Research Article

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Existence and characterization of best φ-approximations by linear subspaces

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Abstract: Given an Orlicz space L^φ, we give very relaxed sufficient conditions on φ to ensure that there exists a best φ-approximation from any finite dimensional bounded linear subspace $S \subset L^\varphi$. In addition, given an operator T, defined from L^φ into itself, we give necessary and sufficient conditions on T to ensure that this is a best φ-approximation operator from a linear subspace S.

Keywords: Orlicz spaces, best approximation, characterization of best approximation operators

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1 Introduction and notations

Let $\mathcal{F}$ be the class of all non decreasing functions φ defined for all real numbers $t \geq 0$, with $\varphi(0) \geq 0$. We also assume a Δ_2 condition for the functions φ, which means that there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \leq \Lambda\varphi(a)$ for $a \geq 0$.

Let $\varphi \in \mathcal{F}$, and let $(B, \mathcal{A}, m)$ be the Lebesgue measure space, where $B \subset \mathbb{R}$ is a bounded set. We denote by $L^\varphi = L^\varphi(B, \mathcal{A}, m)$ the Orlicz space given by the class of all $\mathcal{A}$-measurable functions f defined on B such that $\int_B \varphi(|f|)\,dm < \infty$.

Given a set $S \subset L^\varphi$, an element $s^* \in S$ is called *a best φ-approximation of $f \in L^\varphi$ from the approximation class S* if and only if

$$\int_B \varphi(|f - s^*|)\,dm = \inf_{s\in S} \int_B \varphi(|f - s|)\,dm =: E_S(f),$$

and, in this case, we write $s^* \in \mu_\varphi(f/S)$. The mapping $\mu_\varphi : L^\varphi \to 2^S$ is called the *best φ-approximation operator* from S.

In the present paper we consider two problems. The study of existence of the best φ-approximation (i.e., $\mu_\varphi(f/S) \neq \emptyset$) and the characterization of all operators that behave as the best φ-approximation operator μ_φ defined before.

The problem of existence was extensively treated in [13] for the case where S is a lattice (i.e., if $f, g \in S$, then $\min(f, g) \in S$ and $\max(f, g) \in S$) and φ is a continuous function. Many cases of best approximation by linear subspaces are carried out with this setup. For instance, for $\varphi(t) = t^2$, where the approximation class is given through a sub σ-algebra (the classical conditional expectation) or through a sub σ-lattice was primary considered in [3]. Also [13] covers the case where $\varphi(t) = t^p$, $p > 1$, by considering a sub σ-algebra. The concept of p-predictors was treated by Ando and Amemiya in [1], whereas that of sub σ-lattices was treated

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in the seventies in [5]. Another case covered by [13] is when $\varphi(t) = t$ by considering sub σ-algebras, which is the concept of conditional medians presented in [19]. We also mention the study of the extended best approximation operator in [10], where the class S comes from a σ-lattice. We point out that in all these cases the approximation classes were suitable lattices, and most of them were treated for $\varphi(t) = t^p$.

In this paper, we obtain that $\mu_\varphi(f/S) \neq \emptyset$ for a wider class of functions φ than the ones considered in [13], and with a finite dimensional linear subspace of L^φ as the approximation class, for instance, the real polynomials defined on B with degree at most n. We want to emphasize that the set of polynomials is not in general a lattice. Thus, this approximation class is not considered in [13]. For this case we cannot use the monotonicity argument used in [13] to prove that $\mu_\varphi(f/S) \neq \emptyset$. Also we provide an example where $\mu_\varphi(f/S) = \emptyset$ if the left continuity condition on φ is removed. This result is presented in Section 2.

The characterization problem for the best linear approximation operator has been investigated by many authors. For the classical conditional expectation, i.e., $\varphi(t) = t^2$ and where S is a set of measurable functions with respect to a sub σ-algebra, the first general result appears in [15]. A similar characterization result was given in [2]. These results were also treated by other authors in [8, 16–18] in the sixties.

