair to say that Ewald Latas with an underlying

Heyting algebra provide the natural algebraic semantics for the intuitionistic tense logic IK_t described by Ewald in [13]. In order to state the precise relation, we recall (p. 171 loc. cit.) that IK_t can be axiomatized as follows:

(1) All axioms of the intuitionistic sentential calculus

(2)	$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$	(2′)	$H(p \Rightarrow q) \Rightarrow (Hp \Rightarrow Hq)$
(3)	$\Box(p \wedge q) \Leftrightarrow \Box p \wedge \Box q$	(3′)	$H(p \wedge q) \Leftrightarrow Hp \wedge Hq$
(4)	$\Diamond(p \vee q) \Leftrightarrow (\Diamond p \vee \Diamond q)$	(4′)	$P(p \vee q) \Leftrightarrow (Pp \vee Pq)$
(5)	$\Box(p \Rightarrow q) \Rightarrow (\Diamond p \Rightarrow \Diamond q)$	(5′)	$H(p \Rightarrow q) \Rightarrow (Pp \Rightarrow Pq)$
(6)	$\Box p \wedge \Diamond q \Rightarrow \Diamond(p \wedge q)$	(6′)	$Hp \wedge Pq \Rightarrow P(p \wedge q)$
(7)	$\Box \neg p \Rightarrow \neg \Diamond p$	(7′)	$H \neg p \Rightarrow \neg Pp$
(8)	$\Diamond Hp \Rightarrow p$	(8′)	$P \Box p \Rightarrow p$
(9)	$p \Rightarrow \Box Pp$	(9′)	$p \Rightarrow H \Diamond p$
(10)	$(\Diamond p \Rightarrow \Box q) \Rightarrow \Box(p \Rightarrow q)$	(10′)	$(Pp \Rightarrow Hq) \Rightarrow H(p \Rightarrow q)$
(11)	$\Diamond(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Diamond q)$	(11′)	$P(p \Rightarrow q) \Rightarrow (Hp \Rightarrow Pq)$

plus Modus Ponens, and the usual 'Necessity' rules for $\Box$ and H.

The natural algebraic models suggested by this axiomatization are structures $(A, \Diamond, H, P, \Box)$ where A is a Heyting algebra and the unary operators $\Diamond, H, P, \Box$ satisfy the axioms above. Rows 2, 3, 4, 8 and 9 state that $(A, \Diamond \dashv H)$ and $(A, P \dashv \Box)$ are in **dLata**. Row 6 states that $(A, \Diamond \dashv H, P \dashv \Box)$ is an Ewald Lata. The following simple result implies that IK_t has a much simpler axiomatization.

Proposition 3.12 *Let A be a Heyting algebra equipped with two adjunctions $\Diamond \dashv H$ and $P \dashv \Box$. Then the structure $(A, \Diamond \dashv H, P \dashv \Box)$ satisfies all the axioms of IK_t if and only if it is an Ewald Lata.*

Proof By the discussion above we need only show that rows 5, 7, 10 and 11 follow from the definition of Ewald Lata when A is a Heyting algebra. For example, to prove that axiom (5) holds it is enough to show that $\Box(p \Rightarrow q) \wedge \Diamond p \leq \Diamond q$. But

$$\Box(p \Rightarrow q) \wedge \Diamond p \leq \Diamond((p \Rightarrow q) \wedge p) \leq \Diamond q$$

by hypothesis and monotonicity of $\Diamond$ applied to the counit $(p \Rightarrow q) \wedge p \leq q$. Axiom (5′) and rows 7 and 11 follow analogously. We finish with a proof that (10′) holds. We need to show that $Pp \Rightarrow Hq \leq H(p \Rightarrow q)$. By adjunction this is equivalent to $\Diamond(Pp \Rightarrow Hq) \leq p \Rightarrow q$ and so, equivalent to $\Diamond(Pp \Rightarrow Hq) \wedge p \leq q$ by cartesian closure. Now, $\Diamond(Pp \Rightarrow Hq) \leq \Box(Pp) \Rightarrow \Diamond(Hp)$ by (11) and $p \leq \Box(Pp)$ by the unit of $P \dashv \Box$. Therefore,

$$\Diamond(Pp \Rightarrow Hq) \wedge p \leq (\Box(Pp) \Rightarrow \Diamond(Hp)) \wedge \Box(Pp) \leq \Diamond(Hp) \leq p$$

as we needed to show. $\Box$

The justification of IK_t in [13] is given by a completeness theorem with respect to a 'natural' Kripke-like semantics. The conditions $\Box(p \vee q) \leq \Box p \vee \Diamond q$ and $H(p \vee q) \leq Hp \vee Pq$ correspond to certain restrictions in the models Ewald considers. See section 'Unchanging times' in [13], and rules $(G8^*)$ and $(G8'^*)$ in particular. Further justification for IK_t and relevant technical details about Ewald's work can be found in [45]. In contrast, compare with the Remark just before Section 5 in [9] justifying the condition $\Box(p \vee q) \leq \Box p \vee \Diamond q$.

The reader should also compare the discussion above with that in Section 9 of [10].

