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Entropic entanglement criteria for Fermion systems

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Abstract. Entanglement criteria for general (pure or mixed) states of systems consisting of two identical fermions are introduced. These criteria are based on appropriate inequalities involving the entropy of the global density matrix describing the total system, on the one hand, and the entropy of the one-particle reduced density matrix, on the other hand. A majorization-related relation between these two density matrices is obtained, leading to a family of entanglement criteria based on Rényi's entropic measure. These criteria are applied to various illustrative examples of parametrized families of mixed states. The dependence of the entanglement detection efficiency on Rényi's entropic parameter is investigated. The extension of these criteria to systems of N identical fermions is also considered.

1 Introduction

The entanglement features exhibited by systems consisting of identical fermions have attracted the attention of several researchers in recent years [1–17]. Entanglement in fermion systems has been studied in connection with different problems, such as the entanglement between electrons in a conduction band [9], the entanglement dynamics associated with scattering processes involving two electrons [10], the role played by entanglement in the time-optimal evolution of fermionic systems [11,12], the classification of three-fermion states based on their entanglement features [13], the detection of entanglement in fermion systems through the violation of appropriate uncertainty relations [14], the entanglement features of fractional quantum Hall liquids [15] and the entanglement properties of the eigenstates of soluble two-electrons atomic models [16].

The concept of entanglement in systems of indistinguishable particles exhibits some differences from the corresponding concept as applied to systems consisting of distinguishable parts. There is general consensus among researchers that in systems of identical fermions the minimum quantum correlations between the particles that are required by the antisymmetric character of the fermionic state do not contribute to the state's amount of entanglement [1–17]. This means that the separable (that is, non-entangled) pure states of N fermions are those having Slater rank 1. These are the states whose wave function can be expressed (with respect to an appropriate single-particle basis) as a single Slater determinant [3]. On the other hand, the set of mixed non-entangled states comprises those states that can be written as a statistical mixture of pure states of Slater rank 1. Here, when discussing systems of identical fermions, we are considering entanglement between particles and not entanglement between modes.

The problem of determining whether a given quantum state ρ is separable or entangled is known as "*the separability problem*". It constitutes one of the most fundamental problems in the theory of quantum entanglement and is the subject of a sustained and intense research activity (see [18–25] and references therein). As clearly stated in a comprehensive recent review article on entanglement: "*The fundamental question in quantum entanglement theory is which states are entangled and which are not*" [20]. Besides its intrinsic interest, the development of separability criteria also leads to useful quantitative entanglement indicators: the degree to which a separability criterion is violated constitutes in itself a valuable quantitative indicator of entanglement. For instance, the well-known negativity measure of entanglement (which is one of the most used practical measures of entanglement for mixed states of systems with distinguishable subsystems) is based upon the celebrated Peres' separability criterion [20]. Another interesting recent example of separability criteria leading to quantitative indicators of entanglement concerns separability criteria based upon the violation of appropriate local uncertainty relations. In fact, it has been shown that the amount of violation of these uncertainty relations provides useful lower bounds for the concurrence, which constitutes a quantitative measure of entanglement [26–28].

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In the case of pure states of two identical fermions, necessary and sufficient separability criteria can be formulated in terms of the entropy of the single-particle reduced density matrix [4,8,17]. Alas, no such criteria are known for general, mixed states of two fermions, except for the case of two fermions with a single-particle Hilbert space of dimension four, for which a closed analytical expression for the concurrence (akin to the celebrated Wootters' formula for two qubits [29]) is known. In general, to determine whether a given density matrix of a two-fermion system represents a separable state or not is a notoriously difficult (and largely unexplored) problem. Consequently, there is a clear need for practical separability criteria, or entanglement indicators, which can be extended to systems of higher dimensionality or to scenarios involving more than two fermions [17].

Entropic separability criteria have played a distinguished role in the study of the entanglement-related features of mixed states of multipartite systems constituted by distinguishable subsystems [18–24]. For this kind of composite quantum system, non-entangled states behave classically in the sense that the entropy of a subsystem is always less or equal than the entropy of the whole system. If the entropy of a subsystem happens to be larger than the entropy of the whole system, then we know for sure that the state is entangled (that is, this constitutes a sufficient entanglement criterion). This statement can be formulated mathematically in terms of the Rényi entropic measures,

$$S_q^{(R)}[\rho] = \frac{1}{1-q}\ln(\mathrm{Tr}[\rho^q]), \tag{1}$$

leading to the following family of inequalities satisfied by separable states [18–24],

$$\begin{aligned} S_q^{(R)}[\rho_A] &\le S_q^{(R)}[\rho_{AB}] \\ S_q^{(R)}[\rho_B] &\le S_q^{(R)}[\rho_{AB}]. \end{aligned} \tag{2}$$

In the above equations ρ_{AB} is the joint density matrix describing a bipartite system consisting of the subsystems A and B, and $\rho_{A,B}$ are the marginal density matrices describing the subsystems. The entropic parameter in (1-2) adopts values $q \ge 1$. In the limit $q \to 1$ the Rényi entropy reduces to the von Neumann entropy. Note that the entropic criteria considered in [18–24] and in the present work, which depend on the entropies of the total and reduced density matrices, are different from those studied in [25], which involve entropic uncertainty relations associated with the measurement of particular observables.

The study of entropic entanglement criteria based upon the above considerations has been the focus of a considerable amount of research over the years [18–24]. It would be interesting to extend this approach to systems consisting of identical fermions. The aim of this paper is to investigate entanglement criteria for general (mixed) states of systems of two identical fermions based upon the comparison of the entropy of the global density matrix describing the total system and the entropy of the one-particle reduced density matrix.

The organization of the paper is as follows. A brief review of entanglement between particles in systems of identical fermions is given in Section 2. Entropic entanglement criteria for systems of two identical fermions based on the von Neumann, the linear, and the Rényi entropies are derived in Section 3. These entropic criteria are applied to particular families of states of two-fermion systems in Sections 4 and 5. The extension to systems of N fermions of the entanglement criteria based upon the Rényi entropies is considered in Section 6. Finally, some conclusions are drawn in Section 7.

2 Entanglement between particles in fermionic systems

The concept of entanglement between the particles in a system of identical fermions is associated with the quantum correlations exhibited by quantum states on top of the minimal correlations due to the indistinguishability of the particles and the antisymmetric character of fermionic states. A pure state of Slater rank one of N identical fermions (that is, a state that can be described by one single Slater determinant) must be regarded as separable (non-entangled) [2,3]. The correlations exhibited by such states do not provide a resource for implementing non-classical information transmission or information processing tasks. Moreover, the non-entangled character of states of Slater rank one is consistent with the possibility of assigning complete sets of properties to the parts of the composite system [4]. Consequently, a pure state of two identical fermions of the form

$$|\psi_{sl}\rangle = \frac{1}{\sqrt{2}}\{|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle\}, \tag{3}$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal single-particle states, is regarded as separable.

A pure state $|\psi\rangle$ of a system of N identical fermions has Slater rank 1, and is therefore separable, if and only if

$$\mathrm{Tr}(\rho_1^2) = \frac{1}{N}, \tag{4}$$

where $\rho_1 = \mathrm{Tr}_{2,\ldots,N}(\rho)$ is the single-particle reduced density matrix, $\rho = |\psi\rangle\langle\psi|$, n is the dimension of the single-particle state space and $N \le n$ [17]. On the other hand, entangled pure states satisfy

$$\frac{1}{n} \le \mathrm{Tr}(\rho_1^2) < \frac{1}{N}. \tag{5}$$

Non-entangled mixed states of systems of N identical fermions are those that can be written as a mixture of Slater determinants,

$$\rho_{sl} = \sum_i \lambda_i |\psi_{sl}^{(i)}\rangle\langle\psi_{sl}^{(i)}|, \tag{6}$$

where the states $|\psi_{sl}^{(i)}\rangle$ can be expressed as single Slater determinants, and $0 \le \lambda_i \le 1$ with $\sum_i \lambda_i = 1$.

