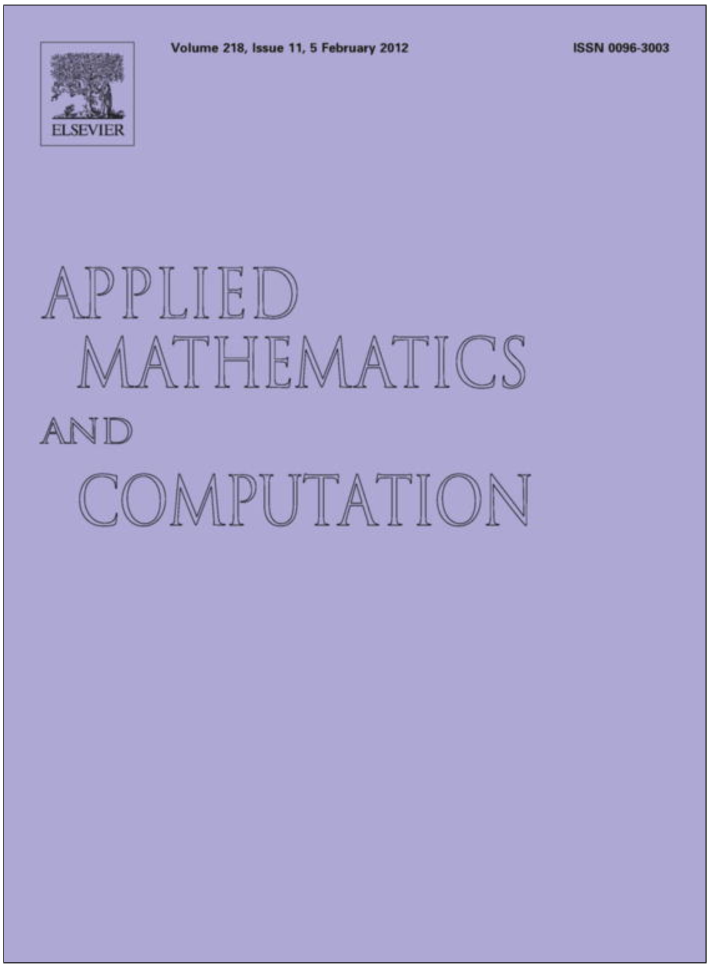
ELSEVIER
Volume 218, Issue 11, 5 February 2012
ISSN 0096-3003
APPLIED MATHEMATICS AND COMPUTATION

Applied Mathematics and Computation 218 (2012) 6351–6374

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical electroseismic modeling: A finite element approach

Juan E. Santos [a,c,d,*], Fabio I. Zyserman [b], Patricia M. Gauzellino [e]

[a] CONICET, Instituto del Gas y del Petróleo, Fac. Ingeniería, Universidad de Buenos Aires, Av. Las Heras 2214 Piso 3, C1127AAR Ciudad Autónoma de Buenos Aires, Argentina
[b] CONICET, Departamento de Geofsica Aplicada, Fac. de Cs. Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, B1900FWA La Plata, Argentina
[c] Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA
[d] Universidad Nacional de La Plata, Av. 7 Nro. 776, La Plata, Argentina
[e] Departamento de Geofísica Aplicada, Fac. de Cs. Astronómicas y Geofísicas, Universidad Nacional de La Plata, Paseo del Bosque s/n, B1900FWA La Plata, Argentina

ARTICLE INFO

Keywords:
Electroseismic modeling
Poroelasticity
Electromagnetics
Finite element methods

ABSTRACT

Electroseismics is a procedure that uses the conversion of electromagnetic to seismic waves in a fluid-saturated porous rock due to the electrokinetic phenomenon. This work presents a collection of continuous and discrete time finite element procedures for electroseismic modeling in poroelastic fluid-saturated media. The model involves the simultaneous solution of Biot's equations of motion and Maxwell's equations in a bounded domain, coupled via an electrokinetic coefficient, with appropriate initial conditions and employing absorbing boundary conditions at the artificial boundaries. The 3D case is formulated and analyzed in detail including results on the existence and uniqueness of the solution of the initial boundary value problem. *Apriori* error estimates for a continuous-time finite element procedure based on parallelepiped elements are derived, with Maxwell's equations discretized in space using the lowest order mixed finite element spaces of Nédélec, while for Biot's equations a nonconforming element for each component of the solid displacement vector and the vector part of the Raviart–Thomas–Nédélec of zero order for the fluid displacement vector are employed. A fully implicit discrete-time finite element method is also defined and its stability is demonstrated. The results are also extended to the case of tetrahedral elements. The 2D cases of compressional and vertically polarized shear waves coupled with the transverse magnetic polarization (PSVTM-mode) and horizontally polarized shear waves coupled with the transverse electric polarization (SHTE-mode) are also formulated and the corresponding finite element spaces are defined. The 1D SHTE initial boundary value problem is also formulated and approximately solved using a discrete-time finite element procedure, which was implemented to obtain the numerical examples presented.

1. Introduction

Electromagnetic waves induced by an artificial electromagnetic source and propagating in the Earth subsurface generate surface measurable seismic disturbances (electroseismic effect) [1,2].

In order to explain this phenomenom, Thompson and Gist [3] and Pride [4] suggested that they are generated by an electrokinetic coupling explained as follows [5,6]. Within a fluid saturated porous medium there exists a nanometer-scale

* Corresponding author at: Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA.
E-mail addresses: santos@math.purdue.edu (J.E. Santos), zyserman@fcaglp.unlp.edu.ar (F.I. Zyserman), gauze@fcaglp.unlp.edu.ar (P.M. Gauzellino).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.12.003

separation of electric charge in which a bound charge existing on the surface of the solid matrix (normally of negative sign) is balanced by adsorbed positive ions of the surrounding fluid, setting an immobile layer. Further from the surface there exists a distribution of mobile counter ions, forming the so called diffuse layer. The effective thickness of this double layer is of about 10 nm. When an electromagnetic wave propagates, the electric field acts on the charge excess of the double layers generating pressure gradients and consequently fluid flow and macroscopic disturbances. This is known as electro-osmosis and is responsible for the electroseismic phenomena.

On the other hand, the reciprocal situation arises when an applied pressure gradient creates fluid flow and hence, an ionic convection current, which in turn produces an electric field. This is known as electrofiltration and is responsible for the so-called seismoelectric phenomena.

Using a volume averaging approach, Pride [4] derived a set of equations describing both electroseismic and seismoelectric effects in electrolyte-saturated porous media. In these equations the coupling mechanism acts through the (generally frequency dependent) electrokinetic coupling coefficient $L(\omega)$. When this coefficient is set to zero, Pride's set of equations turns to the uncoupled Maxwell's and Biot's equations, describing the latter mechanical wave propagation in a fluid saturated porous medium [7,8].

Several works already exist implementing different numerical methods to solve the set of equations modeling both mentioned processes. Among others, Han and Wang [9] used a finite-element algorithm to model diffusive electric fields induced by SH waves, Garambois and Dietrich [10] introduced an extension to the generalized reflection and transmission matrix method to study seismoelectric conversions, Pain et al. [11] used a mixed finite element method to model electric fields induced by acoustic waves in borehole geometries, Haines and Pride [6] developed a finite-difference algorithm capable to model seismoelectric conversions in heterogeneous media; White [12] used seismic ray theory to determine the linear dependence between the magnitude of the electroseismic or seismoelectric responses and the electrokinetic coupling coefficient, and White and Zhou [13] used Ursin's formalism to model electroseismic conversions on homogeneous layered media within the frame of a unified treatment of electromagnetic, acoustic and elastic waves. More recently, analytic and numerical methods were applied to describe induced wave fields in homogeneous fluid-saturated porous media in the case of cylindrical symmetry, [14–16]. In the last reference, a time domain finite-difference (FDTD) with perfectly matched layers (PML) as boundary conditions was presented to model electroseismics as well as acoustic well logging. Santos [17] presented and analyzed a collection of global and domain-decomposed finite element methods, formulated in the space-frequency domain, to solve the fully coupled Maxwell's and Biot's equations as formulated by Pride [4].

The objective of this paper is to define and analyze a collection of continuous and discrete-time continuous-time finite element procedures for the approximate solution of the coupled Maxwell's and Biot's equations of motion in an isotropic bounded domain, with absorbing boundary conditions at the artificial boundaries. The results presented include existence and uniqueness of the solution of the continuous-time initial boundary value problem, *apriori* error estimates for the continuous-time finite element method and the stability analysis for a discrete-time finite element procedure.

In the 3D case, the electromagnetic fields are computed employing the lowest order mixed finite element space of Nédélec [19], Monk [22], while the nonconforming space defined in [24] was used to approximate each component of the displacement vector in the solid phase. The displacement in the fluid phase is approximated using the vector part of the Raviart–Thomas–Nédélec mixed finite element space of zero order [18,19]. In 2D, there are two possible cases, compressional and vertically polarized seismic waves coupled with the transverse magnetic polarization (PSVTM-mode) and horizontally polarized shear waves coupled with the transverse electric polarization (SHTE-mode). The 2D finite element procedures to discretize the PSVTM-mode employ the following spaces. In the case of rectangular elements the vector electric field and the scalar magnetic field are computed using the rotated Raviart–Thomas–Nédélec spaces of zero order [18,19], while for triangular elements the 2D mixed finite element space of Nédélec [19], Monk and Parrot [23] of lowest order are used . Also, both for rectangular and triangular elements, the nonconforming spaces defined in [24] are used to approximate each component of the displacement vector in the solid phase and the displacement in the fluid phase is approximated using the vector part of the Raviart–Thomas–Nédélec mixed finite element space of zero order. The 2D finite element spaces for the SHTE-mode are identical to those of the PSVTM-mode, except that in this mode the solid and fluid displacements are scalar functions in H^1 and L^2, respectively. Consequently, the solid displacement is approximated using the nonconforming spaces defined in [24] and the fluid displacement employing piecewise constants.

The organization of the paper is as follows. In Section 2 the differential system describing the propagation of coupled electromagnetic and seismic waves are stated, with corresponding initial conditions and absorbing boundary conditions at the artificial boundaries. Section 3 gives a variational formulation and a result on the existence and uniqueness of the solution the initial boundary value problem. In Section 4 the 3D finite element spaces based on parallelepipeds used for the spatial discretization are presented and their approximation properties are stated. Also, the continuous-time finite element procedure is formulated and results on the existence and uniqueness of the approximate solution are derived. In Section 5 *apriori* error estimates for the procedure are derived. In Section 6 a fully implicit discrete-time finite element procedure is presented and its unconditional stability is demonstrated. In Section 7 the results are extended to the case of tetrahedral elements. In Sections 8 and 9 the 2D possible cases for this coupled system, *i.e.*, the PSVTM and SHTE cases are defined, and the corresponding initial boundary value problems are formulated, as well as their weak formulations and the corresponding continuous-time finite element methods. Finally in Section 10 the initial boundary value problem for 1D SHTE modeling is defined, a weak formulation is given and a discrete-time finite element method is formulated, which was implemented to obtain the numerical results included in this section.

2. The differential model

Consider a 3D-rectangular domain $\Omega = \Omega_a \cup \Omega_p$ where Ω_a and Ω_p are associated with the air and subsurface poroelastic (disjoint) parts of Ω, respectively.

Let us denote by E, H the electric and magnetic fields in Ω, respectively, and by u^s, u^f the solid and relative fluid displacement vectors in Ω_p.

Following [4,6], for electroseismic modeling the electric and magnetic fields E and H and the displacement vectors u^s and u^f satisfy the coupled electromagnetic-poroelastic equations, stated in the space-time domain as follows:

$$\varepsilon \frac{\partial E}{\partial t} + \sigma E - \nabla \times H = J_e^s, \quad \Omega, \tag{1}$$

$$\nabla \times E + \mu \frac{\partial H}{\partial t} = 0, \quad \Omega, \tag{2}$$

$$\rho_b \frac{\partial^2 u^s}{\partial t^2} + \rho_f \frac{\partial^2 u^f}{\partial t^2} - \nabla \cdot \tau(u) = F^{(s)}, \quad \Omega_p, \tag{3}$$

$$\rho_f \frac{\partial^2 u^s}{\partial t^2} + m \frac{\partial^2 u^f}{\partial t^2} + \frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t} u^f - L_0 \frac{\eta}{\kappa_0} E + \nabla p_f = F^{(f)}, \quad \Omega_p, \tag{4}$$

$$\tau_{lm}(u) = 2G\varepsilon_{lm}(u^s) + \delta_{lm}\left(\lambda_c \nabla \cdot u^s + \alpha K_{av} \nabla \cdot u^f\right), \quad \Omega_p, \tag{5}$$

$$p_f(u) = -\alpha K_{av}\ \nabla \cdot u^s - K_{av} \nabla \cdot u^f, \quad \Omega_p. \tag{6}$$

In the equations above $u = (u^s, u^f)$ and $\tau_{lm}(u)$ is the stress tensor of the bulk material and $p_f(u)$ the fluid pressure, while $\varepsilon_{lm}(u^s)$ denotes the strain tensor of the solid frame. Also, J_e^s is an external applied current density, ε is the electric permitivity, μ the magnetic permeability and σ the electric conductivity.

The coefficients in the stress strain relations (5) and (6) can be determined as follows. The coefficient G is equal to the elastic shear modulus of the dry matrix. Also:

$$\lambda_c = K_c - \frac{2}{3}G, \tag{7}$$

with K_c being the bulk modulus of the saturated material. The coefficients in (5) and (6) can be obtained from the relations [30,31]:

$$\alpha = 1 - \frac{K_m}{K_s}, \quad K_{av} = \left[\frac{\alpha - \phi}{K_s} + \frac{\phi}{K_f}\right]^{-1}, \qquad K_c = K_m + \alpha^2 K_{av}, \tag{8}$$

where ϕ denotes the effective porosity and K_s, K_m and K_f denote the bulk modulus of the solid grains composing the solid matrix, the dry matrix and the saturant fluid, respectively. Furthermore:

$$\rho_b = \phi\rho_f + (1-\phi)\rho_s, \tag{9}$$

where ρ_s and ρ_f denote the mass densities of the solid grains composing the solid matrix and the saturant fluid. On the other hand, η is the fluid viscosity, κ_0 the permeability and m is the mass coupling coefficient between the solid and fluid phases in Ω_p. The coefficient m can be written in the form:

$$m = \frac{\alpha_\infty \rho_f}{\phi}, \tag{10}$$

with α_∞being the formation tortuosity.

The positive coupling coefficient L_0 is defined as [32] by

$$L_0 = -\frac{\phi}{\alpha_\infty}\frac{\epsilon_0 k_f \zeta}{\eta}\left(1 - 2\alpha_\infty \frac{\tilde{d}}{\Lambda}\right), \tag{11}$$

with $\zeta = 0.008 + 0.026\log_{10}(C_e)$ denoting the zeta potential and C_e being the electrolyte molarity. In (11) ϵ_0 and k_f are the vacuum and fluid permitivities and

$$\tilde{d} = \frac{\epsilon_0 k_f k_B T}{e^2 z^2 N_{ic}} \tag{12}$$

is the Debye length in meters. In (12) e is the electronic charge, k_B is the Boltzman constant, T is the absolute temperature (so that $k_B T$ is the thermal energy) z is the ionic valence and N_{ic} the ionic concentration in ions per meters cubed.

To solve Eqs. (1)–(6) in Ω on a time interval $J = (0, T)$ we need a collection of initial and boundary conditions. Let Γ denote the boundary of Ω and let $\Gamma_{a,p} = \overline{\Omega}_a \cap \overline{\Omega}_p$ denote the free surface. Also let $\Gamma_a = \partial\Omega_a \backslash \Gamma_{a,p}$, $\Gamma_p = \partial\Omega_p \backslash \Gamma_{a,p}$ denote the artificial boundaries of Ω_a and Ω_p, respectively; in Fig. 1 a scheme of the used domain and boundaries is shown. Also, if Γ_s is either an inner interface in Ω or a part of the boundaries Γ, Γ_p or $\Gamma_{a,p}$, set:

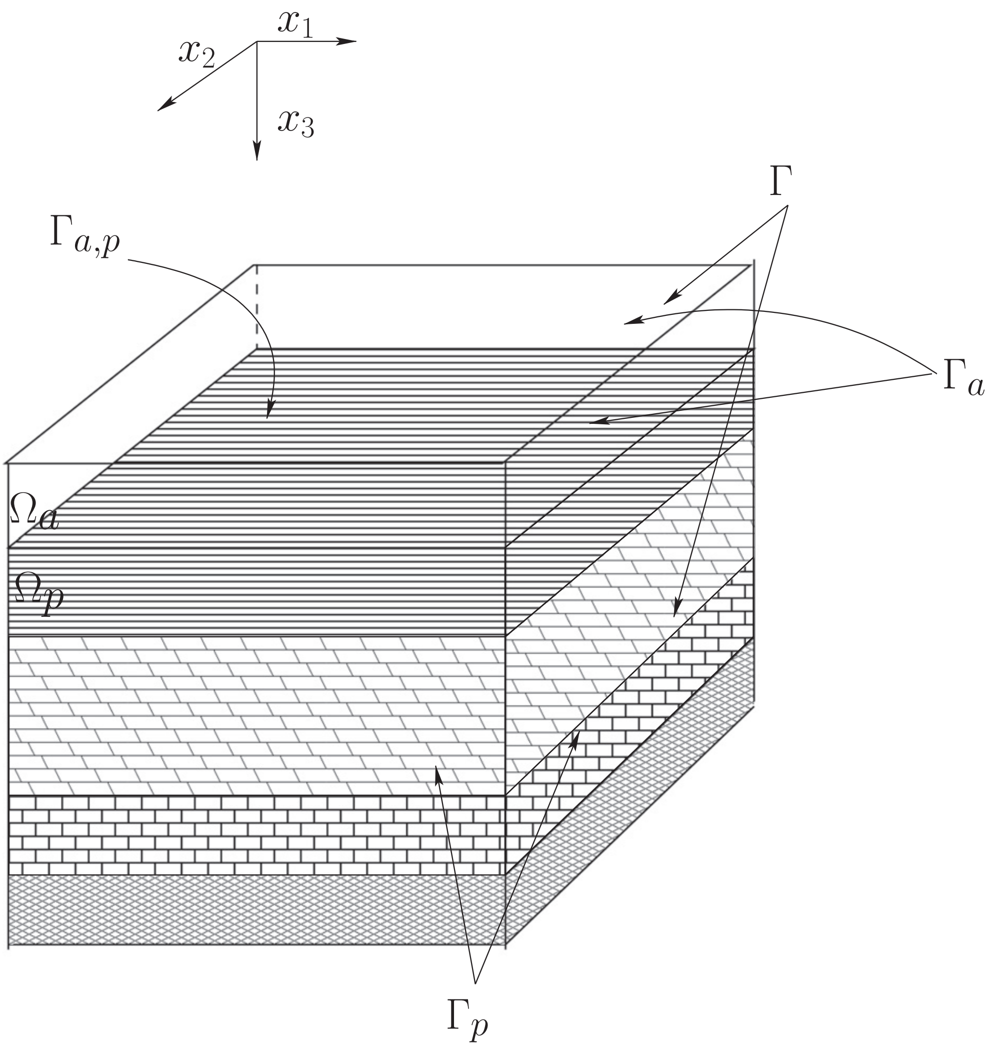

Fig. 1. Scheme of the domain and boundaries used in this work.

$$\mathcal{G}_{\Gamma_s}(u) = \left(\tau(u)\nu \cdot \nu, \tau(u)\nu \cdot \chi^1, \tau(u)\nu \cdot \chi^2, p_f(u)\right)^t, \tag{13a}$$

$$S_{\Gamma_s}(u) = \left(u^s \cdot \nu, u^s \cdot \chi^1, u^s \cdot \chi^2, u^f \cdot \nu\right)^t, \tag{13b}$$

where ^{t} denotes transposition, ν is the unit outer normal on Γ_s and χ^1, χ^2 are two unit tangents on Γ_s such that $\{\nu, \chi^1, \chi^2\}$ is an orthonormal set on Γ_p.

