q-Fourier Transform and its Inversion-Problem

A. Plastino & M. C. Rocca

Milan Journal of Mathematics
Issued by the Seminario Matematico e Fisico di Milano

ISSN 1424-9286
Volume 80
Number 1

Milan J. Math. (2012) 80:243-249
DOI 10.1007/s00032-012-0179-6

Springer

Milan J. Math. Vol. 80 (2012) 243–249
DOI 10.1007/s00032-012-0179-6
Published online April 21, 2012

Milan Journal of Mathematics

q-Fourier Transform and its Inversion-Problem

A. Plastino and M.C. Rocca

Abstract. Tsallis' q-Fourier transform is not generally one-to-one. It is shown here that, if we eliminate the requirement that q be fixed, and let it instead "float", a simple extension of the F_q-definition, this procedure restores the one-to-one character.

Mathematics Subject Classification (2010). Primary 60F05; Secondary 60E05, 60E10, 82Cxx.

Keywords. q-Fourier transform, generalization, one to one character, statistical mechanics, nonextensive statistical mechanics.

1. Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the BoltzmannGibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index q (with q = 1 recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation's applications [16], etc.

NEXT can be advantageously expressed via q-generalizations of standard mathematical concepts (the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [17, 22, 25]). The q-Fourier transform F_q exhibits the nice property of transforming q-Gaussians into q-Gaussians [17]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [18, 19, 20, 21].

A serious problem afflicts F_q. It is not generally one-to-one. A detailed example is discussed below. In this work we show that by recourse to a rather simple but efficient stratagem that consists in

- eliminating the requirement that q be fixed and instead

Birkhäuser

- let it "float",

one restores the one-to-one character.

2. Generalizing the q-Fourier transform

We define, following [17], a q-Fourier transform of $f(x) \in L^1(\mathbb{R})$, $f(x) \geq 0$ as

$$F(k,q) = [H(q-1) - H(q-2)] \times \int\limits_{-\infty}^{\infty} f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\, dx \tag{2.1}$$

where $H(x)$ is the Heaviside step function.

The only difference between this definition and that given in [17] *is that* q *is not fixed and varies within the interval* $[1, 2)$. Herein lies the hard-core of our presentation. This simple change of perspective makes it is easy to find the inversion-formula for (2.1) by recourse to the inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\lim_{\epsilon\to 0^+}\int\limits_{1}^{2} F(k,q)\delta(q-1-\epsilon)\, dq\right] e^{-ikx}\, dk. \tag{2.2}$$

As a consequence, we see that this q-Fourier transform is one-to-one, unlike what happens in [23],[24]. The link between Eqs. (2.1)- (2.2) is discussed in more detail in the illustrative example presented below (next Section).

3. Example

As an illustration we discuss the example given by Hilhorst in Ref. ([22]). Let $f(x)$ be

$$f(x) = \begin{cases} \left(\frac{\lambda}{x}\right)^{\beta}\ ;\ x \in [a, b]\ ;\ 0 < a < b\ ;\ \lambda > 0 \\ 0\ ;\ x \text{ outside } [\mathrm{a, b}]. \end{cases} \tag{3.1}$$

The corresponding q-Fourier transform is

$$F(k,q) = \lambda^{\beta}\int\limits_{a}^{b} x^{-\beta}\{1 + i(1-q)k\lambda^{\beta(q-1)}x^{1-\beta(q-1)}\}^{\frac{1}{1-q}}\, dx. \tag{3.2}$$

Effecting the change of variables

$$y = x^{1-\beta(q-1)},$$

we have for (3.2)

$$F(k,q) = [H(q-1) - H(q-2)] \times \frac{\lambda^{\beta}}{1-\beta(q-1)}\int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}}\{1 + i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy. \tag{3.3}$$

Now, (3.3) can be rewritten in the useful form

$$\begin{aligned}F(k,q) &= [H(q-1)-H(q-2)] \\ &\times \left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\right. \\ &\times \frac{\lambda^\beta}{1-\beta(q-1)} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy \\ &+ \left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\} \\ &\times \left.\frac{\lambda^\beta}{\beta(q-1)-1} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy\right\}. \end{aligned} \tag{3.4}$$

Taking into account that the involved integrals are defined in a finite interval, we can cast (3.4) as

$$\begin{aligned}&F(k,q) \\ &= [H(q-1)-H(q-2)] \times \left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\right. \\ &\times \frac{\lambda^\beta}{1-\beta(q-1)} \lim_{\epsilon\to 0^+} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy \\ &+ \left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\} \\ &\times \left.\frac{\lambda^\beta}{\beta(q-1)-1} \lim_{\epsilon\to 0^+} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy\right\}. \end{aligned} \tag{3.5}$$