Systems of identical fermions with a single-particle Hilbert space of dimension $2k$ (with $k \geq 2$) can be formally regarded as systems consisting of spin-s particles, with $s = (2k-1)/2$. The members $\{|i\rangle,\ i = 1,\ldots,2k\}$ of an orthonormal basis of the single-particle Hilbert space can be identified with the states $|s, m_s\rangle$, with $m_s = s - i + 1,\ i = 1,\ldots,2k$. We can use for these states the shorthand notation $\{|m_s\rangle,\ m_s = -s,\ldots,s\}$, because each particular example discussed here will correspond to a given value of k (and s). According to this angular momentum representation, the antisymmetric joint eigenstates $\{|j,m\rangle,\ -j \leq m \leq j,\ 0 \leq j \leq 2s\}$ of the total angular momentum operators J^2 and J_z constitute a basis for the Hilbert space associated with a system of two identical fermions. The antisymmetric states $|j,m\rangle$ are those with an even value of the quantum number j.

A closed analytical expression for the concurrence of general (pure or mixed) states of two identical fermions sharing a single-particle Hilbert space of dimension four (corresponding to $s = 3/2$) was discovered by Eckert et al. (ESBL) in [2]. The ESBL concurrence formula is

$$\mathcal{C}_{\mathcal{F}}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6\}, \qquad (7)$$

where the λ_i's are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order of magnitude. Here $\tilde{\rho} = \mathcal{D}\rho\mathcal{D}^{-1}$, with the operator $\mathcal{D}$ given by

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{K}, \qquad (8)$$

where $\mathcal{K}$ stands for the complex conjugation operator and (8) is written with respect to the total angular momentum basis, ordered as $|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$ and $i|0,0\rangle$.

In what follows we are going to consider systems comprising a given, fixed number of identical fermions. Therefore, we are going to work within the first quantization formalism.

3 Entropic entanglement criteria for systems of two identical fermions

In this Section we are going to derive the main results of the present paper. We shall advance new entropic criteria for mixed states of systems constituted by identical fermions. In Section 3.1 we derive entropic criteria for mixed states of two fermions (based on inequality (10)) and N fermions (based on inequality (12)) formulated in terms of the von Neumann entropy, and an entropic criterion for two fermions based upon the linear entropy. In Section 3.2 we introduce a full family of entropic criteria based on the Rényi entropy.

3.1 Entanglement criteria based on the von Neumann and the linear entropies

Let ρ be a density matrix describing a quantum state of two identical fermions and ρ_r be the corresponding single-particle reduced density matrix, obtained by computing the partial trace over one of the two particles.

If $\rho = |\psi_{sl}\rangle\langle\psi_{sl}|$, where $|\psi_{sl}\rangle$ represents a separable pure state of the form (3), and

$$S_{\rm vN}[\rho] = -\mathrm{Tr}(\rho \ln \rho) \qquad (9)$$

is the von Neumann entropy of ρ, we have that $S_{\rm vN}[\rho] = 0$ and $S_{\rm vN}[\rho_r] = \ln 2$. That is, for separable pure states we have $S_{\rm vN}[\rho] - S_{\rm vN}[\rho_r] = -\ln 2$. It then follows from the concavity property of the quantum conditional entropy [30] that, for a separable mixed state ρ of the form (6), $S_{\rm vN}[\rho] - S_{\rm vN}[\rho_r] \geq -\ln 2$. Consequently, all separable states (pure or mixed) of a system of two identical fermions satisfy the inequality

$$S_{\rm vN}[\rho_r] \leq S_{\rm vN}[\rho] + \ln 2. \qquad (10)$$

Hence, if the quantity

$$D_{\rm vN} = S_{\rm vN}[\rho_r] - S_{\rm vN}[\rho] - \ln 2 \qquad (11)$$

is positive the state ρ is necessarily entangled. Indeed, in the particular case of pure states this quantity has been used as a measure of entanglement in some applications (see, for instance, [15] and references therein).

The argument leading to inequality (10) can be extended to the more general case of systems of N identical fermions. A separable pure state $\rho = |\psi_{sl}\rangle\langle\psi_{sl}|$ of N identical fermions (that is, a pure state expressible as a single Slater determinant) satisfies $S_{\rm vN}[\rho] = 0$ and $S_{\rm vN}[\rho_r] = \ln N$. Therefore, for this kind of state we have $S_{\rm vN}[\rho] - S_{\rm vN}[\rho_r] = -\ln N$. The concavity property of the quantum conditional entropy then implies that for a mixed state ρ of N fermions having the form (6) we have $S_{\rm vN}[\rho] - S_{\rm vN}[\rho_r] \geq -\ln N$. Consequently, a separable state of N fermions (that is, a state that can be written as a statistical mixture of pure states each having the form of single Slater determinant) satisfies the inequality

$$S_{\rm vN}[\rho_r] \leq S_{\rm vN}[\rho] + \ln N. \qquad (12)$$

Consequently, a state of N fermions violating inequality (12) is necessarily entangled. In the case of pure states of N fermions this entanglement criteria reduces to one of the entanglement criteria previously discussed in [17]. The special case of this criterion corresponding to pure states of two fermions was first analyzed in [4]. That is, our present result (12) constitutes a generalization to arbitrary mixed states of an inequality that has been previously known and shown to be useful for the study of fermionic entanglement in the special case of pure states. When deriving the inequalities (10) and (12) we have used the concavity of the quantum conditional entropy. This property is usually discussed in connection with composite systems comprising distinguishable subsystems. However, within the first quantization formalism, any density matrix of two identical fermions has mathematically also the form of a density matrix describing distinguishable subsystems (in fact, it is just a density matrix that happens to be expressible as a statistical mixture of antisymmetric pure states). Consequently, any mathematical property that is satisfied by general density matrices describing distinguishable subsystems is also satisfied by the special subset of density matrices that can describe a system of identical fermions.

An entanglement criterion for states of two fermions similar to the one already discussed can be formulated in terms of the linear entropy,

$$S_L[\rho] = 1 - \mathrm{Tr}(\rho^2). \tag{13}$$

Given a quantum state ρ of two fermions, let's consider the quantity

$$c[\rho] \;=\; \inf \sum_i p_i c[|\phi_i\rangle], \tag{14}$$

where $c[|\phi_i\rangle] = \sqrt{2\left[1 - \mathrm{Tr}[(\rho_r^{(i)})^2]\right]}$, $\rho_r^{(i)}$ is the one-particle reduced density matrix corresponding to $|\phi_i\rangle$, $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$, and the infimum is taken over all the possible decompositions of ρ as a statistical mixture $\{p_i, |\phi_i\rangle\}$ of pure states (note that $c[\rho]$ adopts values in the range $[0, \sqrt{2}]$). The quantity defined in (14) satisfies the inequality [31]

$$c[\rho]^2 \;\geq\; 2\left[\mathrm{Tr}(\rho^2) - \mathrm{Tr}\left(\rho_r^2\right)\right]. \tag{15}$$

If ρ corresponds to a separable state of the two fermions, we have that $\rho = \sum_i p_i |\psi_{\mathrm{sep}}^{(i)}\rangle\langle\psi_{\mathrm{sep}}^{(i)}|$ with $c[|\psi_{\mathrm{sep}}^{(i)}\rangle] = 1$ for all i. Therefore, for a separable state we have $c[\rho] \leq 1$ and, from (15), $1 \geq (c[\rho])^2 \geq 2\left[\mathrm{Tr}(\rho^2) - \mathrm{Tr}\left(\rho_r^2\right)\right]$. Consequently, separable states (pure or mixed) of a system of two identical fermions comply with the inequality,

$$S_L[\rho_r] \leq S_L[\rho] + \frac{1}{2}. \tag{16}$$

In other words, states for which the quantity

$$D_L = S_L[\rho_r] - S_L[\rho] - \frac{1}{2} \tag{17}$$

is positive are necessarily entangled. In the particular case of pure states of two identical fermions, the positivity of (17) becomes both a necessary and sufficient entanglement criterion ([17] and references therein). Moreover, a quantity basically equal to (17) has been proposed as an entanglement measure for pure states of two fermions and indeed constitutes one of the most useful entanglement measures for these states [10].