Example 3.13 The *intuitionistic Galois algebras* introduced in [46] can be defined as Ewald latas with an underlying complete Heyting algebra. The axioms highlighted in Definition 3.11 are called the 'Best-Of-Both-Worlds' laws. See Definition 2.5.1 loc. cit.

3.3 Monads and Comonads

Because of the close relationship between monads and adjunctions [34] we will consider monads to be part of the general picture we are discussing. The relationship will be made more explicit in Section 4. Recall that a *monad* m on a poset P is simply a monotone function $m : P \to P$ such that $x \leq mx$ and $mmx \leq mx$. Monads on a poset are idempotent.

Example 3.14 (Closure Algebras) A *closure algebra* is a Boolean algebra B equipped with a monad $\mathsf{C} : B \to B$ such that $\mathsf{C}\bot = \bot$ and $\mathsf{C}(x \vee y) = (\mathsf{C}x) \vee (\mathsf{C}y)$. Closure algebras were introduced in [37] as an "attempt" to create "an algebraic apparatus adequate for the treatment of portions of point-set topology". (See second paragraph loc. cit.) This work is one of the main examples motivating the development of Boolean algebras with operators. (See footnote 1 in p. 891 of [26].)

Of course, it is natural to consider monads on more general algebraic structures.

Example 3.15 (Local Algebras) In the first paragraph of [30], Lawvere mentions that "a Grothendieck 'topology' appears most naturally as a modal operator". Goldblatt analyses this quotation in [18] and in Section 6.5 loc. cit. introduces *local algebras*. These are pairs (H, j) where H is a Heyting algebra and $j : H \to H$ is a function satisfying $x \leq jx$, $j(jx) = jx$ and $j(x \wedge y) = (jx) \wedge (jy)$. In other words, j is a lex (i.e. finite-limit preserving) monad on H.

A *comonad* on a poset P is a monotone function $c : P \to P$ such that $cx \leq x$ and $cx \leq ccx$. Of course, comonads on a poset are idempotent. This is also a natural structure and we will encounter concrete examples below. We are not aware, though, of a modal-logical study of such structures analogous to that of local algebras in [18].

Example 3.16 (MAO Couples) Following [42], we define a *MAO lattice* to be a triple $(A, \Diamond, \blacksquare)$ with A a lattice, $\Diamond : A \to A$ a monad on A and $\blacksquare : A \to A$ a comonad on A such that $\Diamond \dashv \blacksquare$. ('MAO' stands for Modal Adjoint Operators.) Simple calculations imply the following properties: $\blacksquare \leq id_P \leq \Diamond$, $\blacksquare^2 = \blacksquare$, $\Diamond^2 = \Diamond$, $\Diamond\blacksquare = \blacksquare$ and $\blacksquare\Diamond = \Diamond$.

More information about these examples appears in the next section.

4 Two-sorted Variations

So far we have discussed different examples of modal operators that appear as adjoints on a partially ordered algebraic structure. Some examples suggest that two-sorted variations of **Lata** may also be useful. We define a category $\mathbf{Lata}_2$ whose objects are structures $(E, B, \mathsf{L} \dashv \mathsf{R})$ where E and B are lattices, and $\mathsf{L} : B \to E$ and $\mathsf{R} : E \to B$ are monotone functions such that $\mathsf{L} \dashv \mathsf{R}$. A morphism $(E, B, \mathsf{L} \dashv \mathsf{R}) \to (E', B', \mathsf{L}' \dashv \mathsf{R}')$ is a pair (f, g) of lattice morphisms $f : E \to E'$ and $g : B \to B'$ such that the following diagrams

$$\begin{tikzcd} E \arrow[r, "f"] & E' \\ B \arrow[u, "\mathsf{L}"] \arrow[r, "g"'] & B' \arrow[u, "\mathsf{L}'"'] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\mathsf{R}"'] & E' \arrow[d, "\mathsf{R}'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

commute. There is an obvious forgetful functor $\mathbf{Lata}_2 \to \mathbf{Set} \times \mathbf{Set}$ which maps the object $(E, B, \mathsf{L} \dashv \mathsf{R})$ to the pair (E, B) consisting of the underlying sets of the lattices E and B. A trivial variation on the presentation of **Lata** provides a presentation of $\mathbf{Lata}_2$ as a 2-sorted algebraic category.

This is a good point to come back to one of the main examples in the introduction.

Example 4.1 (Quantifiers) If we let $\mathcal{C}$ be a category with finite limits, pulling back along a morphism $f : X \to Y$ induces a monotone morphism $f^* : \mathbf{Sub}Y \to \mathbf{Sub}X$ between posets of subobjects. If $\mathcal{C}$ is a topos, then $\mathbf{Sub}X$ and $\mathbf{Sub}Y$ are Heyting algebras and the functor f^* has both a left adjoint $\exists_f : \mathbf{Sub}X \to \mathbf{Sub}Y$ and a right adjoint $\forall_f : \mathbf{Sub}X \to \mathbf{Sub}Y$. The adjoints $\forall_f$ and $\exists_f$ provide a sound semantics for the quantifiers in intuitionistic first order logic (see [30, 32] or [25]).

