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A Note on the Model Theory for Positive Modal Logic

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Abstract. The minimum system of Positive Modal Logic $\mathcal{S}_{\mathbf{K}^+}$ is the $(\wedge, \vee, \Box, \Diamond, \bot, \top)$-fragment of the minimum normal modal logic $\mathbf{K}$ with local consequence. In this paper we develop some of the model theory for $\mathcal{S}_{\mathbf{K}^+}$ along the yet standard lines of the model theory for classical normal modal logic. We define the notion of positive bisimulation between two models, and we study the notions of m-saturated models and replete models. We investigate the positive maximal Hennessy-Milner classes. Finally, we present a Keisler-Shelah type theorem for positive bisimulations, a characterization of the first-order formulas invariant for positive bisimulations, and two definability theorems by positive modal sequents for classes of pointed models.

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1. Introduction

In [7] M. Dunn began the study of Positive Modal Logic, i.e. modal logic in the propositional language with only the logical symbols $\wedge, \vee, \Box, \Diamond, \bot, \top$, that is, the negation-free, or positive, propositional modal language. Since this language lacks a conditional, the systems of Positive Modal Logic, in contrast with the usual modal systems, are not defined as sets of theorems but as consequence relations, or, equivalently, as sets of sequents with certain closure conditions. This second way of defining them suggests that sequents are the syntactic objects that will express semantic constrains. The minimum system of Positive Modal Logic, called $\mathcal{S}_{\mathbf{K}^+}$ in the present paper, is the $(\wedge, \vee, \Box, \Diamond, \bot, \top)$-fragment of the minimum normal modal logic $\mathbf{K}$ with local consequence, that is, the relation between both is the following one: for any set of negation-free modal formulas Γ and any negation-free modal formula φ, φ is deducible from Γ in $\mathcal{S}_{\mathbf{K}^+}$ iff φ is a local consequence (with the usual Kripke semantics) of Γ. The minimum system of Positive Modal Logic is also the $(\wedge, \vee, \Box, \Diamond, \bot, \top)$-fragment of a suitable intuitionistic modal logic.

Extensions of $\mathcal{S}_{\mathbf{K}^+}$ can be obtained by adding new sequents. Moreover, sequents can be used to define clases of frames. For instance the class of frames where the sequent $\Box p \vdash \Box\Box p$ is valid is the class of transitive frames and so is the class of sequents where $\Diamond\Diamond p \vdash \Diamond p$ is valid. Therefore the notions of validity of a sequent in a frame and in a model are of importance for the development of Positive Modal Logic. It seems then natural to ask for sequents in the negation-free modal language the questions typical of the model theory of modal logic. We address some of this questions in the present paper. In it we develop some of the model theory for Positive Modal Logic along the yet standard lines of the model theory for classical normal modal logic, see [1], [9], [13], [14], introducing the basic tools needed to generalize the main results obtained for classical modal logic.

One of the main tools in the model theory of classical modal logic are bisimulations. In our way to define positive bisimulations (p-bisimulations for short) in order to obtain preservation and definability results for negation-free modal sequents we have come to the notion of directed simulation also introduced in [12] where N. Kurtonina and M. De Rijke also adress the problem of defining for the setting of the negation-free modal language a suitable analog of bisimulations and where related results to ours are obtained for negation-free modal formulas instead of sequents. A positive bisimulation between a model $\mathcal{M}_1$ and a model $\mathcal{M}_2$ is the intersection of a directed simulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ and the inverse of a directed simulation between $\mathcal{M}_2$ and $\mathcal{M}_1$, in the sense of [12]. The results for sequents are very natural in the context of Positive Modal Logic since, as we said, sequents are what correspond to conditions on frames and models, as witnessed by the example mentioned before. The standard translation of modal formulas into first-order formulas can be extended to sequents in a straightforward way, and, therefore, sequents can be used to express first-order properties of models. From this perspective we prove, among other things, a Keisler-Shelah type theorem that says that two pointed models satisfy the same negation-free modal sequents iff they have p-bisimilar ultrapowers.

The paper is divided into six sections. In the preliminaries section, the semantics for Positive Modal Logic is defined as well as the minimun deductive system. Section 3 is devoted to introduce the notion of positive bisimulation. In Section 4 the concepts of m-saturated model and of replete model are generalized to PML and the notion of positive Hennessy-Milner class is introduced. Moreover, positive maximal Hennessy-Milner classes are discused. Section 5 presents the already mentioned Keisler-Shelah type theorem for positive bisimulations, a characterization of the first-order formulas invariant for positive bisimulations, two definability theorems by positive modal sequents for classes of pointed models and a separation theorem. Finally in Section 6 we justify that all the results of the paper can be adapted to the new semantics for PML introduced in [5].

2. Preliminaries

The negation-free, or positive, modal language $\mathcal{L}_{\mathbf{K}^+}$ is defined by using a denumerable set of propositional variables $Var = \{p_0, p_1, \ldots, p_n, \ldots\}$, the binary connectives $\vee$ and $\wedge$, two propositional constants $\bot$ and $\top$, and two unary modal operators $\Box$ and $\Diamond$. The set of formulas as well as the formula algebra are denoted by Fm. We will refer indistinctly to the variables by $p, q, \ldots$ A *substitution* is any homomorphism from the formula algebra into itself. A *sequent* is a pair (Γ, φ), usually denoted by $\Gamma \vdash \varphi$, where Γ is a finite set of formulas and φ is a formula. A *substitution instance* of a sequent $\Gamma \vdash \varphi$ is any sequent $\sigma[\Gamma] \vdash \sigma(\varphi)$ obtained from $\Gamma \vdash \varphi$ by a substitution σ. A (finitary) *deductive system* is a set of sequents $\mathcal{S}$ that satisfies the following conditions:

1. If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi \in \mathcal{S}$.
2. If $\Gamma \vdash \varphi \in \mathcal{S}$ and for every $\psi \in \Gamma$, $\Delta \vdash \psi \in \mathcal{S}$, then $\Delta \vdash \varphi \in \mathcal{S}$.
3. If $\Gamma \vdash \varphi \in \mathcal{S}$, then any of its substitution instances belongs to $\mathcal{S}$.

From (1) and (2) it follows that:

(4) If $\Gamma \vdash \varphi \in \mathcal{S}$ then for any ψ, $\Gamma \cup \{\psi\} \vdash \varphi \in \mathcal{S}$.

We say that a formula φ is *deducible* in a deductive system $\mathcal{S}$ from a set of formulas Δ, in symbols $\Delta \vdash_{\mathcal{S}} \varphi$, if there is a finite set of formulas $\Gamma \subseteq \Delta$ such that the sequent $\Gamma \vdash \varphi$ belongs to $\mathcal{S}$. A deductive system $\mathcal{S}'$ is an *extension* of a deductive system $\mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{S}'$. Deductive systems are frequently axiomatized by Gentzen style calculi with all the structural rules. Any such Gentzen calculus defines a deductive system, the one whose elements are the derivable sequents.

In [7] and [5] the minimum system of Positive Modal Logic is introduced. It is the negation-free modal fragment of the minimum normal modal logic $\mathbf{K}$ with the local consequence relation. In this paper we call it $\mathcal{S}_{\mathbf{K}^+}$. In [5] that deductive system is axiomatized by means of the Gentzen calculus $\mathcal{G}_m$ with the following rules:

$$\frac{}{\varphi \vdash \varphi} \qquad \frac{}{\vdash \top} \qquad \frac{}{\Diamond\bot \vdash \bot}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma, \varphi, \psi \vdash \alpha}{\Gamma, \varphi \wedge \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \vdash \alpha \quad \Gamma, \psi \vdash \alpha}{\Gamma, \varphi \vee \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$[\Box\Diamond]\ \frac{\Gamma, \varphi \vdash \psi \vee \alpha}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\psi \vee \Diamond\alpha} \qquad [\Diamond\Box]\ \frac{\Gamma \vdash \varphi \vee \psi}{\Box\Gamma \vdash \Box\varphi \vee \Diamond\psi}$$

and some of its extensions are studied obtaining completeness theorems and some correspondence results for a frame semantics introduced there.

The pairs of sequents $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are denoted by $\varphi \dashv\vdash \psi$. The sequents

1. $\Box(\varphi \wedge \psi) \dashv\vdash \Box\varphi \wedge \Box\psi$,
2. $\Diamond(\varphi \vee \psi) \dashv\vdash \Diamond\varphi \vee \Diamond\psi$,
3. $\Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$,
4. $\Box\varphi \wedge \Diamond\psi \vdash \Diamond(\varphi \wedge \psi)$,
5. $\Box\top \dashv\vdash \top$
6. $\Diamond\bot \dashv\vdash \bot$

are derivable sequents of $\mathcal{G}_m$ and the rules

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi} \qquad \frac{\Gamma \vdash \varphi}{\Diamond\Gamma \vdash \Diamond\varphi}$$

are derived rules. We will call a deductive system that is an extension of $\mathcal{S}_{\mathbf{K}^+}$ *normal* if it is closed under the rules of the Gentzen calculus. We consider only normal deductive systems.
A *frame* is a relational structure $\mathcal{F} = \langle X, R\rangle$ where $X \neq \emptyset$ and R is a binary relation on X. Given a binary relation R on a set X, let for $x \in X$,

$$R(x) = \{y \in X \mid (x, y) \in R\}.$$

The power set of a set X will be denoted by $\mathcal{P}(X)$.

A *valuation* V on a frame $\mathcal{F} = \langle X, R\rangle$ is a function $V : Var \to \mathcal{P}(X)$. A valuation V can be extended recursively to the set of all formulas by means of the following clauses

1. $V(\bot) = \emptyset$, $V(\top) = X$
2. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$, $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$
3. $V(\Box\varphi) = \{x \in X : R(x) \subseteq V(\varphi)\}$,
4. $V(\Diamond\varphi) = \{x \in X : R(x) \cap V(\varphi) \neq \emptyset\}$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a frame and V is a valuation on it. For a set $\Gamma \subseteq Fm$, we write $V(\Gamma) = \bigcap_{\varphi\in\Gamma} V(\varphi)$, if Γ is non-empty, and $V(\Gamma) = X$, if $\Gamma = \emptyset$.

Let $\mathcal{M} = \langle \mathcal{F}, V\rangle$ be a model and $x \in X$. The notions of truth at a point, validity in a model and validity in a frame for formulas and sequents are defined as follows:

- φ is *true* at point x in a model $\langle \mathcal{F}, V\rangle$, in symbols $\langle \mathcal{F}, V\rangle \vDash_x \varphi$, if $x \in V(\varphi)$.
- $\Gamma \vdash \varphi$ is *true* at point x in a model $\langle \mathcal{F}, V\rangle$, in symbols $\langle \mathcal{F}, V\rangle \vDash_x \Gamma \vdash \varphi$, if $x \notin V(\Gamma)$ or $x \in V(\varphi)$.
- φ is *valid* in the model $\langle \mathcal{F}, V\rangle$, in symbols $\langle \mathcal{F}, V\rangle \vDash \varphi$, if $V(\varphi) = X$.
- $\Gamma \vdash \varphi$ is *valid* in the model $\langle \mathcal{F}, V\rangle$, in symbols $\langle \mathcal{F}, V\rangle \vDash \Gamma \vdash \varphi$, if $V(\Gamma) \subseteq V(\varphi)$.

The deductive system $\mathcal{S}_{\mathbf{K}^+}$ is defined as follows: for every sequent $\Gamma \vdash \varphi$, $\Gamma \vdash \varphi \in \mathcal{S}_{\mathbf{K}^+}$ iff $\Gamma \vdash \varphi$ is valid in every model. The proof of the fact that the Gentzen calculus $\mathcal{G}_m$ axiomatizes $\mathcal{S}_{\mathbf{K}^+}$ can be obtained as in [7].