3.2 Entropic entanglement criteria based on the Rényi entropies

On the basis of the Rényi family of entropies we are going to derive now a generalization of the separability criterion associated with inequality (10). We are going to prove that a (possibly mixed) quantum state ρ of a system of two identical fermions satisfying the inequality

$$S_q^{(R)}[\rho] + \ln 2 < S_q^{(R)}[\rho_r], \tag{18}$$

for some $q \geq 1$, is necessarily entangled. Here $S_q^{(R)}$ stands for the Rényi entropy,

$$S_q^{(R)}[\rho] = \frac{1}{1-q}\ln(\mathrm{Tr}[\rho^q]). \tag{19}$$

The inequality (18) leads to an entropic entanglement criterion that detects entanglement whenever the quantity

$$R_q = S_q^{(R)}[\rho_r] - S_q^{(R)}[\rho] - \ln 2 \tag{20}$$

is strictly positive. In the limit $q \to 1$ the Rényi measure reduces to the von Neumann entropy and we recover the entanglement criterion given by inequality (10). When $q \to \infty$ the Rényi entropy becomes

$$S_\infty^{(R)}[\rho] = -\ln\left(\lambda_{\max}\right), \tag{21}$$

where $\lambda_{\max}$ is the largest eigenvalue of ρ. In this limit case, the entropic criterion says that any state satisfying

$$2\,\lambda_{\max}^{(\rho_r)} < \lambda_{\max}^{(\rho)} \tag{22}$$

is entangled, where $\lambda_{\max}^{(\rho)}$ and $\lambda_{\max}^{(\rho_r)}$ are, respectively, the largest eigenvalues of ρ and ρ_r.

3.3 Proof of the entropic criteria based on the Rényi entropies

The following proof is based on the powerful techniques related to the majorization concept [32,33] that were introduced to the field of quantum entanglement by Nielsen and Kempe in [32]. These authors proved that non-entangled states of quantum systems having distinguishable subsystems are such that the total density matrix is always majorized by the marginal density matrix associated with one of the subsystems. In the case of non-entangled states of a system of identical fermions the total density matrix ρ is not necessarily majorized by the one-particle reduced density matrix ρ_r. However, as we are going to prove, *there is still a definite majorization-related relation between ρ and ρ_r that yields a family of inequalities between the Rényi entropies of these two matrices, which leads in turn to a family of entropic entanglement criteria.*

In our proof of the entropic criterion associated with the inequality (18) we are going to use the following fundamental property of quantum statistical mixtures. If $\rho = \sum_i p_i |a_i\rangle\langle a_i| = \sum_j q_j |b_j\rangle\langle b_j|$ are two statistical mixtures representing the same density matrix ρ, then there exists a unitary matrix $\{U_{ij}\}$ such that [32,34]

$$\sqrt{p_i}|a_i\rangle = \sum_j U_{ij}\sqrt{q_j}|b_j\rangle. \tag{23}$$

Let us now consider a separable state of two identical fermions,

$$\rho = \sum_j \frac{p_j}{2}\left(|\psi_1^{(j)}\rangle|\psi_2^{(j)}\rangle - |\psi_2^{(j)}\rangle|\psi_1^{(j)}\rangle)(\langle\psi_1^{(j)}|\langle\psi_2^{(j)}| - \langle\psi_2^{(j)}|\langle\psi_1^{(j)}|\right) \quad (24)$$

where $0 \leq p_j \leq 1$, $\sum_j p_j = 1$ and $|\psi_1^{(j)}\rangle$, $|\psi_2^{(j)}\rangle$ are normalized single-particle states with $\langle\psi_1^{(j)}|\psi_2^{(j)}\rangle = 0$. Equation (24) represents the standard definition of a non-entangled mixed state of two identical fermions. Notice that in (24) no special relation between states $|\psi_i^{(j)}\rangle$ with different values of the label j is assumed. In particular, the overlap between two states with different labels j is not necessarily equal to 0 or 1. This, in turn, means that the overlap between two different members of the family of (separable) two-fermion pure states participating in the statistical mixture leading to (24) may be non-zero.

Let us consider now a spectral representation

$$\rho = \sum_k \lambda_k |e_k\rangle\langle e_k| \quad (25)$$

of ρ. That is, the $|e_k\rangle$ constitute an orthonormal basis of eigenvectors of ρ and the λ_k are the corresponding eigenvalues. Then, (24) and (25) are two different representations of ρ as a mixture of pure states. Therefore, there is a unitary matrix U with matrix elements $\{U_{kj}\}$ such that

$$\sqrt{\lambda_k}|e_k\rangle = \sum_j U_{kj}\sqrt{\frac{p_j}{2}}\left(|\psi_1^{(j)}\rangle|\psi_2^{(j)}\rangle - |\psi_2^{(j)}\rangle|\psi_1^{(j)}\rangle\right). \quad (26)$$

The single-particle reduced density matrix corresponding to the two-fermion density matrix (24) is

$$\rho_r = \sum_j \frac{p_j}{2}\left(|\psi_1^{(j)}\rangle\langle\psi_1^{(j)}| + |\psi_2^{(j)}\rangle\langle\psi_2^{(j)}|\right), \quad (27)$$

admitting a spectral representation

$$\rho_r = \sum_l \alpha_l |f_l\rangle\langle f_l|. \quad (28)$$

We now define,

$$q_{2j} = q_{2j-1} = \frac{1}{2}p_j \qquad (j = 1, 2, 3, \ldots) \quad (29)$$

$$|\phi_{2j-1}\rangle = |\psi_1^{(j)}\rangle$$
$$|\phi_{2j}\rangle = |\psi_2^{(j)}\rangle \qquad (j = 1, 2, 3, \ldots). \quad (30)$$

Now, since (27) and (28) correspond to two statistical mixtures yielding the same density matrix, there must exist a unitary matrix W with matrix elements $\{W_{il}\}$ such that,

$$\sqrt{q_i}|\phi_i\rangle = \sum_l W_{il}\sqrt{\alpha_l}|f_l\rangle \qquad (i = 1, 2, 3, \ldots). \quad (31)$$

Now, equation (26) can be rewritten as

$$\sqrt{\lambda_k}|e_k\rangle = \sum_j U_{kj}\left(\sqrt{q_{2j-1}}|\phi_{2j-1}\rangle|\phi_{2j}\rangle - \sqrt{q_{2j}}|\phi_{2j}\rangle|\phi_{2j-1}\rangle\right). \quad (32)$$

Combining (31) and (32) gives

$$\sqrt{\lambda_k}|e_k\rangle = \sum_l\left[\sum_j U_{kj}\left(W_{2j-1,l}|\phi_{2j}\rangle - W_{2j,l}|\phi_{2j-1}\rangle\right)\right] \times \sqrt{\alpha_l}|f_l\rangle. \quad (33)$$

Therefore, since $\langle e_k|e_{k'}\rangle = \delta_{kk'}$ and $\langle f_l|f_{l'}\rangle = \delta_{ll'}$, we have that

$$\lambda_k = \sum_l M_{kl}\alpha_l, \quad (34)$$

where

$$M_{kl} = \left(\sum_{j'} U^*_{kj'}\left\{W^*_{2j'-1,l}\langle\phi_{2j'}| - W^*_{2j',l}\langle\phi_{2j'-1}|\right\}\right) \times \left(\sum_{j''} U_{kj''}\left\{W_{2j''-1,l}|\phi_{2j''}\rangle - W_{2j'',l}|\phi_{2j''-1}\rangle\right\}\right). \quad (35)$$

We now investigate the properties of the matrix M with matrix elements $\{M_{kl}\}$. First of all, we have

$$M_{kl} \geq 0, \quad (36)$$

since the matrix elements of M are of the form $M_{kl} = \langle\Sigma|\Sigma\rangle$, with

$$|\Sigma\rangle = \sum_j U_{kj}\left(W_{2j-1,l}|\phi_{2j}\rangle - W_{2j,l}|\phi_{2j-1}\rangle\right). \quad (37)$$