Consider the solution of (1)–(6) with the absorbing boundary conditions [37,34]:

$$\epsilon^{1/2} P_\chi E + \mu^{1/2} \nu \times H = 0, \quad \text{on } \Gamma, \tag{14}$$

$$-\mathcal{G}_{\Gamma_p}(u) = \mathcal{D} S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right), \quad \text{on } \Gamma_p, \tag{15}$$

the free surface condition:

$$-\mathcal{G}_{\Gamma_p}(u) = 0, \quad \text{on } \Gamma_{a,p}, \tag{16}$$

and the initial conditions:

$$\begin{aligned} &E(t=0) = E_0, \quad H(t=0) = H_0, \\ &u^s(t=0) = u^s_0, \quad u^f(t=0) = u^f_0, \\ &\frac{\partial u^s}{\partial t}(t=0) = u^s_1, \quad \frac{\partial u^f}{\partial t}(t=0) = u^f_1. \end{aligned} \tag{17}$$

Here:

$$P_\chi E = E - \nu(\nu \cdot E) = -\nu \times (\nu \times E)$$

is the 3-D orthogonal projection of the trace of E into the tangential plane perpendicular to the normal vector ν.

The matrix $\mathcal{D}$ in (15) is defined as: $\mathcal{D} = \mathcal{R}^{\frac{1}{2}}\mathcal{S}^{\frac{1}{2}}\mathcal{R}^{\frac{1}{2}}$, where $\mathcal{S} = \mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}}$ and

$$\mathcal{R} = \begin{pmatrix} \rho_b & 0 & 0 & \rho_f \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ \rho_f & 0 & 0 & m \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \lambda_c + 2G & 0 & 0 & \alpha K_{av} \\ 0 & G & 0 & 0 \\ 0 & 0 & G & 0 \\ \alpha K_{av} & 0 & 0 & K_{av} \end{pmatrix}, \tag{18}$$

where

$$b = \rho_b - \frac{(\rho_f)^2}{m}. \tag{19}$$

Remark. The boundary conditions (14) and (15) impose that seismic and electromagnetic waves arriving normally to the artificial boundaries be completely absorbed.

Remark. Note that since $\alpha_\infty \geqslant 1$, the matrix $\mathcal{R}$ is positive definite. Also, we will require that the following conditions be satisfied by the coefficients defining the matrix $\mathcal{M}$:

$$G > 0, \tag{20a}$$

$$\lambda_c + 2G - \alpha^2 K_{av}^2 > 0, \tag{20b}$$

$$K_{av} > 0. \tag{20c}$$

Conditions (20a)–(20c) are necessary and sufficient for the matrix $\mathcal{M}$ to be positive definite. In particular, the condition (20b) imposes that the inverse of the jacketed compressibility coefficient be strictly positive, see [33]. As a consequence of the positive definitess of the matrices $\mathcal{R}$ and $\mathcal{M}$, the matrix $\mathcal{D}$ is also positive definite.

3. A weak formulation

For $X \subset \mathbb{R}^d, d = 1, 2, 3$ with boundary ∂X, let $(\cdot,\cdot)_X$ denote the $L^2(X)$ inner product for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\|\cdot\|_{s,X}$ will denote the usual norm for the Sobolev space $H^s(X)$. In addition, if $X = \Omega$ or $X = \Gamma$, the subscript X may be omitted such that $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ or $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\Gamma$. Set:

$$H(\mathrm{curl}, \Omega) = \left\{\psi \in [L^2(\Omega)]^d : \ \nabla \times \psi \in L^2(\Omega)\right\},$$

$$H(\mathrm{div}, \Omega_p) = \left\{\psi \in [L^2(\Omega_p)]^d : \ \nabla \cdot \psi \in L^2(\Omega_p)\right\},$$

$$H^1(\mathrm{div}, \Omega_p) = \left\{\psi \in [H^1(\Omega_p)]^d : \ \nabla \cdot \psi \in H^1(\Omega_p)\right\},$$

($d = 3$ here) provided with the natural norms:

$$\|\psi\|_{H(\mathrm{curl},\Omega)} = \left(\|\psi\|_0^2 + \|\nabla \times \psi\|_0^2\right)^{\frac{1}{2}},$$

$$\|\psi\|_{H(\mathrm{div},\Omega_p)} = \left(\|\psi\|_{0,\Omega_p}^2 + \|\nabla \cdot \psi\|_{0,\Omega_p}^2\right)^{\frac{1}{2}},$$

$$\|\psi\|_{H^1(\mathrm{div},\Omega_p)} = \left(\|\psi\|_{1,\Omega_p}^2 + \|\nabla \cdot \psi\|_{1,\Omega_p}^2\right)^{\frac{1}{2}}.$$

Recall the integration by parts formulas [35,36]:

$$(\nabla \times U, V) - (U, \nabla \times V) = \langle \nu \times U, V\rangle = \langle \nu \times U, P_\chi V\rangle, \quad \forall\ U, V \in H(\mathrm{curl}, \Omega), \tag{21}$$

$$(\nabla \cdot \psi, \varphi) + (\psi, \nabla\varphi) = \langle \psi \cdot \nu, \varphi\rangle, \quad \forall\ \psi \in H(\mathrm{div}, \Omega), \quad \varphi \in H^1(\Omega). \tag{22}$$

Note that as indicated in [36] since $U, V \in H(\mathrm{curl}, \Omega)$, $\nu \times U$ and $P_\chi V$ belong to $H^{-1/2}(\partial\Omega)$, and the boundary integral in (21) is understood as $\langle \nu \times U \cdot P_\chi V, 1\rangle$ the duality pair between $\nu \times U \cdot P_\chi V \in [\mathrm{Lip}(\partial\Omega)]'$ and $1 \in \mathrm{Lip}(\partial\Omega)$. Here $[\mathrm{Lip}(\partial\Omega)]'$ is the dual space of Lipschitz-continuous functions on $\partial\Omega$.

To obtain a variational formulation, multiply (1) by $\psi \in H(\mathrm{curl}, \Omega)$, integrate in Ω and use (21) and (14). Also multiply (2) by $\varphi \in [L^2(\Omega)]^3$ and integrate in Ω. Finally, multiply (3) by $v^s \in [H^1(\Omega)]^3$ and (4) by $v^f \in H(\mathrm{div}, \Omega)$, add the resulting equations, integrate in Ω_p and use (22) and (15). We obtain the weak formulation: for $t \in J$, find $(E, H, u^s, u^f)(t) \in H(\mathrm{curl}, \Omega) \times [L^2(\Omega)]^3 \times [H^1(\Omega)]^3 \times H(\mathrm{div}, \Omega)$ such that:

$$\left(\varepsilon \frac{\partial E}{\partial t}, \psi\right) + (\sigma E, \psi) - (H, \nabla \times \psi) + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi E, P_\chi \psi \right\rangle = (J_e^s, \psi), \quad \psi \in H(\mathrm{curl}, \Omega), \tag{23}$$

$$(\nabla \times E, \varphi) + \left(\mu \frac{\partial H}{\partial t}, \varphi\right) = 0, \quad \varphi \in [L^2(\Omega)]^3, \tag{24}$$

$$\left(\mathcal{P}\frac{\partial^2 u}{\partial t^2}, v\right)_{\Omega_p} + \left(\frac{\eta}{\kappa_0}\frac{\partial u^f}{\partial t}, v^f\right)_{\Omega_p} + \mathcal{A}(u,v) - \left(L_0\frac{\eta}{\kappa_0}E, v^f\right)_{\Omega_p} + \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right), S_{\Gamma_p}(v)\right\rangle_{\Gamma_p} = 0, \quad v = (v^s, v^f) \in [H^1(\Omega_p)]^3 \times H(\mathrm{div}, \Omega_p). \tag{25}$$

In (25):

$$\mathcal{P} = \begin{pmatrix} \rho_b I_d & \rho_f I_d \\ \rho_f I_d & m I_d \end{pmatrix}, \tag{26}$$

where I_d is the identity matrix in $\mathbf{R}^{d\times d}$ and $A(u,v)$ is the bilinear form defined as:

$$\mathcal{A}(u,v) = \sum_{l,m}(\tau_{lm}(u), \varepsilon_{lm}(v^s))_{\Omega_p} - (p_f(u), \nabla\cdot v^f)_{\Omega_p} = (\mathbf{M}\tilde{\epsilon}(u), \tilde{\epsilon}(v))_{\Omega_p}, u, v \in \left[H^1(\Omega_p)\right]^2 \times H(\mathrm{div}, \Omega_p). \tag{27}$$

The non-zero entries of the symmetric 7×7 matrix $\mathbf{M} = (m_{ij})$ in (27) have the values $m_{11} = m_{22} = m_{33} = \lambda_c + 2G$, $m_{12} = m_{13} = m_{23} = \lambda_c$, $m_{14} = m_{24} = m_{34} = \alpha K_{av}$, $m_{44} = K_{av}$, $m_{55} = m_{66} = m_{77} = G$ and

$$\tilde{\varepsilon}(u) = \left(\varepsilon_{11}(u^{(s)}), \varepsilon_{22}(u^{(s)}), \varepsilon_{33}(u^{(s)}), \nabla\cdot u^{(f)}, \varepsilon_{12}(u^{(s)}), \varepsilon_{13}(u^{(s)}), \varepsilon_{23}(u^{(s)})\right)^t.$$

Furthermore, we assume that $\mathcal{P}$ and $\mathbf{M}$ are positive definite since they are associated with the kinetic and strain energy densities, respectively. We also assume that the entries in these two matrices are bounded below and above by positive constants.

Our continuous weak formulation is stated as follows: find $(E, H, u^s, u^f) \in H(\mathrm{curl}, \Omega) \times [L^2(\Omega)]^3 \times [H^1(\Omega_p)]^3 \times H(\mathrm{div}, \Omega_p)$ satisfying (23)–(25).

Let us analyze the uniqueness of the solution of our continuous problem. Set $J_e^s = 0$ in (24) and set to zero the initial conditions in (17). Then choose $\varphi = H$ in (24) to get:

$$(\nabla \times E, H) + \left(\mu\frac{\partial H}{\partial t}, H\right) = 0. \tag{28}$$

Also, choose $\psi = E$ in (23) and use (25) to obtain:

$$\left(\varepsilon\frac{\partial E}{\partial t}, E\right) + (\sigma E, E) + \left(\mu\frac{\partial H}{\partial t}, H\right) + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi E, P_\chi E\right\rangle = 0. \tag{29}$$

Next, choose $v^s = \frac{\partial u^s}{\partial t}$, $v^f = \frac{\partial u^f}{\partial t}$ in (25) and add the resulting equation to (29) to get:

$$\frac{1}{2}\frac{d}{dt}\left[\left(\mathcal{P}\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right)_{\Omega_p} + (\mathbf{M}\tilde{\epsilon}(u), \tilde{\epsilon}(u))_{\Omega_p} + (\varepsilon E, E) + (\mu H, H)\right] + \Phi\left(E, \frac{\partial u^f}{\partial t}\right) + (\sigma E, E)_{\Omega_a} + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi E, P_\chi E\right\rangle + \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right)\right\rangle_{\Gamma_p} = 0, \tag{30}$$

where

$$\Phi\left(E, \frac{\partial u^f}{\partial t}\right) = (\sigma E, E)_{\Omega_p} - \left(L_0\frac{\eta}{\kappa_0}E, \frac{\partial u^f}{\partial t}\right)_{\Omega_p} + \left(\frac{\eta}{\kappa_0}\frac{\partial u^f}{\partial t}, \frac{\partial u^f}{\partial t}\right)_{\Omega_p}. \tag{31}$$

Set:

$$A_{\min,p} = \inf\{A(\mathbf{x}), \mathbf{x} \in \Omega_p\}, \quad A = \sigma, \kappa_0, \quad \mathbf{x} = (x_1, x_2, x_3),$$
$$\kappa_{0,\max} = \sup\{\kappa_0(\mathbf{x}),\ \mathbf{x} \in \Omega_p\}.$$

and assume that:

$$\sigma_{\min,p} > 0, \quad \kappa_{0,\min,p} > 0, \quad \kappa_{0,\max,p} < \infty, \tag{32}$$

$$C_1 = \min\left(\sigma_{\min,p} - \frac{L_0\eta}{2\kappa_{0,\max}, p}, \frac{\eta}{\kappa_{0,\max}, p} - \frac{L_0\eta}{2\kappa_{0,\max}, p}\right) > 0. \tag{33}$$

Thus:

$$\Phi\left(E,\frac{\partial u^f}{\partial t}\right) \geqslant C_1\left(\|E\|^2_{0,\Omega_p} + \|u^f\|^2_{0,\Omega_p}\right). \tag{34}$$

Next, since the matrix **M** is positive definite, Korn' second inequality [38,39] implies that:

$$(\mathbf{M}\tilde{\epsilon}(u),\tilde{\epsilon}(u))_{\Omega_p} \geqslant C_2\left(\|u^s\|^2_{1,\Omega_p} + \|u^f\|^2_{H(\mathrm{div},\Omega_p)}\right) - C_3\left(\|u^s\|^2_{0,\Omega_p} + \|u^f\|^2_{0,\Omega_p}\right). \tag{35}$$

Set:

$$\mathcal{Z} = \left[H^1(\Omega_p)\right]^3 \times H(\mathrm{div},\Omega_p),$$

provided with the natural norm:

$$\|u\|_{\mathcal{Z}} = \left(\|u^s\|^2_{1,\Omega_p} + \|u^f\|_{H(\mathrm{div},\Omega_p)}\right)^{1/2}.$$

Then choose a constant ζ such that $\zeta > C_3$ and define the bilinear form:

$$\mathcal{A}_\zeta(u,v) = \mathcal{A}(u,v) + \zeta(u,v),$$

so that $\mathcal{A}_\zeta$ is $\mathcal{Z}$-coercive, i.e.,

$$\mathcal{A}_\zeta(u,u) \geqslant C_2\|u\|^2_{\mathcal{Z}}, \tag{36}$$

Next, add to (30) the inequality:

$$\frac{\zeta}{2}\frac{d}{dt}\|u\|^2_{0,\Omega_p} \leqslant \frac{\zeta}{2}\left(\|u\|^2_{0,\Omega_p} + \left\|\frac{\partial u}{\partial t}\right\|^2_{0,\Omega_p}\right),$$

integrate in time the resulting inequality and use (34) and (36) to obtain:

$$\begin{aligned}&\frac{1}{2}\left[\left(\mathcal{P}\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)_{\Omega_p}(t) + C_2\|u(t)\|^2_{\mathcal{Z}} + (\varepsilon E,E)(t) + (\mu H,H)(t)\right] + C_1\left[\int_0^t\left(\|E(s)\|^2_{0,\Omega_p} + \|u^f(s)\|^2_{0,\Omega_p}\right)ds\right]\\ &+\int_0^t(\sigma E,E)_{\Omega_a}(s)ds + \int_0^t\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}P_\chi E,P_\chi E\right\rangle(s)ds + \int_0^t\left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right),\quad S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right)\right\rangle_{\Gamma_p}(s)ds\\ &\leqslant \frac{\zeta}{2}\int_0^t\left(\|u(s)\|^2_{0,\Omega_p} + \left\|\frac{\partial u}{\partial t}(s)\right\|^2_{0,\Omega_p}\right)ds.\end{aligned} \tag{37}$$

Assume that ε and μ are bounded below and above by positive constants, so that:

$$0 < \varepsilon_* \leqslant \varepsilon \leqslant \varepsilon^* < \infty, \quad 0 < \mu_* \leqslant \mu \leqslant \mu^* < \infty. \tag{38}$$

Thus apply Gronwall's lemma in (37), note that all terms in the left-hand side of (37) are nonnegative and use (38) to conclude that:

$$\|u(t)\|_{\mathcal{Z}} = 0, \quad \|E(t)\|_0 = 0, \quad \|H(t)\|_0 = 0, \quad \forall\, t \in J,$$

so that uniqueness holds for the solution of our differential problem. Assuming sufficient regularity on the initial data and the external sources, existence can be derived using the compactness argument of Lions [28] with an argument similar to that given in [27]. For brevity the argument is omitted. The result is summarized in the following theorem.