Notice that the existential quantifier $\exists_f$ is a left adjoint, just as the modal operators of existential character $\Diamond$ and P. On the other hand, the universal quantifier $\forall_f$ is a right adjoint, just as $\Box$ and H. In contrast with the $\Diamond \dashv H$ or $P \dashv \Box$, the functors $\exists_f$ and $\forall_f$ are adjoints to the 'more basic' functor f^*. It is also worth mentioning that $\forall_f = \neg\exists_f\neg$ and its dual equation only hold in Boolean toposes.

Example 4.2 (Negation) We are not sure if many practitioners would accept 'negation' as a modal operator but if we consider a Heyting algebra H and let $\neg x = x \Rightarrow \bot$ for every $x \in H$ then we obtain a functor $\neg : H^{\mathrm{op}} \to H$ and its opposite $\neg^{\mathrm{op}} : H \to H^{\mathrm{op}}$ which, together, form an object $(H, H^{\mathrm{op}}, \neg^{\mathrm{op}} \dashv \neg)$ in $\mathbf{Lata}_2$.

More generally, if $(L, \cdot, \multimap)$ is a symmetric monoidal closed lattice and $b \in L$ then $(L, L^{\mathrm{op}}, ((\_) \multimap b)^{\mathrm{op}} \dashv (\_) \multimap b)$ is in $\mathbf{Lata}_2$. Compare with the remark below Example 3.1.

Example 4.3 (Idempotent Monads) Let (A, m) be a lattice equipped with a monad as in Section 3.3. Denote by $\mathsf{R} : A_m \to A$ the subcategory determined by those x in

A such that $x = mx$. The assignment $x \mapsto mx$ induces a functor $\mathsf{L} : A \to A_m$ and in this way we obtain an object $(A_m, A, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Lata}_2$. In a context where (A, m) is understood as a lattice of propositions equipped with a modal operator m, the sub-$\wedge$-semilattice $A_m \to A$ may be thought of as the poset of 'modally closed propositions'. (See Section *Modal operators* in Chapter 7 of [28].)

Let $\mathbf{Geo} \to \mathbf{Lata}_2$ be the full subcategory of $\mathbf{Lata}_2$ determined by the objects $(E, B, \mathsf{L}, \mathsf{R})$ such that L preserves finite limits. It can be presented by extending the presentation of $\mathbf{Lata}_2$ suggested above with an axiom saying that L preserves finite infima.

Example 4.4 (Local Algebras Continued) Recall from Example 3.15 that a local algebra is a lex monad j on a Heyting algebra H and hence it induces an object $(H_j, H, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Lata}_2$ as in Example 4.3. But as j preserves finite infima, so does L. In this way, every local algebra induces an object in $\mathbf{Geo}$. In the opposite direction, the well-known relation between adjunctions and monads implies that for any object $(G, H, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Geo}$ with H a Heyting algebra, the pair (H, RL) is a local algebra.

Also related to Example 4.4, let us recall the uncertainty mentioned in the paragraph defining *local operators* in Section 7.6 of [20]:

> Since j is multiplicative and has $j1 = 1$, this will be a normal logic when $\Box$ is interpreted as j, but there has been some uncertainty as to whether a modality modelled by j is of universal or existential character. Note that a local operator has a mixture of the properties of topological interior and closure operators. It fulfills all of the axioms of an interior operator except $\mathsf{I}x \leq x$, satisfying instead the inflationary condition which is possessed by closure operators. But topological closure operators are additive $(\mathsf{C}(x + y) = \mathsf{C}x + \mathsf{C}y)$, a property not required of j.

If we consider this uncertainty as a philosophical problem then, how can we solve it? Consider the following proposal: provide a precise definition of 'universal/existential character' and check if a local operator satisfies either. In this spirit, define that a modal operator is of universal (resp. existential) character if it is a right (resp. left) adjoint. If one accepts this definition then the uncertainty is resolved as follows: the operation j is neither of universal nor of existential character; it is the composition $j = \mathsf{RL}$ of a (lex) operator L of existential character followed by an operator R of universal character. We will not press the issue, each reader will decide if this definition of existential/universal character (and the induced 'clarification' of the uncertainty) is consistent with her or his philosophical convictions.

It is also relevant to mention the categorical dual of the fact relating objects of $\mathbf{Geo}$ and local algebras. Namely, every $(G, H, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Geo}$ induces a pair (G, LR) consisting of a lattice equipped with a lex comonad.

Let $\mathbf{Ess} \to \mathbf{Geo}$ be the full subcategory of $\mathbf{Geo}$ determined by those objects $(E, B, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Geo}$ such that L has an eXtra left adjoint that we denote by $\mathsf{X} : E \to B$. An object in $\mathbf{Ess}$ will be denoted by $(E, B, \mathsf{X} \dashv \mathsf{L} \dashv \mathsf{R})$. For such an object we have, on top of the comonad $\blacksquare = \mathsf{LR} : E \to E$ mentioned above, also a monad $\Diamond : E \to E$ defined as the composition LX. Every object $(E, B, \mathsf{X} \dashv \mathsf{L} \dashv \mathsf{R})$ in **Ess** induces a MAO lattice $(E, \Diamond \dashv \blacksquare)$. (Recall Example 3.16.) Objects of **Ess** and MAO lattices are used in [42] to study modal logics associated to geometric morphisms between toposes. We recall some of this material in Section 4.1.