We now consider the sum of the elements within a given row or column of M. The sum of a row yields,

$$\begin{aligned}\sum_k M_{kl} &= \sum_{j'j''}\delta_{j'j''}\left(W^*_{2j'-1,l}\langle\phi_{2j'}| - W^*_{2j',l}\langle\phi_{2j'-1}|\right) \\ &\quad\times\left(W_{2j''-1,l}|\phi_{2j''}\rangle - W_{2j'',l}|\phi_{2j''-1}\rangle\right) \\ &= \sum_j\left(W^*_{2j-1,l}W_{2j-1,l} + W^*_{2j,l}W_{2j,l}\right) \\ &= \sum_i\left(W^\dagger\right)_{li}W_{il} = 1,\end{aligned} \quad (38)$$

while the sum of a column is,

$$\sum_l M_{kl} = \sum_{j'j''} U^*_{kj'} U_{kj''} \Bigg(\langle \phi_{2j'} | \phi_{2j''} \rangle \left[\sum_l W^*_{2j'-1,l} W_{2j''-1,l} \right] + \langle \phi_{2j'-1} | \phi_{2j''-1} \rangle \left[\sum_l W^*_{2j',l} W_{2j'',l} \right] - \langle \phi_{2j'} | \phi_{2j''-1} \rangle \left[\sum_l W^*_{2j'-1,l} W_{2j'',l} \right] - \langle \phi_{2j'-1} | \phi_{2j''} \rangle \left[\sum_l W^*_{2j',l} W_{2j''-1,l} \right] \Bigg)$$
$$= \sum_{j'j''} U^*_{kj'} U_{kj''} \left(\langle \phi_{2j'} | \phi_{2j''} \rangle \delta_{j'j''} + \langle \phi_{2j'-1} | \phi_{2j''-1} \rangle \delta_{j'j''} \right) = 2 \sum_j \left(U^\dagger \right)_{jk} U_{kj} = 2. \quad (39)$$

When deriving the above two equations we made use of the unitarity of the matrices $\{U_{kj}\}$ and $\{W_{il}\}$. Summing up, we have,

$$\sum_k M_{kl} = 1$$
$$\sum_l M_{kl} = 2. \quad (40)$$

We now define a new set of variables $\{\lambda'_n\}$ and a new matrix M' with elements M'_{nl}, respectively given by,

$$\lambda'_{2k-1} = \lambda'_{2k} = \frac{1}{2}\lambda_k \qquad (k = 1, 2, 3, \ldots) \quad (41)$$
$$M'_{2k-1,l} = M'_{2k,l} = \frac{1}{2} M_{kl} \qquad (k = 1, 2, 3, \ldots), \quad (42)$$

and so we have

$$\lambda'_n = \sum_l M'_{nl} \alpha_l. \quad (43)$$

By construction, then, we have

$$\{\lambda_k\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$$
$$\{\lambda'_n\} = \left\{ \frac{\lambda_1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \frac{\lambda_3}{2}, \ldots \right\}. \quad (44)$$

Let us now compare the matrices $\{M_{kl}\}$ and $\{M'_{nl}\}$. The matrix $\{M'_{nl}\}$ has twice as many rows as $\{M_{kl}\}$, but the rows of $\{M'_{nl}\}$ can be grouped in pairs of consecutive rows such that within each pair the rows are equal to 1/2 a row of $\{M_{kl}\}$. It follows that

$$\sum_k M_{kl} = 1 \implies \sum_n M'_{nl} = 1$$
$$\sum_l M_{kl} = 2 \implies \sum_l M'_{nl} = 1. \quad (45)$$

Thus,

$$\sum_n M'_{nl} = \sum_l M'_{nl} = 1 \quad (46)$$

and, therefore, $\{M'_{nl}\}$ is a doubly stochastic matrix. Interpreting the λ'_n's and the α_l's as probabilities, it follows from (43) and (46) that the probability distribution $\{\lambda'_n\}$ is more "mixed" than the probability distribution $\{\alpha_l\}$ [30] (or, alternatively that $\{\alpha_l\}$ majorizes $\{\lambda'_n\}$ [32]). This, in turn, implies that for any Rényi entropy $S_q^{(R)}$ with $q \geq 1$, we have

$$S_q^{(R)}[\lambda'_n] \geq S_q^{(R)}[\alpha_l]. \quad (47)$$

Thus,

$$S_q^{(R)}[\lambda'_n] = \frac{1}{1-q} \ln \left(2 \sum_k \left(\frac{\lambda_k}{2} \right)^q \right) = \ln 2 + S_q^{(R)}[\lambda_k]. \quad (48)$$

Therefore, all separable states of the two-fermion system comply with the inequality $S_q^{(R)}[\lambda_k] + \ln 2 \geq S_q^{(R)}[\alpha_l]$ and since $\{\lambda_k\}$ and $\{\alpha_l\}$ are the eigenvalues of ρ and ρ_r respectively,

$$S_q^{(R)}[\rho] + \ln 2 \geq S_q^{(R)}[\rho_r]. \quad (49)$$

The above inequality leads to an entanglement criterion that detects entanglement when the indicator R_q defined in equation (19) is strictly positive.

3.4 Connection with a quantitative measure of entanglement

As already mentioned in the Introduction, the development of separability criteria often leads to useful entanglement indicators. In particular, when the separability criterion takes the form of an inequality, such that entanglement is detected when the inequality is not verified, the degree of violation of the inequality constitutes an entanglement indicator. In the case of the entropic indicators advanced in the present work, it is indeed a reasonable expectation that states with larger values of the indicators $D_{\rm vN}$ and R_q tend to be more entangled. In the next Section we shall illustrate this behaviour in the case of two-fermion systems with a single-particle Hilbert space of dimension four, where the exact amount of entanglement can be evaluated analytically.

Now we shall discuss two general aspects of the connection between the abovementioned entanglement indicators and a quantitative measure of entanglement. First of all, it is worth emphasizing that in the case of pure states, the indicators $D_{\rm vN}$ and D_L themselves coincide (up to unessential multiplicative constants) with useful quantitative measures of entanglement for fermion systems that have already been applied to the study of fermionic entanglement. In particular, let us focus on the indicator $D_{\rm vN}$ of a two-fermion system, which is based on the von Neumann entropies of the total and single-particle density matrices, ρ and ρ_r. For a pure state $|\Phi\rangle$ of the two-fermion system we have $\rho = |\Phi\rangle\langle\Phi|$ and $S_{\rm vN}[\rho] = 0$. Consequently, in this case we have $D_{\rm vN} = S_{\rm vN}[\rho_r] - \ln 2$. As already mentioned, this quantity constitutes a quantitative entanglement measure for pure states,

$$\varepsilon[|\Phi\rangle] \;=\; S_{\rm vN}[\rho_r] - \ln 2. \tag{50}$$

The extension of this quantitative entanglement measure to mixed two-fermion states ρ is obtained via the standard convex roof construction,

$$\varepsilon[\rho] \;=\; \inf \sum_i p_i\, \varepsilon[|\Phi_i\rangle], \tag{51}$$

where the infimum is taken over all the possible mixtures $\{p_i, |\Phi_i\rangle\}$ of pure states $|\Phi_i\rangle$ (with weights p_i, $0 \le p_i \le 1$, $\sum_i p_i = 1$) generating the mixed state under consideration, $\rho = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|$. Now, given a particular decomposition $\rho = \sum_i p_i |\Phi_i\rangle\langle\Phi_i|$ of the two-fermion state ρ, let $\rho^{(i)} = |\Phi_i\rangle\langle\Phi_i|$ be the total density matrix corresponding to the pure state $|\Phi_i\rangle$ and $\rho_r^{(i)}$ be the corresponding single-particle reduced density matrix. Then, using the concavity property of the quantum conditional entropy (see Sect. 3.1) one obtains,

$$\begin{aligned} D_{\rm vN}[\rho] \;&=\; S_{\rm vN}[\rho_r] - S_{\rm vN}[\rho] - \ln 2 \\ &\le\; \sum_i p_i \Big[S_{\rm vN}[\rho_r^{(i)}] - S_{\rm vN}[\rho^{(i)}] - \ln 2 \Big] \end{aligned} \tag{52}$$

which implies that

$$D_{\rm vN}[\rho] \;\le\; \inf \sum_i p_i \Big[S_{\rm vN}[\rho_r^{(i)}] - S_{\rm vN}[\rho^{(i)}] - \ln 2 \Big] \;=\; \varepsilon[\rho], \tag{53}$$

which leads to an inequality directly linking the entropic indicator $D_{\rm vN}[\rho]$ with the quantitative entanglement measure $\varepsilon[\rho]$,

$$\varepsilon[\rho] \;\ge\; D_{\rm vN}[\rho]. \tag{54}$$

Summing up, the entropic indicator $D_{\rm vN}[\rho]$ provides a lower bound for the quantitative entanglement measure $\varepsilon[\rho]$. In the case of pure states of a systems of two fermions this lower bound is saturated and the inequality (54) becomes an equality.