A characterization of a non linear operator as a conditional expectation with respect to a sub σ-lattice appears in [9]. In [12] it was given a characterization of the best approximation operator from a sub σ-lattice in the L^p space, $1 < p < \infty$. The same authors in [14] extended these results considering Orlicz spaces L^φ, and also gave a characterization of the best approximation operator for σ-algebras as approximation classes. In [6] Carrizo, Favier and Zó studied the characterization of the extended best φ-approximation operator.

In this paper we obtain, in Theorems 3.4 and 3.12, a characterization of the best φ-approximation operator considering linear subspaces of L^φ. Note that in these theorems, the polynomials can be included as an approximation class. Also, in Theorem 3.12 we get a characterization of best φ-approximation operators considering fewer requirements on φ and different hypotheses on the operator T to those in [14], to ensure that it is a best φ-approximation operator for given a sub σ-algebra.

2 Existence of best φ-approximations

In this section we prove the existence of the best φ-approximation when the class S is a finite-dimensional subspace from $L^\infty(B)$. For this purpose, we first make some considerations. For $\varphi \in \mathcal{F}$, we consider the convex function $\Phi(x) := \int_0^x \varphi(t)dt$, and using the Δ_2 condition on φ it is easy to see that

$$x\frac{\varphi(x)}{2\Lambda_\varphi} \leq \Phi(x) \leq x\varphi(x). \tag{2.1}$$

Next we prove an auxiliary result.

Lemma 2.1. *Let $\varphi \in \mathcal{F}$, $f \in L^\varphi(B)$, and let $S \subset L^\infty(B)$ be a finite dimensional space. Then there exists a positive constant K such that*

$$\varphi(\|s\|_\infty) \leq \frac{K}{|B|}\int_B \varphi(|s|)\,dm$$

for every $s \in S$.

Proof. If $\|s\|_\infty = 0$ the lemma follows at once for $K = 1$. Suppose that $\|s\|_\infty \neq 0$. Using the equivalence of norms in S, there exists a constant $C > 0$ such that

$$\Phi(\|s\|_\infty) \leq \Phi\Big(\frac{C}{|B|}\int_B |s|\,dm\Big)$$

for all $s \in S$. Now, by Jensen's inequality, we have

$$\Phi(\|s\|_\infty) \leq \tilde{K}\frac{1}{|B|}\int_B \Phi(|s|)\,dm, \tag{2.2}$$

for a constant $\tilde{K}$ which depends only on C and Λ_φ. Then we use (2.1), to get $\|s\|_\infty\varphi(\|s\|_\infty) \le 2\Lambda_\varphi\Phi(\|s\|_\infty)$. Now, using (2.2) and (2.1), we obtain

$$\begin{aligned}\|s\|_\infty\varphi(\|s\|_\infty) &\le 2\Lambda_\varphi\tilde{K}\frac{1}{|B|}\int_B \Phi(|s|)\,dm\\ &\le 2\Lambda_\varphi\tilde{K}\frac{1}{|B|}\int_B |s|\varphi(|s|)\,dm\\ &\le 2\Lambda_\varphi\frac{\tilde{K}}{|B|}\|s\|_\infty\int_B \varphi(|s|)\,dm,\end{aligned}$$

and the proof is complete. □

The following example shows that the above lemma does not remain valid if the assumption $S \subset L^\infty(B)$ is not required.

Example 2.2. Let $B = [0, 1]$, $\varphi(x) = \sqrt{x}$, and let S be the subspace spanned by the function $g(x) = \frac{1}{x}$. Clearly, $g \in L^\varphi$, however, $g \notin L^\infty(B)$.