Given a frame $\langle X, R\rangle$ we consider the operations $\Box_R$ and $\Diamond_R$ defined on subsets of X by

$$\Box_R(U) = \{x \in X \mid R(x) \subseteq U\},$$

$$\Diamond_R(U) = \{x \in X \mid R(x) \cap U \neq \emptyset\}.$$

For each subset D of $\mathcal{P}(X)$ we define the subsets $\Box_R^{-1}(D)$=$\{U \in \mathcal{P}(X) \mid \Box_R(U) \in D\}\}$, and $\Diamond_R^{-1}(D) = \{U \in \mathcal{P}(X) \mid \Diamond_R(U) \in D\}\}$. Given a model $\langle \mathcal{F}, V\rangle$ the set $D_V = \{V(\varphi) \mid \varphi \in Fm\}$ of subsets of X is closed under intersection, union and contains X and $\emptyset$; therefore it is a bounded distributive lattice of sets. Moreover, it is closed under the operations $\Box_R$ and $\Diamond_R$, because for every formula φ, $\Box_R(V(\varphi)) = V(\Box\varphi)$ and $\Diamond_R(V(\varphi)) = V(\Diamond\varphi)$.

For every model $\langle \mathcal{F}, V\rangle$ we define the function H from the set X into the set of prime filters $X(D_V)$ of the bounded distributive lattice D_V by

$$H(x) = \{V(\varphi) \mid x \in V(\varphi)\}.$$

Definition 2.1. We will say that a model $\langle \mathcal{F}, V\rangle$ is *surjective* if its function H is onto $X(D_V)$.

Given a model $\langle \mathcal{F}, V\rangle$ we define a new model with universe the set $X(D_V)$ and binary relation the relation R_V on $X(D_V)$ defined by

$$\langle P, Q\rangle \in R_V \quad \text{iff} \quad \Box_R^{-1}(P) \subseteq Q \subseteq \Diamond_R^{-1}(P),$$

for every $P, Q \in X(D_V)$. Notice that

$$\langle P, Q\rangle \in R_V \text{ iff for all } \varphi,\ (V(\Box\varphi) \in P \Rightarrow V(\varphi) \in Q) \text{ and } (V(\varphi) \in Q \Rightarrow V(\Diamond\varphi) \in P).$$

The valuation of the new model is defined by

$$V_{D_V}(p) = \{P \in X(D_V) \mid V(p) \in P\}.$$

The model $\langle X(D_V), R_V, V_{D_V}\rangle$ will be called *the valuation model* of $\langle X, R, V\rangle$. It is not difficult to prove the following lemma.

Lemma 2.1. Let $\mathcal{M}$ be a model. Then for any $\varphi \in Fm$,

$$V_{D_V}(\varphi) = \{P \in X(D_V) \mid V(\varphi) \in P\}.$$

Proof:
The proof is by induction on the complexity of φ. We consider only the case $\Box\varphi$. The case $\Diamond\varphi$ is similar and the other cases are easy.
Let $P \in X(D_V)$. If $V(\Box\varphi) \in P$ and $(P, Q) \in R_V$ then, as $\Box_R(V(\varphi)) \in P$, $V(\varphi) \in Q$. Therefore, by the inductive hypothesis, $Q \in V_{D_V}(\varphi)$. Hence $P \in V_{D_V}(\Box\varphi)$. For the other direction suppose that $V(\Box\varphi) \notin P$. The set $\Box_R^{-1}(P) = \{U \in D_V \mid \Box_R(U) \in P\}$ is a filter of D_V. Let us consider the ideal I generated by the set $\{V(\varphi)\} \cup \{V(\psi) \mid V(\psi) \notin \Diamond_R^{-1}(P)\}$. We prove that $\Box_R^{-1}(P) \cap I = \emptyset$. Suppose the contrary. Then there exists $U \in \Box_R^{-1}(P)$ such that $U \subseteq V(\varphi) \cup V(\psi)$, with $V(\psi) \notin \Diamond_R^{-1}(P)$. Then, $\Box_R(U) \subseteq V(\Box(\varphi \vee \psi))$. Since the sequent $\Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$ is valid in every model, $\Box_R(U) \subseteq V(\Box\varphi \vee \Diamond\psi)$. Therefore, since by assumption $V(\Box\varphi) \notin P$, $V(\Diamond\psi) = \Diamond_R V(\psi) \in P$, which is impossible. Now, by Birkhoff-Stone's theorem there is a prime filter $Q \in X(D_V)$ such that $\Box_R^{-1}(P) \subseteq Q \subseteq \Diamond_R^{-1}(P)$ and $V(\varphi) \notin Q$. Therefore, $(P, Q) \in R_V$. Hence, $P \notin V_{D_V}(\Box\varphi)$. $\Box$

3. Positive Bisimulations

In classical modal logic bisimulations provide a tool to establish an equivalence relation between pointed models. Modal formulas are invariant under bisimulations and two bisimilar pointed models are modally equivalent. Moreover, for finite pointed models (and some other classes of models) the relations of bisimilarity and modal equivalence are equal. In some respects, the bisimilarity relation plays for modal logic a similar role the partial isomorphism relation plays for first-order logic (with equality) and the notion of partial relativeness plays for equality-free logic ([4]). One theorem that sustains this claim is de Rijke's theorem in [13] that says that two pointed models are modally equivalent iff they have bisimilar ultrapowers.

In this section we introduce a notion of positive bisimulation for Kripke models with the help of directed simulations (defined in Definition 3.1). A positive bisimulation is the intersection of two directed simulations, an up-simulation and a down-simulation. Directed simulations have been introduced independently of us by N. Kurtonina and M. de Rijke in [12]. The directed simulations of [12] are our up-simulations and their inverses our down-simulations. We will obtain analogous results to the ones for classical modal logic but now sequents will play the role of formulas.

In order to establish some facts on the just promised notion we need some notations. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and let $x \in X$. We define the set

$$F_x^{\mathcal{M}} = \{\varphi \in Fm \mid x \in V(\varphi)\}.$$

We also say (abusing notation) that a sequent $\Gamma \vdash \varphi \in F_x^{\mathcal{M}}$ iff $x \notin V(\Gamma)$ or $x \in V(\varphi)$.

Consider two models $\mathcal{M}_1$ and $\mathcal{M}_2$ and points $x \in X_1$ and $y \in X_2$. We say that x and y are *modally equivalent for positive formulas*, p-modally equivalent for short, and in symbols $x \approx_p y$, if $F_x^{\mathcal{M}_1} = F_y^{\mathcal{M}_2}$. Notice that if x and y are p-modally equivalent, then for any sequent $\Gamma \vdash \varphi$, $\Gamma \vdash \varphi \in F_x^{\mathcal{M}_1}$ iff $\Gamma \vdash \varphi \in F_y^{\mathcal{M}_2}$. We will also say that two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are *p-modally equivalent* if the sets of sequents valid in them are the same.

Definition 3.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. A relation $B^{\rightarrow} \subseteq X_1 \times X_2$ is an *up-simulation* if the following conditions hold:

- **V1** If $(a,b) \in B^{\rightarrow}$, then $a \in V_1(p)$ implies that $b \in V_2(p)$, for every $p \in Var$.
- **F** If $(a,b) \in B^{\rightarrow}$ and $(a,x) \in R_1$, then there exists $y \in X_2$ such that $(b,y) \in R_2$ and $(x,y) \in B^{\rightarrow}$, i.e., $(B^{\rightarrow})^{-1} \circ R_1 \subseteq R_2 \circ (B^{\rightarrow})^{-1}$.
- **B** If $(a,b) \in B^{\rightarrow}$ and $(b,y) \in R_2$, there exists $x \in X_1$ such that $(a,x) \in R_1$ and $(x,y) \in B^{\rightarrow}$, i.e., $B^{\rightarrow} \circ R_2 \subseteq R_1 \circ B^{\rightarrow}$.

A *down-simulation* is a relation $B^{\leftarrow} \subseteq X_1 \times X_2$ such that conditions **F** and **B** above hold and also the condition:

- **V2** If $(a,b) \in B^{\leftarrow}$, then $b \in V_2(p)$ implies that $a \in V_1(p)$, for every $p \in Var$.

A *positive bisimulation*, p-bisimulation for short, is a relation $B \subseteq X_1 \times X_2$ for which there exits an up-simulation $B^{\rightarrow}$ and a down-simulation $B^{\leftarrow}$ such that $B = B^{\rightarrow} \cap B^{\leftarrow}$.

Notice that the inverse of an up-simulation is a down-simulation and conversely. The notion of positive bisimulation does not coincide with the usual notion of bisimulation between models, as defined for example in [2]. Clearly, any bisimulation (in the usual sense) between two models is a p-bisimulation, because it is at the same time an up-simulation and a down-simulation. However the converse does not hold as shown in Example 3.3 in [12].

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models and let $x \in X_1$ and $y \in X_2$. We say that x and y are *up-similar* (*down-similar*) if there exists an up-simulation (down-simulation) $B^{\rightarrow}$ $(B^{\leftarrow})$ between $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $(x, y) \in B^{\rightarrow}$ ($(x, y) \in B^{\leftarrow}$). We shall say that x and y are *p-bisimilar*, if there is a p-bisimulation B such that $(x, y) \in B$. In this case we write

$$\mathcal{M}_1, x \underline{\leftrightarrow}_p \mathcal{M}_2, y.$$

We will also say that two models $\mathcal{M}_1$ and $\mathcal{M}_2$ are *p-bisimilar* if there is a p-bisimulation B such that its domain is X_1 and its range is X_2. Such a p-bisimulation is said to be a *total p-bisimulation*. The p-bisimilarity relation is symmetric because the inverse of a p-bisimulation is a p-bisimulation. It is also easy to see that it is transitive.

Lemma 3.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models. If there is an up-simulation $B^{\rightarrow}$ between $\mathcal{M}_1$ and $\mathcal{M}_2$, then for all $(a, b) \in B^{\rightarrow}$, $F_a^{\mathcal{M}_1} \subseteq F_b^{\mathcal{M}_2}$. And if there is a down-simulation $B^{\leftarrow}$ between $\mathcal{M}_1$ and $\mathcal{M}_2$, then for all $(a, b) \in B^{\leftarrow}$, $F_b^{\mathcal{M}_2} \subseteq F_a^{\mathcal{M}_1}$.

Proof:
The proof is by induction on the complexity of the formulas. The non-modal cases are standard. We will only deal with the modal cases and $B^{\rightarrow}$. Let $\Box\varphi \in F_a^{\mathcal{M}_1}$ and assume that $(a, b) \in B^{\rightarrow}$ and that $(b, y) \in R_2$. As $B^{\rightarrow}$ is an up-simulation, by condition **B** there is an element $y \in X_2$ such that $(b, y) \in R_2$ and $(x, y) \in B^{\rightarrow}$. Since $x \in V_1(\varphi)$ and $(x, y) \in B^{\rightarrow}$, by the inductive hypothesis, $y \in V_2(\varphi)$. Thus, for all $y \in R_2(b)$, $y \in V_2(\varphi)$, i.e., $b \in V_2(\Box\varphi)$. The case of formulas of the form $\Diamond\varphi$ can be dealt with similarly, using condition **F**. $\Box$

Corollary 3.1. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models and let B be a p-bisimulation between them such that $(a, b) \in B$. Then a and b are p-modally equivalent and therefore, for every sequent $\Gamma \vdash \varphi$, $\Gamma \vdash \varphi \in F_a^{\mathcal{M}_1}$ iff $\Gamma \vdash \varphi \in F_b^{\mathcal{M}_2}$.

From this corollary the next fact follows immediately.

Corollary 3.2. Any two p-bisimilar models are p-modally equivalent.

4. Saturation

Our goal in this section is to generalize for the positive modal language the well known notion of m-saturated model ([11]), introduced by K.Fine in [8] under the label of modally saturated$_2$, its equivalent of image compact model ([3]), and the also well known notion of replete model ([9]). Each one of these notions can be generalized in (at least) two ways depending on the topologies we consider. Given a model we can consider the topology having as a subbasis the family of the sets of points that are values of negation-free formulas or are their complements, and demand that the sets of the form $R(x)$ have to be compact in this topology. We can also consider two topologies, the topology having as a subbasis the sets of points that are the values of negation-free formulas, and the topology having as a subbasis the complements of these sets, and demand that the sets of the form $R(x)$ are compact in both topologies. Accordingly, we introduce the notions of positive m-saturated model (and its equivalent of R-compact model), weakly R-compact model, positive replete model and weakly positive replete model.