4 Two-fermion systems with a single-particle Hilbert space of dimension four

In this and the next sections we are going to illustrate our entanglement criteria by recourse to examples of fermion systems with a single-particle Hilbert space of low dimensionality. In this section we are going to focus on systems of two fermions with a single-particle Hilbert space of dimension four. This case is of considerable relevance both from the conceptual and the practical points of view, and has been the subject of various recent research efforts [2,8,14]. It is the fermionic system of lowest dimensionality admitting the phenomenon of entanglement and it has profound physical and mathematical relationships with the celebrated two-qubits system of paramount importance in quantum information science [2]. It is worth mentioning that, in spite of its low dimensionality, this system is of considerable complexity, its generic (mixed) state depending on 35 (real) parameters. This system can be realized when one has spin-$\frac{1}{2}$ particles confined by an external potential well such that, within the range of energies available in the experimental setting, there are only two relevant eigenfunctions, $\Psi_a(\mathbf{x})$ and $\Psi_b(\mathbf{x})$ [2] corresponding, for instance, to the ground and first excited states of the confining potential. In such a scenario, the relevant single-particle Hilbert space is spanned by the single-particle states $|\Psi_a,+\rangle$, $|\Psi_a,-\rangle$, $|\Psi_b,+\rangle$, $|\Psi_b,-\rangle$ (here we use standard, self-explanatory notation, the $\pm$ signs corresponding to the spin degree of freedom).

Now we are going to apply our above-derived entropic entanglement criteria to some parametrized families of states of two fermions with a single-particle Hilbert space of dimension four. In this case there is an exact, analytical expression for the state's concurrence. It is then possible to compare the range of parameters for which entanglement is detected by the criteria with the exact range of parameters for which the states under consideration are entangled. As mentioned in Section 2, in this case the two-fermion states can be formally mapped onto the states of two $s = \frac{2}{3}$ spins. The antisymmetric eigenstates $|j,m\rangle$ of the total angular momentum operators J^2 and J_z constitute then a basis of the system's Hilbert space. These states are $|0,0\rangle$, $|2,-2\rangle$, $|2,-1\rangle$, $|2,0\rangle$, $|2,1\rangle$ and $|2,2\rangle$.

4.1 Werner-like states

First we are going to consider a family of states consisting of a mixture of the maximally entangled state $|0,0\rangle$ and a totally mixed state. These states are of the form,

$$\rho_W = p|0,0\rangle\langle 0,0| + \frac{1-p}{6}\, I \tag{55}$$

where $0 \le p \le 1$, and

$$I \;=\; |0,0\rangle\langle 0,0| + \sum_{m=-2}^{2} |2,m\rangle\langle 2,m| \tag{56}$$

is the identity operator acting on the six-dimensional Hilbert space corresponding to the two-fermion system. Evaluation of the concurrence shows that these states are entangled when $p > 0.4$. For these states we have,

$$\begin{aligned} D_{\rm vN}[\rho_W] &= \ln 2 + \frac{5}{6}(1-p)\ln\left(\frac{1-p}{6}\right) \\ &\quad + \frac{1}{6}(1+5p)\ln\left(\frac{1}{6}(1+5p)\right) \\ D_L[\rho_W] &= -\frac{7}{12} + \frac{5p^2}{6} \\ R_2[\rho_W] &= \ln\left(\frac{1+5p^2}{3}\right) \\ R_\infty[\rho_W] &= \ln\left(\frac{1+5p}{3}\right). \end{aligned} \tag{57}$$

The minimum values $p_{\min}$ of the parameter p such that for $p > p_{\min}$ the entanglement indicators D_{vN}, D_L, R_2 and R_∞ are positive (and thus entanglement is detected by the corresponding criteria) are given in the following table (that is, in each case, entanglement is detected when p is larger than the listed value):

	$D_{\mathrm{vN}} > 0$	$D_L > 0$	$R_{q=2} > 0$	$R_{q\to\infty} > 0$
$p_{\min}$	$\approx$0.809	$\sqrt{0.7} \approx 0.837$	$\approx$0.632	0.4

The entanglement detection efficiency of the entropic criterion based upon Rényi entropy increases with q. Indeed, in the limit $q \to \infty$ the Rényi entropic criterion detects all the entangled states within the family of states (55). The behaviour of the minimum value of p for which entanglement is detected as a function of the entropic parameter q is depicted in Figure 1.

4.2 θ-state

As second illustration we consider the following pure state,

$$|\psi\rangle = \frac{\sin\theta}{\sqrt{2}}\left[\left|-\frac{3}{2}\frac{3}{2}\right\rangle - \left|\frac{3}{2} - \frac{3}{2}\right\rangle\right] + \frac{\cos\theta}{\sqrt{2}} \times \left[\left|-\frac{1}{2}\frac{1}{2}\right\rangle - \left|\frac{1}{2} - \frac{1}{2}\right\rangle\right], \quad (58)$$

for which

$$D_{\mathrm{vN}}[|\psi\rangle\langle\psi|] = -\ln 2 - \cos^2\theta \ln\left(\frac{\cos^2\theta}{2}\right) - \sin^2\theta \ln\left(\frac{\sin^2\theta}{2}\right)$$
$$D_L[|\psi\rangle\langle\psi|] = \cos^2\theta \sin^2\theta. \quad (59)$$

Thus, both D_{vN} and $D_L = 0$ for $\theta = 0, \frac{\pi}{2}, \pi$ and both quantities are positive for all other values of θ in the interval $[0, \pi]$. We also have $S_q^{(R)}[\rho] + \ln 2 < S_q^{(R)}[\rho_r]$ for all $\theta \in (0, \pi), \theta \neq \frac{\pi}{2}$. Therefore, comparing this with the concurrence, one sees that all entangled states are detected.

4.3 Gisin-like states

As a final example let us consider the parametrized family of mixed states given by,

$$\rho_G = p|0,0\rangle\langle 0,0| + \frac{1-p}{2}\left(|2,-2\rangle\langle 2,-2| + |2,2\rangle\langle 2,2|\right), \quad (60)$$

with $0 \le p \le 1$. In this case we have,

$$D_{\mathrm{vN}}[\rho_G] = (1-p)\ln(1-p) + p\ln(2p)$$
$$D_L[\rho_G] = \frac{1}{4}\left(-1 - 4p + 6p^2\right)$$
$$R_2[\rho_G] = \ln(1 - 2p + 3p^2)$$
$$R_\infty[\rho_G] = \begin{cases} \ln(1-p) & 0 \le p \le \frac{1}{3} \\ \ln(2p) & \frac{1}{3} \le p \le 1. \end{cases} \quad (61)$$

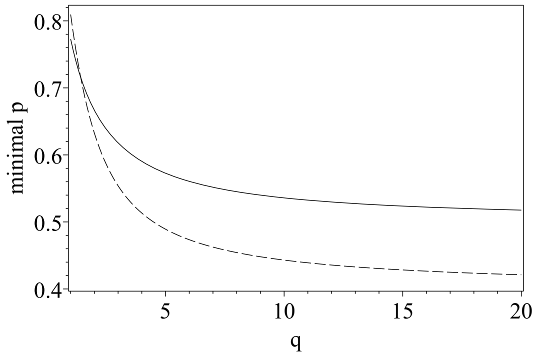


Fig. 1. Minimum p-value for which entanglement is detected in the case of the state ρ_W defined in equation (55) (dashed line) and ρ_G given by equation (60) (solid line).