Theorem 1. *Assume the validity of (33) and (38) and that the matrices* **M** *and* $\mathcal{D}$ *are positive definite. Then there exists a unique solution of problem (1)–(4) with the boundary conditions (14)–(16) and the initial conditions (17).*

4. A continuous-time finite element method. Parallelepiped elements

Let $\mathcal{T}^h$ be a quasiregular partition of Ω into parallelepipeds Ω_j of diameter bounded by h. Denote by ξ_j and ξ_{jk} the centroids of $\Gamma_j = \partial\,\Omega_j \cap \Gamma$ and $\Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k$, respectively.

To approximate the electromagnetic fields E, H we will employ the 3D mixed finite element space $\mathcal{V}^h \times \mathcal{W}^h$, defined as follows [19,22]:

$$\mathcal{V}^h = \left\{\psi \in H(\mathrm{curl},\Omega):\ \psi|_{\Omega_j} \in \mathcal{V}^h_j \equiv P_{0,1,1} \times P_{1,0,1} \times P_{1,1,0}\right\},$$
$$\mathcal{W}^h = \left\{\varphi \in [L^2(\Omega)]^3:\ \varphi|_{\Omega_j} \in \mathcal{W}^h_j \equiv P_{1,0,0} \times P_{0,1,0} \times P_{0,0,1}\right\}.$$

Here, $P_{r,s,t}$ denote the polynomials of degree not greater than r in x_1, not greater than s in x_2 and not greater than t in x_3 on Ω_j. Note that functions in $\mathcal{V}^h$ have continuous projections of their traces across the interelement boundaries Γ_{jk}. Also:

$$\nabla \times \mathcal{V}^h \subset \mathcal{W}^h.$$

For any $\psi \in \mathcal{V}^h$ and any $\varphi \in \mathcal{W}^h$ the local degrees of freedom in Ω_j are [22]:

$$\sum_j^{\psi} = \left\{(\psi \cdot \chi_p^j)(m_p), \quad \text{where } m_p \text{ is the mid-point of the p-edge } e_p^j \text{ of } \Omega_j \text{ of unit tangent } \chi_p^j, 1 \leqslant p \leqslant 12\right\}, \tag{39}$$

$$\sum_j^{\varphi} = \left\{(\varphi \cdot \nu_p^j)(G_p), \quad \text{where } G_p \text{ is the centroid of the p-face } f_p^j \text{ of } \Omega_j \text{ of unit normal } \nu_p^j, 1 \leqslant p \leqslant 6\right\}. \tag{40}$$

To approximate the solid displacement vector we will employ the nonconforming finite element space $\mathcal{NC}^h$ presented in [24]. This choice is made based on the numerical dispersion analysis presented in [25], where it is shown that using this non-conforming space allows for using about half the number of points per wavelength as compared with standard bilinear elements to have a desired tolerance in numerical dispersion. See also [26] for the analysis of the numerical dispersion of waves in fluid-saturated poroelastic media when employing this non-conforming element to represent the solid displacement vector. The space $\mathcal{NC}^h$ is defined as follows. On the reference element $\widehat{R} = [-1,1]^3$ set:

$$Q(\widehat{R}) = \text{Span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_3, \tilde{\alpha}(\hat{x}_1) - \tilde{\alpha}(\hat{x}_2), \tilde{\alpha}(\hat{x}_1) - \tilde{\alpha}(\hat{x}_3)\}, \quad \tilde{\alpha}(z) = z^2 - \frac{5}{3}z^4.$$

Then let $\mathcal{NC}_j^h = [Q(\Omega_j)]^3$ and

$$\mathcal{NC}^h = \left\{v : v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h, v_j(\xi_{jk}) = v_k(\xi_{jk}) \quad \forall\ (j,k)\right\}.$$

The six local degrees of freedom are the values at the centroids ξ_{jk} of the faces of Ω_j.

To approximate the fluid displacement, we employ the vector part of the Raviart–Thomas–Nédélec space of zero order [18,19], defined as follows:

$$\mathcal{M}^h = \left\{w \in H(\text{div}, \Omega_p) : \ w|\Omega_j \in \mathcal{M}_j^h \equiv P_{1,0,0} \times P_{0,1,0} \times P_{0,0,1}\right\}.$$

The approximating properties of the finite element spaces defined above can be stated as follows. Let:

$$\left[H_h^1(\Omega)\right]^3 = \left\{\psi : \psi|\Omega_j \in \left[H^1(\Omega_j)\right]^3\right\},$$

with $\left[H_h^1(\Omega_p)\right]^3$ defined in similar fashion. Also, if $\Gamma_{jk,p}$ denotes any inner interface Γ_{jk} in Ω_p let:

$$\widetilde{\Lambda}^h = \left\{\tilde{\lambda}^h : \tilde{\lambda}_{jk}^h = \text{tr}_{\Gamma_{jk,p}}\left(\tilde{\lambda}^h|_{\Omega_j}\right) \in \left[P_0(\Gamma_{jk,p})\right]^3 \equiv \widetilde{\Lambda}_{jk}^h, \quad \tilde{\lambda}_{jk}^h + \tilde{\lambda}_{kj}^h = 0\right\},$$

where $P_0(\Gamma_{jk,p})$ denotes the constant functions defined on $\Gamma_{jk,p}$.

Remark. Note that there are two copies of $[P_0(\Gamma_{jk,p})]^3$ assigned to each $\Gamma_{jk,p}$, one from Ω_j to Ω_k and another from Ω_k to Ω_j.

Then we define the projections:

$$\Pi_h : H(\text{curl}, \Omega) \cap [H_h^1(\Omega)]^3 \to \mathcal{V}^h : \int_{e_j^p} (\psi - \Pi_h \psi) \cdot \chi_p^j ds = 0, \quad 1 \leqslant p \leqslant 12, \tag{41}$$

for each p-edge e_p^j with tangent χ_p^j of Ω_j:

$$P_h : [L^2(\Omega)]^3 \to \mathcal{W}^h : \quad (P_h w - w, \varphi) = 0, \quad \varphi \in \mathcal{W}^h, \tag{42}$$

$$R_h : [H^2(\Omega_p)]^3 \to \mathcal{NC}^h : \quad \left(v_i^s - R_h v_i^s\right)(\xi) = 0, \xi = \xi_{jk} \quad \text{or} \quad \xi_j, \quad \text{for } v^s = \left(v_1^s, v_2^s, v_3^s\right), \tag{43}$$

$$Q_h : [H^1(\Omega_p)]^3 \to \mathcal{M}^h : \langle\left(v^f - Q_h v^f\right) \cdot \nu, 1\rangle_B = 0; \quad B = \Gamma_{jk,p} \quad \text{or} \quad \Gamma_j, \tag{44}$$

$$S_h : [H^2(\Omega_p)]^3 \times H^1(\text{div}; \Omega_p) \to \widetilde{\Lambda}^h : \quad \langle\tau(v)\nu - S_h(v), 1\rangle_B = 0, v = \left(v^s, v^f\right), B = \Gamma_{jk,p} \quad \text{or} \quad \Gamma_j. \tag{45}$$

Let us define the broken norms:

$$\|v\|_{s,h,\Omega_p}^2 = \sum_{\Omega_j \subset \Omega_p} \|v\|_{s,\Omega_j}^2.$$

The approximation properties of these operators can be stated as follows [19,29]:

$$\|\psi - \Pi_h \psi\|_0 \leqslant Ch\ \|\psi\|_1, \quad \psi \in [H^1(\Omega)]^3, \tag{46}$$

$$\|\nabla\times(\psi-\Pi_h\psi)\|_0 \leqslant Ch\ \|\nabla\times\psi\|_1,\quad \psi\in[H^1(\Omega)]^3,\quad \nabla\times\psi\in H^1(\Omega), \tag{47}$$

$$\|P_h\varphi-\varphi\|_0 \leqslant Ch\ \|\varphi\|_1,\quad \forall\ \varphi\in H^1(\Omega), \tag{48}$$

$$\Bigg[\|v^s-R_hv^s\|_{\Omega_p}+h\|v^s-R_hv^s\|^2_{1,h,\Omega_p}+h^2\|v^s-R_hv^s\|^2_{2,h,\Omega_p}+h^{\frac12}\left(\sum_{\Omega_j\subset\Omega_p}\|v^s-R_hv^s\|^2_{0,\partial\Omega_j}\right)^{\frac12}$$
$$+h^{\frac32}\left(\sum_{\Omega_j\subset\Omega_p}\|\tau(v_j)v_j-S_hv_j\|^2_{0,\partial\Omega_j}\right)^{1/2} \leqslant Ch^2\ \left(\|v^s\|_{2,\Omega_p}+\|\nabla\cdot v^f\|_{1,\Omega_p}\right),\quad v=(v^s,v^f)\in[H^2(\Omega_p)]^3\times H^1(\mathrm{div},\Omega_p),\Bigg] \tag{49}$$

$$\|Q_hv^f-v^f\|_{0,\Omega_p} \leqslant Ch\ \|\mathbf{v}^f\|_{1,\Omega_p},\quad v^f\in\left[H^1(\Omega_p)\right]^3, \tag{50}$$

$$\|\nabla\cdot(v^f-Q_hv^f)\|_{0,\Omega_p} \leqslant Ch\ \|\nabla\cdot v^f\|_{1,\Omega_p},\quad v^f\in H^1(\mathrm{div},\Omega_p). \tag{51}$$

Note that since $\nabla\times\psi\in\mathcal{W}^h\forall\ \psi\in\mathcal{V}^h$, it follows from (42) that:

$$(P_hf-f,\nabla\times\psi)=0,\quad \forall\ \psi\in\mathcal{V}^h. \tag{52}$$

Also note the orthogonality property for functions on $\mathcal{NC}^h$:

$$\left\langle v^s_j-v^s_k,1\right\rangle_{\Gamma_{jk}}=0\quad \text{for all interior interfaces } \Gamma_{jk},\quad v^s\in\mathcal{NC}^h. \tag{53}$$

Set:

$$\mathcal{A}_h(u,v)=\sum_{\Omega_j\subset\Omega_p}\left[\sum_{l,m}\left(\tau_{lm}(u),\varepsilon_{lm}(v^{(s)})\right)_{\Omega_j}-\left(p_f(u),\nabla\cdot v^f\right)_{\Omega_j}\right]=\sum_{\Omega_j\subset\Omega_p}(\mathbf{M}\tilde\epsilon(u),\tilde\epsilon(v))_{\Omega_j}, \tag{54}$$

and

$$\Theta_h\big((E,H,u^s,u^f),(\psi,\varphi,v^s,v^f)\big)=\left(\varepsilon\frac{\partial E}{\partial t},\psi\right)+(\sigma E,\psi)-(H,\nabla\times\psi)+\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}P_\chi E,P_\chi\psi\right\rangle+(\nabla\times E,\varphi)$$
$$+\left(\mu\frac{\partial H}{\partial t},\varphi\right)+\left(\mathcal{P}\frac{\partial^2u}{\partial t^2},v\right)_{\Omega_p}+\left(\frac{\eta}{\kappa_0}\frac{\partial u^f}{\partial t},v^f\right)_{\Omega_p}+\mathcal{A}_h(u,v)-\left(L_0\frac{\eta}{\kappa_0}E,v^f\right)_{\Omega_p}$$
$$+\left\langle\mathcal{D}S_\Gamma\left(\frac{\partial u}{\partial t}\right),S_\Gamma(v)\right\rangle_{\Gamma_p}. \tag{55}$$

Let:

$$\mathcal{Y}^h=\mathcal{V}^h\times\mathcal{W}^h\times\mathcal{NC}^h\times\mathcal{M}^h.$$

Then the continuous-time Galerkin procedure is defined as follows: find $(E^h,H^h,u^{s,h},u^{f,h}):J\to\mathcal{Y}^h$ such that:

$$\Theta_h\Big((E^h,H^h,u^{s,h},u^{f,h}),(\psi,\varphi,v^s,v^f)\Big)=(J^s_e,\psi),(\psi,\varphi,v^s,v^f)\in\mathcal{Y}^h, \tag{56}$$

with the initial conditions:

$$E^h(t=0)\approx E_0,\quad H^h(t=0)\approx H_0, \tag{57}$$

$$u^{s,h}(t=0)\approx u^s_0,\quad u^{f,h}(t=0)\approx u^f_0. \tag{58}$$

To analyze uniqueness for (56)–(58), let us introduce the space:

$$\mathcal{Z}^h=\left[H^1_h(\Omega_p)\right]^3\times H(\mathrm{div},\Omega_p)$$

provided with the norm:

$$\|u\|^2_{\mathcal{Z}^h}=\left(\|u^s\|^2_{1,h,\Omega_p}+\|u^f\|_{H(\mathrm{div}^2,\Omega_p)}\right)^{1/2}.$$

As in the continuous case, let us define the bilinear form:

$$\mathcal{A}_{\zeta,h}(u,v)=\mathcal{A}_h(u,v)+\zeta(u,v),\quad u,v\in\mathcal{Z}^h, \tag{59}$$

with $\zeta > C_3$, so that:

$$\mathcal{A}_{\zeta,h}(u,u) \geqslant C_2\|u\|^2_{\mathcal{Z}^h}, \quad (60)$$
$$\mathcal{A}_{\zeta,h}(u,v) \leqslant C\|u\|_{\mathcal{Z}^h}\|v\|_{\mathcal{Z}^h}. \quad (61)$$

Then if $u^h = (u^{s,h}, u^{f,h})$, a repetition of the argument leading to (37) implies that:

$$\|u^h\|_{\mathcal{Z}^h} = 0, \quad \|E^h(t)\|_0 = 0, \quad \|H^h(t)\|_0 = 0, \quad \forall\ t,$$

so that uniqueness holds for (56)–(58). Existence follows from finite-dimensionality. Thus we can state the following theorem:

Theorem 2. *Under the hypothesis of the positive definitess of the matrices* **M** *and* $\mathcal{D}$ *and the validity of* (33) and (38), *there exists a unique solution of* (56) *for every choice of the initial conditions* (57) and (58).

5. *Apriori* error estimates

Now we derive the error estimate for the procedure (56) stated in the following theorem:

Theorem 3. *Assume that the matrices* **M** *and* $\mathcal{D}$ *are positive definite and the validity of* (33) and (38). *Also assume that* E, H, u^s, u^f *are smooth enough so that the following quantities are finite:*

$$M_1^2 = \|u^s\|^2_{L^\infty(J,H^2(\Omega_p))} + \left\|\frac{\partial u^s}{\partial t}\right\|^2_{L^\infty(J,H^2(\Omega_p))} + \|u^f\|^2_{L^\infty(J,H^1(\mathrm{div},\Omega_p))} + \left\|\frac{\partial u^f}{\partial t}\right\|^2_{L^2(J,H^1(\ \mathrm{div},\Omega_p))} + \|E\|^2_{L^2(J,H^1(\Omega))},$$
$$N_1^2 = \|E\|^2_{L^\infty(J,H^1(\Omega))} + \|\nabla\times E\|^2_{L^\infty(J,H^1(\Omega))} + \left\|\frac{\partial E}{\partial t}\right\|^2_{L^\infty(J,H^1(\Omega))} + \left\|\frac{\partial H}{\partial t}\right\|^2_{L^2(J,H^1(\Omega))},$$
$$M_0^2 = \|u^s(0)\|^2_{2,\Omega_p} + \|u^f(0)\|^2_{H^1(\mathrm{div},\Omega_p)} + \left\|\frac{\partial u^s}{\partial t}(0)\right\|^2_{2,\Omega_p} + \left\|\frac{\partial u^f}{\partial t}(0)\right\|^2_{H^1(\mathrm{div},\Omega_p)},$$
$$N_0^2 = \|E(0)\|_1^2 + \|H_2(0)\|_1^2.$$

Then the following a priori error estimate holds:

$$\|E-E^h\|_{L^\infty(J,L^2(\Omega))} + \|H-H^h\|_{L^\infty(J,L^2(\Omega))} + \|u^s-u^{s,h}\|_{L^\infty(J,H^1_h(\Omega_p))} + \|u^f-u^{f,h}\|_{L^\infty(J,H(\mathrm{div},\Omega_p))} + \left\|\frac{\partial(u^s-u^{s,h})}{\partial t}\right\|_{L^\infty(J,L^2(\Omega_p))}$$
$$+ \left\|\frac{\partial(u^f-u^{f,h})}{\partial t}\right\|_{L^\infty(J,L^2(\Omega_p))} + \|u^s-u^{s,h}\|_{L^2(J,L^2(\Gamma_p))} + \|(u^f-u^{f,h})\cdot\nu\|_{L^2(J,L^2(\Gamma_p))} \leqslant Ch^{1/2}\ [N_0+N_1+M_0+M_1]. \quad (62)$$

Proof. Set:

$$\delta = \left(E-E^h, H-H^h, u^s-u^{s,h}, u^f-u^{f,h}\right) \equiv \left(\delta^E, \delta^H, \delta^s, \delta^f\right),$$
$$\gamma = \left(\Pi_h E - E^h, P_h H - H^h, \mathcal{R}_h u^s - u^{s,h}, Q_h u^f - u^{f,h}\right) \equiv \left(\gamma^E, \gamma^H, \gamma^s, \gamma^f\right),$$

and

$$\delta^B = \left(\delta^s, \delta^f\right), \quad \gamma^B = \left(\gamma^s, \gamma^f\right).$$

