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Facets of the polytope of legal sequences [1]

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Abstract

A sequence of vertices in a graph is called a *(total) legal dominating sequence* if every vertex in the sequence (totally) dominates at least one vertex not dominated by the ones that precedes it, and at the end all vertices of the graph are (totally) dominated. The *Grundy (total) domination number* of a graph is the size of the largest (total) legal dominating sequence. In this work, we present integer programming formulations for obtaining the Grundy (total) domination number of a graph, we study some aspects of the polyhedral structure of one of them and we test the performance of some new valid inequalities as cuts.

Keywords: Grundy (total) domination number, Legal dominating sequence, Facet-defining inequality, Web graph.

1 Introduction

Domination set problems are among the most studied problems in Combinatorial Optimization due to a large number of applications. Two of them are

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Grundy domination [1] and Grundy total domination [2] problems.

Let $G = (V, E)$ be a graph and C be a subset of vertices such that no vertex of $V \backslash C$ is isolated in G. Define $N\langle \_ \rangle : V \to \mathcal{P}(V)$ as follows: $N\langle v \rangle \doteq N(v)$ if $v \notin C$ and $N\langle v \rangle \doteq N[v]$ if $v \in C$. Let $S = (v_1, \ldots, v_k)$ be a sequence of distinct vertices of G. The sequence S is called a *legal sequence* of $G; C$ if $W_i \doteq N\langle v_i \rangle \backslash \bigcup_{j=1}^{i-1} N\langle v_j \rangle \neq \emptyset$ holds for every $i = 2, \ldots, k$. If S is a legal sequence, then we say that v_i *footprints* the vertices from W_i. That is, v_i footprints a vertex $u \in N\langle v_i \rangle$ if u does not belong to $N\langle v_j \rangle$, $j = 1, 2, \ldots, i-1$.

If $\cup_{j=1}^{k} W_j = V$ then S is called a *dominating sequence*. It is easy to see that, for given G and C, any legal sequence of maximum length is also a dominating sequence. This maximum length is denoted by $\gamma_{\text{gr}}(G; C)$.

The Grundy domination and total domination problems consist in finding $\gamma_{\text{gr}}(G; V)$ and $\gamma_{\text{gr}}(G; \emptyset)$, respectively ($N\langle \_ \rangle$ denotes the closed or the open neighborhood, respectively). Both are $\mathcal{NP}$-hard problems [1,2].

In order to reduce the size of the input graph, two simple rules can be applied. First, if G is the disjoint union of graphs G_1 and G_2, $\gamma_{\text{gr}}(G; C) = \gamma_{\text{gr}}(G_1; C \cap V(G_1)) + \gamma_{\text{gr}}(G_2; C \cap V(G_2))$. Therefore, we can restrict ourselves to connected graphs. In addition, if there exist vertices u, v such that $N\langle u \rangle = N\langle v \rangle$ (such vertices are called *twins*), $\gamma_{\text{gr}}(G; C) = \gamma_{\text{gr}}(G - v; C \backslash \{v\})$, where $G - v$ is the graph obtained from G by deleting v. Hence, we can suppose that $G; C$ does not have twin vertices.

In this work, we present integer programming formulations for obtaining $\gamma_{\text{gr}}(G; C)$, we study some aspects of the polyhedral structure of one of them and we test the performance of some new valid inequalities as cuts. Also, we give below the value of $\gamma_{\text{gr}}(G; C)$ for two families of graphs.

An upper bound on $\gamma_{\text{gr}}(G; C)$ is $m \doteq n - \delta(G; C) + 1$ [1,2], where n is the number of vertices of G and $\delta(G; C) = \min_{v \in V} |N\langle v \rangle|$. If $G = P_n$ (a path on n vertices) with $V = \{1, \ldots, n\}$, we say that C is a good configuration for P_n if (i) $n = 0$ (empty path), (ii) $n = 1$ and $1 \in C$, or (iii) $n \geq 2$ and either (iii.1) $1 \notin C$ and C is a good configuration for the subpath $(3, \ldots, n)$ or (iii.2) $n \notin C$ and C is a good configuration for the subpath $(1, \ldots, n-2)$.