Theorem 2.3. *Let $\varphi \in \mathcal{F}$ be a left continuous function, and let $S \subset L^\infty(B)$ be a finite-dimensional subspace. Then, for $f \in L^\varphi$, there exists $s^* \in S$ such that*

$$\int_B \varphi(|f - s^*|)\,dm = E_S(f).$$

Proof. Let $\{s_k\}_{k\in\mathbb{N}} \subset S$ be such that

$$\int_B \varphi(|f - s_k|)\,dm \le E_S(f) + \frac{1}{k}.$$

Then, using the Δ_2 condition on φ, we obtain

$$\int_B \varphi(|s_k|)\,dm \le \Lambda_\varphi\Big(\int_B \varphi(|f - s_k|)\,dm + \int_B \varphi(|f|)\,dm\Big) \le \Lambda_\varphi\Big(E_S(f) + 1 + \int_B \varphi(|f|)\,dm\Big).$$

From Lemma 2.1 we get that $\{\varphi(\|s_k\|_{L^\infty(B)})\}_{k\in\mathbb{N}}$ is bounded. Now, if $\lim_{x\to\infty}\varphi(x) = \infty$, then we have that $\{\|s_k\|_{L^\infty(B)}\}_{k\in\mathbb{N}}$ is uniformly bounded and there exists $s^* \in S$ such that $\lim_{j\to\infty}\|s_{k_j} - s^*\|_{L^\infty(B)} = 0$ for some subsequence $\{s_{k_j}\}_{j\in\mathbb{N}}$. Thus, $s_{k_j} - s^*$ converges to 0 a.e. on B. Finally, by the left continuity of φ and Fatou's Lemma, we have

$$\int_B \varphi(|f - s^*|)\,dm \le \int_B \liminf_{j\to\infty}\varphi(|f - s_{k_j}|)\,dm \le \liminf_{j\to\infty}\int_B \varphi(|f - s_{k_j}|)\,dm \le E_S(f). \tag{2.3}$$

On the other hand, if $\varphi(x)$ is a bounded function, set $\varphi(\infty) = \lim_{x\to\infty}\varphi(x)$ and suppose that $\|s_{k_j}\|_\infty \nearrow \infty$ as $j \to \infty$. For $R_{k_j} := s_{k_j}/\|s_{k_j}\|_\infty$, we get $R_{k_j} \to R_0$, with $R_0 \in S$ and $\|R_0\|_\infty = 1$, for a subsequence that we call again by R_{k_j}. Then, using Lebesgue's dominated theorem, we obtain

$$\lim_{j\to\infty}\int_B \varphi(|f - s_{k_j}|)\,dm = \lim_{j\to\infty}\int_{B-\{R_0=0\}} \varphi\Big(\frac{|f - s_{k_j}|}{\|s_{k_j}\|_\infty}\|s_{k_j}\|_\infty\Big)\,dm = \varphi(\infty)|B|.$$

Then

$$E_S(f) = \varphi(\infty)|B| \le \int_B \varphi(|f|)\,dm \le \varphi(\infty)|B|,$$

which implies $0 \in \mu_\varphi(f/S)$ and the proof is complete. □

The next example shows that the left continuity condition on φ has to be enforced.

Example 2.4. Let φ be the following non decreasing and non left continuous function

$$\varphi(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2}, \\ 2x + 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Set $B = [0, 1]$, $f(x) = \chi_{[0,\frac{1}{2}]}(x)$ and, as the approximation class, $S = \Pi^0$. Then the approximation problem is to minimize the function $F(x) = \varphi(x) + \varphi(1 - x)$ for $0 \le x \le 1$. Then $F(x) = |x - \frac{1}{2}| + \frac{5}{2}$ for $x \ne \frac{1}{2}$ and $F(\frac{1}{2}) = 4$, so the minimum is not reached.

In the next example we show that the best polynomial approximation may not be unique.

Example 2.5. Let $B = [0, 1]$ and $S = \Pi^0$ be the set of all constant functions, and let $f(x) = \chi_{[0,1/2]}(x)$ and $\varphi(x) = \sqrt{x}$. Then a direct calculation gives $\mu_\varphi(f/S) = \{0, 1\}$.