First, use integration by parts element by element and the boundary conditions (14)–(16) to see that, for $\psi \in H(\mathrm{curl},\Omega)$, $\varphi \in [L^2(\Omega)]^3$ and $v = (v^s, v^f) \in [L^2(\Omega_p)]^6$ such that $v^s|_{\Omega_j} \in [H^1(\Omega_j)]^3$ and $v^f|_{\Omega_j} \in H(\mathrm{div},\Omega_j)$:

$$\Theta_h\big((E,H,u^s,u^f),(\psi,\varphi,v^s,v^f)\big) = (J_e^s,\psi) + \sum_{\Omega_j\subset\Omega_p}\left[\langle\tau(u)\nu,\ v^s\rangle_{\partial\Omega_j\setminus\Gamma_p} - \langle p_f(u),\ v^f\cdot\nu\rangle_{\partial\Omega_j\setminus\Gamma_p}\right]. \quad (63)$$

Now subtract (63) from (56) to see that, for $(\psi,\varphi,v^s,v^f) \in \mathcal{Y}^h$:

$$\Theta_h\left((\delta^E,\delta^H_y,\delta^s,\delta^f),(\psi,\varphi,v^s,v^f)\right) = \sum_{\Omega_j\subset\Omega_p}\left[\langle\tau(u)\nu,\ v^s\rangle_{\partial\Omega_j\setminus\Gamma_p} - \langle p_f(u),\ v^f\cdot\nu\rangle_{\partial\Omega_j\setminus\Gamma_p}\right]. \quad (64)$$

Next, by hypothesis $u^s \in [H^2(\Omega_p)]^3$, $u^f \in H^1(\mathrm{div},\Omega_p)$ and consequently $p_f(u) \in H^{1/2}(\partial\Omega_j)$. Also, $v^f \in \mathcal{M}^h \subset H(\mathrm{div},\Omega)$ and consequently $v^f\cdot\nu$ is continuous across the interior interfaces of the elements Ω_j. Thus:

$$\sum_{\Omega_j \subset \Omega_p} \langle p_f(u),\ v^f \cdot \nu \rangle_{\partial\Omega_j \setminus \Gamma_p} = 0. \tag{65}$$

Also use that $v^s \in [\mathcal{NC}^h]^3$ is constant on Γ_{jk} and that $S_h(u)$ changes sign on each side of Γ_{jk} (c.f. (45)) to see that:

$$\sum_{\Omega_j \subset \Omega_p} \langle S_h(u),\ v^s \rangle_{\partial\Omega_j \setminus \Gamma_p} = 0. \tag{66}$$

Thus, (63) becomes:

$$\Theta_h\left(\left(\delta^E, \delta^H, \delta^s, \delta^f\right), \left(\psi, \varphi, v^s, v^f\right)\right) = \sum_{\Omega_j \subset \Omega_p} \langle \tau(u)\nu - S_h(u),\ v^s \rangle_{\partial\Omega_j \setminus \Gamma_p}, \quad (\psi, \varphi, v^s, v^f) \in \mathcal{Y}^h. \tag{67}$$

Now (67) yields:

$$\begin{aligned}\Theta_h((\gamma^E, \gamma^H, \gamma^s, \gamma^f), (\psi, \varphi, v^s, v^f)) = {} & \Theta_h((\Pi_h E - E, P_h H - H, R_h u^s - u^s, Q_h u^f - u^f), (\psi, \varphi, v^s, v^f)) \\ & + \sum_{\Omega_j \subset \Omega_p} \langle \tau(u)\nu - S_h(u), v^s \rangle_{\partial\Omega_j \setminus \Gamma_p}, \quad (\psi, \varphi, v^s, v^f) \in \mathcal{Y}^h.\end{aligned} \tag{68}$$

Next, choose $\psi = 0, \varphi = 0, v^s = \frac{\partial \gamma^s}{\partial t}, v^f = \frac{\partial \gamma^f}{\partial t}$ in (68) and add to the resulting equation the inequality:

$$\frac{\zeta}{2}\frac{d}{dt}\|\gamma^B\|^2_{0,\Omega_p} \leqslant \frac{\zeta}{2}\left(\|\gamma^B\|^2_{0,\Omega_p} + \left\|\frac{\partial \gamma^B}{\partial t}\right\|^2_{0,\Omega_p}\right),$$

to obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left(\mathcal{P}\frac{\partial \gamma^B}{\partial t}, \frac{\partial \gamma^B}{\partial t}\right)_{\Omega_p} + \mathcal{A}_h(\gamma^B, \gamma^B)\right] + \left(\frac{\eta}{\kappa_0}\frac{\partial \gamma^f}{\partial t}, \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p} + \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial \gamma^B}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial \gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p} \\ &\quad \leqslant \frac{\zeta}{2}\left(\|\gamma^B\|^2_{0,\Omega_p} + \left\|\frac{\partial \gamma^B}{\partial t}\right\|^2_{0,\Omega_p}\right) + \left(\mathcal{P}(R_h u^s - u^s, Q_h u^f - u^f), \frac{\partial \gamma^B}{\partial t}\right)_{\Omega_p} + \left(\frac{\eta}{\kappa_0}\frac{\partial (Q_h u^f - u^f)}{\partial t}, \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p} \\ &\qquad + \mathcal{A}_h\left((R_h u^s - u^s, Q_h u^f - u^f), \frac{\partial \gamma^B}{\partial t}\right) + \left\langle \mathcal{D}S_{\Gamma_p}((R_h u^s - u^s, Q_h u^f - u^f)), S_{\Gamma_p}\left(\frac{\partial \gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p} + \left(L_0\frac{\eta}{\kappa_0}\gamma^E, \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p} \\ &\qquad - \left(L_0\frac{\eta}{\kappa_0}[\Pi_h E - E], \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p} + \sum_{\Omega_j \subset \Omega_p}\left\langle \tau(u)\nu - S_h(u), \frac{\partial \gamma^s}{\partial t}\right\rangle_{\partial\Omega_j \setminus \Gamma_p},\end{aligned} \tag{69}$$

which will be integrated in time from 0 to t, but first we will bound the time integral of the last eight terms in the right-hand side of (69) First, using the approximation properties (49) and (50):

$$\left|\left(\mathcal{P}(R_h u^s - u^s, Q_h u^f - u^f), \frac{\partial \gamma^B}{\partial t}\right)_{\Omega_p}\right| \leqslant C\left(h^4\|u^s\|^2_{2,\Omega_p} + h^2\|u^f\|^2_{1,\Omega_p} + \left\|\frac{\partial \gamma^B}{\partial t}\right\|^2_{0,\Omega_p}\right),$$

so that:

$$\int_0^t \left|\left(\mathcal{P}(R_h u^s - u^s, Q_h u^f - u^f), \frac{\partial \gamma^B}{\partial t}\right)_{\Omega_p}(s)\right| ds \leqslant C\left[\int_0^t \left\|\frac{\partial \gamma^B}{\partial t}(s)\right\|^2_{0,\Omega_p} ds + h^2 M_1^2\right]. \tag{70}$$

Next, using again (50):

$$\int_0^t \left|\left(\frac{\eta}{\kappa_0}\frac{\partial (Q_h u^f - u^f)}{\partial t}, \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p}(s)\right| ds \leqslant \hat{\epsilon}\int_0^t \left(\frac{\eta}{\kappa_0}\frac{\partial \gamma^f}{\partial t}, \frac{\partial \gamma^f}{\partial t}\right)_{\Omega_p}(s) ds + Ch^2\ M_1^2. \tag{71}$$

Also, using integration by parts in time:

$$\begin{aligned}\int_0^t \mathcal{A}_h\left((R_h u^s - u^s, Q_h u^f - u^f),\ \frac{\partial \gamma^B}{\partial t}\right)(s) ds = {} & \mathcal{A}_h((R_h u^s - u^s, Q_h u^f - u^f), \gamma^B)|_0^t \\ & - \int_0^t \mathcal{A}_h\left(\left(\frac{\partial R_h u^s - u^s}{\partial t}, \frac{\partial Q_h u^f - u^f}{\partial t}\right), \gamma^B\right)(s) ds.\end{aligned} \tag{72}$$

Let us bound each term in the right-hand side of (72). First:

$$\begin{aligned}|\mathcal{A}_h((R_h u^s - u^s, Q_h u^f - u^f),\ \gamma^B)(0)| & \leqslant C\left(\|(R_h u^s - u^s)(0)\|_{1,h,\Omega_p} + \|(Q_h u^f - u^f)(0)\|_{H(\mathrm{div},\Omega_p)}\right)\|\gamma^B(0)\|_{\mathcal{Z}^h} \\ & \leqslant C\left(\|\gamma^B(0)\|^2_{\mathcal{Z}^h} + h^2 M_0^2\right).\end{aligned} \tag{73}$$

Now choose the initial condition $u^{s,h}(0) \in \mathcal{NC}^h, u^{f,h}(0) \in \mathcal{M}^h$ such that:

$$A_{\zeta,h}(\delta^B(0), v) = 0, \quad v = (v^s, v^f) \in \mathcal{NC}^h \times \mathcal{M}^h. \tag{74}$$

Then choose $v = \delta^B(0) + (R_h u^s(0), Q_h u^f(0)) - (u^s(0), u^f(0))$ in (74) and use the coercivity of $A_{\zeta,h}$ in $\mathcal{Z}^h$ to see that:

$$\|\delta^B(0)\|_{\mathcal{Z}^h} \leqslant Ch\, M_0, \tag{75}$$

so that using the triangle inequality and (49) and (50) we obtain the bound:

$$\|\gamma^B(0)\|_{\mathcal{Z}^h} \leqslant Ch\, M_0. \tag{76}$$

Thus, (73) becomes:

$$\left|\mathcal{A}_h\left((R_h u^s - u^s, Q_h u^f - u^f),\ \gamma^B\right)(0)\right| \leqslant Ch^2\, M_0^2. \tag{77}$$

Next:

$$\left|\mathcal{A}_h\left((R_h u^s - u^s, Q_h u^f - u^f), \gamma^B\right)|(t)\right| \leqslant \hat{\epsilon}\|\gamma^B(t)\|^2_{\mathcal{Z}^h} + Ch^2\ \left(\|u^s(t)\|^2_{2,\Omega_p} + \|u^f(t)\|^2_{H^1(\mathrm{div},\Omega_p)}\right), \tag{78}$$

and

$$\int_0^t \left|\mathcal{A}_h\left(\frac{\partial(R_h u^s - u^s)}{\partial t}, \frac{\partial(Q_h u^f - u^f)}{\partial t}, \gamma^B\right)(s)ds\right| \leqslant C\left(\int_0^t \|\gamma^B(s)\|_{\mathcal{Z}^h} ds + h^2 M_1^2\right). \tag{79}$$

Thus, collecting the estimates (77)–(79) we get the bound:

$$\int_0^t \left|\mathcal{A}_h\left((R_h u^s - u^s, Q_h u^f - u^f), \frac{\partial\gamma^B}{\partial t}\right)(s)\right| ds \leqslant \hat{\epsilon}\|\gamma^B(t)\|^2_{\mathcal{Z}^h} + C\left(\int_0^t \|\gamma^B(s)\|_{\mathcal{Z}^h} ds + h^2\left(M_0^2 + M_1^2\right)\right). \tag{80}$$

Note that the next term in the right-hand side of (69) can be bounded as follows:

$$\begin{aligned}\left|\left\langle \mathcal{D}S_{\Gamma_p}\left((R_h u^s - u^s, Q_h u^f - u^f)\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p}\right| &\leqslant C\sum_j h^{1/2}\left(\|u^s\|_{1/2,\Gamma_j} + \|u^f\cdot v_j\|_{1/2,\Gamma_j}\right)\left(\left\|\frac{\partial\gamma^s}{\partial t}\right\|_{0,\Gamma_j} + \left\|\frac{\partial\gamma^f\cdot v_j}{\partial t}\right\|_{0,\Gamma_j}\right)\\ &\leqslant \hat{\epsilon}\left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p} + Ch\ \left(\|u^s\|^2_{1,\Omega_p} + \|u^f\|^2_{1,\Omega_p}\right),\end{aligned}$$

so that:

$$\int_0^t \left|\left\langle \mathcal{D}S_{\Gamma_p}\left((R_h u^s - u^s, Q_h u^f - u^f)\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p}(s)ds\right| \leqslant \hat{\epsilon}\int_0^t \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p}(s)ds + Ch\ M_1^2. \tag{81}$$

Next, using (46), the integral in time of the sixth and seventh terms in the right-hand side of (69) can be bounded as:

$$\begin{aligned}&\int_0^t \left|\left(L_0\frac{\eta}{\kappa_0}\gamma^E, \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)\right| ds + \int_0^t \left|\left(L_0\frac{\eta}{\kappa_0}[\Pi_h E - E], \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)\right| ds\\ &\leqslant \hat{\epsilon}\int_0^t \left(\frac{\eta}{\kappa_0}\frac{\partial\gamma^f}{\partial t}, \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)ds + C\left[h^2\int_0^t \|E(s)\|^2_{1,\Omega_p} ds + \int_0^t \|\gamma^E(s)\|^2_{0,\Omega_p} ds\right]\\ &\leqslant \hat{\epsilon}\int_0^t \left(\frac{\eta}{\kappa_0}\frac{\partial\gamma^f}{\partial t}, \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)ds + C\left[h^2 N_1^2 + \int_0^t \|\gamma^E(s)\|^2_0 ds\right].\end{aligned} \tag{82}$$

Next, using integration by parts in time in the last term in the right-hand side of (69):

$$\int_0^t \left\langle \tau(u)v - S_h(u), \frac{\partial\gamma^s}{\partial t}\right\rangle_{\partial\Omega_j\setminus\Gamma_p}(s)ds = \langle \tau(u)v - S_h(u),\ \gamma^s\rangle_{\partial\Omega_j\setminus\Gamma_p}|_0^t - \int_0^t \left\langle \tau\left(\frac{\partial u}{\partial t}\right)v - S_h\left(\frac{\partial u}{\partial t}\right), \gamma^s\right\rangle_{\partial\Omega_j\setminus\Gamma_p}(s)ds. \tag{83}$$

Next, since $(\tau(u)v - S_h(u))(s)$ has average value zero on $\partial\Omega_j\setminus\Gamma_p$, if $q^s(s)$ is the average value of $\gamma^s(s)$ on Ω_j for $s = 0,\ldots,t$, using the trace inequality and Korn's second inequality [38,39]:

$$\begin{aligned}|\langle \tau(u)v - S_h(u), \gamma^s\rangle_{\partial\Omega_j\setminus\Gamma_p}(0)| &= |\langle \tau(u)v - S_h(u), \gamma^s - q^s\rangle_{\partial\Omega_j\setminus\Gamma_p}(0)| \leqslant \|(\tau(u)v - S_h(u))(0)\|_{0,\partial\Omega_j}\|(\gamma^s - q^s)(0)\|_{0,\partial\Omega_j}\\ &\leqslant Ch^{1/2}(\|u^s(0)\|_{2,\Omega_j} + \|\nabla\cdot u^f(0)\|_{1,\Omega_j})\|(\gamma^s - q^s)(0)\|^{1/2}_{0,\Omega_j}\|(\gamma^s - q^s)(0)\|^{1/2}_{1,\Omega_j}\\ &\leqslant Ch(\|u^s(0)\|_{2,\Omega_j} + \|\nabla\cdot u^f(0)\|_{1,\Omega_j})\|\gamma^s(0)\|_{1,\Omega_j}.\end{aligned} \tag{84}$$

Thus, adding over j in (84) and using (76) we obtain:

$$\sum_j \left|\langle \tau(u)v - S_h(u),\ \gamma^s\rangle_{\partial\Omega_j\setminus\Gamma_p}(0)\right| \leqslant Ch\ M_0\|\gamma^s\|_{1,h} \leqslant Ch^2\ M_0^2. \tag{85}$$

Similarly:

$$\left|\langle \tau(u)v - S_h(u),\ \gamma^s - q^s\rangle_{\partial\Omega_j\setminus\Gamma_p}(t)\right| \leqslant \hat{\epsilon}\|\gamma^s(t)\|^2_{1,\Omega_j} + Ch^2\ \left(\|u^s(t)\|^2_{2,\Omega_j} + \|\nabla\cdot u^f(t)\|^2_{1,\Omega_j}\right), \tag{86}$$

so that adding over j in (86) we get:

$$\sum_j \left|\langle \tau(u)v - S_h(u),\ \gamma^s - q^s\rangle_{\partial\Omega_j\setminus\Gamma_p}(t)\right| \leqslant \hat{\epsilon}\|\gamma^s(t)\|^2_{1,h} + Ch^2\ M_1^2. \tag{87}$$

In a similar fashion:

$$\sum_j \left|\int_0^t \left\langle \tau\left(\frac{\partial u}{\partial t}\right)v - S_h\left(\frac{\partial u}{\partial t}\right), \gamma^s\right\rangle_{\partial\Omega_j\setminus\Gamma_p}(s)ds\right| \leqslant C\left(\int_0^t \|\gamma^s(s)\|^2_{1,h}ds + h^2M_1^2\right). \tag{88}$$

Combining the estimates (85), (87) and (88) we conclude that:

$$\left|\int_0^t \left\langle \tau(u)v - S_h(u), \frac{\partial\gamma^s}{\partial t}\right\rangle_{\partial\Omega_j\setminus\Gamma_p}(s)ds\right| \leqslant \hat{\epsilon}\|\gamma^s(t)\|^2_{1,h} + C\left(\int_0^t \|\gamma^s(s)\|^2_{1,h}ds + h^2M_1^2\right). \tag{89}$$