We will also introduce in this section the positive Hennessy-Milner classes and show that each one of the classes of R-compact models, weakly R-compact models, positive replete models and weakly positive replete models are one of them. Moreover, we will study the maximal positive Hennessy-Milner classes and see that the class of weakly R-compact models is a maximal positive Hennessy-Milner class and that it can be obtained by closing each one of the classes of R-compact models, positive replete models or weakly positive replete models under the operations of taking positive generated submodels and taking p-bisimilar models.

4.1. Positive m-saturated models

Let $\mathcal{M}$ be a model and let $\Gamma \subseteq Fm$. Recall that $V(\Gamma) = \bigcap_{\varphi\in\Gamma} V(\varphi)$ when Γ is non-empty and $V(\Gamma) = X$ when $\Gamma = \emptyset$. We will write

$$V(\Gamma)^{ic} = \bigcap \{X \setminus V(\varphi) \mid \varphi \in \Gamma\},$$

when Γ is non-empty, and $V(\Gamma)^{ic} = X$, when $\Gamma = \emptyset$.

Definition 4.1. Let $\mathcal{M} = \langle X, R, V\rangle$ be a model. We say that it is a *positive m-saturated model*, a p-m-saturated model, for short, if for all $x \in X$ and for all sets $\Gamma, \Delta \subseteq Fm$ it holds that $R(x) \cap V(\Gamma) \cap V(\Delta)^{ic} \neq \emptyset$ whenever for all finite $\Gamma_0 \subseteq \Gamma$ and all finite $\Delta_0 \subseteq \Delta$, $R(x) \cap V(\Gamma_0) \cap V(\Delta_0)^{ic} \neq \emptyset$.

Clearly the notion of p-m-saturated model introduced in the above definition is a generalization of the known notion of m-saturated model for classical modal logic. Therefore any m-saturated model is a p-m-saturated model.

Analogoulsly as it can be done with the notion of m-saturated model, the notion of positive m-saturated model can be reformulated in topological terms. Let $\langle \mathcal{F}, V\rangle$ be a model. Let us consider the family of sets

$$\{V(\varphi) \mid \varphi \in Fm\} \cup \{X \setminus V(\varphi) \mid \varphi \in Fm\},$$

and the topology $\mathcal{T}$ on X generated by taking this set as a subbasis. By the Alexander's subbasis theorem, to say that a set $Y \subseteq X$ is compact is equivalent to saying that for any two sets Γ, Δ of formulas such that $Y \subseteq \bigcup_{\varphi\in\Gamma} V(\varphi) \cup \bigcup_{\varphi\in\Delta} (X \setminus V(\varphi))$ there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $Y \subseteq \bigcup_{\varphi\in\Gamma'} V(\varphi) \cup \bigcup_{\varphi\in\Delta'} (X \setminus V(\varphi))$.

Definition 4.2. We shall say that a model $\langle \mathcal{F}, V\rangle$ is *R-compact* if for all $x \in X$ the set $R(x)$ is compact in the topology $\mathcal{T}$, which is equivalent to saying that for all $x \in X$ and all sets of formulas Γ, Δ such that $R(x) \subseteq \bigcup_{\varphi\in\Gamma} V(\varphi) \cup \bigcup_{\varphi\in\Delta} (X \setminus V(\varphi))$ there are finite sets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $R(x) \subseteq \bigcup_{\varphi\in\Gamma'} V(\varphi) \cup \bigcup_{\varphi\in\Delta'} (X \setminus V(\varphi))$.

It is straighforward to show that this notion of R-compact model is equivalent to the notion of p-m-saturated model; we state this fact in the next theorem.

Theorem 4.1. For every model $\mathcal{M}$, $\mathcal{M}$ is p-m-saturated if and only if $\mathcal{M}$ is R-compact.

Examples of R-compact models are the image-finite models. Recall that a model $\langle \mathcal{F}, V \rangle$ is *image-finite* if for all $x \in X$, $R(x)$ is a finite set. Clearly, then, any open covering of $R(x)$ must have a finite subcovering and therefore the image-finite models are R-compact.

The canonical models of the normal deductive systems of PML are also R-compact, which can be proved using the next proposition. They are defined as follows. Let $\mathcal{S}$ be a normal deductive system of Positive Modal Logic (see [5] or [7]). A *theory of* $\mathcal{S}$ is a set of negation-free modal formulas T such that for any sequent $\Delta \vdash \varphi \in \mathcal{S}$ such that $\Delta \subseteq T$, $\varphi \in T$. A theory T of $\mathcal{S}$ is a *prime theory* if for every formulas φ, ψ, if $\varphi \vee \psi \in T$ then $\varphi \in T$ or $\psi \in T$. Let $X_{\mathcal{S}}$ be the set of all prime theories of $\mathcal{S}$. Define the binary relation $R_{\mathcal{S}}$ in $X_{\mathcal{S}}$ by

$$(T_1, T_2) \in R_{\mathcal{S}} \quad \text{iff} \quad \{\varphi \mid \Box\varphi \in T_1\} \subseteq T_2 \subseteq \{\varphi \mid \Diamond\varphi \in T_1\},$$

for all $T_1, T_2 \in X_{\mathcal{S}}$, and the valuation $V_{\mathcal{S}}$ by

$$V_{\mathcal{S}}(p) = \{T \in X_{\mathcal{S}} \mid p \in T\},$$

for all propositional variables p. Then it follows that for every formula φ,

$$V_{\mathcal{S}}(\varphi) = \{T \in X_{\mathcal{S}} \mid \varphi \in T\}.$$

See [7] or [5] for a proof.

Proposition 4.1. Let $\langle \mathcal{F}, V \rangle$ be a surjective model such that for all $x, y \in X$, if $(H(x), H(y)) \in R_V$, implies $(x, y) \in R$. Then $\langle \mathcal{F}, V \rangle$ is R-compact.

Proof:
Let Γ, Δ be sets of formulas. Assume that for every finite set $\Gamma_0 \subseteq \Gamma$ and every finite set $\Delta_0 \subseteq \Gamma$,

$$R(x) \cap \bigcap_{\varphi \in \Gamma_0} V(\varphi) \cap \bigcap_{\varphi \in \Delta_0} (X \setminus V(\varphi)) \neq \emptyset.$$

Consider the filter F of D_V generated by the set $\Box_R^{-1}(H(x)) \cup \{V(\varphi) : \varphi \in \Gamma\}$ and the ideal I of D_V generated by the set $\{V(\varphi) \mid V(\varphi) \notin \Diamond_R^{-1}(H(x))\} \cup \{V(\varphi) : \varphi \in \Delta\}$. Let us see that $F \cap I = \emptyset$. If we suppose the contrary there are $\varphi, \psi, \delta, \varepsilon$ such that $V(\varphi) \in \Box_R^{-1}(H(x))$, $\psi \in \Gamma$, $V(\delta) \notin \Diamond_R^{-1}(H(x))$, $\varepsilon \in \Delta$ and $V(\varphi) \cap V(\psi) \subseteq V(\delta) \cup V(\varepsilon)$. By assumption we have $R(x) \cap V(\psi) \cap (X \setminus V(\varepsilon)) \neq \emptyset$. So, there exists $z \in R(x)$ such that $z \in V(\psi)$ and $z \notin V(\varepsilon)$. Since $V(\Box\varphi) \in H(x)$, $x \in V(\Box\varphi)$; therefore $R(x) \subseteq V(\varphi)$. Hence $z \in V(\varphi) \cap V(\psi)$. And since $z \notin V(\varepsilon)$, $z \in V(\delta)$. Thus $R(x) \cap V(\delta) \neq \emptyset$. Hence $V(\Diamond\delta)) \in H(x)$, which is absurd. We conclude that $F \cap I = \emptyset$. Then by Birkhoff-Stone's Theorem there is $P \in X(D_V)$ such that $(H(x), P) \in R_V$, $\{V(\varphi) \mid \varphi \in \Gamma\} \subseteq P$ and $\{V(\varphi) \mid \varphi \in \Delta\} \cap P = \emptyset$. Since H is surjective, let $y \in X$ be such that $H(y) = P$. Then, as $(H(x), H(y)) \in R_V$, $(x, y) \in R$. So, $y \in R(x) \cap \bigcap_{\varphi \in \Gamma} V(\varphi) \cap \bigcap_{\varphi \in \Delta} (X \setminus V(\varphi))$; hence $R(x) \cap \bigcap_{\varphi \in \Gamma} V(\varphi) \cap \bigcap_{\varphi \in \Delta} (X \setminus V(\varphi) \neq \emptyset$, as desired. □

We can introduce a notion of positive descriptive model analogous to the one for classical modal logic. A model $\langle X, R, V\rangle$ is a *positive descriptive model* when the function H is an isomorphism between it and its valuation model $\langle X(D_V), R_V, V_{D_V}\rangle$. Therefore positive descriptive models are R-compact. Canonical models are a special kind of positive descriptive models.

The notion of image compact model can also be generalized in another way obtaining a weaker notion than the one of R-compactness by considering two topologies, one having as a subbasis the values of the formulas and the other one having as a subbasis their complements.

Definition 4.3. A model $\langle \mathcal{F}, V\rangle$ is *weakly R-compact* if the following two conditions hold:

1. For every $x \in X$ and every set of formulas Γ such that for every finite $\Gamma' \subseteq \Gamma$, $R(x) \cap \bigcap_{\varphi\in\Gamma'} V(\varphi) \neq \emptyset$, it also holds that $R(x) \cap \bigcap_{\varphi\in\Gamma} V(\varphi) \neq \emptyset$.
2. For every $x \in X$ and every set of formulas Γ such that for every finite $\Gamma' \subseteq \Gamma$, $R(x) \cap \bigcap_{\varphi\in\Gamma'} X \setminus V(\varphi) \neq \emptyset$, it also holds that $R(x) \cap \bigcap_{\varphi\in\Gamma} X \setminus V(\varphi) \neq \emptyset$.

Clearly, a model $\langle \mathcal{F}, V\rangle$ is weakly R-compact iff for every $x \in X$, $R(x)$ is a compact set in the topology generated by $\{V(\varphi) \mid \varphi \in Fm\}$ as a subbasis and is also a compact set in the topology generated by $\{X \setminus V(\varphi) \mid \varphi \in Fm\}$ as a subbasis. Therefore, every R-compact model is weakly R-compact.

An example of a weakly R-compact model that is not R-compact is the following:

Example 4.1. Let $p_0, p_1, p_2, \ldots, p_n, \ldots$ be an enumeration of the propositional variables and consider the model whose universe is the set $X := \omega \cup \{a\}$, where $a \notin \omega$, whose relation is $R = \{\langle a, n\rangle : n \in \omega\}$ and whose valuation is defined as follows:

$$V(p_n) = \begin{cases} \{n\} & \text{if } n \text{is even} \\ \omega \setminus \{n\} & \text{if } n \text{is odd} \end{cases}$$

The family $\mathcal{A}$ of finite unions of sets in the collection

$$\{\{2n\} \mid n \in \omega\} \cup \{\omega \setminus \{2n+1\} \mid n \in \omega\} \cup \{\{a\}, \omega, \emptyset\}$$

is closed under finite intersections and under the operations $\Diamond_R$ and $\Box_R$, because for $Z \subseteq X$,

$$\Box_R(Z) = \begin{cases} \omega & \text{if } \omega \not\subseteq Z \\ \omega \cup \{a\} & \text{if } \omega \subseteq Z \end{cases}$$

and

$$\Diamond_R(Z) = \begin{cases} \omega & \text{if } \omega \cap Z = \emptyset \\ \{a\} & \text{otherwise.} \end{cases}$$

Therefore, the set $\{V(\varphi) : \varphi \in Fm\}$ is a subset of this family. Each covering of the set $R(a) (= \omega)$ by elements of the family $\mathcal{A}$ has a finite subcovering, and each covering of $R(a)$ by complements of elements of the family $\mathcal{A}$ has a finite subcovering. Therefore the model is weakly R-compact. Moreover, the covering

$$\bigcup_{n\in\omega} V(p_{2n}) \cup \bigcup_{n\in\omega} X \setminus V(p_{2n+1})$$

is a covering of $R(a)$ without any finite subcovering. Hence the model is not R-compact.

4.2. Replete models

Now we introduce the weakly positive replete models and the positive replete models. The definitions are two generalizations of the definition of replete model give in [9].