The critical p values at which the entropic criteria based on the indicators D_{vN}, D_L, R_2 and R_∞ begin to detect entanglement are listed in the table below:

	$D_{\mathrm{vN}} > 0$	$D_L > 0$	$R_{q=2} > 0$	$R_{q\to\infty} > 0$
$p_{\min}$	$\approx$0.773	$\frac{2+\sqrt{10}}{6} \approx 0.860$	$\approx$0.667	0.5

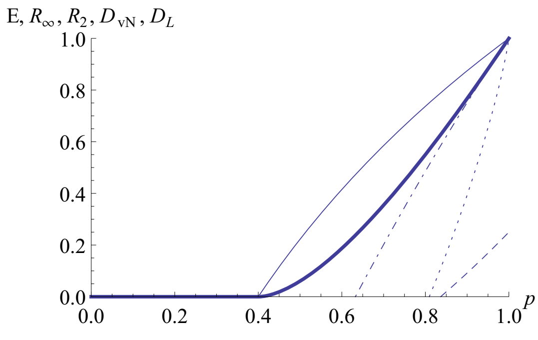
From the evaluation of the concurrence it follows that the Gisin-like states are entangled for $p > 0.5$. Thus, once again, the Rényi-based entropic criterion based on the indicator $R_{q\to\infty}$ detects all the entangled states in the family (60). The behaviour of the minimum value of p for which entanglement is detected as a function of the entropic parameter q is depicted in Figure 1.

We shall now illustrate the fact that the quantities D_{vN}, D_L and R_q involved in the entanglement criteria advanced here can also be regarded as entanglement indicators, in the sense that states exhibiting large values of these quantities tend to have higher entanglement. Two-fermion states with a single-particle Hilbert space of dimension four allow for the illustration of this, because in the case of these systems we have a closed analytical expression for the amount of entanglement of mixed states [2],

$$E[\rho] = h\left(\frac{1 + \sqrt{1 - \mathcal{C}_{\mathcal{F}}[\rho]^2}}{2}\right),$$
$$h(x) = -x\log_2 x - (1-x)\log_2(1-x), \quad (62)$$

where the concurrence $\mathcal{C}_{\mathcal{F}}$ was defined in equation (7). In Figure 2 we compare, for two parametrized families of mixed states, the behaviour of the entanglement measure with the behaviour of the abovementioned quantities. Note that in order to compare the entanglement measure with our entanglement indicators, the logarithms in the entanglement indicators are taken to the base 2 in Figure 2.

It transpires from Figure 2 that for these families of states the indicators associated with our entropic entanglement criteria do indeed tend to increase with the amount of entanglement exhibited by these states.



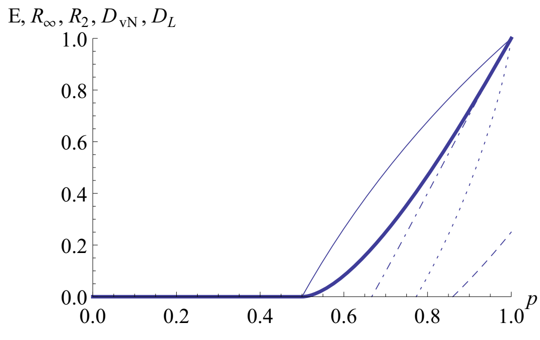


Fig. 2. (Color online) Entanglement measure (thick line) and entanglement indicators R_∞ (solid line), R_2 (dash-dotted line), $D_{\rm vN}$ (dotted line) and D_L (dashed line) for the states (a) ρ_W defined in equation (55) and (b) ρ_G given by equation (60). The logarithms in the entanglement indicators are taken to the base 2.

5 Two-fermion systems with a single-particle Hilbert space of dimension six

Two identical fermions with a four-dimensional single-particle Hilbert space (the simplest fermionic system admitting the phenomenon of entanglement) constitutes the only fermion system for which an exact analytical formula for the concurrence has been obtained. It is thus of interest to apply the entropic entanglement criteria to systems of higher dimensionality, for which such an expression for the concurrence is not known. Here we are going to consider a system consisting of two identical fermions with a single-particle Hilbert space of dimension six. The Hilbert space of this system is 15-dimensional. The generic (mixed) state of this system depends on 224 (real) parameters. The entanglement features of mixed states of this system are (up to now) basically "terra incognita". Here we are going to identify, for some parametrized families of mixed states, the range of values of the relevant parameters for which the states are entangled.

Using the angular momentum representation, the two-fermion system considered in this Section can be mapped onto a system of two spins with $s = \frac{5}{2}$. It is useful to introduce the following notation,

$$|m_1 m_2| = \frac{1}{\sqrt{2}}\left[|m_1\rangle|m_2\rangle - |m_2\rangle|m_1\rangle\right]. \tag{63}$$

We are going to study three particular families of mixed states of the form

$$\rho_i = p|\varphi_i\rangle\langle\varphi_i| + \frac{1-p}{15}\, I, \tag{64}$$

where $0 \le p \le 1$ and

$$I \;=\; |0,0\rangle\langle 0,0| + \sum_{m=-2}^{2} |2,m\rangle\langle 2,m| + \sum_{m=-4}^{4} |4,m\rangle\langle 4,m| \tag{65}$$

is the identity operator acting on the 15-dimensional Hilbert space describing the two-fermion system, and $|\varphi_i\rangle$ is an entangled two-fermion pure state. We consider three particular instances of $|\varphi_i\rangle$. In each case we provide the expressions for the indicators $D_{\rm vN}$, D_L, R_2 and R_∞, and give the minimum values $p_{\min}$ of the parameter p such that for $p > p_{\min}$ entanglement is detected by the criteria based on the positivity of the entanglement indicators.

The first illustration corresponds to

$$|\varphi_1\rangle = \frac{1}{\sqrt{3}}\left[\left|\frac{5}{2}\frac{3}{2}\right| + \left|\frac{1}{2} - \frac{1}{2}\right| - \left|-\frac{3}{2} - \frac{5}{2}\right|\right], \tag{66}$$

for which

$$\begin{aligned} D_{\rm vN}[\rho_1] &= \ln 3 + \frac{14}{15}(1-p)\ln\left(\frac{1-p}{15}\right) \\ &\quad + \frac{1}{15}(1+14p)\ln\left(\frac{1}{15}(1+14p)\right) \\ D_L[\rho_1] &= \frac{1}{15}\left(-9+14p^2\right) \\ R_2[\rho_1] &= \ln\left(\frac{1}{5}(1+14p^2)\right) \\ R_\infty[\rho_1] &= \ln\left(\frac{1}{15}(1+14p)\right) + \ln 3, \end{aligned} \tag{67}$$

resulting in

	$D_{\rm vN} > 0$	$D_L > 0$	$R_{q=2} > 0$	$R_{q\to\infty} > 0$
$p_{\min}$	$\approx$0.767	$\frac{3}{\sqrt{14}} \approx 0.802$	$\approx$0.535	$\frac{2}{7} \approx 0.286$