Then integrate (69) from 0 to t and use the bounds (70) (71) (76) (80) (81) and (89) to obtain:

$$\begin{aligned}
&\frac{1}{2}\left(\mathcal{P}\frac{\partial\gamma^B}{\partial t}, \frac{\partial\gamma^B}{\partial t}\right)_{\Omega_p}(t) + \mathcal{A}_{\zeta,h}(\gamma^B,\gamma^B)(t) + \int_0^t \left(\frac{\eta}{\kappa_0}\frac{\partial\gamma^f}{\partial t}, \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)ds + \int_0^t \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p}(s)ds \\
&\quad\leqslant \hat{\epsilon}\left(\|\gamma^B(t)\|^2_{\mathcal{Z}^h} + \int_0^t \left(\frac{\eta}{\kappa_0}\frac{\partial\gamma^f}{\partial t}, \frac{\partial\gamma^f}{\partial t}\right)_{\Omega_p}(s)ds + \int_0^t \left\langle \mathcal{D}S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right), S_{\Gamma_p}\left(\frac{\partial\gamma^B}{\partial t}\right)\right\rangle_{\Gamma_p}(s)ds\right) \\
&\qquad + C\left(\int_0^t \left(\|\gamma^B(s)\|_{\mathcal{Z}^h} + \left\|\frac{\partial\gamma^B}{\partial t}(s)\right\|_{0,\Omega_p}\right)ds + h(M_0^2+M_1^2) + \left\|\frac{\partial\gamma^B}{\partial t}(0)\right\|^2_{0,\Omega_p}\right) + C\int_0^t \|\gamma^E(s)\|_0^2 ds.
\end{aligned} \tag{90}$$

Next choose $\frac{\partial u^{s,h}}{\partial t}(0) \in \mathcal{NC}^h, \frac{\partial u^{f,h}}{\partial t}(0) \in \mathcal{M}^h$ such that:

$$A_{\zeta,h}\left(\frac{\partial\delta^B}{\partial t}(0), v\right) = 0, \quad v = (v^s, v^f) \in \mathcal{NC}^h \times \mathcal{M}^h. \tag{91}$$

to see that, with the argument leading to (76):

$$\left\|\frac{\partial\gamma^B}{\partial t}(0)\right\|_{\mathcal{Z}^h} \leqslant Ch\ M_0. \tag{92}$$

Then use (92), that $\mathcal{P}$ and $\mathcal{D}$ are positive definite, that $\mathcal{A}_{\zeta,h}$ is $\mathcal{Z}^h$-coercive (see (60)), choose $\hat{\epsilon}$ sufficiently small in (90) and use Gronwall's lemma in the resulting inequality to obtain the following estimate:

$$\left\|\frac{\partial\gamma^B}{\partial t}\right\|_{L^\infty(J,L^2(\Omega_p))} + \|\gamma^B\|_{L^\infty(J,\mathcal{Z}^h)} + \left\|\frac{\partial\gamma^s}{\partial t}\right\|_{L^2(J,L^2(\Gamma_p))} + \left\|\frac{\partial\gamma^f\cdot v}{\partial t}\right\|_{L^2(J,L^2(\Gamma_p))} \leqslant C\left[h^{1/2}(M_0+M_1) + \int_0^t \|\gamma^E(s)\|_0^2 ds\right]. \tag{93}$$

Next, choose $\psi = v^s = v^f = 0$, $\varphi = \gamma^H$ in (68) to get:

$$(\nabla\times\gamma^E, \gamma^H) + \frac{1}{2}\frac{d}{dt}(\mu\gamma^H, \gamma^H) = (\nabla\times[\Pi_hE - E], \gamma^H) + \left(\mu\frac{\partial(P_hH - H)}{\partial t}, \gamma^H\right). \tag{94}$$

Thus, taking $\psi = \gamma^E$, $\varphi = 0$, $v^s = v^f = 0$ in (68) and using (94) yields:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[(\varepsilon\gamma^E,\gamma^E) + (\mu\gamma^H,\gamma^H)] + (\sigma\gamma^E,\gamma^E) + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi\gamma^E, P_\chi\gamma^E\right\rangle \\
&\quad= \left(\frac{\partial(\Pi_hE - E)}{\partial t}, \gamma^E\right) + (\nabla\times[\Pi_hE - E], \gamma^H) + \left(\mu\frac{\partial(P_hH - H)}{\partial t}, \gamma^H\right) + (\sigma[\Pi_hE - E], \gamma^E) \\
&\qquad + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi([\Pi_hE - E]), P_\chi\gamma^E\right\rangle - ([P_hH - H], \nabla\times\gamma^E).
\end{aligned} \tag{95}$$

Let us bound each term in the right-hand side of (95). First, the fact that $\nabla \times \gamma^E \in \mathcal{W}^h$ and the orthogonality property (52) imply that the last term in the right-hand side of (95) vanish. The other terms can be bounded using the approximating properties of Π_h and P_h in (46)–(48) as follows:

$$\left|\left(\frac{\partial(\Pi_h E - E)}{\partial t}, \gamma^E\right)\right| + \left|\left(\nabla \times [\Pi_h E - E], \gamma^H\right)\right| + \left|\left(\mu\frac{\partial(P_h H - H)}{\partial t}, \gamma^H\right)\right| + \left|\left(\sigma[\Pi_h E - E], \gamma^E\right)\right| \leqslant C\left(\|\gamma^E\|_0^2 + \|\gamma^H\|_0^2 + h^2\left(\|E\|_1^2 + \|\nabla \times E\|_1^2 + \left\|\frac{\partial E}{\partial t}\right\|_1^2 + \left\|\frac{\partial H}{\partial t}\right\|_1^2\right)\right), \tag{96}$$

and

$$\left|\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi([\Pi_h E - E]), P_\chi\gamma^E\right\rangle\right| \leqslant \|\Pi_h E - E\|_{0,\Gamma}\|P_\chi\gamma^E\|_{0,\Gamma} \leqslant h^{1/2}\|E\|_{1/2,\Gamma}\|P_\chi\gamma^E\|_{0,\Gamma} \leqslant \frac{1}{2}\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi\gamma^E, P_\chi\gamma^E\right\rangle + Ch\ \|E\|_1^2 \tag{97}$$

Thus, apply the bounds (96) and (97) in (95) to get the inequality:

$$\frac{1}{2}\frac{d}{dt}\left[(\varepsilon\gamma^E, \gamma^E) + (\mu\gamma^H, \gamma^H)\right] + (\sigma\gamma^E, \gamma^E) + \frac{1}{2}\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi\gamma^E, P_\chi\gamma^E\right\rangle \leqslant C\left(\|\gamma^E\|_0^2 + \|\gamma^H\|_0^2 + h\left(\|E\|_1^2 + \|\nabla \times E\|_1^2 + \|\frac{\partial E}{\partial t}\|_1^2 + \|\frac{\partial H}{\partial t}\|_1^2\right)\right) \tag{98}$$

Now integrate (98) from 0 to t and use (38) to get:

$$\|\gamma^E(t)\|_0^2 + \|\gamma^H(t)\|^2 + \int_0^t (\sigma\gamma^E, \gamma^E)(s)ds \leqslant C\left(\|\gamma^E(0)\|_0^2 + \|\gamma^H(0)\|_0^2 + \int_0^t \left(\gamma^E(s)\|_0^2 + \|\gamma^H(s)\|_0^2\right)ds + Ch\ N_1^2\right). \tag{99}$$

Next, choose $E^h(0)$ to be the Π_h-projection of $E(0)$ into $\mathcal{V}^h$ and $H^h(0)$ to be the P_h-projection of $H(0)$ into $\mathcal{W}^h$, so that using the triangle inequality we have that:

$$\|\gamma^E(0)\| \leqslant Ch\ \|E(0)\|_1, \quad \|\gamma^H(0)\| \leqslant Ch\ \|H(0)\|_1. \tag{100}$$

Thus use (100) and apply Gronwall's lemma in (99) to obtain the estimate:

$$\|\gamma^E\|_{L^\infty(J,L^2(\Omega))} + \|\gamma^H\|_{L^2(J,L^2(\Omega))} \leqslant Ch^{1/2}\ (N_0 + N_1). \tag{101}$$

Next using (101) in (93) we get:

$$\left\|\frac{\partial\gamma^B}{\partial t}\right\|_{L^\infty(J,L^2(\Omega_p))} + \|\gamma^B\|_{L^\infty(J,\mathcal{Z}^h)} + \left\|\frac{\partial\gamma^s}{\partial t}\right\|_{L^2(J,L^2(\Gamma_p))} + \left\|\frac{\partial\gamma^f\cdot\nu}{\partial t}\right\|_{L^2(J,L^2(\Gamma_p))} \leqslant Ch^{1/2}\ [M_0 + M_1 + N_0 + N_1)]. \tag{102}$$

Finally, using the triangle inequality, the approximating properties (46), (48) (49) and the estimates (101) and (102) we obtain the estimate in (62). This completes the proof. □

Remark. Notice that the loss of half power of h in the error estimate (62) is only due to the error terms associated with the absorbing boundary conditions appearing in the right-hand side of (81) and (97).

6. The discrete-time finite element procedure

Let L a positive integer, $\Delta t = T/L$, $g^n = g(n\Delta t)$. Set $\hat{g}^{n-1/2} = \frac{g^n + g^{n-1}}{2}$ and

$$\partial^2 g^n = \frac{g^{n+1} - 2g^n + g^{n-1}}{\Delta t^2}, \quad \partial g^n = \frac{g^{n+1} - g^{n-1}}{2\Delta t}, \quad d_t g^n = \frac{g^{n+1} - g^n}{\Delta t}.$$

Our fully implicit discrete-time procedure is defined as follows: given $(E^{h,0}, H^{h,1}, u^{s,h,0}, u^{s,h,1}, u^{f,h,0}, u^{f,h,1}) \in \mathcal{V}^h \times \mathcal{W}^h \times (\mathcal{NC}^h)^2 \times (\mathcal{M}^h)^2$, for $n \geqslant 1$ compute $(E^{h,n}, H^{h,n}, u^{s,h,n+1}, u^{f,h,n+1}) \in \mathcal{Y}^h$ such that:

$$\left(\varepsilon d_t E^{h,n-1}, \psi\right) + \left(\sigma\widehat{E}^{h,n-1/2}, \psi\right) - \left(\widehat{H}^{h,n-1/2}, \nabla \times \psi\right) + \left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2} P_\chi\widehat{E}^{h,n-1/2}, P_\chi\psi\right\rangle = (J_e^{s,n}, \psi), \quad \psi \in \mathcal{V}^h, \tag{103}$$

$$\left(\nabla\times\widehat{E}^{h,n-1/2},\varphi\right)+\left(\mu d_t H^{h,n-1},\varphi\right)=0,\quad \varphi\in\mathcal{W}^h, \tag{104}$$

$$(\mathcal{P}\partial^2 u^{h,n},v)_{\Omega_p}+\left(\frac{\eta}{\kappa_0}\partial u^{f,h,n},v^f\right)_{\Omega_p}+\mathcal{A}_h\left(\left(\frac{u^{h,n+1}+u^{h,n-1}}{2}\right),v\right)+\langle\mathcal{D}S_\Gamma(\partial u^{h,n}),S_\Gamma(v)\rangle_{\Gamma_p}-\left(L_0\frac{\eta}{\kappa_0}\widehat{E}^{h,n-1/2},v^f\right)=0,$$
$$v=(v^s,v^f)\in\mathcal{NC}^h\times\mathcal{M}^h. \tag{105}$$

To analyze the procedure (103)–(105), choose $\psi=\widehat{E}^{h,n-1/2}$ in (103) and $\varphi=\widehat{H}^{h,n-1/2}$ in (104) and add the resulting equations to get:

$$\frac{1}{2\Delta t}\left[\left(\varepsilon E^{h,n},E^{h,n}\right)-\left(\varepsilon E^{h,n-1},E^{h,n-1}\right)+\left(\mu H^{h,n},H^{h,n}\right)-\left(\mu H^{h,n-1},H^{h,n-1}\right)\right]+\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}P_\chi\widehat{E}^{h,n-1/2},P_\chi\widehat{E}^{h,n-1/2}\right\rangle$$
$$=\left(J_e^{s,n},\widehat{E}^{h,n-1/2}\right). \tag{106}$$

Next, choose $v=\partial u^{h,n}$ in (105) and add the resulting equation to (106) to obtain:

$$\frac{1}{2\Delta t}\Big[\left(\varepsilon E^{h,n},E^{h,n}\right)-\left(\varepsilon E^{h,n-1},E^{h,n-1}\right)+\left(\mu H^{h,n},H^{h,n}\right)-\left(\mu H^{h,n-1},H^{h,n-1}\right)+(\mathcal{P}d_t u^{h,n},d_t u^{h,n})_{\Omega_p}-(\mathcal{P}d_t u^{h,n-1},d_t u^{h,n-1})_{\Omega_p}$$
$$+\frac{1}{4}\mathcal{A}_h(u^{h,n+1},u^{h,n+1})-\frac{1}{4}\mathcal{A}_h(u^{h,n-1},u^{h,n-1})\Big]+\Phi\left(\widehat{E}^{h,n-1/2},\partial u^{f,h,n}\right)+\left(\sigma\widehat{E}^{h,n-1/2},\widehat{E}^{h,n-1/2}\right)_{\Omega_a}$$
$$+\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}P_\chi\widehat{E}^{h,n-1/2},P_\chi E^{h,n+1/2}\right\rangle+\langle\mathcal{D}S_\Gamma(\partial u^{h,n}),S_\Gamma(\partial u^{h,n})\rangle_{\Gamma_p}=\left(J_e^{s,n},\widehat{E}^{h,n-1/2}\right), \tag{107}$$

where $\Phi(\widehat{E}^{h,n+1/2},\partial u^{f,h,n})$ is defined in (31). Add the inequality:

$$\frac{\zeta}{4\Delta t}\left(\|u^{h,n+1}\|^2_{0.\Omega_p}-\|u^{h,n-1}\|^2_{0.\Omega_p}\right)\leqslant C\left(\|u^{h,n+1}\|^2_{0.\Omega_p}+\|u^{h,n}\|^2_{0.\Omega_p}+\|u^{h,n-1}\|^2_{0.\Omega_p}+\|d_t u^{h,n}\|^2_{0.\Omega_p}+\|d_t u^{h,n-1}\|^2_{0.\Omega_p}\right) \tag{108}$$

to (107), multiply the result by Δt, add from $n=1$ to $n=N$ and use (34) and (60) to conclude that:

$$\left(\varepsilon E^{h,N},E^{h,N}\right)+\left(\mu H^{h,N},H^{h,N}\right)+(\mathcal{P}d_t u^{h,N},d_t u^{h,N})_{\Omega_p}+\frac{C_2}{4}\left(\|u^{h,N+1}\|^2_{\mathcal{Z}^h}+\|u^{h,N}\|^2_{\mathcal{Z}^h}\right)+C_1\sum_{n=1}^{N}\left(\|\widehat{E}^{h,n-1/2}\|^2_{0,\Omega_p}+\|\partial u^{f,h,n})\|^2_{0,\Omega_p}\right)\Delta t$$
$$+\sum_{n=1}^{N}\left(\left(\sigma\widehat{E}^{h,n-1/2},\widehat{E}^{h,n-1/2}\right)_{\Omega_a}+\left\langle\left(\frac{\varepsilon}{\mu}\right)^{1/2}\widehat{P}_\chi E^{h,n-1/2},P_\chi E^{h,n-1/2}\right\rangle+\langle\mathcal{D}S_\Gamma(\partial u^{h,n}),\ S_\Gamma(\partial u^{h,n})\rangle_{\Gamma_p}\right)\Delta t$$
$$\leqslant C\Bigg(\|E^{h,0}\|_0^2+\|H^{h,0}\|_0^2+\|u^{h,0}\|^2_{\mathcal{Z}^h}+\|u^{h,1}\|^2_{\mathcal{Z}^h}+\|d_t u^{h,0}\|^2_{0,\Omega_p}$$
$$+\sum_{n=1}^{N}\left(\|J_e^{s,n}\|_0^2++\|u^{h,n+1}\|^2_{0,\Omega_p}+\|u^{h,n}\|^2_{0,\Omega_p}+\|u^{h,n-1}\|^2_{0,\Omega_p}+\|d_t u^{h,n}\|^2_{0,\Omega_p}+\|d_t u^{h,n-1}\|^2_{0,\Omega_p}\right)\Delta t\Bigg). \tag{109}$$

Next apply Gronwall's lemma in (109), use (38) and that the matrices $\mathcal{P}$ and $\mathcal{D}$ are positive definite to conclude that:

$$\max_{1\leqslant n\leqslant L-1}\left[\|E^{h,n}\|_0^2+\|H^{h,n}\|_0^2+\|d_t u^{h,n}\|^2_{0,\Omega_p}+\|u^{h,n}\|^2_{\mathcal{Z}^h}\right]+\sum_{n=1}^{L-1}\left(\|\widehat{E}^{h,n-1/2}\|^2_{0,\Omega_p}+\|\partial u^{f,h,n}\|^2_{0,\Omega_p}\right)\Delta t$$
$$\times\sum_{n=1}^{L-1}\left((\sigma\widehat{E}^{h,n-1/2},\widehat{E}^{h,n-1/2})_{\Omega_a}+\|P_\chi\widehat{E}^{h,n-1/2}\|^2_{0,\Gamma}+\|\partial u^{s,h,n}\|^2_{0,\Gamma_p}+\|\partial u^{f,h,n}\cdot\nu\|^2_{0,\Gamma_p}\right)\Delta t$$
$$\leqslant C\left(\|E^{h,0}\|_0+\|H^{h,0}\|_0+\|u^{h,0}\|_{\mathcal{Z}^h}+\|u^{h,1}\|_{\mathcal{Z}^h}+\|d_t u^{h,0}\|_{0,\Omega_p}+\sum_{n=1}^{L-1}\|J_e^{s,n}\|^2_{0,\Omega_p}\Delta t\right). \tag{110}$$

The estimate (110) yields uniqueness, existence and unconditional stability for the procedure (103)–(105).