Let $\langle \mathcal{F}, V\rangle$ be a model and let us consider the following property:

R For all $x, y \in X$ if $(H(x), H(y)) \in R_V$, then there are elements $z_1, z_2 \in X$ such that $z_1, z_2 \in R(x)$ and $H(z_1) \subseteq H(y) \subseteq H(z_2)$.

Definition 4.4. A model $\langle \mathcal{F}, V\rangle$ is a *weakly positive replete model* if it satisfies condition **R** and is surjective.

The first result it is worth establishing relates the property of being R-compact with the property of satisfying condition **R**.

Proposition 4.2. Any weakly R-compact model $\langle \mathcal{F}, V\rangle$ has property **R**.

Proof:
Assume that there are elements $x, y \in X$ such that $(H(x), H(y)) \in R_V$. This implies that $R(x) \neq \emptyset$. Assume also that for all $z \in R(x)$, $H(z) \nsubseteq H(y)$. Then, for each $z \in R(x)$ there is a formula φ_z such that $z \in V(\varphi_z)$ and $y \notin V(\varphi_z)$. Then $R(x) \subseteq \bigcup_{z\in R(x)} V(\varphi_z)$ and $y \notin \bigcup_{z\in R(x)} V(\varphi_z)$. Since the model is weakly R-compact, there is a finite sequence $z_1, z_2, ..., z_n$ of elements of $R(x)$ such that $R(x) \subseteq V(\varphi_{z_1}) \cup \ldots \cup V(\varphi_{z_n}) = V(\varphi_{z_1} \vee \ldots \vee \varphi_{z_n})$ and $y \notin V(\varphi_{z_1} \vee \ldots \vee \varphi_{z_n})$. Then $V(\Box(\varphi_{z_1} \vee \ldots \vee \varphi_{z_n})) \in H(x)$ and $V(\varphi_{z_1} \vee \ldots \vee \varphi_{z_n}) \notin H(y)$. Since $(H(x), H(y)) \in R_V$, $V(\varphi_{z_1} \vee \ldots \vee \varphi_{z_n}) \in H(y)$, which is not possible. □

Thus the models that satisfy the conditions in Proposition 4.1 are weakly positive replete models. A typical example of weakly positive replete models are the positive descriptive models.

The next result shows that the weakly positive replete models are just the weakly R-compact models that are surjective.

Theorem 4.2. Let $\langle \mathcal{F}, V\rangle$ be a model. Then the following conditions are equivalent:

1. $\langle \mathcal{F}, V\rangle$ is weakly R-compact and surjective.
2. $\langle \mathcal{F}, D\rangle$ is a weakly positive replete model.

Proof:
That (1) implies (2) follows by Proposition 4.2. We prove that (2) implies (1). Assume that $x \in X$ and let Γ be a set of formulas such that

$$R(x) \cap \{V(\varphi) \mid \varphi \in \Gamma'\} \neq \emptyset,$$

for every finite subset Γ' of Γ. Let us consider the filter F generated by the set $\Box_R^{-1}(H(x)) \cup \{V(\varphi) \mid \varphi \in \Gamma\}$ and prove that

$$F \cap \{V(\varphi) \mid V(\varphi) \notin \Diamond_R^{-1} H(x)\} = \emptyset \qquad (4.1)$$

If we assume the contrary, let φ, ψ, δ be such that $V(\varphi) \in \Box_R^{-1}(H(x))$, $\psi \in \Gamma$, $V(\delta) \notin \Diamond_R^{-1} H(x)$ and $V(\varphi) \cap V(\psi) \subseteq V(\delta)$. By hypothesis, $R(x) \cap V(\psi) \neq \emptyset$. So, let $z \in R(x)$ be such that $z \in V(\psi)$. Since $V(\varphi) \in \Box_R^{-1}(H(x))$, $z \in V(\varphi)$. Thus, $z \in V(\varphi) \cap V(\psi) \subseteq V(\delta)$. So, $z \in V(\delta) \cap R(x)$, but this is impossible because $V(\delta) \notin \Diamond_R^{-1} H(x)$. Therefore, as (4.1) holds, there is a prime filter $P \in X(D)$ such that $F \subseteq P$ and $P \subseteq \Diamond_R^{-1} H(x)$. This implies that $(H(x), P) \in R_V$ and $\{V(\varphi) \mid \varphi \in \Gamma\} \subseteq P$. Since H is surjective there is an element $y \in X$ such that $H(y) = P$. So, $(H(x), H(y)) \in R_V$ and $y \in V(\varphi)$ for all $\varphi \in \Gamma$. As $\langle \mathcal{F}, V \rangle$ is replete, there are $z_1, z_2 \in R(x)$ such that $H(z_1) \subseteq H(y) \subseteq H(z_2)$. But, as for all $\varphi \in \Gamma, y \in V(\varphi)$, then for all $\varphi \in \Gamma, z_2 \in V(\varphi)$, and since $z_2 \in R(x)$, then $z_2 \in R(x) \cap \bigcap_{\varphi \in \Gamma} V(\varphi)$; hence $R(x) \cap \bigcap_{\varphi \in \Gamma} V(\varphi) \neq \emptyset$.
The proof of the other condition in the definition of weakly R-compact model is handled similarly considering the ideal I generated by the set $\{V(\varphi) \mid V(\varphi) \notin \Diamond_R^{-1}(H(x))\} \cup \{V(\varphi) \mid \varphi \in \Gamma\}$ and proving that $\Box_R^{-1}(H(x)) \cap I = \emptyset$. □

In the light of this theorem there can be another generalization of the notion of replete model.

Definition 4.5. A model is a *positive replete model* if it is R-compact and surjective.

Clearly every positive replete model is a weakly positive replete model. The converse does not hold as follows from the fact that the model in Example 4.1 satifies condition **R**. Indeed, for every n, m, $\Box_R^{-1} H(n) \not\subseteq H(m)$ because $\Box_R^{-1} H(n) = \mathcal{A}$; therefore $(H(n), H(m)) \notin R_V$. Moreover, for every n, $H(a) \not\subseteq \Diamond_R^{-1} H(n)$ because $\Diamond_R^{-1} H(n) = \{\emptyset, \{a\}\}$; therefore $(H(a), H(n)) \notin R_V$. Finally $\Box_R^{-1} H(a) \not\subseteq H(a)$ because $\Box_R^{-1} H(a) = \{Z \in \mathcal{A} \mid \omega \subseteq Z\}$; therefore $(H(a), H(a)) \notin R_V$. Hence, if $(H(x), H(y)) \in R_V$ then $x = a$ and $y = m$ for some m, but in this case xRy.

An example of positive replete models are the positive descriptive models. Another interesting example are the ultrapowers of a model by non-principal ultrafilters over ω. Any such model is known to be m-saturated (because it is ω-saturated in the model-theoretic sense), therefore is p-m-saturated. It is easy to see that it is surjective. We state this facts in a proposition for further use.

Proposition 4.3. Let $\mathcal{M}$ be any model and let $\mathcal{U}$ be any non-principal ultrafilter over ω. The ultrapower $\mathcal{M}^\omega/\mathcal{U}$ is p-m-saturated and surjective, that is, replete.

The concepts of weakly R-compact model, of surjective model and of model having property **R**, and therefore the concept of weakly positive replete model, are preserved by total p-bisimulations. This shows that in a certain sense these concepts are the good generalizations of the classical concepts instead of the concepts of R-compact and positive replete that do not seem to be preserved under p-bisimulations. To find an example showing this is an open problem.

Proposition 4.4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models and let B be a total p-bisimulation between them. Then

1. $\mathcal{M}_1$ is weakly R-compact iff $\mathcal{M}_2$ is weakly R-compact.
2. $\mathcal{M}_1$ is surjective iff $\mathcal{M}_2$ is surjective.
3. $\mathcal{M}_1$ has property $\mathbf{R}$ iff $\mathcal{M}_2$ has property $\mathbf{R}$.

Proof:
We prove only the implications from right to left. The other implications follow form the first ones because the inverse of a p-bisimulation is a p-bisimulation. (The inverse of an up-simulation is a down-simulation and conversely.) First of all assume that $B = B^{\rightarrow} \cap B^{\leftarrow}$ is a total p-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$, where $B^{\rightarrow}$ is an up-simulation and $B^{\leftarrow}$ is a down-simulation.

(1) Assume $\mathcal{M}_1$ is weakly R-compact and let $b \in X_2$ and Γ be a set of negation-free formulas such that $R_2(b) \cap \bigcap_{\varphi \in \Gamma'} V_2(\varphi) \neq \emptyset$ for every finite $\Gamma' \subseteq \Gamma$. Since B is total, let $a \in X_1$ be such that $(a, b) \in B$. Suppose that $\Gamma' \subseteq \Gamma$ is finite. Take $y \in R_2(b) \cap \bigcap_{\varphi \in \Gamma'} V_2(\varphi)$. Then, as bR_2y and $(a, b) \in B^{\leftarrow}$, there is $x \in X_1$ such that aR_1x and $(x, y) \in B^{\leftarrow}$. Therefore, since $B^{\leftarrow}$ is a down-simulation, $x \in \bigcap_{\varphi \in \Gamma'} V_1(\varphi)$. We can conclude that $R_1(a) \cap \bigcap_{\varphi \in \Gamma'} V_1(\varphi) \neq \emptyset$ for every finite $\Gamma' \subseteq \Gamma$. Therefore, $R_1(a) \cap \bigcap_{\varphi \in \Gamma} V_1(\varphi) \neq \emptyset$. Let $x \in X_1$ be such that aR_1x and $x \in \bigcap_{\varphi \in \Gamma} V_1(\varphi)$. As $(a, b) \in B^{\rightarrow}$, there is $y \in X_2$ such that bR_2y and $(x, y) \in B^{\rightarrow}$. Then, $y \in \bigcap_{\varphi \in \Gamma} V_2(\varphi)$, because $B^{\rightarrow}$ is an up-simulation. Hence, $R_2(b) \cap \bigcap_{\varphi \in \Gamma} V_2(\varphi) \neq \emptyset$, as desired. The proof of the other condition of weak R-compactness for $\mathcal{M}_2$ is similar and left to the reader.

(2) Assume that $\mathcal{M}_1$ is surjective. Let P be a prime filter of $D_{V_2} = \{V_2(\varphi) \mid \varphi \in Fm\}$. Define $Q = \{V_1(\varphi) \mid V_2(\varphi) \in P\}$. We prove that Q is a prime filter of D_{V_1}. For this, we prove first that $V_2(\varphi) \subseteq V_2(\psi)$ implies $V_1(\varphi) \subseteq V_1(\psi)$. Using this fact it is easy to check that Q has the desired property. Assume that $V_2(\varphi) \subseteq V_2(\psi)$ and $a \in V_1(\varphi)$. Since B is total, let $b \in X_2$ be such that $(a, b) \in B$. Then, as B is a p-bisimulation, $b \in V_2(\varphi)$; hence $b \in V_2(\psi)$ and, therefore, $a \in V_1(\psi)$. Now, since $\mathcal{M}_1$ is surjective, let $a \in X_1$ such that $H_1(a) = Q$. Since B is total, let $b \in X_2$ such that $(a, b) \in B$. Then $V_2(\varphi) \in H_2(b)$ iff $b \in V_2(\varphi)$ iff $a \in V_1(\varphi)$ iff $V_1(\varphi) \in H_1(a)$ iff $V_1(\varphi) \in Q$ iff $V_2(\varphi) \in P$. Therefore, $H_2(b) = P$ and we can conclude that $\mathcal{M}_2$ is surjective.