Theorem 2.6. *Let $\varphi \in \mathcal{F}$ be a left continuous function such that $\lim_{x\to\infty} \varphi(x) = \infty$, and let $S \subset L^\infty(B)$ be a finite-dimensional subspace. Then $\mu_\varphi(f/S)$ is a compact set in $(S, \|\cdot\|_{L^\infty(B)})$.*

Proof. Let $\{s_k\}_{k\in\mathbb{N}} \subset \mu_\varphi(f/S)$ be such that $\int_B \varphi(|f - s_k|)\, dm = E_S(f)$. According to Lemma 2.1, we have that $\{\|s_k\|_{L^\infty(B)}\}_{k\in\mathbb{N}}$ is bounded. So, there exist $s^* \in S$ and a subsequence of $\{s_k\}_{k\in\mathbb{N}}$, denoted again by $\{s_k\}$, such that $\lim_{k\to\infty} \|s^* - s_k\|_{L^\infty(B)} = 0$. Using (2.3), we have that $s^* \in \mu_\varphi(f/S)$, which means that $\mu_\varphi(f/S)$ is sequentially compact in $(S, \|\cdot\|_{L_\infty(B)})$, and this completes the proof. □

Remark 2.7. The hypothesis $\lim_{x\to\infty} \varphi(x) = \infty$ in Theorem 2.6 cannot be removed. In fact, we can consider $\varphi(x) = 1$ and then $\mu_\varphi(f/\Pi^n) = \Pi^n$ for any measurable function f.

3 Characterization of best φ-approximation operators

For $C \subset L^\varphi$, we denote by $\overline{C}^\varphi$ the set of all limits of sequences on C, by considering

$$d_\varphi(f, g) = \int_B \varphi(|f - g|)\, dm.$$

Let $T\colon L^\varphi \to L^\varphi$ be a single-valued operator. We denote the range of T by

$$R_T = \{T(f) : f \in L^\varphi\}.$$

In this section we denote by $\widetilde{\mathcal{F}}$ the set of functions φ in $\mathcal{F}$ which satisfy $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$. We give necessary and sufficient conditions on an operator T to assure that it is a best φ-approximation operator. To this end, we introduce the following definition.

Definition 3.1. A single-valued operator $T\colon L^\varphi \to L^\varphi$ is called:
(i) quasiadditive if $T(f + Tg) = Tf + Tg$ for all $f, g \in L^\varphi$,
(ii) quasihomogeneous if $T(\alpha Tf) = \alpha Tf$ for all $f \in L^\varphi$ and $\alpha \in \mathbb{R}$,
(iii) quasialgebraic if $T(1) = 1$ and $T(Tg\, Tf) = Tg\, Tf$ for all $f, g \in L^\infty(B)$,
(iv) φ-closed if $\lim_{n\to\infty} d_\varphi(f_n, f) = 0$ and $\lim_{n\to\infty} d_\varphi(Tf_n, g) = 0$ imply $Tf = g$,
(v) φ-expectation invariant if $\int_B \varphi(|f - Tf|)\, dm \le \int_B \varphi(|f|)\, dm$ for all $f \in L^\varphi$,
(vi) φ-bounded if $R_T \subset \overline{R_T \cap L^\infty(B)}^\varphi$.

The following examples show that the best φ-approximation operators satisfy most of the conditions given in the last definition.

Example 3.2. It is easy to check the following:
(i) Let φ be any convex function with $\varphi(0) = 0$, and let S be the class of algebraic polynomials with real coefficients of degree at most n defined on any measurable and bounded set B in $\mathbb{R}$. Then the operator T, defined by $T(f) = \mu_\varphi(f/S)$, $f \in L^\varphi$, satisfies (i), (ii) and (v) of Definition 3.1.

(ii) Let $\varphi(t) = t^p$, $1 < p < \infty$, $B = [0,1]$, and $S = \operatorname{span}\{\chi_{[0,\frac{1}{2})}, \chi_{[\frac{1}{2},1]}\}$. Then the operator T, defined by $T(f) = \mu_\varphi(f/S)$, satisfies (i)–(vi) of Definition 3.1.

Next we need some auxiliary results.

Lemma 3.3. *If the operator $T\colon L^\varphi \to L^\varphi$ is quasiadditive and $T0 = 0$, then $R_T = \{f \in L^\varphi : Tf = f\}$. In addition, if T is φ-closed, then $\overline{R_T}^\varphi = R_T$.*

Proof. Set $S = \{f \in L^\varphi : Tf = f\}$. Clearly, $S \subset R_T$. On the other hand, since T is quasiadditive and $T0 = 0$,

$$T(Tf) = Tf \quad \text{for all } f \in L^\varphi.$$

Therefore, $R_T \subset S$.