Proposition 1.1 *Let $G = P_{n \geq 1}$ and $C \subset V$. If $\{1, n\} \subset C$ or if C is a good configuration for G then, $\gamma_{\text{gr}}(G; C) = m$. Otherwise, $\gamma_{\text{gr}}(G; C) = m - 1$.*

Define $\mathcal{H}(G; C)$ as the hypergraph $(V, \mathcal{E})$ with $\mathcal{E} = \{N\langle v \rangle : v \in V\}$. $\mathcal{H}(G; C)$ is a *clutter* if $N\langle u \rangle \backslash N\langle v \rangle \neq \emptyset$ for all $u \neq v$. For example, consider instances $W_n^k; C$ where $n, k \in \mathbb{N}$, $n \geq 2(k+1)$ and W_n^k is the web graph with $V = \{0, \ldots, n-1\}$ and $E = \{(i, j) : 0 < |i - j| \leq k \text{ or } |i - j| \geq n - k\}$.

Proposition 1.2 *Let $G = W_n^k$ and $C \subset V$. $\mathcal{H}(G; C)$ is a clutter if and only* *if $n > 2(k+1)$ or $C = V$. Moreover, $\gamma_{\mathrm{gr}}(G;C) = m$ if (i) $C = V$, or (ii) there exists $i \in V \setminus C$ such that $V \setminus N[i]$ induces a path P and C is a good configuration for P. Otherwise $\gamma_{\mathrm{gr}}(G;C) = m - 1$.*

2 Integer programming formulations

Legal sequences can be modeled as binary vectors as follows. For every $v \in V$ and $i = 1, \ldots, m$, let y_{vi} be a binary value such that $y_{vi} = 1$ if v is chosen in step i. Also, for every $u \in V$ and $i = 1, \ldots, m$, let x_{ui} be a binary value such that $x_{ui} = 1$ if u is available to footprint in step i (i.e. not footprinted by any of the chosen vertices in previous steps). The following formulation computes the parameter $\gamma_{\mathrm{gr}}(G;C)$:

$$\max \sum_{i=1}^{m} \sum_{v \in V} y_{vi}$$

subject to

$$\sum_{v \in V} y_{vi} \leq 1, \qquad \forall \ \ i = 1, \ldots, m \tag{1}$$

$$\sum_{i=1}^{m} y_{vi} \leq 1, \qquad \forall \ \ v \in V \tag{2}$$

$$y_{vi+1} \leq \sum_{u \in N\langle v \rangle} (x_{ui} - x_{ui+1}), \quad \forall \ \ v \in V, \ i = 1, \ldots, m-1 \tag{3}$$

$$x_{ui} + \sum_{v \in N\langle u \rangle} y_{vi} \leq 1, \qquad \forall \ \ u \in V, \ i = 1, \ldots, m \tag{4}$$

$$x_{ui+1} \leq x_{ui}, \qquad \forall \ \ u \in V, \ i = 1, \ldots, m-1 \tag{5}$$

$$x, y \in \{0,1\}^{nm},$$

Constraints (1) ensure that at most one vertex is chosen in each iteration. Constraints (2) guarantee that each vertex is chosen at most once. Constraints (3) specify that v can be chosen in the next iteration only if there is at least one non-footprinted vertex of $N\langle v \rangle$ in the current iteration (that will become footprinted). Constraints (4) force to footprint u if some vertex of $N\langle u \rangle$ is chosen. Finally, constraints (5) tell that footprinted vertices will remain footprinted for the next iterations.

Although this formulation works fine, there exist several integer solutions associated to each legal sequence, since a variable x_{ui} can be set to zero even if it is not footprinted by any vertex. In order to forbid this situation, we also consider these constraints:

$$x_{u1} + \sum_{v \in N\langle u \rangle} y_{v1} \geq 1, \qquad \forall \;\; u \in V \tag{6}$$

$$x_{ui+1} + \sum_{v \in N\langle u \rangle} y_{vi+1} \geq x_{ui}, \qquad \forall \;\; u \in V, \; i = 1, \ldots, m-1 \tag{7}$$

Note that it is still allowed not to choose a vertex in a step, leading to several symmetric solutions. They can be avoided by replacing (1) by

$$\sum_{v \in V} y_{v1} = 1 \quad \text{and} \quad \sum_{v \in V} y_{vi+1} \leq \sum_{v \in V} y_{vi}, \qquad \forall \;\; i = 1, \ldots, m-1 \tag{8}$$

These constraints force to use the first k slots when the solution represents a legal sequence of length k.