(3) Assume that $\mathcal{M}_1$ verifies property $\mathbf{R}$. In order to prove that $\mathcal{M}_1$ verifies also this property, assume that $a_2, b_2 \in X_2$ are such that $(H_2(a_2), H_2(b_2)) \in R_{V_2}$. Since B is total, let $a_1, b_1 \in X_1$ be such that $(a_1, a_2), (b_1, b_2) \in B$. First we prove that $(H_1(a_1), H_1(b_1)) \in R_{V_1}$. Assume that $V_1(\varphi) \in \Box_{R_1}^{-1}(H_1(a_1))$. Then $V_1(\Box\varphi) \in H_1(a_1)$. Therefore, $a_1 \in V_1(\Box\varphi)$. Hence $a_2 \in V_2(\Box\varphi)$, which implies that $b_2 \in V_2(\varphi)$, because $(H_2(a_2), H_2(b_2)) \in R_{V_2}$. Then $b_1 \in V_1(\varphi)$ and $V_1(\varphi) \in H_1(b_1)$. Similarly we can prove that $H_1(b_1) \subseteq \Diamond_{R_1}^{-1}(H_1(a_1)$. So, $(H_1(a_1), H_1(b_1)) \in R_{V_1}$. Now we apply the assumption that $\mathcal{M}_1$ verifies property $\mathbf{R}$ to obtain $x_1, y_1 \in X_1$ such that $a_1R_1x_1$, $a_1R_1y_1$ and $H_1(x_1) \subseteq H_1(b_1) \subseteq H_1(y_1)$. Since $(a_1, a_2) \in B^{\leftarrow}$ and $a_1R_1x_1$ there is $x_2 \in X_2$ such that $a_2R_2x_2$ and $(x_1, x_2) \in B^{\leftarrow}$. Similarly, as $a_1R_1y_1$ there is $y_2 \in X_2$ such that $a_2R_2y_2$ and $(y_1, y_2) \in B^{\rightarrow}$. Now we prove that $H_2(x_2) \subseteq H_2(b_2) \subseteq H_2(y_2)$. If $x_2 \in V_2(\varphi)$ then $x_1 \in V_1(\varphi)$, so $b_1 \in V_1(\varphi)$ and hence $b_2 \in V_2(\varphi)$. Moreover, if $b_2 \in V_2(\varphi)$, $b_1 \in V_1(\varphi)$ and so $y_1 \in V_1(\varphi)$. Therefore $y_2 \in V_2(\varphi)$. We conclude that $\mathcal{M}_2$ verifies property $\mathbf{R}$. $\Box$

Positive Hennessy-Milner classes

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models and let $B \subseteq X_1 \times X_2$ be a p-bisimulation. We proved that if $(x, y) \in B$ then $F_x^{\mathcal{M}_1} = F_y^{\mathcal{M}_2}$, i.e., p-bisimilarity implies positive modal equivalence. The converse direction does not hold in general. However, if the models are weakly R-compact, p-bisimilarity does coincide with positive modal equivalence. In [10] and [11] Goldblatt and Hollenberg, respectively, investigate the Hennessy-Milner classes of models, i.e., classes for whose members the modal equivalence relation is a bisimulation. We will study the similar notion for positive modal logic.

Definition 4.6. Let M be a class of models. We say that M is a *Positive Hennessy-Milner class*, or PHM-class for short, if for every two models $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathsf{M}$ and for any two elements $x \in X_1$ and $y \in X_2$ such that $F_x^{\mathcal{M}_1} = F_y^{\mathcal{M}_2}$ it holds that $\mathcal{M}_1, x \underline{\leftrightarrow}_p \mathcal{M}_2, y$. This condition is equivalent to saying that the relation $\approx_p$ of positive modal equivalence is a p-bisimulation.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two models. The relations $\lesssim, \gtrsim \subseteq X_1 \times X_2$ defined by

$$(x, y) \in \lesssim \text{ iff } F_x^{\mathcal{M}_1} \subseteq F_y^{\mathcal{M}_2}$$

and

$$(x, y) \in \gtrsim \text{ iff } F_y^{\mathcal{M}_2} \subseteq F_x^{\mathcal{M}_1}$$

are not, in general, an up-simulation and a down-simulation, respectively. We will see that in the case of weakly R-compact models these relations $\lesssim$ and $\gtrsim$ are an up-simulation and down-simulation, respectively. This is a generalization of similar results for the classical modal logic case (see, for example, [10] or [11])

Proposition 4.5. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two weakly R-compact models. Then the relations $\lesssim$ and $\gtrsim$ are an up-simulation and a down-simulation, respectively.

Proof:
We shall prove that $\lesssim$ is an up-bisimulation. The proof of the fact that $\gtrsim$ is a down-simulation is similar and left to the reader. Condition **V1** is clear. To prove condition **F**, let $a, x \in X_1$ and $y \in X_2$ be such that $F_a^{\mathcal{M}_1} \subseteq F_b^{\mathcal{M}_2}$ and $(a, x) \in R_1$. We prove that there is an element $y \in X_2$ such that $(b, y) \in R_2$ and $F_x^{\mathcal{M}_1} \subseteq F_y^{\mathcal{M}_2}$. Suppose that for all $y \in R_2(b)$, $F_x^{\mathcal{M}_1} \nsubseteq F_y^{\mathcal{M}_2}$. Then for each $y \in R_2(b)$ let φ_y be a formula such that $\varphi_y \in F_x^{\mathcal{M}_1}$ and $\varphi_y \notin F_y^{\mathcal{M}_2}$. Then

$$R_2(b) \subseteq \bigcup \{V_2(\varphi_y)^c \mid y \in R_2(b)\}.$$

By weak R-compactness, there are $y_1, \ldots, y_n \in R_2(b)$ such that

$$R_2(b) \subseteq V_2(\varphi_{y_1})^c \cup \ldots \cup V_2(\varphi_{y_n})^c.$$

Therefore $b \notin V_2(\Diamond(\varphi_{y_1} \wedge \ldots \wedge \varphi_{y_n}))$. Hence, $a \notin V_1(\Diamond(\varphi_{y_1} \wedge \ldots \wedge \varphi_{y_n}))$. But $(a, x) \in R_1$ and $x \in V_1(\varphi_{y_1} \wedge \ldots \wedge \varphi_{y_n})$. So we have a contradiction.

The proof of condition **B** is done analogously by using $\Box(\varphi_{y_1} \wedge \ldots \wedge \varphi_{y_n})$, and is left to the reader.

$\Box$

As an immediate consequence of the proposition we obtain:

Theorem 4.3. The following classes are Positive Hennessy-Milner classes:

(i) The class of weakly R-compact models.

(ii) The class of R-compact models.

(iii) The class of image-finite models.

(iv) The class of positive replete models.

(v) The class of weakly positive replete models.

4.3. Maximal Hennessy-Milner classes

In the following we will proceed to prove that the class of weakly R-compact models is a maximal positive Hennessy-Milner class, that is, is a positive Hennessy-Milner class not properly included in any positive Hennessy-Milner class. The proof is analogous to the proof of the fact that the class of m-saturated models is Hennessy-Milner, as exposed in [11]. The concept that will play the role of generated submodels is the concept of positive generated submodel that we introduce in the following definition.

A model $\mathcal{M} = \langle X, R, V\rangle$ is a *positively generated submodel*, p-generated submodel for short, of a model $\mathcal{M}' = \langle X', R', V'\rangle$, in symbols $\mathcal{M} \subseteq_{pg} \mathcal{M}'$, if

1. $X \subseteq X'$
2. $R = R' \cap (X \times X)$
3. $V(p) = V'(p) \cap X$, for every propositional variable p
4. If $x \in X$ and $xR'y$, then there are $y_1, y_2 \in X$ such that xRy_1, xRy_2 and $F^{\mathcal{M}'}_{y_1} \subseteq F^{\mathcal{M}'}_{y} \subseteq F^{\mathcal{M}'}_{y_2}$.

The proof of the following lemma is straightforward.

Lemma 4.1. If $\mathcal{M} = \langle X, R, V\rangle$ is a p-generated submodel of $\mathcal{M}' = \langle X', R', V'\rangle$, then for every formula φ,

$$V(\varphi) = V'(\varphi) \cap X.$$

We define the following operations on classes of models. Let M be a class of models.

$$\begin{aligned}\mathbf{S}_p(\mathsf{M}) &= \{\mathcal{M} \mid \exists \mathcal{M}' \in \mathsf{M} \text{ such that } \mathcal{M} \subseteq_{pg} \mathcal{M}'\}\\ \mathbf{B}_p(\mathsf{M}) &= \{\mathcal{M} \mid \exists \mathcal{M}' \in \mathsf{M} \text{ such that } \mathcal{M} \text{ and } \mathcal{M}' \text{ are totally p-bisimilar}\}\end{aligned}$$

Let $\langle X_c, R_c, V_c\rangle$ be the canonical model of $\mathcal{S}_{\mathbf{K}^+}$ as defined in Section 4. This model is also called the Henkin model of $\mathcal{S}_{\mathbf{K}^+}$. A *positive Henkin-like* model is a structure $\mathcal{M} = \langle X_c, R^{\mathcal{M}}, V_c\rangle$ with universe and valuation the sames as in the canonical model of $\mathcal{S}_{\mathbf{K}^+}$ and accessibility relation a relation $R^{\mathcal{M}} \subseteq R_c$ such that for every formula φ and every $T \in X_c$

$$\mathcal{M} \models_T \varphi \quad \text{iff} \quad \varphi \in T.$$

The analogous of the next result for the classical case is due to A.Visser; see [11] for its proof. Our proof is just an adaptation to PML.

Lemma 4.2. If $\mathcal{M}$ is a positive Henkin-like model, then $\mathbf{B}_p\mathbf{S}_p(\mathcal{M})$ is a maximal positive Hennessy-Milner class.

Proof:
Let $\mathcal{M}$ be a Henkin-like model. In $\mathcal{M}$, $T \approx_p T'$ iff $T = T'$. Therefore, $\approx_p$ is a bisimulation in $\mathcal{M}$ and hence a p-bisimulation. Thus, $\mathcal{M}$ has the positive Hennessy-Milner property.

First of all we see that the class $\mathbf{B}_p\mathbf{S}_p(\mathcal{M})$ is a positive Hennessy-Milner class. In $\mathbf{S}_p(\mathcal{M})$ the relation $\approx_p$ between two models is the identity, because this holds in $\mathcal{M}$, and therefore a p-bisimulation. Hence $\mathbf{S}_p(\mathcal{M})$ is a positive Hennessy-Milner class. Moreover, it is easy to see that if M is a positive Hennessy-Milner class, $\mathbf{B}_p(\mathsf{M})$ is so. We conclude that $\mathbf{B}_p\mathbf{S}_p(\mathcal{M})$ is a positive Hennessy-Milner class as desired.

Let us see that $\mathbf{B}_p\mathbf{S}_p(\mathcal{M})$ is maximal among the positive Hennessy-Milner classes. Assume that M is a positive Hennessy-Milner class that includes $\mathbf{B}_p\mathbf{S}_p(\mathcal{M})$. Let $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$. Consider the model $\mathcal{N}' = \langle\{F_x^{\mathcal{N}} : x \in X\}, R', V'\rangle$ where R' is defined by

$$F_x^{\mathcal{N}} R' F_y^{\mathcal{N}} \quad \text{iff} \quad \Box^{-1}(F_x^{\mathcal{N}}) \subseteq F_y^{\mathcal{N}} \subseteq \Diamond^{-1}(F_x^{\mathcal{N}})$$

and $V'(p) = \{F_x^{\mathcal{N}} : p \in F_x^{\mathcal{N}}\}$ for each propositional variable p. Let us see that $\mathcal{N}' \in \mathbf{S}_p(\mathcal{M})$. Clearly $\{F_x^{\mathcal{N}} : x \in X\} \subseteq X_c$ since each $F_x^{\mathcal{N}}$ is a prime theory. Assume that $x \in X, T \in X_c$ and $(F_x^{\mathcal{N}}, T) \in R^{\mathcal{M}}$. Then x is p-bisimilar to $F_x^{\mathcal{N}}$ because $x \approx_p F_x^{\mathcal{N}}$ (in $\mathcal{N}$), M is a PHM class and $\mathcal{M}, \mathcal{N} \in \mathsf{M}$. Therefore, there are $y, z \in X$ such that xRy, xRz and $F_y^{\mathcal{N}} \subseteq T \subseteq F_z^{\mathcal{N}}$. Thus $\mathcal{N}'$ is a p-generated submodel of $\mathcal{M}$. Since $\mathcal{N} \in \mathsf{M}$, $\mathcal{N}' \in \mathbf{B}_p\mathbf{S}_p(\mathcal{M}) \subseteq \mathsf{M}$ and M is a PHM class, we obtain that the relation $x \approx_p F_x^{\mathcal{N}}$ is a p-bisimulation between $\mathcal{N}$ and $\mathcal{N}'$. Therefore, $\mathcal{N} \in \mathbf{B}_p\mathbf{S}_p(\mathcal{M})$. □

Let M be any class of models. We denote by $\mathcal{M}_{\mathsf{M}}$ the positive Henkin-like model whose accesibility relation R_{M} is defined by:

$(T, T') \in R_{\mathsf{M}}$ iff (1) $(T, T') \in R_c$ and $T \neq F_x^{\mathcal{N}}$ for all $\mathcal{N} \in \mathsf{M}$, or
(2) $T = F_x^{\mathcal{N}}$ for some $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$ and some $x \in X$,
and $\exists y, z \in X$ such that xRy, xRz and $F_y^{\mathcal{N}} \subseteq T' \subseteq F_z^{\mathcal{N}}$.