7. The case of tetrahedral elements

Let $\mathcal{T}^{\langle}(\Omega)$ be a nonoverlapping quasiregular partition of $\Omega=\Omega_p\cup\Omega_a$ into tetrahedral elements Ω_j of diameter bounded by h and for $\mathbf{x}=(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ let:

$$R_1=\left\{U:\ u=(\alpha+\beta\mathbf{x},\alpha,\beta\in(P_0)^3\right\},$$

Then following [20,21] the space $\mathcal{V}^h$ to approximate the electric field is:

$$\mathcal{V}^h=\left\{\psi\in H(\mathrm{curl},\Omega):\psi|_{\Omega_j}\in R_1(\Omega_j)\ \forall\ j\right\},$$

while the space to approximate the magnetic field is:

$$\mathcal{W}^h = \left\{\varphi \in (L^2(\Omega))^3 : \varphi|_{\Omega_j} \in (P_0(\Omega_j))^3 \ \forall\, j\right\}.$$

Nédélec [20] showed that the following degrees of freedom:

$$M_j^{\psi} = \left\{\int_{e_j} \psi \cdot \chi^j ds, \quad \text{for the six edges} \ \ e_j \ \text{of} \ \ \Omega_j\right\},$$

where χ_j is a unit vector parallel to e_j, are R_1-solvent and curl-conforming, that curl $\mathcal{V}^h \subset \mathcal{W}^h$ and that (47) holds.

Let us define the projection:

$$\Pi_h : H(\text{curl}, \Omega) \to \mathcal{V}^h : \int_{e_j} (\psi - \Pi_h \psi) \cdot \chi^j ds = 0, \quad 1 \leqslant j \leqslant 6, \tag{111}$$

where χ^j is a unit vector parallel to the edge e_j.

Also, let the L^2-projection P_h be defined as in (42). Then the approximating properties (46)–(48) remain valid.

The nonconforming finite element space to approximate the solid displacement vector is defined as in [24] as follows: Let $\mathcal{NC}_j^h = [P_1(\Omega_j)]^3$ and

$$\mathcal{NC}^h = \left\{v : v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h, \ \ v_j(\xi_{jk}) = v_k(\xi_{jk}) \quad \forall\, (j,k)\right\}.$$

The four local degrees of freedom are the values at the centers ξ_{jk} of the faces of Ω_j. Next, let:

$$S_1 = \{U : U = \alpha + \lambda \mathbf{x}, \alpha \in [P_0]^3, \lambda \in P_0\}, \tag{112}$$

and let:

$$\mathcal{M}^h = \{v^f \in H(\text{div}, \Omega_p) : v_j = v|_{\Omega_j} \in S_1 \ \forall\, j\}. \tag{113}$$

The projections:

$$R_h : [H^2(\Omega_p)]^3 \to [\mathcal{NC}^h]^3,$$
$$Q_h : [H^1(\Omega_p)]^3 \cup \left(H_h^1(\Omega_p)\right)^3 \to \mathcal{M}^h,$$
$$S_h : [H^2(\Omega_p)]^3 \times H^1(\text{div}; \Omega_p) \to \widetilde{\Lambda}^h,$$

are defined identically than in (43)–(45). The degrees of freedom:

$$N_j(v^f) = \{\langle v^f \cdot \nu, 1\rangle_B, B \quad \text{any of the four faces of} \ \ \Omega_j\} \tag{114}$$

are S_1-solvent and conforming in $H(\text{div};\Omega_p)$ (see [20]), so that (44) uniquely defines Q_h.

Setting:

$$\mathcal{Y}^h = \mathcal{V}^h \times \mathcal{W}^h \times \mathcal{NC}^h \times \mathcal{M}^h,$$

the definition of the continuous and discrete-time Galerkin procedures (56)–(58) and (103)–(105) remain unchanged. Also, the existence and uniquess results and *apriori* error estimates in Theorems 2 and 3 and the stability results as derived in (110) remain valid.

8. 2D case. The PSVTM and SHTE modes

In this section we will assume that all physical quantities describing our domains Ω_a and Ωp are independent of the x_2-direction (i.e., x_2 is the symmetry axis) and consider two types of electromagnetic sources. First, if the source is an infinite solenoid J_m^s in the x_2-direction at depth $x_3 = 0$, under the above symmetry asumption, this source term induces electric and magnetic fields of the form $(E_1(x_1,x_3,t),0,E_3(x_1,x_3,t))$, $(0,H_2(x_1,x_3,t),0)$, respectively, and solid and relative fluid displacements of the form $u^s = (u_1^s(x_1,x_3,t), 0, u_3^s(x_1,x_3,t))$ and $u^f = \left(u_1^f(x_1,x_3,t), 0, u_3^f(x_1,x_3,t)\right)$, respectively. Consequently only compressional and vertically polarized shear seismic waves (PSV-waves) are generated. This is a 2D model known as a PSVTM-mode.

On the other hand, if the electromagnetic source is a infinite line source current density J_e^s in the x_2-direction at depth $x_3 = 0$, this source term induces electromagnetic fields $(0,E_2(x_1,x_3,t),0)$ and $(H_1(x_1,x_3,t),0,H_3(x_1,x_3,t))$ and horizontally polarized shear waves (SH-waves), so that $u^s = (0, u_2^s(x_1,x_3,t), 0)$, $u^f = (0, u_2^f(x_1,x_3,t), 0)$, and consequently:

$$\nabla \cdot u^s = \nabla \cdot u^f = 0. \tag{115}$$

Hence, it follows from (6) and (115) that:

$$p_f = 0. \tag{116}$$

In this case we get another 2D model known as SHTE-mode.

For the PSVTM-mode, let us identify the 3D vectors $(E_1(x_1,x_3,t),0,E_3(x_1,x_3,t))$ and $(0,H_2(x_1,x_3,t)),0)$ with the 2D vector $E(x_1,x_3,t) = (E_1(x_1,x_3,t),E_3(x_1,x_3,t))$ and the scalar $H_2(x_1,x_3,t)$, respectively. Similarly, for the SHTE-mode, let us identify the 3D vectors $(H_1(x_1,x_3,t),0,H_3(x_1,x_3,t))$ and $(0,E_2(x_1,x_3,t)),0)$ with the 2D vector $H(x_1,x_3,t) = (H_1(x_1,x_3,t),H_3(x_1,x_3,t))$ and the scalar $E_2(x_1,x_3,\, t)$, respectively.

Next recall that for a scalar function φ and a 2D vector function $V = (V_1,V_3)$:

$$\text{curl } \varphi = \left(-\frac{\partial\varphi}{\partial x_3},\frac{\partial\varphi}{\partial x_1}\right), \quad \text{curl } V = \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}.$$

Also, let us identify our 3D-rectangular domain Ω with the 2D-rectangular domain $\Omega \cap \{x_2 = 0\}$, so that Ω is the union of the disjoint rectangular subdomains Ω_a and Ω_p.

For 2D PSVTM electroseismic modeling the electric and magnetic fields E and H and the displacement vectors u^s and u^f satisfy the coupled electromagnetic-poroelastic equations, stated in the space-time domain as follows:

$$\varepsilon\frac{\partial E}{\partial t} + \sigma E - \text{curl } H_2 = 0, \quad \Omega, \tag{117}$$

$$\text{curl } E + \frac{\partial H_2}{\partial t} = J^s_m, \quad \Omega, \tag{118}$$

$$\rho_b\frac{\partial^2 u^s}{\partial t^2} + \rho_f\frac{\partial^2 u^f}{\partial t^2} - \nabla\cdot\tau(u) = 0, \quad \Omega_p, \tag{119}$$

$$\rho_f\frac{\partial^2 u^s}{\partial t^2} + m\frac{\partial^2 u^f}{\partial t^2} + \frac{\eta}{\kappa_0}\frac{\partial u^f}{\partial t} - L_0\frac{\eta}{\kappa_0}E + \nabla p_f(u) = 0, \quad \Omega_p, \tag{120}$$

For 2D SHTE electroseismics, the corresponding equations are:

$$\varepsilon\frac{\partial E_2}{\partial t} + \sigma E_2 - \text{curl } H = J^s_e, \quad \Omega, \tag{121}$$

$$\text{curl } E_2 + \frac{\partial H}{\partial t} = 0, \quad \Omega, \tag{122}$$

$$\rho_b\frac{\partial^2 u^s_2}{\partial t^2} + \rho_f\frac{\partial^2 u^f_2}{\partial t^2} - \nabla\cdot\left(G\nabla u^s_2\right) = 0, \quad \Omega_p, \tag{123}$$

$$\rho_f\frac{\partial^2 u^s_2}{\partial t^2} + m\frac{\partial^2 u^f_2}{\partial t^2} + \frac{\eta}{\kappa_0}\frac{\partial u^f_2}{\partial t} - L_0\frac{\eta}{\kappa_0}E_2 = 0, \quad \Omega_p, \tag{124}$$

Set:

$$\mathcal{G}_{\Gamma_s}(u) = \left(\tau(u)\nu\cdot\nu, \tau(u)\nu\cdot\chi, p_f(u)\right)^t, \tag{126a}$$

$$S_{\Gamma_s}(u) = \left(u^s\cdot\nu, u^s\cdot\chi, u^f\cdot\nu\right)^t, \tag{126b}$$

where Γ_s is any subset of $\partial\Omega_p$, ν is the unit outer normal on Γ_s and χ is a unit tangent on Γ_s oriented counterclockwise.

Then, for the PSVTM-mode, consider the solution of (117)–(120) with the absorbing boundary conditions:

$$-\varepsilon^{1/2}E\cdot\chi + H_2 = 0, \quad \text{on } \Gamma, \tag{127}$$

$$-\mathcal{G}_{\Gamma_p}(u) = \mathcal{D}S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right), \quad \text{on } \Gamma_p, \tag{128}$$

the free surface condition:

$$-\mathcal{G}_{\Gamma_p}(u) = 0, \quad \text{on } \Gamma_{a,p}, \tag{129}$$

and the initial conditions:

$$\begin{aligned}
&E(x_1,x_3,t=0) = E_0, \quad H_2(x_1,x_3,t=0) = H_{2,0},\\
&u^s(x_1,x_3,t=0) = u^s_0, \quad u^f(x_1,x_3,t=0) = u^f_0,\\
&\frac{\partial u^s}{\partial t}(x_1,x_3,t=0) = u^s_1, \quad \frac{\partial u^f}{\partial t}(x_1,x_3,t=0) = u^f_1.
\end{aligned} \tag{130}$$

The matrix $\mathcal{D}$ in (132) is defined as in the 3D case, changing the definition of the matrices $\mathcal{R}$ and $\mathcal{M}$ in the obvious fashion.

For the SHTE-mode, consider the solution of (121)–(124) with the absorbing boundary conditions:

$$\mu^{1/2}H\cdot\chi - \varepsilon^{1/2}E_2 = 0, \quad \text{on } \Gamma, \tag{131}$$

$$-G\nabla u^s_2\cdot\nu = \alpha\frac{\partial u^s_2}{\partial t}, \quad \alpha = \left(G(\rho_b - \rho_f^2/m)\right)^{1/2}, \quad \text{on } \Gamma_p, \tag{132}$$

the free surface condition:

$$G\nabla u_2^s \cdot \nu = 0, \quad \text{on } \Gamma_{a,p}, \tag{133}$$

and the initial conditions:

$$\begin{aligned} &H(x_1, x_3, t=0) = H_0, \quad E_2(x_1, x_3, t=0) = E_{2,0} \\ &u_2^s(x_1, x_3, t=0) = u_{2,0}^s, \quad u_2^f(x_1, x_3, t=0) = u_{2,0}^f, \\ &\frac{\partial u_2^s}{\partial t}(x_1, x_3, t=0) = u_{2,1}^s, \quad \frac{\partial u_2^f}{\partial t}(x_1, x_3, t=0) = u_{2,1}^f. \end{aligned} \tag{134}$$

Remark. The absorbing boundary conditions employed for Maxwell's and Biot's equations were obtained from the general 3D case.

Using the integration by parts formula[35]:

$$(\psi, \text{curl } \varphi) - (\text{curl}\psi, \varphi) = \langle \psi \cdot \chi, \varphi \rangle, \quad \forall\ \psi \in H(\text{curl}, \Omega), \varphi \in H^1(\Omega), \tag{135}$$

for the PSVTM-mode we get the weak form: find $(E, H_2, u^s, u^f) \in H(\text{curl}, \Omega) \times L^2(\Omega) \times [H^1(\Omega_p)]^2 \times H(\text{div}, \Omega_p)$ satisfying:

$$\left(\varepsilon \frac{\partial E}{\partial t}, \psi\right) + (\sigma E, \psi) - (H_2, \text{curl}\psi) + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} E \cdot \chi, \psi \cdot \chi \right\rangle = 0,$$
$$\psi \in H(\text{curl}, \Omega), \tag{136}$$

$$(\text{curl } E, \varphi) + \left(\mu \frac{\partial H_2}{\partial t}, \varphi\right) = (J_m^s, \varphi), \quad \varphi \in L^2(\Omega), \tag{137}$$

$$\left(\mathcal{P} \frac{\partial^2 u}{\partial t^2}, v\right)_{\Omega_p} + \left(\frac{\eta}{\kappa_0} \frac{\partial u^f}{\partial t}, v^f\right)_{\Omega_p} + \mathcal{A}(u, v) - \left(L_0 \frac{\eta}{\kappa_0} E, v^f\right)_{\Omega_p} + \left\langle \mathcal{D} S_{\Gamma_p}\left(\frac{\partial u}{\partial t}\right), S_{\Gamma_p}(v) \right\rangle_{\Gamma_p} = 0,$$
$$v = (v^s, v^f) \in [H^1(\Omega_p)]^2 \times H(\text{div}, \Omega_p). \tag{138}$$

Remark. The matrix $\mathcal{P}$ in (138) is defined as in (26) for $d = 2$ while the bilinear form $\mathcal{A}$ is given by (27) deleting rows and columns 3, 6, and 7 in the matrix **M**.

Similarly, for the SHTE-mode, the weak form is: find $(E_2, H, u_2^s, u_2^f) \in L^2(\Omega) \times H(\text{curl}, \Omega) \times H^1(\Omega_p) \times L^2(\Omega_p)$ satisfying:

$$\left(\varepsilon \frac{\partial E_2}{\partial t}, \varphi\right) + (\sigma E_2, \varphi) - (\text{curl } H, \varphi) = (J_e^s, \varphi), \quad \varphi \in L^2(\Omega), \tag{139}$$

$$(E_2, \text{curl}\psi) + \left(\mu \frac{\partial H}{\partial t}, \psi\right) + \left\langle \left(\frac{\mu}{\varepsilon}\right)^{1/2} H \cdot \chi, \psi \cdot \chi \right\rangle = 0, \quad \psi \in H(\text{curl}, \Omega), \tag{140}$$

$$\left(\rho_b \frac{\partial^2 u_2^s}{\partial t^2}, v^s\right)_{\Omega_p} + \left(\rho_f \frac{\partial^2 u_2^f}{\partial t^2}, v^s\right)_{\Omega_p} + (G\nabla u_2^s, \nabla v^s)_{\Omega_p} + \left\langle \alpha\left(\frac{\partial u_2^s}{\partial t}\right), v^s \right\rangle_{\Gamma_p} + \left(\rho_f \frac{\partial^2 u_2^s}{\partial t^2}, v^f\right)_{\Omega_p} + \left(m \frac{\partial^2 u_2^f}{\partial t^2}, v^f\right)_{\Omega_p}$$
$$+ \left(\frac{\eta}{\kappa_0} \frac{\partial u_2^f}{\partial t}, v^f\right)_{\Omega_p} - \left(L_0 \frac{\eta}{\kappa_0} E_2, v^f\right)_{\Omega_p} = 0, \quad v = (v^s, v^f) \in H^1(\Omega_p) \times L^2(\Omega_p). \tag{141}$$

Uniqueness for the solution of 136,23,24 and (139)–(141) follows with the same argument than for the 3D case.

9. Finite element methods for the PSVTM and SHTE modes

First consider the case that $\mathcal{T}^h(\Omega)$ is a nonoverlapping quasiregular partition of $\Omega = \Omega_p \cup \Omega_a$ into rectangles Ω_j of diameter bounded by h such that $\overline{\Omega} = \cup_{j=1}^J \overline{\Omega}_j$.

Let us consider first the PSVTM-mode. To approximate the electromagnetic fields E and H_2 we will employ the mixed finite element space $\mathcal{V}^h \times \mathcal{W}^h$, defined as follows [19,23]:

$$\mathcal{V}^h = \left\{\psi \in H(\text{curl}, \Omega) : \ \psi|\Omega_j \in \mathcal{V}_j^h \equiv P_{0,1}(\Omega_j) \times P_{1,0}(\Omega_j)\right\},$$
$$\mathcal{W}^h = \left\{\varphi \in L^2(\Omega) : \ \varphi|\Omega_j \in \mathcal{W}_j^h \equiv P_0(\Omega_j)\right\}.$$

The functions in $\mathcal{V}^h$ have continuous tangential components across the internal boundaries Γ_{jk}. Also, curl $\mathcal{V}^h \subset \mathcal{W}^h$.