Now, let $g \in \overline{R_T}^\varphi$. Then there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset R_T$ such that $\lim_{k\to\infty} d_\varphi(Tg_k, g) = 0$. Since $T(Tg_k) = Tg_k$, $\lim_{k\to\infty} d_\varphi(T(Tg_k), g) = 0$. As T is φ-closed, we have $Tg = g$, and so $g \in R_T$. □

The following theorem gives sufficient conditions for an operator T to be a best φ-approximation operator.

Theorem 3.4. *Let $\varphi \in \widetilde{\mathcal{F}}$. If the operator $T\colon L^\varphi \to L^\varphi$ is quasiadditive, quasihomogeneous and φ-expectation invariant, then $Tf \in \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.*

Proof. We have $T0 = 0$, since T is quasihomogeneous. So, Lemma 3.3 shows that $R_T = \{f \in L^\varphi : Tf = f\}$. As T is also quasiadditive, we obtain

$$Tf - Tg = Tf + T(-Tg) = T(f + T(-Tg)) = T(f - Tg), \quad f, g \in L^\varphi. \tag{3.1}$$

Let $f \in L^\varphi$ and $P \in R_T \setminus \{Tf\}$. Since $TP = P$, according to (3.1), we have $T(f - TP) = Tf - TP = Tf - P \neq 0$. As T is φ-expectation invariant, we get

$$\begin{aligned}
\int_B \varphi(|f - Tf|)\,dm &= \int_B \varphi(|f - TP - (Tf - TP)|)\,dm \\
&= \int_B \varphi(|f - TP - T(f - TP)|)\,dm \\
&\leq \int_B \varphi(|f - TP|)\,dm \\
&= \int_B \varphi(|f - P|)\,dm.
\end{aligned}$$

Therefore, $Tf \in \mu_\varphi(f/R_T)$. □

Remark 3.5. Under the same hypotheses of Theorem 3.4, if $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$, then $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$. This is the case, for example, if Definition 3.1 (v) is considered with a strict inequality.

If we also assume the uniqueness of the best φ-approximation for all $f \in L^\varphi$, we get the following characterization result.

Theorem 3.6. *Let $\varphi \in \widetilde{\mathcal{F}}$, and let $T\colon L^\varphi \to L^\varphi$ be an operator. Assume that $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$. Then the following statements are equivalent:*
(i) *R_T is a subspace of L^φ, and $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.*
(ii) *T is quasiadditive, quasihomogeneous and φ-expectation invariant.*

Proof. (i) ⇒ (ii) Let $f, g \in L^\varphi$ and $\alpha \in \mathbb{R}$. Clearly, T is φ-expectation invariant. Since $\int_B \varphi(|\alpha f - \alpha T(f)|)\,dm = 0$ for $f \in R_T$, T is quasihomogeneous. Finally, an easy computation shows that $Tf + Tg = \mu_\varphi((f + Tg)/R_T)$ and consequently $T(f + Tg) = Tf + Tg$, i.e., T is quasiadditive.

(ii) ⇒ (i) Let $f \in L^\varphi$. By hypothesis and Theorem 3.4, we have

$$Tf = \mu_\varphi(f/R_T).$$

Now, we claim that R_T is a subspace of L^φ. Indeed, let $P, Q \in R_T$ and $\alpha, \beta \in \mathbb{R}$. Since T is quasihomogeneous, $T(\beta Q) = \beta Q$, which implies

$$T(\alpha P + \beta Q) = T(\alpha P + T(\beta Q)) = T(\alpha P) + T(\beta Q) = \alpha P + \beta Q,$$

because T is quasiadditive. This completes the proof. □

The next corollary provides special cases of best φ-approximation operators assuming additional properties on φ.