4.1 Geometric Morphisms

Objects of **Geo** appear naturally in the context of [15] and [41, 42] where toposes are used to interpret first order modal logic. We briefly present the setting of this work assuming that the reader has the necessary background on topos theory [25]. We follow mainly [42] which gives a generalization of both [15] and [41].

Let $F : \mathcal{E} \to \mathcal{B}$ be a geometric morphism between toposes. We use standard notation so that the functor $F_* : \mathcal{E} \to \mathcal{B}$ is the *direct image* of F, and $F^* : \mathcal{B} \to \mathcal{E}$ is its finite-limit-preserving left adjoint. (See Definition A4.1.1 in [25].) We denote by $\mathbf{Sub}_{\mathcal{B}} S$ the Heyting algebra of subobjects of the object S in $\mathcal{B}$ and similarly for objects of $\mathcal{E}$. Restricting F^* to subobjects gives, for each S in $\mathcal{B}$, a morphism $\mathsf{L}_S : \mathbf{Sub}_{\mathcal{B}} S \to \mathbf{Sub}_{\mathcal{E}}(F^* S)$. If S is clear from the context we may write L instead of L_S.

Lemma 4.5 *For each S in $\mathcal{B}$, $\mathsf{L} : \mathbf{Sub}_{\mathcal{B}} S \to \mathbf{Sub}_{\mathcal{E}}(F^* S)$ has a right adjoint.*

Proof As explained in [42] the right adjoint $\mathsf{R} : \mathbf{Sub}_{\mathcal{E}}(F^* S) \to \mathbf{Sub}_{\mathcal{B}} S$ assigns to each subobject $k : K \to F^* S$ the subobject of S given by the following pullback

$$
\begin{tikzcd}
\mathsf{R}K \arrow[r] \arrow[d] & F_* K \arrow[d, "F_* k"] \\
S \arrow[r, "\eta_S"'] & F_* F^* S
\end{tikzcd}
$$

where η denotes the unit of the adjunction $F^* \dashv F_*$. $\square$

The right adjoint is denoted by $\mathsf{R}_S : \mathbf{Sub}_{\mathcal{E}}(F^* S) \to \mathbf{Sub}_{\mathcal{B}} S$ or by R. We therefore have that every S in $\mathcal{B}$ induces an object $(\mathbf{Sub}(F^* S), \mathbf{Sub} S, \mathsf{L} \dashv \mathsf{R})$ in **Geo**. We will also be interested in the lex comonad $\blacksquare_S = \mathsf{LR}$ on the Heyting algebra $\mathbf{Sub}(F^* S)$.

Example 4.6 (Example 1 in Section 4 of [42]) Fix a set I and let $\mathcal{E}$ be the slice category $\mathbf{Set}/I$. Also, let $\mathcal{B} = \mathbf{Set}$ and $\mathcal{E} \to \mathcal{B}$ be the canonical geometric morphism. So that $F^* S$ is the projection $S \times I \to I$ and F_* applied to an object $f : X \to I$ of $\mathcal{E}$ is the set of sections $I \to X$ of f. A subobject of $F^* S$ in $\mathcal{E}$ is determined by a subobject $\phi : K \to S \times I$ (in **Set**) and we write $i \Vdash \phi(s)$ instead of $(s, i) \in K$. It follows that for a subobject $\psi : S_0 \to S$ of S and $s \in S$, $i \Vdash (\mathsf{L}\psi)s$ if and only if $s \in S_0$. Similarly, the functor $\mathsf{R}_S : \mathbf{Sub}_{\mathcal{E}}(F^* S) \to \mathbf{Sub}_{\mathcal{B}} S$ has the following description: $s \in \mathsf{R}\phi$ if and only if $(\forall j \in I)(j \Vdash \phi(s))$. So we can write $\blacksquare_S : \mathbf{Sub}_{\mathcal{E}}(S \times I) \to \mathbf{Sub}_{\mathcal{E}}(S \times I)$ and conclude that for a subobject $\phi : K \to S \times I$, $i \Vdash (\blacksquare\phi)(s)$ if and only if for all $j \in I$, $j \Vdash \phi(s)$.

Notice that at the propositional extreme (i. e. with $S = 1$) we have that $i \Vdash \blacksquare\phi$ if and only if for all $j \in I$, $j \Vdash \phi$. This is the *global box*, denoted by A in section *The global modality* of 7.1 in [3].

We now look at a different source of examples given in [15]. Let us recall a useful piece of notation. If P is a presheaf on $\mathcal{C}$, $h : C' \to C$ is a map in $\mathcal{C}$ and $x \in PC$ then we write $x \cdot h$ instead of $(Ph)x$.