Following [23], the degrees of freedom for $\mathcal{V}^h$ are defined in the following way. Let Ω_j be a general element of the partition $\mathcal{T}^{\langle}(\Omega)$ and let $\psi \in [H^1(\Omega_j)]^2$. Then define the following moments on Γ_{jk}:

$$M_{\Gamma_{jk}}(\psi) = \left\{ \langle \psi \cdot \tau, f \rangle_{\Gamma_{jk}}; \quad f \in P_0(\Gamma_{jk}) \right\}. \tag{142}$$

Note that (142) are curl–conforming and unisolvent for elements in $\mathcal{V}^h$.

To approximate each component of the solid displacement vector we employ the nonconforming finite element space $\mathcal{NC}^h$ as in [24], while to approximate the fluid displacement vector we choose $\mathcal{M}^h$, the vector part of the Raviart–Thomas–Nédélec space [18,19] of zero order. Thus:

$$\widehat{R} = [-1,1]^2, \quad Q(\widehat{R}) = \text{Span}\{1, \hat{x}_1, \hat{x}_3, \tilde{\alpha}(\hat{x}_1) - \tilde{\alpha}(\hat{x}_3)\}, \quad \tilde{\alpha}(z) = z^2 - \frac{5}{3}z^4,$$

with the degrees of freedom being the values at the midpoint of each edge of $\widehat{R}$. Next let $\mathcal{NC}_j^h = [Q(\Omega_j)]^2$ and

$$\mathcal{NC}^h = \left\{ v : v_j = v|_{\Omega_j} \in \mathcal{NC}_j^h, v_j(\xi_{jk}) = v_k(\xi_{jk}) \quad \forall\ (j,k) \right\}, \tag{143}$$

$$\mathcal{M}^h = \left\{ w \in H(\text{div}, \Omega_p) : \ w|\Omega_j \in \mathcal{M}_j^h \equiv P_{1,0}(\Omega_j) \times P_{0,1}(\Omega_j) \right\}. \tag{144}$$

Next consider the case in which $\mathcal{T}^h$ is a quasiregular partition of Ω into triangles Ω_j of diameter bounded by h. Set:

$$R_1 = \{U : u = (a + bx_3, c + dx_1), \quad a, b, c \text{ constants}\},$$

and let us define the spaces to approximate the electric and magnetic fields $\mathcal{V}^h$ and $\mathcal{V}^h$ as:

$$\mathcal{V}^h = \left\{ \psi \in H(\text{curl}, \Omega) : \psi|_{\Omega_j} \in R_1(\Omega_j)\ \forall\ j \right\},$$

$$\mathcal{W}^h = \left\{ \varphi \in L^2(\Omega) : \varphi|_{\Omega_j} \in P_0(\Omega_j)\ \forall\ j \right\}.$$

The local degrees of freedom for $\mathcal{V}^h$ and $\mathcal{W}^h$ can be taken to be the values of the tangential components at the mid-points of each edge of the triangle Ω_j and the values at the centroids of Ω_j, respectively.

Next, if:

$$\mathcal{NC}_j^h = [P_1(\Omega_j)]^2,$$

the space $\mathcal{NC}^h$ to approximate each component of the solid displacement in Ω_p is defined as in the rectangular case in (143). with the local degrees of freedom being the values at the mid-point of the edges of each Ω_j.

Finally if:

$$S_1 = \{U : u = (a + bx_1, c + dx_3), \quad a, b, c \text{ constants}\},$$

the space to approximate the fluid displacement vector is:

$$\mathcal{M}^h = \left\{ v^f \in H(\text{div}, \Omega_p) : v^f|_{\Omega_j} \in S_1(\Omega_j)\ \forall\ j \right\}.$$

The local degrees of freedom for $\mathcal{M}^h$ are the values of normal components at the mid-points of each edge of the triangle Ω_j.

The approximating properties of the 2D finite element spaces defined above are the same than those stated in the 3D case.

Let:

$$\mathcal{Y}^h = \mathcal{V}^h \times \mathcal{W}^h \times \mathcal{NC}^h \times \mathcal{M}^h.$$

Then the continuous-time Galerkin procedure for the PSVTM-mode is defined as follows: find $\left(E^h, H_2^h, u^{s,h}, u^{f,h}\right) : J \to \mathcal{Y}^h$ such that:

$$\left(\varepsilon \frac{\partial E^h}{\partial t}, \psi\right) + (\sigma E^h, \psi) - \left(H_2^h, \text{curl}\psi\right) + (\text{curl } E^h, \varphi) + \left\langle \left(\frac{\varepsilon}{\mu}\right)^{1/2} E^h \cdot \chi, \psi \cdot \chi \right\rangle + \left(\mu \frac{\partial H_2^h}{\partial t}, \varphi\right) + \left(\mathcal{P}\frac{\partial^2 u^h}{\partial t^2}, v\right)_{\Omega_p}$$
$$+ \left(\frac{\eta}{\kappa_0}\frac{\partial u^{f,h}}{\partial t}, v^f\right)_{\Omega_p} + \mathcal{A}_h(u^h, v) - \left(L_0 \frac{\eta}{\kappa_0} E^h, v^f\right)_{\Omega_p} + \left\langle \mathcal{D}S_\Gamma\left(\frac{\partial u^h}{\partial t}\right), S_\Gamma(v)\right\rangle_{\Gamma_p} = (J_e^s, \varphi), \quad (\psi, \varphi, v^s, v^f) \in \mathcal{Y}^h, \tag{145}$$

with the initial conditions:

$$E^h(t=0) \approx E_0, \quad H_2^h(t=0) \approx H_{2,0}, \tag{146}$$

$$u^{s,h}(t=0) \approx u_0^s, \quad u^{f,h}(t=0) \approx u_0^f. \tag{147}$$

Remark. In (145) the bilinear form $\mathcal{A}_h$ is defined as in (54), changing the definition of the matrix **M** as indicated in the remark below (138).

Next we consider the SHTE-mode. In this case we will employ the spaces $\mathcal{V}^h$ and $\mathcal{W}^h$ to approximate the magnetic vector field H and the scalar field E_2. Also, to approximate the solid displacement u_2^s in Ω_p we employ the (scalar) nonconforming finite element space $\mathcal{NC}^h$ (*i.e.* in (143) change the definition of $\mathcal{NC}_j^h$to $\mathcal{NC}_j^h = Q(\Omega_j)$ (resp. $\mathcal{NC}_j^h = P_1(\Omega_j)$, depending on whether we have rectangular of triangular elements). To approximate the fluid displacement u_2^s we choose the space of piecewise constants over the restriction of the partition $\mathcal{T}^\langle(\Omega)$ to Ω_p. Thus, for the fluid displacement, the space $\widehat{\mathcal{M}}_|^h$ is defined as:

$$\widehat{\mathcal{M}}_j^h = \left\{ w : \; w|_{\Omega_j} \in P_0(\Omega_j) \right\}, \quad \widehat{\mathcal{M}}^h = \left\{ w \in L^2(\Omega) : w_j = w|_{\Omega_j} \in \widehat{\mathcal{M}}_j^h \right\}$$

Let:

$$\widehat{\mathcal{Y}}^h = \mathcal{W}^h \times \mathcal{V}^h \times \mathcal{NC}^h \times \widehat{\mathcal{M}}^h.$$

Then the global finite element procedure for the approximate solution of the SHTE-mode is defined as follows: Find $\left(E_2^h, H^h, u_2^{s,h}, u_2^{f,h}\right) \in \widehat{\mathcal{Y}}^h$ such that:

$$\begin{aligned}
&\left(\varepsilon \frac{\partial E_2^h}{\partial t}, \varphi\right) + \left(\sigma E_2^h, \varphi\right) - \left(\text{curl } H^h, \varphi\right) + \left(E_2^h, \text{curl } \psi\right) + \left(\mu \frac{\partial H^h}{\partial t}, \psi\right) + \left\langle \left(\frac{\mu}{\varepsilon}\right)^{1/2} H^h \cdot \chi, \psi \cdot \chi \right\rangle \\
&\quad + \left(\mu \frac{\partial H^h}{\partial t}, \varphi\right) \left(\rho_b \frac{\partial^2 u_2^{s,h}}{\partial t^2}, v^s\right)_{\Omega_p} + \left(\rho_f \frac{\partial^2 u_2^{f,h}}{\partial t^2}, v^s\right)_{\Omega_p} + \left(G \nabla u_2^{s,h}, \nabla v^s\right)_{\Omega_p} + \left\langle \alpha \left(\frac{\partial u_2^{s,h}}{\partial t}\right), v^s \right\rangle_{\Gamma_p} \\
&\quad + \left(\rho_f \frac{\partial^2 u_2^{s,h}}{\partial t^2}, v^f\right)_{\Omega_p} + \left(m \frac{\partial^2 u_2^{f,h}}{\partial t^2}, v^f\right)_{\Omega_p} + \left(\frac{\eta}{\kappa_0} \frac{\partial u_2^{f,h}}{\partial t}, v^f\right)_{\Omega_p} - \left(L_0 \frac{\eta}{\kappa_0} E_2^h, v^f\right)_{\Omega_p} \\
&\quad = \left(J_e^s, \varphi\right), \quad \left(\varphi, \psi, v^s, v^f\right) \in \mathcal{Y}^h,
\end{aligned} \tag{148}$$

with the initial conditions:

$$E_2^h(t=0) \approx E_{2,0}, \quad H^h(t=0) \approx H_0, \tag{149}$$

$$u_2^{s,h}(t=0) \approx u_{2,0}^s, \quad u_2^{f,h}(t=0) \approx u_{2,0}^f. \tag{150}$$

Existence and uniqueness results for (145)–(150) follow with the argument given for the 3D case.

Discrete time finite element procedures for the PSVTM and SHTE-modes can be defined and analyzed similarly to that in (103)–(105) for the 3D case and for brevity are not stated here.

10. Numerical experiments for electroseismic modeling

Consider the ideal case of an infinite plane of current density in the x_2-direction. In this case the electromagnetic and displacement fields depend only on the x_3-direction and have components $(0, E_2(x_3, t), 0)$ and $(H_1(x_3, t), 0, 0)$, $u^s = \left(0, u_2^s(x_3, t), 0\right)$, $u^f = \left(0, u_2^f(x_3, t), 0\right)$, and we have a 1D SHTE model. Thus, if $\Omega_a = (0 = \Gamma^T, \Gamma_{a,p})$, $\Omega_p = (\Gamma_{a,p}, \Gamma^B)$ and the x_3-axis is positive downward (1)–(6) reduce to:

$$\varepsilon \frac{\partial E_2}{\partial t} + \sigma E_2 - \frac{\partial H_1}{\partial x_3} = J_e^s = \delta_{x_3} g(t), \quad \Omega, \tag{151}$$

$$-\frac{\partial E_2}{\partial x_3} + \frac{\partial H_1}{\partial t} = 0, \quad \Omega, \tag{152}$$

$$\rho_b \frac{\partial^2 u_2^s}{\partial t^2} + \rho_f \frac{\partial^2 u_2^f}{\partial t^2} - \frac{\partial}{\partial x_3}\left(G \frac{\partial u_2^s}{\partial x_3}\right) = 0, \quad \Omega_p, \tag{153}$$

$$\rho_f \frac{\partial^2 u_2^s}{\partial t^2} + m \frac{\partial^2 u_2^f}{\partial t^2} + \frac{\eta}{\kappa_0} \frac{\partial u_2^{f,h}}{\partial t} - L_0 \frac{\eta}{\kappa_0} E_2 = 0, \quad \Omega_p, \tag{154}$$

with the boundary conditions:

$$\mu^{1/2} H_1 + v \varepsilon^{1/2} E_2 = 0, \quad \text{on } \Gamma \tag{155}$$

$$-vG \frac{\partial u_2^s}{\partial x_3} = \alpha \frac{\partial u_2^s}{\partial t}, \quad \text{on } \Gamma_p, \tag{156}$$

where $v = -1$ on Γ^T and $v = 1$ on Γ^B, and initial conditions:

$$H_1(x_3,t=0)=H_{1,0},\quad E_2(x_3,t=0)=E_{2,0}$$
$$u_2^s(x_3,t=0)=u_{2,0}^s,\quad u_2^f(x_3,t=0)=u_{2,0}^f, \tag{157}$$
$$\frac{\partial u_2^s}{\partial t}(x_3,t=0)=u_{2,1}^s,\quad \frac{\partial u_2^f}{\partial t}(x_3,t=0)=u_{2,1}^f.$$

A weak formulation ca be stated as follows: for $t\in J$, find $\left(E_2,H_1,u_2^s,u_2^f\right)(t)\in L^2(\Omega)\times H^1(\Omega)\times H^1(\Omega_p)\times L^2(\Omega_p)$ satisfying:

$$\left(\varepsilon\frac{\partial E_2}{\partial t},\varphi\right)+(\sigma E_2,\varphi)-\left(\frac{\partial H_1}{\partial x_3},\varphi\right)=(J_e^s,\varphi),\quad \varphi\in L^2(\Omega), \tag{158}$$

$$\left(E_2,\frac{\partial\psi}{\partial x_3}\right)+\left(\mu\frac{\partial H_1}{\partial t},\psi\right)+\left\langle\left(\frac{\mu}{\varepsilon}\right)^{1/2}H_1,\psi\right\rangle=0,\quad \psi\in H(\mathrm{curl},\Omega), \tag{159}$$

$$\left(\rho_b\frac{\partial^2 u_2^s}{\partial t^2},v^s\right)_{\Omega_p}+\left(\rho_f\frac{\partial^2 u_2^f}{\partial t^2},v^s\right)_{\Omega_p}+\left(G\frac{\partial u_2^s}{\partial x_3},\frac{\partial v^s}{\partial x_3}\right)_{\Omega_p}+\left\langle\alpha\left(\frac{\partial u_2^s}{\partial t}\right),v^s\right\rangle_{\Gamma_p}+\left(\rho_f\frac{\partial^2 u_2^s}{\partial t^2},v^f\right)_{\Omega_p}+\left(m\frac{\partial^2 u_2^f}{\partial t^2},v^f\right)_{\Omega_p}$$
$$+\left(\frac{\eta}{\kappa_0}\frac{\partial u_2^{f,h}}{\partial t},v^f\right)_{\Omega_p}-\left(L_0\frac{\eta}{\kappa_0}E_2,v^f\right)_{\Omega_p}=0,\quad v=(v^s,v^f)\in H^1(\Omega_p)\times L^2(\Omega_p). \tag{160}$$

Uniqueness for the solution of (158)–(160) follows with the argument given for the 3D case.

10.1. A discrete time finite element procedure

Let $\mathcal{T}^{\langle}(\Omega)$ is a nonoverlapping partition of $\Omega=\Omega_p\cup\Omega_a$ into subintervals Ω_j of diameter bounded by h such that $\overline{\Omega}=\cup_{j=1}^{J}\overline{\Omega}_j$. Let:

$$\mathcal{V}^h=\left\{\psi\in H^1(\Omega):\ \psi|\Omega_j\in P_1(\Omega_j)\right\},\quad \mathcal{W}^h=\left\{\varphi\in L^2(\Omega):\ \varphi|\Omega_j\in P_0(\Omega_j)\right\},$$
$$\mathcal{Z}^h=\left\{v^s\in H^1(\Omega_p):\ v^s|\Omega_j\in P_1(\Omega_j)\right\},\quad \mathcal{M}^h=\left\{v^f\in L^2(\Omega_p):\ \varphi|\Omega_j\in P_0(\Omega_j)\right\}.$$

Our fully implicit discrete-time Galerkin procedure for 1D SHTE electroseismics is defined as follows.

Given $\left(E_2^{h,0},H_1^{h,1},u_2^{s,h,0},u_2^{s,h,1},u_2^{f,h,0},u_2^{f,h,1}\right)\in\mathcal{V}^h\times\mathcal{W}^h\times(\mathcal{Z}^h)^2\times(\mathcal{M}^h)^2$, for $n\geqslant 1$ compute $\left(E_2^{h,n},H_1^{h,n},u_2^{s,h,n+1},u_2^{f,h,n+1}\right)\in\mathcal{V}^h\times\mathcal{W}^h\times\mathcal{Z}^h\times\mathcal{M}^h$ such that:

$$\left(\varepsilon d_t E_2^{h,n-1},\varphi\right)+\left(\sigma\widehat{E}_2^{h,n-1/2},\varphi\right)-\left(\frac{\partial\widehat{H}_1^{h,n-1/2}}{\partial x_3},\varphi\right)=(J_e^{s,n},\varphi)\quad \varphi\in\mathcal{W}^h, \tag{161}$$

$$\left(E^{h,n-1/2},\frac{\partial\psi}{\partial x_3}\right)+\left(\mu d_t H_1^{h,n-1},\psi\right)+\left\langle\left(\frac{\mu}{\varepsilon}\right)^{1/2}\widehat{H}^{h,n-1/2},\psi\right\rangle=0,\quad \psi\in\mathcal{V}^h, \tag{162}$$

$$\left(\rho_b\partial^2 u_2^{s,h,n},v^s\right)_{\Omega_p}+\left(\rho_f\partial^2 u_2^{f,h,n},v^s\right)_{\Omega_p}+\left(\rho_f\partial^2 u_2^{s,h,n},v^f\right)_{\Omega_p}+\left(m\partial^2 u_2^{f,h,n},v^f\right)_{\Omega_p}+\left(\frac{\eta}{\kappa_0}\partial u_2^{f,h,n},v^f\right)_{\Omega_p}$$
$$+\left(G\nabla\left(\frac{u^{h,n+1}+u^{h,n-1}}{2}\right),\nabla v^s\right)_{\Omega_p}+\langle\alpha\partial u^{h,n}),v^s\rangle_{\Gamma_p}-\left(L_0\frac{\eta}{\kappa_0}\widehat{E}_2^{h,n-1/2},v^f\right)=0,\quad v=(v^s,v^f)\in\mathcal{Z}^h\times\mathcal{M}^h. \tag{163}$$

The fact that (161)–(163) is unconditionally stable follows with a similar argument to that given in the analysis of (103)–(105) for the 3D case.