The space of feasible solutions can be even smaller if we impose that every solution represents a dominating sequence:

$$\sum_{i=1}^{m} \sum_{v \in N\langle u \rangle} y_{vi} \geq 1, \qquad \forall \;\; u \in V \tag{9}$$

A preliminary experiment over random instances showed that considering constraints (8) greatly improve the performance of the optimization. Constraints (6) and (7) also help to improve the performance, while constraints (9) do not seem to be useful. According to our experiment, the best formulation consists of constraints (2) to (8).

3 The polytope of legal sequences

From now on, suppose that $n \geq 3$, G is connected and $G;C$ does not have twin vertices. Let P be the convex hull of the set of binary feasible solutions satisfying constraints (1) to (5). We choose not to consider (6) to (9) for two reasons. On the one hand, every valid inequality of P is still valid for polytopes that consider more constraints. On the other hand, the dimension of P just depends on the size of the instance, which is an interesting feature for polyhedral studies. This fails for the other polytopes. For instance, if constraints (8) were considered, the dimension of the polytope would be $nm - m|V_0| - (m-1)|V_1| + \sum_{v \in V} i(G; C, v) - 1$, where $i(G; C, v)$ is the index of largest step where v can occur in any legal sequence of $G;C$ (clearly, an $\mathcal{NP}$-hard parameter), $V_0 = \{v \in V : N\langle v \rangle = V\}$, and $V_1 = \{v \in V : N\langle v \rangle = V \backslash \{v\}\}$.

The following results give the dimension of P and prove the facet-definition of non-negativity inequalities and constraints (3), (4) and (5).

Proposition 3.1 *The polytope P is full dimensional, and the following inequalities define facets of P: (i) $y_{vi} \geq 0$ for all $v \in V$ and $i = 1, \dots, m$; (ii) $x_{ui} \geq 0$ for all $u \in V$ if and only if $i = m$; (iii) Constraints (5), i.e. $x_{ui+1} \leq x_{ui}$, if and only if for all $v \in V$, $N\langle v \rangle \neq \{u\}$ (i.e. every leaf of G adjacent to u must belong to C).*

Corollary 3.2 *If $\mathcal{H}(G; C)$ is a clutter, (5) always define facets of P.*

Proposition 3.3 *Let $W = \{w_1, \dots, w_t\}$ be a non-empty set of vertices satisfying the property $N\langle w_j \rangle \subset N\langle w_1 \rangle$ for all $j = 2, \dots, t$. Then, for all $i = 1, \dots, m-1$, the following is a valid inequality: $\sum_{w \in W} y_{wi+1} \leq \sum_{u \in N\langle w_1 \rangle}(x_{ui} - x_{ui+1})$. Moreover, it is facet defining if and only if every $v \in V \backslash W$ satisfies $N\langle v \rangle \backslash N\langle w_1 \rangle \neq \emptyset$ (W is* maximal *with respect to the property).*

Corollary 3.4 *If $\mathcal{H}(G; C)$ is a clutter, (3) always define facets of P.*

Proposition 3.5 *Constraint (4) defines a facet of P if and only if $i = 1$ or for every $v \in N\langle u \rangle$ there exists $w \in N\langle u \rangle \backslash \{v\}$ such that $N\langle w \rangle \backslash N\langle v \rangle \neq \emptyset$.*

Corollary 3.6 *If $\mathcal{H}(G; C)$ is a clutter, (4) always define facets of P.*

The following results provide two families of valid inequalities. The former dominates constraints (2) and, sometimes, constraints (4).

Proposition 3.7 *Let $u \in V$, $i = 2, \dots, m$, $N \subset N\langle u \rangle$ and $W = \{w_1, \dots, w_t\} \subset N\langle u \rangle \backslash N$ be a non-empty set of vertices, such that* (i) *$N\langle w_{r+1} \rangle \subset N\langle w_r \rangle$ for all $r = 1, \dots, t-1$,* (ii) *$N\langle v \rangle \subset N\langle w_t \rangle$ for all $v \in N$. Let $\{j_1, \dots, j_{t+1}\} \subset \{1, \dots, i\}$ be such that $j_1 = 1$, $j_{t+1} = i$ and $j_r \leq j_{r+1}$ for all $r = 1, \dots, t$. Then, the inequality $x_{ui} + \sum_{v \in N} y_{vi} + \sum_{r=1}^{t} \sum_{j=j_r}^{j_{r+1}} y_{w_r j} \leq 1$ is valid. Moreover, it defines a facet of P if and only if* (iii) *$N \neq \emptyset$ or $N\langle w_t \rangle \neq \{u\}$ (i.e. if w_t is a leaf of G then $w_t \in C$),* (iv) *for every $v \in N\langle u \rangle \backslash (N \cup W)$, the sets $R^{\supset}(v) \doteq \{r : N\langle v \rangle \backslash N\langle w_r \rangle \neq \emptyset\}$ and $R^{\subset}(v) \doteq \{r : N\langle w_r \rangle \backslash N\langle v \rangle \neq \emptyset\}$ have non-empty intersection, and $j_r < j_{r+1}$ for some $r \in R^{\supset}(v) \cap R^{\subset}(v)$.*