It is easy to check that this model is a positive Henkin-like model.

Lemma 4.3. Let M be a positive Hennessy-Milner class of models. Then $(T, T') \in R_{\mathsf{M}}$ implies that for all $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$ and every $x \in X$, if $T = \mathcal{F}_x^{\mathcal{N}}$, then there are $y, z \in R(x)$ such that $\mathcal{F}_y^{\mathcal{N}} \subseteq T' \subseteq \mathcal{F}_z^{\mathcal{N}}$.

Proof:
Assume that $(T, T') \in R_{\mathsf{M}}$. If there is no $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$ such that for some $x \in X$, $T = \mathcal{F}_x^{\mathcal{N}}$, then we are done. On the contray, by the definition we have a $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$, an $x \in X$ and $y, z \in R(x)$ such that $\mathcal{F}_y^{\mathcal{N}} \subseteq T' \subseteq \mathcal{F}_z^{\mathcal{N}}$. Consider any $\mathcal{N}' = \langle X', R', V'\rangle \in \mathsf{M}$ and any $x' \in X'$ such that $T = \mathcal{F}_{x'}^{\mathcal{N}'}$. Since M is a positive Hennessy-Milner class, let $B^{\rightarrow}$ and $B^{\leftarrow}$ and up and a down simulation such that $(x, x') \in B^{\rightarrow} \cap B^{\leftarrow}$. Then it is easy to find $y', z' \in R'(x')$ such that $\mathcal{F}_{y'}^{\mathcal{N}'} \subseteq T' \subseteq \mathcal{F}_{z'}^{\mathcal{N}'}$. □

Lemma 4.4. Let M be a positive Hennessy-Milner class whose elements are weakly R-compact models. Then $\mathcal{M}_{\mathsf{M}}$ is also weakly R-compact.

Proof:
We prove that the first condition of the definition of weakly R-compact model holds and leave the analogous proof of the other condition to the reader. Assume that Γ is a set of formulas such that for every finite subset Γ_0, $R_{\mathsf{M}}(T) \cap V_c(\Gamma_0) \neq \emptyset$, for a prime theory T. We reason by cases. If T is not of the form $\mathcal{F}_x^{\mathcal{N}}$ for some $\mathcal{N} \in \mathsf{M}$ and some x on the domain of $\mathcal{N}$, given a finite set $\Gamma_0 \subseteq \Gamma$ let $T' \in R_{\mathsf{M}}(T) \cap V_c(\Gamma_0)$. Then $T' \in R_c(T)$. Therefore, since the canonical model is weakly R-compact, there is $T'' \in R_c(T)$ such that $T'' \in V_c(\Gamma)$. As $T'' \in R_{\mathsf{M}}(T)$ we conclude that $R_{\mathsf{M}}(T) \cap V_c(\Gamma) \neq \emptyset$.

If $T = \mathcal{F}_x^{\mathcal{N}}$ for some $\mathcal{N} = \langle X, R, V\rangle \in \mathsf{M}$ and some $x \in X$, consider any finite set $\Gamma_0 \subseteq \Gamma$. As $R_{\mathsf{M}}(T) \cap V_c(\Gamma_0) \neq \emptyset$, let $T' \in R_{\mathsf{M}}(T) \cap V_c(\Gamma_0)$. By the previous lemma take $y, z \in R(x)$ be such that $\mathcal{F}_y^{\mathcal{N}} \subseteq T' \subseteq \mathcal{F}_z^{\mathcal{N}}$. Hence $z \in R(x) \cap V(\Gamma_0)$. We can conclude that for any finite set $\Gamma_0 \subseteq \Gamma$, $R(x) \cap V(\Gamma_0)$ is non-empty. Since $\mathcal{N}$ is weakly R-compact, $R(x) \cap V(\Gamma)$ is non-empty. Consider now $u \in R(x) \cap V(\Gamma)$. Then $(\mathcal{F}_x^{\mathcal{N}}, \mathcal{F}_u^{\mathcal{N}}) \in R_{\mathsf{M}}$ and $\mathcal{F}_u^{\mathcal{N}} \in V_c(\Gamma)$. Therefore $R_{\mathsf{M}}(T) \cap V_c(\Gamma) \neq \emptyset$. □

The next lemma and its proof are a positive version of an analogous lemma for arbitrary Hennessy-Milner classes due to A.Visser, see [11] for a proof. It is open whether the lemma holds for arbitrary positive Hennessy-Milner classes.

Lemma 4.5. Let M be a positive Hennessy-Milner class whose elements are weakly R-compact models. Then $\mathsf{M} \subseteq \mathbf{B}_p\mathbf{S}_p(\mathcal{M}_{\mathsf{M}})$.

Proof:
Let $\mathcal{M} = \langle X, R, V\rangle \in \mathsf{M}$ and let us consider the model

$$\mathcal{M}' = \langle \{F_x^{\mathcal{M}} \mid x \in X\}, R'_{\mathsf{M}}, V\rangle,$$

where

$$R'_{\mathsf{M}} = R_{\mathsf{M}} \cap \{F_x^{\mathcal{M}} \mid x \in X\}^2$$

and

$$V(p) = V_c(p) \cap \{F_x^{\mathcal{M}} \mid x \in X\},$$

for every propositional variable p. Then $\mathcal{M}'$ is a p-generated submodel of $\mathcal{M}_{\mathsf{M}}$, for if $F_x^{\mathcal{M}} R_{\mathsf{M}} T$ then there is $\mathcal{N} = \langle Y, S, V_1\rangle \in \mathsf{M}$ and $y, y_1, y_2 \in Y$ such that $F_x^{\mathcal{M}} = F_y^{\mathcal{N}}$, xSy_1, xSy_2 and $F_{y_1}^{\mathcal{N}} \subseteq T \subseteq F_{y_2}^{\mathcal{N}}$. Then, as M is Hennesy-Milner and $x \approx_p y$, x is p-bisimlar to y. So there are $z_1, z_2 \in X$ such that z_1 is p-bisimilar to y_1, z_2 is p-bisimilar to y_2 and xRz_1, xRz_2. It follows that $F_x^{\mathcal{M}} R'_{\mathsf{M}} F_{z_1}^{\mathcal{M}}$ and $F_x^{\mathcal{M}} R'_{\mathsf{M}} F_{z_2}^{\mathcal{M}}$. Moreover, $F_{z_1}^{\mathcal{M}} \subseteq T \subseteq F_{z_2}^{\mathcal{M}}$. This shows that $\mathcal{M}'$ is a p-generated submodel of $\mathcal{M}_{\mathsf{M}}$. By the previous lemma, $\mathcal{M}_{\mathsf{M}}$ is weakly R-compact. Moroever its p-generated submodels are also weakly R-compact as it is easy to show. Therefore, $\mathcal{M}'$ is weakly R-compact. By Proposition 4.5, the relation $\sim = \lesssim \cap \gtrsim$ is a p-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Moreover, it is total because for each $x \in X$, $x \approx_p F_x^{\mathcal{M}}$; this follows from the fact that $\mathcal{M}'$ is a p-generated submodel of a Henkin-like model. Hence, $\mathcal{M} \in \mathbf{B}_p\mathbf{S}_p(\mathcal{M}_{\mathsf{M}})$. □

Theorem 4.4. Let $\mathcal{M}_c$ be the canonical model of $\mathcal{S}_{\mathbf{K}^+}$. The class M_{wc} of all weakly R-compact models is precisely $\mathbf{B}_p\mathbf{S}_p(\mathcal{M}_c)$. Thus, it is a maximal positive Hennessy-Milner class.

Proof:
Since $\mathcal{M}_c \in \mathsf{M}_{wc}$, it is easy to check that $\mathcal{M}_{\mathsf{M}_{wc}} = \mathcal{M}_c$. The conditions of the above lemma apply to M_{wc}. Hence, $\mathsf{M}_{wc} \subseteq \mathbf{B}_p\mathbf{S}_p(\mathcal{M}_c)$. The other inclusion follows form the fact that $\mathcal{M}_c \in \mathsf{M}_{wc}$ and M_{wc} is closed under $\mathbf{B}_p$ and $\mathbf{S}_p$. □

Corollary 4.1. Let M_s, M_r and M_{wr} be, repectively, the classes of positively m-saturated models, replete models and weakly replete models. Then the three classes $\mathbf{B}_p\mathbf{S}_p(\mathsf{M}_s)$, $\mathbf{B}_p\mathbf{S}_p(\mathsf{M}_r)$, $\mathbf{B}_p\mathbf{S}_p(\mathsf{M}_w r)$ and the class M_{wc} of all weakly R-compact models are equal.

Proof:
The canonical model $\mathcal{M}_c$ belongs to the three classes. We show that $\mathbf{B}_p\mathbf{S}_p(\mathcal{M}_s) = \mathsf{M}_{wc}$. The proofs of the other equalities are analogous. Since $\mathcal{M}_c \in \mathsf{M}_s$, $\mathsf{M}_{wc} = \mathbf{B}_p\mathbf{S}_p(\mathcal{M}_c) \subseteq \mathbf{B}_p\mathbf{S}_p(\mathsf{M}_s) \subseteq \mathbf{B}_p\mathbf{S}_p(\mathsf{M}_{wc}) = \mathsf{M}_{wc}$. □

5. Positive bisimulations, positive modal equivalence and first-order translations

Let $\mathcal{M} = \langle X, T, V\rangle$ be a model. Let I be a set, and $\mathcal{U}$ be a ultrafilter over I. Let $\mathcal{M}^I/U$ be the ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$. For each element $a \in X$ let $f_a \in \mathcal{M}^I$ be the constant function such that $f_a(i) = a$, for all$i \in I$. Recall that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are models, $a \in X_1$ and $b \in X_2$, a and b are said to be p-modally equivalent if they satisfy the same positive modal formulas; in this case we write $a \approx_p b$, or $\mathcal{M}_1, a \approx_p \mathcal{M}_2, b$. We say that $\mathcal{M}_1, a$ and $\mathcal{M}_2, b$ have *p-bisimilar ultrapowers* if there is a set I and an ultrafilter $\mathcal{U}$ over I such that $\mathcal{M}_1^I/\mathcal{U}, f_a/\mathcal{U} \underline{\leftrightarrow}_p \mathcal{M}_2^I/\mathcal{U}, f_b/\mathcal{U}$.

Theorem 5.1. (p-Bisimulation Theorem)
Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models, and let $a \in X_1$ and $b \in X_2$. Then $a \approx_p b$ iff they have p-bisimilar ultrapowers.

Proof:
$\Rightarrow$) Assume that $\mathcal{M}_1, a \approx_p \mathcal{M}_2, b$. Let $\mathcal{U}$ be a non-principal ultrafilter over ω. Consider the ultrapowers $\mathcal{M}_1^\omega/\mathcal{U}$ and $\mathcal{M}_2^\omega/\mathcal{U}$, that, by Proposition 4.3, are p-m-saturated, and the objects $f_a/\mathcal{U}$ and $f_b/\mathcal{U}$. Then $\mathcal{M}_1, a \approx_p \mathcal{M}_1^\omega/\mathcal{U}, f_a/\mathcal{U}$ and $\mathcal{M}_2, b \approx_p \mathcal{M}_2^\omega/\mathcal{U}, f_b/\mathcal{U}$. Since the ultrapowers are p-m-saturated, the corresponding relations $\lesssim$ and $\gtrsim$, as defined after Definition 4.6, are an up-simulation and a down-simulation, respectively. Moreover, $(f_a/\mathcal{U}, f_b/\mathcal{U}) \in \lesssim \cap \gtrsim$. Therefore, $\mathcal{M}_1^\omega/\mathcal{U}, f_a/\mathcal{U} \underline{\leftrightarrow}_p \mathcal{M}_2^\omega/\mathcal{U}, f_b/\mathcal{U}$.