Example 4.7 For a small category $\mathcal{C}$ we denote the topos of presheaves by $\widehat{\mathcal{C}}$. Let $\mathcal{C}_0$ be the discrete subcategory of objects of $\mathcal{C}$. The inclusion $i : \mathcal{C}_0 \to \mathcal{C}$ induces a geometric morphism $i^* \dashv i_* : \widehat{\mathcal{C}_0} \to \widehat{\mathcal{C}}$. If P is an object of $\widehat{\mathcal{C}}$ then a subobject $\psi \in \mathbf{Sub}(i^*P)$ is nothing but a collection $\{\psi C \subseteq PC\}_{C \in \mathcal{C}_0}$ of subsets indexed by the objects of $\mathcal{C}$. It is clear from the role of $\mathsf{R}_P : \mathbf{Sub}(i^*P) \to \mathbf{Sub}P$ as a right adjoint that $\mathsf{R}\psi$ is the largest subpresheaf $S \to P$ such that $i^*S \leq \psi$ in $\mathbf{Sub}(i^*P)$. The following more explicit 'Kripke-style' definition is due to Ghilardi and Meloni: for C in $\mathcal{C}$, $x \in (\mathsf{R}\psi)C$ if and only if for all $h : C' \to C$ in $\mathcal{C}$, $x \cdot h \in \psi C'$. We can write $s \Vdash_C \psi$ instead of $s \in \psi C$ and it follows that $s \Vdash_C \blacksquare\psi$ if and only if for every $f : D \to C$ in $\mathcal{C}$, $s \cdot f \Vdash_D \psi$.

Consider the following more concrete case.

Example 4.8 Let $\mathcal{C}$ be the category with parallel maps $d, c : N \to E$. The topos $\widehat{\mathcal{C}}$ can be identified with the category of directed graphs. A presheaf G has a set GE of edges, a set GN of nodes and for each edge x, $x \cdot d$ is the domain of x and $x \cdot c$ its codomain. The topos $\widehat{\mathcal{C}_0}$ can be identified, in this case, with $\mathbf{Set} \times \mathbf{Set}$. We then have $i^*G = (GN, GE)$. A subobject $\phi : (K_0, K_1) \to i^*G$ is given by a subset K_0 of nodes of G and a subset K_1 of edges of G. We then have that $n \Vdash_N \blacksquare\phi$ if and only if $n \Vdash_N \phi$. On the other hand, $e \Vdash_E \blacksquare\phi$ if and only if $e \Vdash_E \phi$ and $e \cdot d \Vdash_N \phi$ and $e \cdot c \Vdash_N \phi$. That is, an edge e is in $\blacksquare\phi$ iff itself and both its domain and codomain are in ϕ.

Back to general facts: if the geometric morphism $F : \mathcal{E} \to \mathcal{B}$ is *open* (as in Example 4.9 below) or *essential* (as in Examples 4.7 and 4.8 above) then for every S in $\mathcal{B}$, the object $(\mathbf{Sub}(F^*S), \mathbf{Sub}S, \mathsf{L} \dashv \mathsf{R})$ in $\mathbf{Geo}$ is actually in $\mathbf{Ess}$. (See [42].) It follows that every such S induces a MAO lattice $(\mathbf{Sub}_{\mathcal{E}}(F^*S), \Diamond_S \dashv \blacksquare_S)$ as in Example 3.16.

Example 4.9 (Example 4.6 Continued) Recall that for the geometric morphism $\mathbf{Set}/I \to \mathbf{Set}$ we have that F^*S is the projection $S \times I \to I$. We leave the reader to calculate $\mathsf{X}_S : \mathbf{Sub}_{\mathcal{E}}(S \times I) \to \mathbf{Sub}_{\mathcal{B}}S$. As a hint, we state the fact that when $S = 1$, $i \Vdash \Diamond\phi$ if and only if there exists $j \in I$ such that $j \Vdash \phi$. That is, in the propositional case, $\Diamond$ is the *global diamond* discussed in Section 7.1 in [3].

Example 4.10 (Example 4.7 Continued) The role of X as an adjunction implies that $\mathsf{X}\psi$ is the smallest subpresheaf of P including the elements of ψ. It follows that $s \Vdash_C \Diamond\psi$ if and only if there exist a map $g : C \to B$ in $\mathcal{C}$ and $r \in PB$ such that $r \cdot g = s$ and $r \Vdash_B \psi$.

So far we have only discussed the most elementary part of [15] and [42]. Our purpose was to present an interesting class of adjoint operators that appears when investigating certain geometric morphisms $\mathcal{E} \to \mathcal{B}$. But the stress in [15] and [42] is that the logic associated to such geometric morphisms is a *first order* modal logic.

Toposes are Heyting categories so one can interpret intuitionistic first order logic in the hyperdoctrine that assigns the monotone $f^* : \mathbf{Sub}Y \to \mathbf{Sub}X$ to each morphism $f : X \to Y$ in the category.

If $F : \mathcal{E} \to \mathcal{B}$ is a geometric morphism inducing a family $\{(F^*S, \Diamond_S \dashv \blacksquare_S)\}_{S\in\mathcal{B}}$ of MAO lattices then the relation with first order logic can be studied in the hyperdoctrine that assigns $(F^* f)^* : \mathbf{Sub}_{\mathcal{E}}(F^*T) \to \mathbf{Sub}_{\mathcal{E}}(F^*S)$ to each $f : S \to T$ in $\mathcal{B}$.