Corollary 3.8 *Let $u \in V$, $i = 2, \dots, m$, $w \in N\langle u \rangle$. If $\mathcal{H}(G; C)$ is a clutter, the following inequalities are facet-defining:*

$$x_{ui} + \sum_{j=1}^{i} y_{wj} \leq 1. \tag{10}$$

Proposition 3.9 *Let u_1, u_2 be distinct vertices such that $N\langle u_1 \rangle \cap N\langle u_2 \rangle \neq \emptyset$, $i = 2, \dots, m$, $k = 1, \dots, i$ and $w \in N\langle u_1 \rangle \cap N\langle u_2 \rangle$. The following inequality* *is valid:*

$$x_{u_1 i} + x_{u_2 i} + \sum_{j=1}^{i} y_{wj} + \sum_{v \in N\langle u_1\rangle \cup N\langle u_2\rangle} y_{vk} \leq 2. \tag{11}$$

(note that variable y_{wk} has coefficient 2 in the left hand side). Moreover, it defines a facet of P if (i) *there exist $v_1 \in N\langle u_1\rangle \backslash N\langle u_2\rangle$ and $v_2 \in N\langle u_2\rangle \backslash N\langle u_1\rangle$ such that $N\langle v_1\rangle \backslash N\langle w\rangle \neq \emptyset$ and $N\langle v_2\rangle \backslash N\langle w\rangle \neq \emptyset$,* (ii) *there exist $\tilde{v_1} \in N\langle u_1\rangle \backslash N\langle u_2\rangle$ and $\tilde{v_2} \in N\langle u_2\rangle \backslash N\langle u_1\rangle$ such that $N\langle w\rangle \backslash (\{u_2\} \cup N\langle \tilde{v_1}\rangle) \neq \emptyset$ and $N\langle w\rangle \backslash (\{u_1\} \cup N\langle \tilde{v_2}\rangle) \neq \emptyset$,* (iii) *for every $v \in (N\langle u_1\rangle \cap N\langle u_2\rangle) \backslash \{w\}$ we have $N\langle v\rangle \backslash N\langle w\rangle \neq \emptyset$ and $N\langle w\rangle \backslash N\langle v\rangle \neq \emptyset$, and* (iv) *if $k < i$ then for every $v \in (N\langle u_1\rangle \backslash N\langle u_2\rangle) \cup (N\langle u_2\rangle \backslash N\langle u_1\rangle)$ we have $N\langle v\rangle \backslash N\langle w\rangle \neq \emptyset$.*

Corollary 3.10 *If $\mathcal{H}(G;C)$ is a clutter and hypothesis* (ii) *of Proposition 3.9 holds, then constraints* (11) *define facets of P.*

We also carried out an experiment to see the performance of the new valid inequalities (10) and (11) when they are embedded as cuts. We used a computer equipped with an Intel i5 CPU 2.67GHz, Ubuntu 16.04, IBM ILOG CPLEX 12.7, formulation with constraints (2) to (8) and 24 instances (random graphs of 15, 20 and 30 vertices with edge densities from 20% to 80%, and $C \in \{\emptyset, V\}$). The following table reports averages obtained:

	All instances		Only high density	
Algorithm	Nodes	Time (sec.)	Nodes	Time (sec.)
pure B&B	64106	425.18	18596	362.52
B&C with ineq. (10)	**29658**	**287.38**	23327	443.79
B&C with ineq. (10), (11)	50779	409.59	**17970**	**300.41**

We conclude that embedding inequalities (10) as cuts helps to decrease the number of nodes and consumed time. Moreover, inequalities (11) are useful for instances with high density.

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