$\Leftarrow$) Assume that we have ultrapowers $\mathcal{M}_1^I/\mathcal{U}$ and $\mathcal{M}_2^I/\mathcal{U}$, for an ultrafilter $\mathcal{U}$ over I, such that $\mathcal{M}_1^I/\mathcal{U}$, $f_a/\mathcal{U} \underline{\leftrightarrow}_p \mathcal{M}_2^I/\mathcal{U}, f_b/\mathcal{U}$. Since $\mathcal{M}_1, a \approx_p \mathcal{M}_1^I/\mathcal{U}, f_a/\mathcal{U}$ and $\mathcal{M}_2, b \approx_p \mathcal{M}_2^I/\mathcal{U}, f_b/\mathcal{U}$ we conclude that $\mathcal{M}_1, a \approx_p \mathcal{M}_2, b$. □

Now we introduce the usual standard translation of modal formulas into first-order formulas, but restricted to the positive modal language; we will also translate sequents into first-order formulas. In order to do it we have to fix our first order-language. It has a denumerable set of predicate symbols, $P_0, P_1, \ldots$ and a binary relation symbol R.

The translation of modal formulas is defined as follows:

$$ST(p_i) = P_i x \qquad ST(\varphi \vee \psi) = ST(\varphi) \vee ST(\psi)$$

$$ST(\top) = x \approx x$$

$$ST(\Box\varphi) = \forall y(Rxy \to ST(\varphi)(y/x))$$

$$ST(\bot) = \neg x \approx x$$

$$ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\psi) \qquad ST(\Diamond\varphi) = \exists y(Rxy \wedge ST(\varphi)(y/x))$$

The translation of a sequent $\Gamma \vdash \varphi$ is defined by:

$$ST(\Gamma \vdash \varphi) = \neg ST(\bigwedge \Gamma) \vee ST(\varphi).$$

Notice that the translation of a formula and the translation of a sequent have just one free variable, the variable x.

Modal models correspond to first-order structures in the fixed first-order language. Given a model $\mathcal{M} = \langle X, R, V\rangle$, its associated first-order structure is denoted by $\mathfrak{A}(\mathcal{M})$ and given a first-order structure $\mathfrak{A}$, we denote by $\mathcal{M}(\mathfrak{A})$ its associated model.

For any positive modal formula φ, any modal sequent $\Gamma \vdash \varphi$, any (modal) model $\mathcal{M}$ and any $a \in X$:

1. $\mathcal{M} \models_a \varphi$ iff $\mathfrak{A}(\mathcal{M}) \models ST(\varphi)[a]$.
2. $\mathcal{M} \models_a \Gamma \vdash \varphi$ iff $\mathfrak{A}(\mathcal{M}) \models ST(\Gamma \vdash \varphi)[a]$.

From now on we will not distinguish typographically either a model $\mathcal{M}$ from its associated first-order structure $\mathfrak{A}(\mathcal{M})$, neither a first-order structure $\mathfrak{A}$ from its associated model $\mathcal{M}(\mathfrak{A})$.

We will characterize the first-order formulas, in at most one free variable, that are invariant for p-bisimulations. They are the formulas equivalent to conjunctions of translations of modal sequents in the negation-free modal language. A first-order formula $\alpha(x)$, in at most one free variable x, is *invariant for p-bisimulations* iff for any two models $\mathcal{M}_1$ and $\mathcal{M}_2$ and any two objects $a \in X_1$ and $b \in X_2$ such that $\mathcal{M}_1, a \underline{\leftrightarrow}_p \mathcal{M}_2, b$ it holds that $\mathcal{M}_1 \models \alpha[a]$ iff $\mathcal{M}_2 \models \alpha[b]$.

Theorem 5.2. A first-order formula $\alpha(x)$ is equivalent to a conjunction of translations of positive modal sequents iff it is invariant for p-bisimulations.

Proof:
The proof of the implication from left to right is easy. To prove the other implication assume that α is invariant for p-bisimulations and let us consider the set

$$T(\alpha) = \{ST(\Gamma \vdash \varphi) \mid \Gamma \vdash \varphi \text{ is a modal sequent and } \alpha \models ST(\Gamma \vdash \varphi)\}.$$

If $T(\alpha) \models \alpha$ then by the Compactness Theorem for first-order logic we will obtain that $\alpha(x)$ is equivalent to a conjunction of translations of modal sequents. So let us show that $T(\alpha) \models \alpha$. Assume that $\mathfrak{A} \models T(\alpha)[a]$. Let

$$T = \{ST(\varphi) \mid \mathfrak{A} \models ST(\varphi)[a]\} \cup \{\neg ST(\varphi) \mid \mathfrak{A} \models \neg ST(\varphi)[a]\}.$$

We will prove that $T \cup \{\alpha\}$ is satisfiable. Assume that it is not, then, by the Compactness theorem for first-order logic, there are finite subsets of T

$$T_0 = \{ST(\varphi_1), \ldots, ST(\varphi_n)\} \text{ and } T_1 = \{\neg ST(\psi_1), \ldots, \neg ST(\psi_m)\}$$

such that $T_0 \cup T_1 \cup \{\alpha\}$ is unsatisfiable. Then

$$\alpha \models \neg ST(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee ST(\psi_1 \vee \ldots \vee \psi_m).$$

Clearly $\beta = \neg ST(\varphi_1 \wedge \ldots \wedge \varphi_n) \vee ST(\psi_1 \vee \ldots \vee \psi_m)$ is the translation of the sequent $\{\varphi_1, \ldots, \varphi_n\} \vdash \psi_1 \vee \ldots \vee \psi_m$ and so belongs to $T(\alpha)$. Hence $\mathfrak{A} \models \beta[a]$. Therefore $\mathfrak{A} \not\models ST(\varphi_1 \wedge \ldots \wedge \varphi_n)[a]$ or $\mathfrak{A} \models ST(\psi_1 \vee \ldots \vee \psi_m)[a]$, which is absurd. We conclude that $T \cup \{\alpha\}$ is satisfiable. So, let $\mathfrak{B}$ be a structure and $b \in B$ such that $\mathfrak{B} \models T \cup \{\alpha\}[b]$. It follows that a and b are modally equivalent. Therefore by the p-Bisimulation Theorem there is a set I and an ultrafilter $\mathcal{U}$ over I and ultrapowers $\mathfrak{A}^I/\mathcal{U}$ and $\mathfrak{B}^I/\mathcal{U}$ such that $f_a/\mathcal{U}$ and $f_b/\mathcal{U}$ are p-bisimilar. Since $\mathfrak{B} \models \alpha[b]$, $\mathfrak{B}^I/\mathcal{U} \models \alpha[f_b/\mathcal{U}]$. Therefore, by the hypothesis, $\mathfrak{A}^I/\mathcal{U} \models \alpha[f_a/\mathcal{U}]$. Hence, $\mathfrak{A} \models \alpha[a]$, concluding the proof. □

In [12] the following characterization of the first-order formulas that are equivalent to the standard translation of a negation-free modal formula is given. It is interesting to compare it with the preceding theorem.

A first-order formula $\alpha(x)$, in at most one free variable x, is said to be *preserved by up-simulations* iff for any two models $\mathcal{M}_1$ and $\mathcal{M}_2$ and any two objects $a \in X_1$ and $b \in X_2$ such that there is an up-simulation $B^{\rightarrow}$ such that $(a, b) \in B^{\rightarrow}$ it holds that $\mathcal{M}_1 \models \alpha[a]$ implies $\mathcal{M}_2 \models \alpha[b]$.

Theorem 5.3. (Kurtonina, de Rijke)
A first-order formula $\alpha(x)$ is equivalent to the standard translation of a negation-free modal formula iff it is preserved by up-simulations.

Now we present some results on the modal definability of classes of pointed models. A *pointed model* is a structure of the form $(\mathcal{M}, w)$ where w belongs to the domain of $\mathcal{M}$. We say that a class M of pointed models is *definable by a set of negation-free modal sequents* if there is a set $\mathbf{\Gamma}$ of positive modal sequents such that $\mathsf{M} = \{(\mathcal{M}, w) \mid \mathcal{M} \models_w \Gamma \vdash \varphi \text{ for all } \Gamma \vdash \varphi \in \mathbf{\Gamma}\}$.

Theorem 5.4. Let M be a class of pointed models. Then M is definable by a set of negation-free modal sequents iff M is closed under p-bisimulations and ultraproducts and its complement is closed under ultrapowers.

Proof:
The implication from left to right is routine. To prove the other implication let

$$T = \{\Gamma \vdash \varphi \mid \text{ for all } (\mathcal{M}, w) \in \mathsf{M}, \mathcal{M} \models_w \Gamma \vdash \varphi\}.$$

We claim that T defines M. Clearly M is included in the class of pointed models defined by T, so assume that $(\mathcal{M}, w) \models T$ in order to see that $(\mathcal{M}, w) \in \mathsf{M}$. Consider the following two sets

$$\Sigma_1 = \{\varphi \mid \mathcal{M} \models_w \varphi\} \qquad \text{and} \qquad \Sigma_2 = \{\varphi \mid \mathcal{M} \not\models_w \varphi\}.$$

For each finite $\Delta \subseteq \Sigma_1 \times \Sigma_2$ let

$$\Delta_1 = \mathrm{dom}(\Delta) = \{\varphi \mid \exists\psi\ (\varphi, \psi) \in \Delta\} \text{ and } \Delta_2 = \mathrm{ran}(\Delta) = \{\varphi \mid \exists\psi\ (\psi, \varphi) \in \Delta\}.$$

Then there is $(\mathcal{N}, v) \in \mathsf{M}$ such that $(\mathcal{N}, v) \models \bigwedge \Delta_1$ and $(\mathcal{N}, v) \not\models \bigvee \Delta_2$, because on the contrary the sequent $\Delta_1 \vdash \bigvee \Delta_2$ would belong to T and this is not possible. Let $I = \mathcal{P}_f(\Sigma_1 \times \Sigma_2)$ the set of all finite subsets of $\Sigma_1 \times \Sigma_2$ and choose for each $i \in I$ a model $(\mathcal{N}_i, v_i) \in \mathsf{M}$ such that

$$(\mathcal{N}_i, v_i) \models \bigwedge i_1 \text{ and } (\mathcal{N}_i, v_i) \not\models \bigvee i_2.$$

For each $(\varphi, \psi) \in \Sigma_1 \times \Sigma_2$ consider the set $\widehat{(\varphi, \psi)} = \{i \in I \mid (\varphi, \psi) \in i\}$. Then the set $\{\widehat{(\varphi, \psi)} \mid (\varphi, \psi) \in \Sigma_1 \times \Sigma_2\}$ has the finite intersection property, so let $\mathcal{U}$ be an ultrafilter over I that includes that set. Consider the ultraproduct $\Pi_{i\in I}(\mathcal{N}_i, v_i)/\mathcal{U}$. It is the pointed model $(\Pi_{i\in I}\mathcal{N}_i/\mathcal{U}, f/\mathcal{U})$ where f is the function defined by $f(i) = v_i$, for each $i \in I$. Then $(\Pi_{i\in I}\mathcal{N}_i/\mathcal{U}, f/\mathcal{U}) \models \Sigma_1$ and for each $\psi \in \Sigma_2$, $(\Pi_{i\in I}\mathcal{N}_i/\mathcal{U}, f/\mathcal{U}) \not\models \psi$. So w and $f/\mathcal{U}$ are modally equivalent. Hence, by the P-Bisimulation Theorem there are p-bisimilar ultrapowers, say $((\Pi_{i\in I}(\mathcal{N}_i/\mathcal{U})^J/\mathcal{U}', g_{f/\mathcal{U}}/\mathcal{U}')$ and $(\mathcal{M}^J/\mathcal{U}', g_w/\mathcal{U}')$. Since M is closed under ultraproducts $((\Pi_{i\in I}(\mathcal{N}_i/\mathcal{U})^J/\mathcal{U}', g_{f/\mathcal{U}}/\mathcal{U}') \in \mathsf{M}$, and since it is closed under p-bisimulations $(\mathcal{M}^J/\mathcal{U}', g_w/\mathcal{U}') \in \mathsf{M}$. Therefore, as the complement of M is closed under ultrapowers, we obtain that $(\mathcal{M}, w) \in \mathsf{M}$, as desired. □

Theorem 5.5. Let M be a class of pointed models. Then M is definable by a finite set of negation-free modal sequents iff M is closed under p-bisimulations and ultraproducts and its complement is closed under ultraproducts.