We now use results of the Integral's table [26] to evaluate (3.5) and get

$$\begin{aligned}&\lim_{\epsilon\to 0^+} \int\limits_{a^{1-\beta(q-1)}}^{\infty} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\, dy \\ &= \frac{(q-1)[1-\beta(q-1)]a^{\frac{q-2}{q-1}}}{(2-q)[(1-q)i(k+i0)\lambda^\beta]^{\frac{1}{q-1}}} \\ &\quad \times F\left(\frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right. \\ &\quad \left. -\frac{1}{(1-q)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right), \end{aligned} \tag{3.6}$$

and

$$\lim_{\epsilon\to 0^+}\int\limits_0^{a^{1-\beta(q-1)}} y^{\frac{\beta(2-q)}{\beta(q-1)-1}}\{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy$$
$$=\frac{[\beta(q-1)-1]a^{1-\beta}}{\beta-1}$$
$$\times F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right), \quad (3.7)$$

where $F(a,b,c;z)$ is the hypergeometric function. Thus we obtain for $F(k,q)$

$$\begin{aligned}F(k,q)=[H(q-1)-H(q-2)]\times\Bigg\{\bigg\{&H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\bigg\}\\
&\times\frac{(q-1)\lambda^\beta}{(2-q)[(1-q)i(k+i0)\lambda^\beta]^{\frac{1}{q-1}}}\\
&\times\left\{a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\right.\\
&\left.\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right)\\
&-b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\\
&\left.\left.\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}}\right)\right\}\\
&+\left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\}\frac{\lambda^\beta}{\beta-1}\\
&\times\left\{a^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};\right.\right.\\
&\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right)\\
&-b^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};\right.\\
&\left.\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}\right)\right\}\Bigg\}. \quad (3.8)\end{aligned}$$

As we can see from (3.8), $F(k,q)$ *depends on* a *and* b, and, as consequence, is one-to-one, as shown in Section 2.

However, and *this is the crucial issue*, if we **fix** q and select $\beta=1/(q-1)$ (3.8) simplifies and adopts the appearance

$$F(k,q) = \lambda^{\frac{1}{q-1}}\frac{q-1}{2-q}\left[H(q-1)-H(q-2)\right]$$
$$\times\left[a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{q-1},\frac{2-q}{q-1};(q-1)i(k+i0)\lambda\right)\right.$$
$$\left.-\,b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{q-1},\frac{2-q}{q-1};(q-1)i(k+i0)\lambda\right)\right]. \qquad (3.9)$$

With the help of the result given in [27] for

$$F(-a,b,b,-z) = (1+z)^a,$$

we obtain for (3.9):

$$F(k,q) = \lambda^{\frac{1}{q-1}}\frac{q-1}{2-q}\left[H(q-1)-H(q-2)\right]\left(a^{\frac{q-2}{q-1}}-b^{\frac{q-2}{q-1}}\right)\left[1+(1-q)ik\lambda\right]^{\frac{1}{1-q}}. \qquad (3.10)$$

Using now the expression for λ of [22], i.e.,

$$\lambda = \left[\left(\frac{q-1}{2-q}\right)\left(a^{\frac{q-2}{q-1}}-b^{\frac{q-2}{q-1}}\right)\right]^{1-q},$$

we have, finally,

$$F(k,q) = \left[H(q-1)-H(q-2)\right]\left[1+(1-q)ik\lambda\right]^{\frac{1}{1-q}}, \qquad (3.11)$$

which is the result given by Hilhorst in [22], that is *independent* of the values adopted by a, b. Such independence is evidence that $F(k,q)$ is not one-to-one. All infinite $F(k,q,a,b)$ associated to each possible pair a, b coalesce now in a infinitely degenerate solution $F(k,q)$. As a conclusion we can say that for fixed q the q-Fourier transform is NOT one-to-one for fixed q. On the contrary, as we have shown in section 2, when q is NOT fixed, the q-Fourier transform is indeed one-to-one.

Conclusions

In the present communication we have discussed the NOT one-to-one nature of the q-Fourier transform F_q. We have shown that, if we eliminate the requirement that q be fixed and let it"float" instead, such simple extension of the F_q-definition restores the desired one-to-one character.

Acknowledments. The authors thank Prof. C. Tsallis for having called our attention to the present problem.

References

[1] C. Tsallis, J. Stat. Phys. 52 (1988) 479.

[2] M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy Interdisciplinary Applications, Oxford University Press, New York, 2004; C. Tsallis, Introduction to Nonextensive Statistical Mechanics Approaching a Complex World, Springer, New York, 2009.

[3] A. R. Plastino, A. Plastino, Phys. Lett A **177** (1993) 177.