Table 1
Parameters characterizing the model used in Example 1.

	Homogeneous region brine saturated	Layer 1: brine saturated	Layer 2: 75% gas – 25% brine saturation
σ^c (S/m)	0.01	0.1	0.001
ϕ (–)	0.2	0.25	0.2
K_s (Pa)	3.7×10^{10}	2.5×10^{10}	3.7×10^{10}
v_s (m/s)	1400	1450	1800
ρ_s (kg/m^3)	2650	2650	2650
k_0 (m^2)	10^{-13}	10^{-16}	10^{-13}
L_0 (A/(Pa m))	3.2×10^{-15}	1.5×10^{-9}	3.3×10^{-9}
ρ_f (kg/m^3)	1000	1000	0.88
η (kg/(m s))	0.001	0.001	1×10^{-5}
K_f (Pa)	2.25×10^9	2.25×10^9	0.1×10^9
S_f (–)	1	1	0.75

10.2. Numerical example

Let us consider a model comprising two layers of two hundred and one hundred meters depth referred to as Layer 1 and Layer 2, respectively, immersed in an otherwise homogeneous Earth. The top of Layer 1 is located at 500 m below the surface; Layer 2 is immediately beneath the former. Above the Earth surface a 100 m thick air layer is considered, with electric conductivity of 10^{-7} S/m. The electric permittivity is taken to be equal to that of the vacuum in the whole model. Both the homogeneous portion of the Earth model and Layer 1 are fully saturated with brine while Layer 2 is partially saturated with gas (75% gas saturation), the remaining portal space is occupied by brine. This model corresponds to a partially gas-saturated sandstone located beneath a seal layer. As Biot's equations admit a single-phase saturating fluid, an effective one is built in the gas-bearing region. The effective fluid properties are calculated as follows: the bulk modulus is calculated by means of the Reuss average [40] $1/K_{eff} = S_w/K_w + S_g/K_g$, where K_w, K_g denote the bulk modulus of brine and gas, respectively and S_w, S_g their respective saturations; the effective fluid density is the weighted average $\rho_{eff} = S_w\rho_w + S_g\rho_g$ and the effective viscosity is given by $\eta_{eff} = \eta_g(\eta_w/\eta_g)^{S_w}$ [41]. The electric conductivity is considered the mean value of the Hashin–Strikman bounds of a two-phase composite [40]; the upper one is $\sigma^{+}_{eff} = \sigma_w + (1 - S_w)/((\sigma_g - \sigma_w)^{-1} + S_w/(3\sigma_w))$ and the lower one is obtained interchanging the g and w subscripts in the last formula. The electric conductivity of the different subsurface regions used in Maxwell's equations is obtained from the electric conductivity of the effective fluid by means of the expression $\sigma = \sigma_{eff}\phi^2$ [40,42]. Finally, it must be noticed that the electrokinetic coupling coefficient L_0 is calculated considering that only brine is present. As in gas/oil reservoirs brine is always the wetting phase, it is here assumed that the electrokinetic coupling takes place through it, and it is not modified by the presence of gas.

Table 1 displays the values of the model parameters used in this example. The electromagnetic source is located on the Earth's surface, and has the following time dependence:

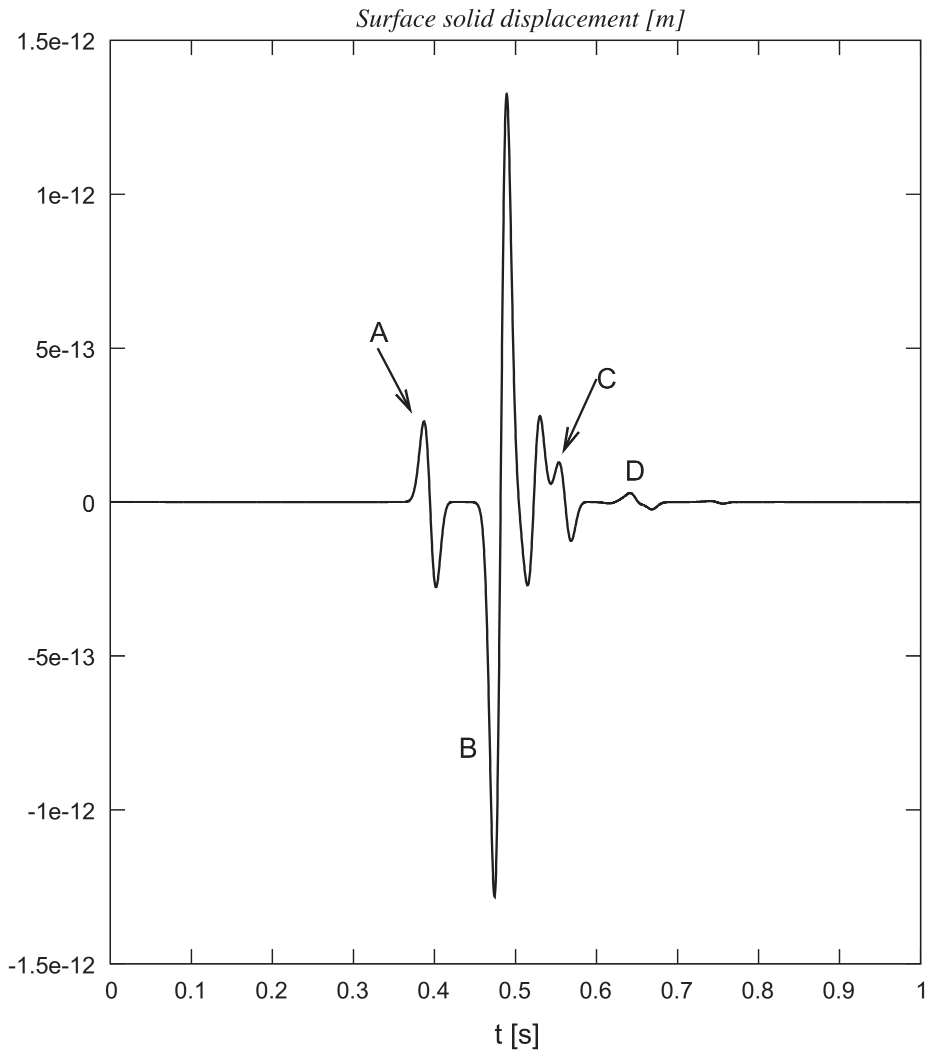

Fig. 2. Trace of solid displacements measured on the surface. A'', B'' and C'' are direct arrivals of waves originated at the 500 m, 700 m and 800 m depth interfaces respectively. D'' is a reflection on the lower boundary of the 200 m width layer of a downwards travelling wave originated on the top boundary of the same layer.

Table 2
Material parameters for the second model.

	First and Third Layers (200 mts thick) brine saturated	Second layer (100 mts 75% gas-25% brine saturation thick)
σ^c (S/m)	0.01	0.001
ϕ (−)	0.2	0.2
v_s (m/s)	1400	1800
ρ_s (kg/m^3)	2650	2650
k_0 (m^2)	10^{-13}	10^{-16}
L_0 (A/(Pa m))	3.2×10^{-15}	1.5×10^{-9}
ρ_f (kg/m^3)	1000	0.88
η (kg/(m s))	0.001	1×10^{-5}
S_f (−)	1	0.75

Table 3
Numerical estimate of the order of approximation of the studied finite element method, 1D case.

$\mathcal{E}$	α
$\Vert E - E^h\Vert_0$	0.98
$\Vert H - H^h\Vert_0$	1.06
$\Vert u^s - u^{s,h}\Vert_1$	1.0
$\Vert u^f - u^{f,h}\Vert_0$	1.05
$\Vert\frac{\partial}{\partial t}(u^s - u^{s,h})\Vert_0$	1.0
$\Vert\frac{\partial}{\partial t}(u^f - u^{f,h})\Vert_0$	1.06

$$f(t) = \left(1 - 2(\pi f_0(t - t_0))^2\right) \exp\left(-(\pi f_0(t - t_0))^2\right), \tag{164}$$

where f_0 = 30 Hz and t_0 = 0.06 s. In this example h = 0.5 m, $dt = 2.5 \times 10^{-4}$ s and the numerical domain comprises 2750 elements. Fig. 2 shows a trace recorded by a surface geophone. In the figure the arrival times corresponding to seismic waves originated by the conversion of electromagnetic to mechanic energy at the boundaries of the different layers are shown; the arrival time values are in good agreement with the expected ones.

Next the order of spatial approximation given in Theorem 3 will be numerically tested in this one dimensional model; in this case a simpler model will be considered, namely a partially gas saturated (75% gas–25% brine) single layer 100 m thick surrounded by two brine saturated 200 m thick slabs. The model parameters are given in Table 2, and the same source as in the previous model is used. It must be here noticed that in our choice of the finite element spaces employed, the two points associated with the boundary of our domain are nodal points of the finite element discretization into C^0-piecewise linear functions and consequently the error terms appearing in the right-hand side of (81) and (97) (associated with the absorbing boundary conditions) are of order h and not of order $h^{1/2}$ as in the 3D case. Therefore the power of h in the *apriori* error estimate in Theorem 3 is one.

Since analytical solutions for this model are not available, four different solutions were calculated, using h_1 = 0.025, h_2 = 0.25, h_3 = 0.5 and h_4 = 1 with the same time step 2.5×10^{-4} in all cases; the solution corresponding to the finest mesh (h = 0.025) was associated with the analytical one. Three different snapshots were taken at three different times. Then, for each snapshot the errors $\Vert E - E^h\Vert_0$, $\Vert H - H^h\Vert_0$, $\Vert u^s - u^{s,h}\Vert_1$, $\Vert u^f - u^{f,h}\Vert_0$, $\Vert\frac{\partial}{\partial t}(u^s - u^{s,h})\Vert_0$ and $\Vert\frac{\partial}{\partial t}(u^f - u^{f,h})\Vert_0$ were calculated. The time partial derivatives were approximated by a second order approximation at each snapshot. Let $\mathcal{E}_i, i = 2, 3, 4$ be any of the errors calculated for h_i, i = 2,3,4, and consider the system of equations:

$$\begin{aligned} \mathcal{E}_2 &= Cdt^\beta + Dh_2^\alpha, \\ \mathcal{E}_3 &= Cdt^\beta + Dh_3^\alpha, \\ \mathcal{E}_4 &= Cdt^\beta + Dh_4^\alpha, \end{aligned} \tag{165}$$

where of course the seeked solution is the exponent α. In Table 3 its values obtained for the different errors using the first snapshot are shown; the values or the other snapshots did not show significative differences. It can be clearly seen that the numerical results are in very good agreement with the theoretical value (one) for the exponent α.

Acknowledgements

This work was partially supported by CONICET through PIP 112-200801-00952.

References

[1] A.H. Thompson, Electromagnetic-to-seismic conversion: successful developments suggest viable applications in exploration and production, in: 75th SEG Annual Meeting, Expanded Abstracts, Houston, 2005.
[2] S. Hornbostel, A.H. Thompson, Waveform design for electroseismic exploration, Geophysics 72 (2) (2007) Q1–Q10.
[3] A.H. Thompson, G. Gist, Geophysical applications of electrokinetic conversion, The Leading Edge 12 (1993) 1169–1173.
[4] S.R. Pride, Governing equations for the coupled electromagnetics and acoustics of porous media, Phys. Rev. B 50 (1994) 15678–15696.
[5] G.I. Block, J.G. Harris, Conductivity dependence of seismoelectric wave phenomena in fluid-saturated sediments, J. Geophys. Res. 111 (2006) B01304, doi:10.1029/2005JB003798.
[6] S.H. Haines, S.R. Pride, Seismoelectric numerical modeling on a grid, Geophysics 71 (6) (2006) N57–N65.
[7] M.A. Biot, Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low frequency range, J. Acoust. Soc. Am. 28 (1956) 168–178.
[8] M.A. Biot, Theory of propagation of elastic waves in a fluid-saturated porous solid. II. High frequency range, J. Acoust. Soc. Am. 28 (1956) 179–191.
[9] Q. Han, Z. Wang, Time-domain simulation of SH-wave induced electromagnetic field in heterogeneous porous media: a fast finite element algorithm, Geophysics 66 (2) (2001) 448–461.
[10] S. Garambois, M. Dietrich, Full waveform numerical simulations of seismoelectromagnetic wave conversions in fluid-saturated stratified porous media, J. Geophys. Res. 107 (2002) 40–58.
[11] C.C. Pain, J.H. Saunders, M.H. Wortington, J.M. Singer, W. Stuart-Bruges, G. Mason, A. Goddard, A mixed finite element method for solving the poroelastic Biot equations with electrokinetic coupling, Geophys. J. Int. 160 (2005) 592–608.
[12] B.S. White, Asymptotic theory of electroseismic prospecting, SIAM J. Appl. Math. 65 (4) (2005) 1443–1462.
[13] B.S. White, M. Zhou, Electroseismic prospecting in layered media, SIAM J. Appl. Math. 67 (1) (2006) 69–98.
[14] H. Hu, W. Guan, J. Harris, Theoretical simulation of electroacoustic borehole logging in fluid-saturated porous formation, J. Acoust. Soc. Am. 122 (2007) 135–145.
[15] W. Guan, H. Hu, Finite difference modeling of electroacoustic logging response in fluid-saturated porous formation, in: Annual Meeting, Society of Exploration Geophysicists, San Antonio, USA, 2007, pp. 511–515.
[16] W. Guan, H. Hu, Finite-difference modeling of electroseismic logging in a fluid saturated porous formation, J. Comput. Phys. 228 (2008) 5633–5648.
[17] J.E. Santos, Finite element approximation of coupled seismic and electromagnetic waves in fluid-saturated poroviscoelastic media, Numer. Methods Part. Diff. Equat. (2010), doi:10.1002/num.20527.
[18] P.A. Raviart, J.M. Thomas, A mixed finite element method for second order elliptic problems, in: I. Galligani, E. Magenes (Eds.), Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, Berlin, New York, 1977, p. 292.
[19] J.C. Nédélec, Mixed finite elements in R^3, Numer. Math. 35 (1980) 315–341.
[20] J.C. Nédélec, A new family of mixed finite elements in R^3, Numer. Math. 50 (1986) 57–81.
[21] P.B. Monk, A mixed method for approximating Maxwell's equations, SIAM J. Numer. Anal. 28 (6) (1991) 1610–1634.
[22] P.B. Monk, An analysis of Nédélec method for the spatial discretization of Maxwell's equations, J. Comput. Appl. Math. 47 (1993) 101–121.
[23] P.B. Monk, A.K. Parrot, A dispersion analysis of finite element methods for Maxwell's equations, SIAM J. Sci. Stat. Comput. 15 (4) (1994) 916–937.
[24] J. Douglas Jr., J.E. Santos, D. Sheen, X. Ye, Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems, RAIRO Math. Model. Numer. Anal. (M2AN) 33 (1999) 747–770.
[25] F.I. Zyserman, P.M. Gauzellino, J.E. Santos, Dispersion analysis of a non-conforming finite element method for the Helmholtz and elastodynamic equations, J. Numer. Meth. Eng. 58 (2003) 1381–1395.
[26] F.I. Zyserman, J.E. Santos, Analysis of the numerical dispersion of waves in saturated poroelastic media, Comput. Methods Appl. Mech. Eng. 196 (2007) 4644–4655.
[27] J.E. Santos, Elastic wave propagation in fluid-saturated porous media. Part I. The existence and uniqueness results, Math. Model. Numer. Anal. (M2AN) 20 (1) (1986) 113–128.
[28] J.L. Lions, Quelques methodes dde resolution des problems aux limites nonlineaires, Dunod, Gauthier-Villars, Paris, 1969.
[29] J.E. Santos, D. Sheen, Finite element methods for the simulation of waves in composite saturated poroviscoelastic materials, SIAM J. Numer. Anal. 45 (1) (2007) 389–420.
[30] F. Gassmann, Uber die elastizität poröser medien, Vierteljahrsschrift der Naturforschenden Gessellschaft in Zurich 96 (1951) 1–23.
[31] J.E. Santos, J.M. Corberó, C.L. Ravazzoli, J.L. Hensley, Reflection and transmission coefficients in fluid-saturated porous media, J. Acoust. Soc. Am. 91 (1992) 1911–1923.
[32] S.H. Haartsen, S. Pride, Electroseismic waves from point sources in layered media, J. Geophys. Res. 102 (24) (1997) 745–769.
[33] M.A. Biot, Mechanics of deformation and acoustic propagation in porous media, J. Appl. Phys. 33 (4) (1962) 1482–1498.
[34] J.E. Santos, J. Douglas Jr., M.E. Morley, O.M. Lovera, Finite element methods for a model for full waveform acoustic logging, IMA J. Numer. Anal. 8 (1998) 415–433.
[35] V. Girault, P. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer-Verlag, Berlin, 1986.
[36] D. Sheen, A generalized Green's theorem, Appl. Math. Lett. 5 (1992) 95–98.
[37] D. Sheen, Approximation of electromagnetic fields: part I. Continuous problems, SIAM J. Appl. Math. 6 (1997) 1716–1736.
[38] G. Duvaut, J.L. Lions, Les Inéqualities en Méchanique et en Physique, Dunod, Paris, 1972.
[39] J.A. Nitsche, On Korn's second inequality, RAIRO Anal. Numer. 15 (1981) 237–248.
[40] G. Mavko, T. Mukerji, J. Dvorkin, The Rock Physics Handbook, Cambridge University Press, 1998.
[41] J.M. Carcione, S. Picotti, D. Gei, G. Rossi, Physics and seismic modelling for monitoring CO_2 storage, Pure Appl. Geophys. 163 (2006) 175–207.
[42] S. Pride, S. Garambois, Electroseismic wave theory of Frenkel and more recent developments, J. Eng. Mech. 131 (9) (2005) 697–706.