The second example is given by

$$|\varphi_2\rangle = -\frac{2}{3}\left|\frac{5}{2}\frac{3}{2}\right| - \frac{2}{3}\left|\frac{1}{2} - \frac{1}{2}\right| + \frac{1}{3}\left|-\frac{3}{2} - \frac{5}{2}\right|, \tag{68}$$

with,

$$D_{\rm vN}[\rho_2] = \frac{1}{45}\Bigg(-45\ln 2 + 42(1-p)\ln\left(\frac{1-p}{15}\right) - 5(3-2p)\ln\left(\frac{3-2p}{18}\right) - 10(3+p)\ln\left(\frac{3+p}{18}\right) + 3(1+14p)\ln\left(\frac{1}{15}(1+14p)\right)\Bigg)$$

$$D_L[\rho_2] = -\frac{3}{5} + \frac{121p^2}{135}$$

$$R_2[\rho_2] = -\ln(9+2p^2) + \ln\left(\frac{9}{5}(1+14p^2)\right)$$

$$R_\infty[\rho_2] = -\ln\left(\frac{1-p}{6} + \frac{2p}{9}\right) + \ln\left(\frac{1}{15}(1+14p)\right) - \ln 2, \quad (69)$$

and

	$D_{\rm vN} > 0$	$D_L > 0$	$R_{q=2} > 0$	$R_{q\to\infty} > 0$
$p_{\rm min}$	≈ 0.788	$\frac{9}{11} \approx 0.818$	≈ 0.557	$\frac{12}{37} \approx 0.324$

As a third instance we tackle,

$$|\varphi_3\rangle = \frac{1}{\sqrt{2}}\left[\left|\frac{5}{2}\frac{3}{2}\right| + \left|\frac{1}{2} - \frac{1}{2}\right|\right], \quad (70)$$

leading to,

$$D_{\rm vN}[\rho_3] = -\ln 2 - \frac{1-p}{3}\ln\left(\frac{1-p}{6}\right) - \frac{2+p}{3}\ln\left(\frac{2+p}{12}\right) + \frac{14}{15}(1-p)\ln\left(\frac{1-p}{15}\right) + \frac{1+14p}{15} \times \ln\left(\frac{1}{15}(1+14p)\right)$$

$$D_L[\rho_3] = -\frac{3}{5} + \frac{17p^2}{20}$$

$$R_2[\rho_3] = -\ln(2+p^2) + \ln\left(\frac{2}{5}(1+14p^2)\right)$$

$$R_\infty[\rho_3] = -\ln\left(\frac{1-p}{6} + \frac{p}{4}\right) + \ln\left(\frac{1}{15}(1+14p)\right) - \ln 2, \quad (71)$$

and

	$D_{\rm vN} > 0$	$D_L > 0$	$R_{q=2} > 0$	$R_{q\to\infty} > 0$
$p_{\rm min}$	≈ 0.825	$2\sqrt{\frac{3}{17}} \approx 0.840$	≈ 0.590	$\frac{8}{23} \approx 0.348$

For the above three cases, the behaviour of the minimum value of p for which entanglement is detected, as a function of the entropic parameter q, is depicted in Figure 3.

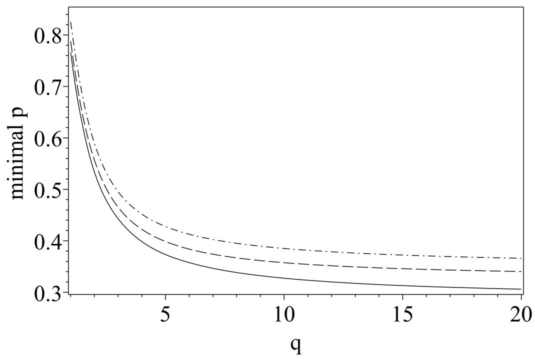


Fig. 3. Minimum value of p, as a function of the entropic parameter q, for entanglement detection in the states (64) with $|\varphi_1\rangle$ (solid line), $|\varphi_2\rangle$ (dashed line) and $|\varphi_3\rangle$ (dash-dotted line).

6 Systems of N identical fermions

Let us consider the general case of N fermions with single-particle Hilbert space of general (even) dimension $n > N$. The dimension of the Hilbert space associated with the N-fermion system is then $d = \frac{n!}{(n-N)!N!}$. The Rényi-based entropic criterion for two fermions that we derived in Section 3 can be extended to the case of N fermions. According to the extended criterion a state ρ of N identical fermions satisfying the inequality

$$S_q^{(R)}[\rho_r] > S_q^{(R)}[\rho] + \ln N, \quad (72)$$

for some $q \geq 1$ is necessarily entangled, where ρ_r is the single-particle reduced density matrix. This criterion can be derived following a procedure that is a straightforward generalization to the case of N fermions of the one detailed in Section 3.3 for the case of two fermions. One starts with a state of the N fermions that is a statistical mixture of pure states, each one represented by a single Slater determinant. Then one considers two equivalent representations for the total density matrix ρ: the spectral one, and the abovementioned one as a mixture of separable pure states. On the other hand, one considers two equivalent representations for the single-particle reduced density matrix ρ_r: again, the spectral one, and the one derived from the representation of the total state as a mixture of separable states. The two representations for ρ and the two ones for ρ_r are then related via appropriate unitary transformations according to equation (23). Following the same steps as in Section 3.3 it is then possible to obtain a majorization relation connecting ρ and ρ_r, which finally leads to the inequality (72).

As an illustration of the entanglement criterion based on the inequality (72) let us consider a family of states of a system of N fermions having the form

$$p\,|\Phi\rangle\langle\Phi| + \frac{(1-p)}{d}\,I_d, \quad (73)$$

where $0 \leq p \leq 1$, I_d is the identity operator acting on the N-fermion Hilbert space, and the single-particle Hilbert space has dimension $n = kN$, with $k \geq 2$ integer when N is even and for N odd $k \geq 2t$ ($t \geq 1$ integer). We also assume that the (pure) N-fermion state $|\Phi\rangle$ is of the form

$$|\Phi\rangle = \frac{1}{\sqrt{k}}\Big(|1,2,\ldots,N| + |N+1,N+2,\ldots,2N| + \ldots + |(k-1)N+1,(k-1)N+2,\ldots,kN|\Big), \quad (74)$$

where $|i_1, i_2, \ldots, i_N|$ denotes the Slater determinant (as in Eq. (63)) constructed with N different members $\{|i_1\rangle, \ldots, |i_N\rangle\}$ of an orthonormal basis $\{|1\rangle, \ldots, |n\rangle\}$ of the single-particle Hilbert space. The single-particle, reduced density matrix associated with the (pure) state $|\Phi\rangle$ corresponds to the totally mixed (single-particle) state, $\frac{1}{n}I_n$, where I_n is the identity operator corresponding to the single-particle Hilbert space. On the basis of the Rényi entropic criterion corresponding to $q \to \infty$ we identify as entangled the states of the form (73) satisfying the inequality,

$$\ln n + \ln\left(p + \frac{(1-p)}{d}\right) - \ln N > 0 \quad (75)$$

and hence entanglement is detected for

$$p > \frac{N\,(n-1)! - (n-N)!\,N!}{n! - (n-N)!\,N!}. \quad (76)$$

With N fixed, we find that the efficiency of the entanglement criterion grows as the dimension of the single-particle states, n, increases (that is, $p_{\min}$ decreases with n).

6.1 Full multi-particle entanglement: the case of systems of three fermions

When studying entanglement criteria for composite systems with more than two distinguishable subsystems a new problem arises: how to distinguish states exhibiting full multipartite entanglement from those that, although being entangled, are such that a subset of the parts constituting the system is disentangled from the rest of the system. A problem somewhat similar to this one also arises in the case of systems of N identical fermions with $N > 2$, although in the fermionic case this problem is much more subtle than in the case of distinguishable subsystems [5]. Although the analysis of this problem is beyond the scope of the present work, we shall now discuss it (in connection with our entropic entanglement criteria) for the case of systems of three identical fermions.

In the case of three fermions, a separable pure state (Slater determinant) is of the form

$$|\phi_{sl}\rangle = |\phi_1,\phi_2,\phi_3| = \frac{1}{\sqrt{6}}\big[|\phi_1\rangle|\phi_2\rangle|\phi_3\rangle - |\phi_1\rangle|\phi_3\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle|\phi_3\rangle + |\phi_2\rangle|\phi_3\rangle|\phi_1\rangle + |\phi_3\rangle|\phi_1\rangle|\phi_2\rangle - |\phi_3\rangle|\phi_2\rangle|\phi_1\rangle\big], \quad (77)$$

with $|\phi_1\rangle, |\phi_2\rangle, \phi_3\rangle$ being three orthonormal single-particle states. A general, separable mixed state is a state that can be expressed as a statistical mixture of states like (77).