[4] C. Tsallis, M. Gell-Mann, Y. Sato, Proc. Natl. Acad. Sci. USA 102 (2005) 15377; F. Caruso, C. Tsallis, Phys. Rev. E 78 (2008) 021102.

[5] P. Douglas, S. Bergamini, F. Renzoni, Phys. Rev. Lett. 96 (2006) 110601; G.B. Bagci, U. Tirnakli, Chaos 19 (2009) 033113.

[6] B. Liu, J. Goree, Phys. Rev. Lett. 100 (2008) 055003.

[7] R.G. DeVoe, Phys. Rev. Lett. 102 (2009) 063001.

[8] R.M. Pickup, R. Cywinski, C. Pappas, B. Farago, P. Fouquet, Phys. Rev. Lett. 102 (2009) 097202.

[9] L.F. Burlaga, N.F. Ness, Astrophys. J. 703 (2009) 311.

[10] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, A. Rapisarda, Phys. Rev. E 75 (2007) 055101(R); B. Bakar, U. Tirnakli, Phys. Rev. E 79 (2009) 040103(R); A. Celikoglu, U. Tirnakli, S.M.D. Queiros, Phys. Rev. E 82 (2010) 021124.

[11] V. Khachatryan, et al., CMS Collaboration, J. High Energy Phys. 1002 (2010) 041; V. Khachatryan, et al., CMS Collaboration, Phys. Rev. Lett. 105 (2010) 022002.

[12] Adare, et al., PHENIX Collaboration, Phys. Rev. D 83 (2011) 052004; M. Shao, L. Yi, Z.B. Tang, H.F. Chen, C. Li, Z.B. Xu, J. Phys. G 37 (8) (2010) 085104.

[13] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53; E.P. Borges, C. Tsallis, G.F.J. Ananos, P.M.C. de Oliveira, Phys. Rev. Lett. 89 (2002) 254103; G.F.J. Ananos, C. Tsallis, Phys. Rev. Lett. 93 (2004) 020601; U. Tirnakli, C. Beck, C. Tsallis, Phys. Rev. E 75 (2007) 040106(R); U. Tirnakli, C. Tsallis, C. Beck, Phys. Rev. E 79 (2009) 056209.

[14] L. Borland, Phys. Rev. Lett. 89 (2002) 098701.

[15] A. R. Plastino, A. Plastino, Phys. Lett A **174** (1993) 834.

[16] A. R. Plastino, A. Plastino, Physica A **222** (1995) 347.

[17] S. Umarov, C. Tsallis, S. Steinberg, Milan J. Math. 76 (2008) 307; S. Umarov, C. Tsallis, M. Gell-Mann, S. Steinberg, J. Math. Phys. 51 (2010) 033502.

[18] M. Jauregui, C. Tsallis, J. Math. Phys. 51 (2010) 063304.

[19] A. Chevreuil, A. Plastino, C. Vignat, J. Math. Phys. 51 (2010) 093502.

[20] M. Mamode, J. Math. Phys. 51 (2010) 123509.

[21] A. Plastino and M.C.Rocca: J. Math. Phys **52**, (2011) 103503.

[22] H.J.Hilhorst: J. Stat. Mech. (2010) P10023

[23] M.Jauregui and C.Tsallis: Phys. Lett. A **375**, (2011) 2085.

[24] M.Jauregui, C.Tsallis and E.M.F. Curado: J. Stat. Mech. P10016 (2011).

[25] M. Jauregui, C, Tsallis, Phys. Lett. A **375** (2011) 2085.

[26] L. S. Gradshtein and I. M. Ryzhik : *Table of Integrals, Series, and Products*. Fourth edition, Academic Press (1965) 3.194 1 and 3.194 2 pages 284 and 285.

[27] M.Abramowitz and I.A.Stegun: *Handbook of Mathematical Functions*. National Bureau of Standards. Applied Mathematical Series 55 Tenth Printing (1972), 15.1.8 page 556.

A. Plastino
Instituto de Física (IFLP-CCT-Conicet)
Universidad Nacional de La Plata (UNLP)
C.C. 67 (1900)
La Plata
Argentina
e-mail: plastino@fisica.unlp.edu.ar
angeloplastino@gmail.com

M.C. Rocca
Instituto de Física (IFLP-CCT-Conicet)
Universidad Nacional de La Plata (UNLP)
C.C. 67 (1900)
La Plata
Argentina
and
Departamento de Física
Fac. de Ciencias Exactas
Universidad Nacional de La Plata (UNLP)
La Plata
Argentina
e-mail: rocca@fisica.unlp.edu.ar
mariocarlosrocca@gmail.com

Received: February 10, 2012.