Lemma 4.11 *The family $\{\blacksquare_S\}_{S\in\mathcal{B}}$ is natural in S; in the sense that for every $f : S \to T$ in $\mathcal{B}$, $\blacksquare_S(F^* f)^* = (F^* f)^*\blacksquare_T$.*

Lemma 4.11 follows from the fact that both $\mathsf{L}_S : \mathbf{Sub}_{\mathcal{B}}S \to \mathbf{Sub}_{\mathcal{E}}(F^*S)$ and $\mathsf{R}_S : \mathbf{Sub}_{\mathcal{E}}(F^*S) \to \mathbf{Sub}_{\mathcal{B}}S$ are natural in S. One of the key observations in [15, 42] is that, in general, $\Diamond_S$ is only *lax*-natural, in the sense that the inequality $\Diamond_S f^* \leq f^*\Diamond_T$, which holds for general reasons, may be strict.

The relation between naturality and 'internalization' is well explained in [42]. The fact that $\mathsf{L}_S : \mathbf{Sub}_{\mathcal{B}}S \to \mathbf{Sub}_{\mathcal{E}}(F^*S)$ and $\mathsf{R}_S : \mathbf{Sub}_{\mathcal{E}}(F^*S) \to \mathbf{Sub}_{\mathcal{B}}S$ are natural implies that there are morphisms $\delta : \Omega_{\mathcal{B}} \to F_*(\Omega_{\mathcal{E}})$ and $\gamma : F_*(\Omega_{\mathcal{E}}) \to \Omega_{\mathcal{B}}$ in $\mathcal{B}$ (where Ω denotes the subobject classifier in the corresponding topos) and δ is internally left adjoint to γ. So $\blacksquare = \delta\gamma : F_*(\Omega_{\mathcal{E}}) \to F_*(\Omega_{\mathcal{E}})$ is an endomorphism of the object resulting from applying $F_* : \mathcal{E} \to \mathcal{B}$ to the object of 'truth values' of the topos $\mathcal{E}$. Cases where $\Diamond$ can also be internalized are discussed in [42].

The above also explains why we defined **Ess** as a full subcategory of **Geo**. If $(f, g) : (E, B, \mathsf{X} \dashv \mathsf{L} \dashv \mathsf{R}) \to (E', B', \mathsf{X}' \dashv \mathsf{L}' \dashv \mathsf{R}')$ is a map in **Ess**, we always have that $\mathsf{X}' f \leq g\mathsf{X}$, for general reasons. But we do *not* require equality to hold. The reason is that, as explained above, every map $h : S \to T$ in $\mathcal{B}$ induces monotone maps $h^* : \mathbf{Sub}T \to \mathbf{Sub}S$ and $(F^*h)^* : \mathbf{Sub}(F^*T) \to \mathbf{Sub}(F^*S)$ and, together, they induce a morphism

$$(\mathbf{Sub}(F^*T), \mathbf{Sub}T, \mathsf{X} \dashv \mathsf{L} \dashv \mathsf{R}) \to (\mathbf{Sub}(F^*S), \mathbf{Sub}S, \mathsf{X} \dashv \mathsf{L} \dashv \mathsf{R})$$

in **Ess** which, in general, does not satisfy the equality mentioned above.

5 Non-posetal Examples

In the previous sections we considered several categories whose objects are adjunctions between partially ordered structures. In this section we discuss two examples of non-posetal categorical structures used in modal logic. The work reviewed in Section 5.1 uses categories equipped with monads as models for the proof theory of intuitionistic variants of the logic S4. In Section 5.2 we discuss an account of how Kripke semantics can naturally be seen as an extension of the 'internal logic' of the category **Set** of (sets and functions) to the category **Rel** of sets and relations between them.

5.1 Monads, Comonads and Intuitionistic S4

The work we review here is motivated by the study of the behavior of computer programs. This motivation is well-explained in, for example, [1, 21, 38] and references therein. Suffice it to say here that programs written in many programming languages can be seen in correspondence with proofs in different logics. The behavior of such programs may then be studied in terms of the relation between proofs. The fact that different proofs of the same statement may reflect the behavior of different programs calculating the same function implies that the typical algebraic structures used in the semantics of logics are not suitable to study the behavior of programs in the way referred to above. Categories serve this purpose better because, intuitively, two different arrows with same domain and codomain may represent two different programs calculating the same function. The best known examples of this are the theorems relating intuitionistic propositional logic, typed λ-calculus and cartesian closed categories [29].

Analogous results exist for other logics such as those investigated in [1]. The logic called *Propositional Lax Logic* (PLL) can be presented as an extension of intuitionistic propositional logic with a single unary operation $\Diamond$ satisfying the axioms: $\Diamond T : A \Rightarrow \Diamond A$, $\Diamond 4 : \Diamond\Diamond A \Rightarrow \Diamond A$ and $\Diamond F : (A \Rightarrow B) \Rightarrow (\Diamond A \Rightarrow \Diamond B)$. The axioms immediately suggest the type of categorical structure the logic is referring to: a cartesian closed category $\mathcal{H}$ with finite coproducts (to interpret the propositional connectives) equipped with a strong monad $(\Diamond, \mathbf{u}, \mathbf{m}, st)$ with functor $\Diamond : \mathcal{H} \to \mathcal{H}$, unit $\mathbf{u}_A : A \to \Diamond A$, multiplication $\mathbf{m}_A : \Diamond\Diamond A \to \Diamond A$ and strength $st_{A,B} : B^A \to (\Diamond B)^{\Diamond A}$.