Now, let us consider a pure state of three fermions of the form,

$$|\Psi\rangle = \sum_{1<i<j} c_{ij}\,|1,i,j|, \quad (78)$$

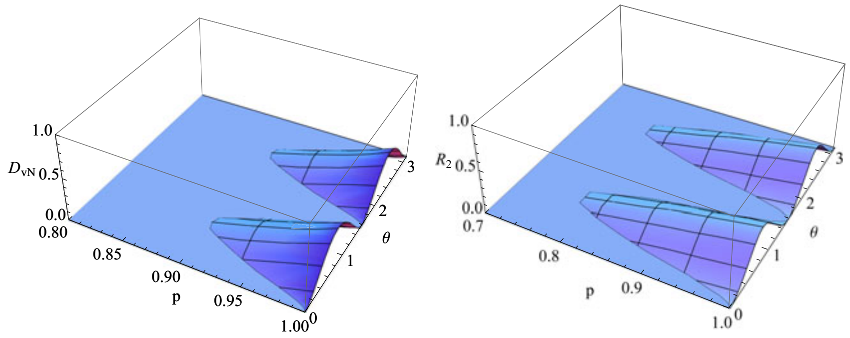
where $|1,i,j|$ stands for the Slater determinant constructed with the three normalized and orthogonal single-particle states $|1\rangle, |i\rangle, |j\rangle$, and $\{|k\rangle,\ k = 1,2,\ldots\}$ is a single-particle orthonormal basis. Now, in general, pure states of the above form are entangled in the sense that they cannot be written as one, single Slater determinant (that is, they are not "fully separable"). However, these states are special because they are a superposition of Slater determinants each of them involving the single-particle state $|1\rangle$. This means that it is physically sensible to say that when the system is in a state like (78) one of the particles is in the state $|1\rangle$ (although it does not make sense to ask which particle is in the state $|1\rangle$). Consequently, according to the analysis made in [4], where separability is associated to the possibility of assigning complete set of properties to the constituting particles, the state (78) can be regarded as describing a physical situation where one of the particles is disentangled from the other two. The same considerations apply to mixed states that are a mixture of states like (78) (each one involving the same "privileged" single-particle state $|1\rangle$).

The above discussion raises the following question: can the entropic entanglement criteria advanced here be used to discriminate between entangled states that are mixtures of states of the form (78) (having one "disentangled" particle in a given, single-particle state $|1\rangle$), on the one hand, and entangled states that cannot be expressed as (78) (or cannot be written as statistical mixtures of states like (78)) on the other hand? To address this problem let us first notice that, as can be verified after some algebra, the single-particle, reduced density matrix ρ_r corresponding to states of the form (78) (or to mixtures of such states) always has its largest eigenvalue equal to $\frac{1}{3}$. This implies that $S_\infty^{(R)}(\rho_r) = \ln 3$. Consequently, if a three-fermion state satisfies the (strict) inequality

$$R_\infty = S_\infty^{(R)}(\rho_r) - S_\infty^{(R)}(\rho) - \ln 3 > 0, \quad (79)$$

which implies $S_\infty^{(R)}(\rho_r) - \ln 3 > S_\infty^{(R)}(\rho) \geq 0$, one then knows for sure that this state is entangled and that it cannot be written as a statistical mixture of states like (78) (all with the same "privileged" single-particle state $|1\rangle$). In other words, for three-fermion systems, the entropic entanglement criterion based on the Rényi entropy with $q \to \infty$ is not just a sufficient entanglement criterion, but also a sufficient criterion for full, three-particle entanglement.

To illustrate the above discussion we choose the minimum single-particle dimension compatible with three-fermion entanglement, namely the single-particle Hilbert space of dimension six. As examples of entangled three-fermion states that do not exhibit full three-particle entanglement, let us consider the following family of



(a) (b)

Fig. 4. (Color online) Entanglement indicators (a) D_{vN} and (b) R_2 for the state (80).

states,

$$\rho = p|\phi\rangle\langle\phi| + (1-p)\rho_{\text{mix}}, \qquad (80)$$

where

$$|\phi\rangle = \cos\theta|1,2,3| + \sin\theta|1,4,5|, \qquad (81)$$

ρ_{mix} is a mixture (with equal weights) of the projectors corresponding to the ten Slater states containing a "1", that is, $|1,2,3|$, $|1,2,4|$, $|1,2,5|$, $|1,2,6|$, $|1,3,4|$, $\ldots$, $|1,5,6|$, where $|1\rangle, |2\rangle, \ldots, |6\rangle$ are normalized and mutually orthogonal single-particle states that form a basis for the single-particle state space. It is clear that one particle is in the state $|1\rangle$ whereas the other two particles are entangled (although it does not make sense to ask which particle is in the state $|1\rangle$), which means this is a multipartite system that is neither fully separable, nor fully entangled in the sense that all three particles are entangled. In order to evaluate the entanglement indicators D_{vN}, R_2 and R_∞, one has to find the eigenvalues of ρ and ρ_r. These are, $\{0,\ldots,0,\frac{1-p}{10},\ldots,\frac{1-p}{10},\frac{1+9p}{10}\}$ and $\{\frac{1}{3},\frac{p\cos^2\theta}{3}+\frac{2}{15}(1-p),\frac{p\cos^2\theta}{3}+\frac{2}{15}(1-p),\frac{p\sin^2\theta}{3}+\frac{2}{15}(1-p),\frac{p\sin^2\theta}{3}+\frac{2}{15}(1-p),\frac{2}{15}(1-p)\}$ respectively. In this case $R_\infty = \ln(\frac{1+9p}{10}) \leq 0$ and consequently full three-particle entanglement is (correctly) not detected. However, the entanglement indicators D_{vN} and R_2 do detect entanglement and Figures 4a and 4b show the results. Hence entanglement is detected for this multipartite state where not all particles are entangled with each other. However, full multi-particle entanglement is (correctly) not detected.

Summing up, we have seen that the entropic criterion based on the Rényi entropy $S_\infty^{(R)}$ is, in the three-fermion case, also a sufficient criterion for full three-particle entanglement. Incidentally, this is another manifestation of the fact that the most powerful entropic entanglement criterion based upon the Rényi entropy corresponds to the limit $q \to \infty$. The case of N-fermion systems with $N \geq 4$ is much more complex and certainly deserves further research. Previous experience with composite quantum systems with distinguishable subsystems (see [35] and references therein) suggests that in this case, besides the entropies of the single-particle reduced density matrix, the entropies of M-particle reduced density matrices (with $2 \leq M < N$) are going to be necessary to tackle this problem.

7 Summary

In the present work new entropic entanglement criteria for systems of two identical fermions have been advanced. These criteria have the form of appropriate inequalities involving the entropy of the density matrix associated with the total system, on the one hand, and the entropy of the single-particle reduced density matrix, on the other hand. We obtained entanglement criteria based upon the von Neumann, the linear, and the Rényi entropies. The criterion associated with the von Neumann entropy constitutes a special instance, corresponding to the particular value $q \to 1$ of the Rényi entropic parameter, of the more general criteria associated with the Rényi family of entropies. Extensions of these criteria to systems constituted by N identical fermions were also considered.

We applied our entanglement criteria to various illustrative examples of parametrized families of mixed states, and studied the dependence of the entanglement detection efficiency on the entropic parameter q. For the two-fermion states we considered, the entanglement criterion improves as q increases and is the most efficient in the limit $q \to \infty$.

In the three-fermion case we have seen that the entropic criterion based on the Rényi entropy $S_\infty^{(R)}$ is also a sufficient criterion for full three-particle entanglement. Incidentally, this is another manifestation of the fact that the most powerful entropic entanglement criterion based upon the Rényi entropy corresponds to the limit $q \to \infty$.

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