Proof:
From Theorem 5.4we have that M is definable by a set of sequents, say $\mathbf{\Gamma}$, and since the complement of M is clearly closed under p-bisimulations, it also follows that it is definable by a set of sequents, say $\mathbf{\Delta}$. Then $\mathbf{\Gamma} \cup \mathbf{\Delta}$ is unsatisfiable on pointed models. By a standard ultraproduct argument one can show that $\mathbf{\Gamma} \cup \mathbf{\Delta}$ must have an unsatisfiable finite subset. Let $\mathbf{\Gamma}_0 \subseteq \mathbf{\Gamma}$ and $\mathbf{\Delta}_0 \subseteq \mathbf{\Delta}$ be finite and such that $\mathbf{\Gamma}_0 \cup \mathbf{\Delta}_0$ is unsatisfiable. If $(\mathcal{M}, w) \models \mathbf{\Gamma}_0$ then must belong to M. Therefore the finite set of sequents $\mathbf{\Gamma}_0$ defines M. □

In [12] the definability of pointed classes of models by negation-free modal formulas is studied. We state the results in order to be compared with the preceding ones.

Theorem 5.6. (Kurtonina, de Rijke)
Let M be a class of pointed models.

(i) M is definable by a set of negation-free modal formulas iff M is closed under up-simulations and ultraproducts and its complement is closed under ultrapowers.

(ii) M is definable by a negation-free modal formula iff M is closed under up-simulations and ultraproducts and its complement is closed under ultraproducts.

From the previous theorems we can obtain separation theorems in the standard way.

Theorem 5.7. Let K and L be two disjoint classes of pointed models.

(i) If K is closed under p-bisimulations and ultrapowers and L is closed under p-bisimulations and ultrapowers, then there exists a class of pointed models M that is definable by a set of negation-free modal sequents and such that $\mathsf{K} \subseteq \mathsf{M}$ and $\mathsf{L} \cap \mathsf{M} = \emptyset$.

(ii) If K is closed under p-bisimulations and ultrapowers and L is closed under p-bisimulations and ultraproducts, then there exists a class of pointed models M that is definable by a finite set of negation-free modal sequents and such that $\mathsf{K} \subseteq \mathsf{M}$ and $\mathsf{L} \cap \mathsf{M} = \emptyset$.

Proof:
(i) Let K and L be disjoint classes of pointed models both closed under p-bisimulations and the first one closed under ultraproducts and the second one closed under ultrapowers. Consider the classes

$$\mathsf{K}' = \{(\mathcal{M}, w) \mid \exists(\mathcal{N}, v) \in \mathsf{K}, \mathcal{M}, w \approx_p \mathcal{N}, v\}$$

$$\mathsf{L}' = \{(\mathcal{M}, w) \mid \exists(\mathcal{N}, v) \in \mathsf{L}, \mathcal{M}, w \approx_p \mathcal{N}, v\}.$$

Then, $\mathsf{K}' \cap \mathsf{L}' = \emptyset$. On the contrary there will be two pointed models $(\mathcal{N}_1, v_1) \in \mathsf{K}$ and $(\mathcal{N}_2, v_2) \in \mathsf{L}$ such that $\mathcal{N}_1, v_1 \approx_p \mathcal{N}_2, v_2$. Then by Theorem 5.1 these models have p-bisimilar ultrapowers and therefore $\mathsf{K} \cap \mathsf{L} \neq \emptyset$, which is against the assumption.

It is easy to see that the complement of K' is closed under p-bisimulations and ultrapowers. Therefore, by Theorem 5.4, K' is definable by a set of negation-free modal sequents. So K' is the desired class M.

To prove (ii) we argue as in (i), but now we obtain that both K' and L' are definable by a set of negation-free modal sequents. Let $Seq(\mathsf{K}')$ and $Seq(\mathsf{L}')$ be respectively the set of first-order translations of the sequents true at each model of K' and the set of first-order translations of the sequents true at each model of L'. Then the union of these two sets is unsatisfiable. By the Compactness Theorem for first-order logic there is a finite subset $\mathbf{\Delta}$ of this union which is unsatisfiable. Then the class defined by the negation-free modal sequents whose first-order translation belongs to $\mathbf{\Delta} \cap Seq(\mathsf{K}')$ defines the desired class M. □

From the theorem for standard modal logic analogous to the last separation theorem, a Craig interpolation theorem can be proved. If we adapt the argument to the present setting we obtain the next theorem where a sequent $\Gamma \vdash \varphi$ is said to be a consequence of a set of sequents $\mathbf{\Delta}$, in symbols $\mathbf{\Delta} \models \Gamma \vdash \varphi$, if for every pointed model $(\mathcal{M}, w)$ where all the sequents in $\mathbf{\Delta}$ are true at w it holds that $\Gamma \vdash \varphi$ is also true at w. The relation between this consequence relation between sequents and the consequence between sequents that can be defined in terms of validity is the same as the relation that exists at the level of formulas between the local consequence relation and the global one.

Before stating and proving the theorem we have to notice that the theorems of this section can be proved relativized to a negation-free modal language with set of variables any fixed subset of the set Var of propositional variables.

Theorem 5.8. If $\Gamma_1 \vdash \varphi_1 \models \Gamma_2 \vdash \varphi_2$ then there is a finite set of sequents $\mathbf{\Delta}$ with propositional variables among the set of common propositional variables in $\Gamma_1 \vdash \varphi_1$ and $\Gamma_2 \vdash \varphi_2$ such that for every $\Gamma \vdash \varphi \in \mathbf{\Delta}$, $\Gamma_1 \vdash \varphi_1 \models \Gamma \vdash \varphi$ and $\mathbf{\Delta} \models \Gamma_2 \vdash \varphi_2$.

Proof:
Consider the classes of pointed models

$$\mathsf{Mod}(\Gamma_1 \vdash \varphi_1) = \{(\mathcal{M}, w) \mid \Gamma_1 \vdash \varphi_1 \text{ is true at } w \text{ in } (\mathcal{M}, w)\}$$

and

$$\mathsf{Mod}(\Gamma_2 \vdash \varphi_2) = \{(\mathcal{M}, w) \mid \Gamma_2 \vdash \varphi_2 \text{ is false at } w \text{ in } (\mathcal{M}, w)\}.$$

Let $\mathcal{V}$ be the set of propositional variables common to $\Gamma_1 \vdash \varphi_1$ and $\Gamma_2 \vdash \varphi_2$. The class of the $\mathcal{V}$-reducts of the elements of $\mathsf{Mod}(\Gamma_1 \vdash \varphi_1)$ and the class of $\mathcal{V}$-reducts of the elements of $\mathsf{Mod}(\Gamma_2 \vdash \varphi_2)$ are closed under ultraproducts and p-bisimulations. Moreover they are disjoint. Arguing as before there is a class M of pointed-models of the appropriate type $\mathcal{V}$, definable by a finite set $\mathbf{\Delta}$ of sequents in the variables in $\mathcal{P}$. This set $\mathbf{\Delta}$ is the desired one. □

6. Final remarks

In [5] a different semantics for positive modal logic is introduced. In it frames are triples $\langle X, \leq, R\rangle$ where $\langle X, R\rangle$ is a Kripke frame and $\leq$ is a quasi-ordering of X, that is, a reflexive and transitive relation on W, such that $\leq \circ R \subseteq R \circ \leq$ and $\leq^{-1} \circ R \subseteq R \circ \leq^{-1}$. Models are pairs $\langle \mathcal{F}, V\rangle$ where $\mathcal{F}$ is a frame and V is a valuation that for each propositional variable p, $V(p)$ is an $\leq$-increasing set. This semantics is inherited from a suitable intuitionistic modal logic from which $\mathcal{S}_{\mathbf{K}^+}$ is the positive fragment. Reasons in favor of this semantics are given in [5] and [6]. The notions and results of this paper can be transferred to the new semantics almost without change by dealing with the order in the obvious way. The notions of positive m-saturated model, R-compact model, weakly R-compact model, positive replete model and weakly positive replete model, are defined exactly (the quasi-order has not to be considered in the definition). Given a model $\langle X, \leq, R, V\rangle$, the model of prime filters of D_V is defined as before and its quasi-ordering is inclusion. The descriptive models are defined also in the same way but in addition H must be an order isomorphism. When defining the ultrapowers the quasi-ordering has to be defined in the natural way. Finally, in the definiton of positively generated submodel, the quasi-order of the submodel $\mathcal{N}$ of $\mathcal{M}$ is the restriction of the quasi-ordering of $\mathcal{M}$ to the universe of $\mathcal{N}$. Condition 4 in the definition of positively generated submodel can be replaced by the following condition: if $x \in X$ and $xR'y$, then there are $y_1, y_2 \in X$ such that xRy_1, xRy_2 and $y_1 \leq y \leq y_2$. This condition is not equivalent to the original one but is more elegant and for it and the new models holds that a model $\mathcal{M}_1$ is a positive generated submodel of a model $\mathcal{M}_2$ iff $\leq_1$ is an up-simulation and its inverse a down-simulation. The results hold also for this new notion of positively generated submodel.

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References

[1] J. van Benthem, **Modal Logic and Classical Logic.** Bibliopolis, Napoles, 1985.

[2] J. van Benthem, J. Bergstra, *Logic of transition systems,* **Journal of Logic, Language and Information**, 3, (1994), pp. 247-283.

[3] M. M. Bosangue and M. Z. Kwiatkowska, *Re-interpreting the Modal μ-Calculus*, **Modal Logic and Process Algebra, A Bisimulation Perspective**. CSLI Lectures Notes No. 53. Center for the Study of Language and Information, Stanford University, Stanford, 1995.

[4] E. Casanovas, P. Dellunde, R. Jansana, *On Elementary Equivalence for Equality-free Logic*, **Notre Dame of Formal Logic**, vol. 37 (1996), pp. 506-522.

[5] S. CELANI AND R. JANSANA, *A New semantics for Positive Modal Logic*, **Notre Dame Journal of Formal Logic**, 38 (1997), pp. 1-18.

[6] S. CELANI AND R. JANSANA, *Priestley Duality, a Sahlqvist Theorem and a Goldblatt-Thomason Theorem for Positive Modal Logic*, Logic Journal of the IGPL, Volume7, Issue 6 (1999), pp. 683-715.

[7] M. DUNN, *Positive Modal Logic*, **Studia Logica**, 55 (1995), pp. 301-317.

[8] K. FINE, *Some connections between elementary and modal logic*, in **Proceedings of the Third Scandinavian Logic Symposium**, Upssala 1973. Amsretdam, North-Holland, 1975.

[9] R. GOLDBLATT, **Mathematics of Modality.** CSLI Lectures Notes No. 43. Center for the Study of Language and Information, Stanford University, 1993.

[10] R. GOLDBLATT, *Saturation and the Hennessy-Milner Property*, in **Modal Logic and Process Algebra, A Bisimulation Perspective**. CSLI Lectures Notes No. 53. Center for the Study of Language and Information, Stanford University, Stanford, 1995.

[11] M. HOLLENBERG, *Hennessy-Milner classes and process algebra*, in **Modal Logic and Process Algebra, A Bisimulation Perspective**. CSLI Lectures Notes No. 53. Center for the Study of Language and Information, Stanford University, Stanford, 1995.

[12] N. KURTONINA, M. DE RIJKE, *Simulating without Negation*, **Journal of Logic and Computation**, 7 (1997), pp. 501-522.

[13] M. DE RIJKE, **Extending Modal Logic**. Doctoral Dissertation. ILLC Dissertation Series 1993-4, University of Amsterdam, Amsterdam, 1993.

[14] M. DE RIJKE AND H. STURM, Global Definability in Basic Modal Logic. In: H. Wansing, editor, Essays on Non-classical Logic, pages 111-135, World Scientific Publishers, 2001.