The categorical models of the logic called *Constructive S4* (CS4) consist of a cartesian closed category $\mathcal{H}$ with finite coproducts and equipped with:

(1) a monad $(\Diamond, \mathbf{u}, \mathbf{m})$ with $\Diamond : \mathcal{H} \to \mathcal{H}$, unit $\mathbf{u}_A : A \to \Diamond A$ and multiplication $\mathbf{m}_A : \Diamond\Diamond A \to \Diamond A$;
(2) a comonad $(\Box, \mathbf{n}, \mathbf{w})$ with $\Box : \mathcal{H} \to \mathcal{H}$ preserving finite products (i.e. the canonical $\Box(A \times B) \to \Box A \times \Box B$ is an iso), counit $\mathbf{n}_A : \Box A \to A$ and comultiplication $\mathbf{w}_A : \Box A \to \Box\Box A$;
(3) a natural transformation $st_{A,B} : \Box A \times \Diamond B \to \Diamond(\Box A \times B)$ satisfying some coherence conditions.

Every categorical model of PLL becomes one for CS4 by adding the trivial comonad to it. On the other hand, if we assume that the counit $\mathbf{n}$ is an iso in a model of CS4, then the natural $st_{A,B} : \Box A \times \Diamond B \to \Diamond(\Box A \times B)$ can be transformed into an actual strength for $\Diamond$ and in this way a model for PLL is obtained.

The authors of [1] stress that the statements $\Diamond(A \vee B) \Rightarrow \Diamond A \vee \Diamond B$ and $\neg\Diamond\bot$ are not theorems of these logics and say that the rejection of these axioms (which are sometimes required in intuitionistic modal logics) is motivated by computer science applications. See also [21] for examples of different categories of spaces equipped with a product preserving comonad. In particular, Proposition 54 loc. cit. describes such a comonad $\Box$ on the category of Kelley spaces which, moreover, has a strong monad $\blacklozenge$ as left adjoint.

5.2 Hermida's Account of Kripke Semantics

In [23] the standard Kripke semantics of the symbols $\Box$ and $\Diamond$ is explained as the "extension of a predicate logic from *functions* to (abstract) *relations*". We give a simplified presentation. We assume familiarity with the interpretation of first order logic in Heyting categories. (See [33] for the fundamental ideas and the first volume of [25] for a recent presentation.) But we briefly discuss the terminology here.

Fix a regular category $\mathcal{C}$. Denote the poset of subobjects of X by $\mathbf{Sub}X$. Pulling back along a map $f : X \to Y$ induces a monotone $f^* : \mathbf{Sub}Y \to \mathbf{Sub}X$. Its left adjoint is denoted by $\exists_f : \mathbf{Sub}X \to \mathbf{Sub}Y$. A *relation from* X *to* Y is a subobject $R \to X \times Y$ in $\mathcal{C}$. The objects of $\mathcal{C}$ together with relations between them form a locally ordered 2-category $\mathbf{Rel}\mathcal{C}$ and the embedding $\mathcal{C} \to \mathbf{Rel}\mathcal{C}$ maps an object X to the object X in $\mathbf{Rel}\mathcal{C}$ and a map $f : X \to Y$ to its associated 'graph' $\langle id, f\rangle : X \to X \times Y$. (See Chapter A3 in [25] for details.)

The analysis of Kripke semantics done in [23] rests on a universal characterization of the embedding $\mathcal{C} \to \mathbf{Rel}\mathcal{C}$ given in Theorem 2.3 loc. cit. The result implies that for certain functors $F : \mathcal{C} \to \mathbf{Cat}$ there exists a canonical extension of F to a pseudo-functor $\mathbf{Rel}\mathcal{C} \to \mathbf{Cat}$. The functors F that extend in this way must satisfy, among other things, that for every map f in $\mathcal{C}$, Ff has a right adjoint (denoted by $(Ff)^*$). Then, on a relation given by $\langle d, c\rangle : R \to X \times Y$, the extension $\hat{F} : \mathbf{Rel}\mathcal{C} \to \mathbf{Cat}$ is defined by $\hat{F}R = (Fc)(Fd)^* : FX \to FY$. The rest of the conditions that F must satisfy in order to extend imply that $\hat{F}$ is a pseudo-functor.

For example, the functor $\mathcal{C} \to \mathbf{Cat}$ which maps every morphism $f : X \to Y$ in $\mathcal{C}$ to $\exists_f : \mathbf{Sub}X \to \mathbf{Sub}Y$ extends to a pseudo functor $\mathbf{Rel}\mathcal{C} \to \mathbf{Cat}$ such that every relation $\langle d, c\rangle : R \to X \times Y$ as on the left below

$$X \xleftarrow{d} R \xrightarrow{c} Y \qquad\qquad \mathbf{Sub}X \xrightarrow{d^*} \mathbf{Sub}R \xrightarrow{\exists_c} \mathbf{Sub}Y$$

is mapped to the functor $\mathbf{Sub}X \to \mathbf{Sub}Y$ as on the right above. Let us call this functor $P_R : \mathbf{Sub}X \to \mathbf{Sub}Y$.

In the particular case of $\mathcal{C} = \mathbf{Set}$ we write $\mathbf{Rel}$ instead of $\mathbf{Rel}(\mathbf{Set})$. We also write ϕ for an object in $\mathbf{Sub}X$ and write ϕx instead of $x \in \phi$. If $R \to X \times Y$ is a relation on sets then $P_R\phi \in \mathbf{Sub}Y$ has the following, easily recognizable, explicit definition: $P_R\phi = \{y \in Y \mid (\exists x \in X)(xRy \wedge \phi x)\}$. In other words: when R is a binary relation on a set X then P_R is the usual Kripke semantics for 'it was true at some Past time'. It is in this precise sense that Kripke semantics (for P in this case) is explained as the canonical extension of functional existential quantification $\exists_f$ to a relational setting.

Precomposing with the equivalence $\mathbf{Rel}^{\mathrm{op}} \to \mathbf{Rel}$ which maps a relation R to its converse R^*, the usual Kripke semantics for $\Diamond$ is obtained: if we define $\Diamond_R = P_{R^*}$ then $\Diamond_R\phi = \{x' \in X \mid (\exists x \in X)(x'Rx \wedge \phi x)\}$.

If $\mathcal{C}$ is a Heyting category then the pullback functors $f^* : \mathbf{Sub}Y \to \mathbf{Sub}X$ have right adjoints which we denote by $\forall_f : \mathbf{Sub}X \to \mathbf{Sub}Y$. In this setting, we easily have that, for a relation $\langle d, c\rangle : R \to X \times X$ in $\mathcal{C}$, $P_R = \exists_c d^* \dashv \forall_d c^*$. We can denote $\forall_d c^*$ by $\Box_R : \mathbf{Sub}X \to \mathbf{Sub}X$. In the case of $\mathcal{C} = \mathbf{Set}$ we obtain that $\Box_R\phi = \{x' \in X \mid (\forall x \in X)(x'Rx \Rightarrow \phi x)\}$. So, even though the setting is quite different from that of BAOs, it is still the case that the adjunction $P_R \dashv \Box_R$ is the fundamental relation between P_R and $\Box_R$.

6 Non-examples

Of course, not all modal operators will fit into an adjunction. For example, if $(B, \Diamond)$ is a BAO then $\Diamond$ need not have any adjoint. A different source of non-examples is given by non-functorial operations. These, trivially, cannot be part of an adjunction. A nice example is the boundary operator in a coHeyting algebra [31]. If A is a coHeyting algebra, the monotone $(\_) \vee x : A \to A$ has a left adjoint that we denote by $(\_)/x : A \to A$. Define $\sim x = \top/x$ and $\partial x = x \wedge (\sim x)$. The element ∂x is called the *boundary* of x. The resulting function $\partial : A \to A$ is not functorial (either covariantly or contravariantly) in general.

In many toposes, the Heyting algebra of subobjects of any object is also a coHeyting algebra. In particular, in presheaf toposes, ∂_S (as an operation on the poset $\mathbf{Sub}S$ of subobjects of S) has a 'Kripke-like' explicit description just like the operators $\Diamond$ and $\blacksquare$ discussed in Section 4.1. It is only lax-natural, so in this sense, is similar to the $\Diamond$s of that section. See [31] for more details.

A similar example appears in a more explicit modal-logic context if we consider the following minimal variant of the *modal logic of agency* discussed in [11] and [22]. Extend classical propositional logic with a unary symbol **Does** satisfying the following axioms:

(1) $\neg(\mathbf{Does}\top)$
(2) $(\mathbf{Does}\ p) \wedge (\mathbf{Does}\ q) \Rightarrow \mathbf{Does}(p \wedge q)$
(3) $\mathbf{Does}\ p \Rightarrow p$

together with the rule of Modus Ponens and the rule saying that from $p \Leftrightarrow q$ you can conclude $\mathbf{Does}\ p \Leftrightarrow \mathbf{Does}\ q$. The intended reading of **Does** p is that 'the agent brings it about that p'. (See Section 2.1 in [11].) Such a logic can be interpreted in a Boolean algebra B extended with a function $\mathbf{Does} : B \to B$ but is not possible to prove in the logic that **Does** is functorial (again, either covariantly or contravariantly).

Finally, we mention another way in which a modality may fail to accept an adjoint role. The situation we have in mind is that given by a logic with two monotone modalities, say M and N, and one attempts to force $M \dashv N$. This may easily collapse the logic or radically restrict the models as, for example, in Proposition 2.7. Also, this is the reason why the comparisons made in Section 9 of [10] among Ewald's IK_t, Dunn's positive logic and the logic IntGC described loc. cit. turn out as they do.

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