Corollary 3.7. *Let $\varphi \in \widetilde{\mathcal{F}}$ be a differentiable strictly convex function with $\varphi'(0) = 0$. Let $T\colon L^\varphi \to L^\varphi$ be a quasiadditive and quasihomogeneous operator that satisfies*

$$\int_B \varphi'(|f - Tf|)\,\mathrm{sgn}(f - Tf)T(f)\,dm \geq 0 \quad \text{for all } f \in L^\varphi.$$

Then R_T is a subspace of L^φ and $Tf = \mu_\varphi(f/R_T)$.

Proof. Let $f \in L^\varphi$ be such that $Tf \neq 0$ and consider $r\colon (0,1] \to \mathbb{R}$, a strictly convex function defined by

$$r(t) = \frac{1}{t}\int_B (\varphi(|f - Tf + tTf|) - \varphi(|f - Tf|))\,dm.$$

By hypothesis, we have

$$\int_B \varphi(|f|)\,dm - \int_B \varphi(|f - Tf|)\,dm = r(1) \geq r(t), \qquad t \in (0,1].$$

So, [11, pp. 16–17] implies that

$$\int_B \varphi(|f|)\,dm - \int_B \varphi(|f - Tf|)\,dm \geq \lim_{t\to 0^+} r(t) \geq \int_B \varphi'(|f - Tf|)\,\mathrm{sgn}(f - Tf)T(f)\,dm \geq 0,$$

i.e., T is φ-expectation invariant. Since $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$, Theorem 3.6 completes the proof. □

We point out that the extended best polynomial approximation operator given in [7], defined on $L^{p-1}(B)$, satisfies the hypothesis of Corollary 3.7.

In the following we consider additional properties on T, which allow us to obtain specific subspaces R_T.

Definition 3.8. Let S be a subspace of L^φ. We say that S is *measurable* if there exists a sub σ-algebra of $\mathcal{A}$, say $\mathcal{L}$, such that S is the class of all $\mathcal{L}$-measurable functions in L^φ, i.e., $S = L^\varphi(B, \mathcal{L}, m)$. The subspace S is called *bounded* if $S \subset \overline{S \cap L^\infty(B)}^\varphi$.

Lemma 3.9. *Let $S \subset L^\varphi$ be a subspace. If S is measurable, then $\overline{S}^\varphi = S$.*

Proof. By hypothesis, $S = L^\varphi(B, \mathcal{L}, m)$ for some σ-subalgebra $\mathcal{L}$ of $\mathcal{A}$. If $g \in \overline{S}^\varphi$, then there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset S$ such that

$$\lim_{k\to\infty} d_\varphi(g_k, g) = 0. \tag{3.2}$$

Hence, there exists a subsequence of $\{g_k\}_{k\in\mathbb{N}}$, which is denoted in the same way, such that $\lim_{k\to\infty} g_k = g$ a.e. on B. So, g is an $\mathcal{L}$-measurable function. In addition, from (3.2) we have $g \in L^\varphi$ and consequently $g \in S$. This completes the proof. □

Example 3.10. The subspace S given in Example 3.2 (b) is bounded and measurable. Indeed, it is easy to check that $S = L^\varphi(B, \mathcal{L}, m) \subset L^\infty(B)$, where $\mathcal{L}$ is the sub σ-algebra $\{\emptyset, [0, \frac{1}{2}), [\frac{1}{2}, 1], B\}$.

Lemma 3.11. *Let $\varphi \in \widetilde{\mathcal{F}}$ and let $T\colon L^\varphi \to L^\varphi$ be a quasiadditive, quasialgebraic and φ-closed operator. Then the set $\mathcal{L} = \{A \subset B : T(\chi_A) = \chi_A\}$ is a sub σ-algebra of $\mathcal{A}$.*

Proof. Clearly, $B \in \mathcal{L}$, since $T(1) = 1$. Let $A_1, A_2 \in \mathcal{L}$. As $\chi_{A_1\cap A_2} = \chi_{A_1}\chi_{A_2} = T(\chi_{A_1})T(\chi_{A_2})$, we have

$$T(\chi_{A_1\cap A_2}) = T(T(\chi_{A_1})T(\chi_{A_2})) = T(\chi_{A_1})T(\chi_{A_1}) = \chi_{A_1}\chi_{A_2} = \chi_{A_1\cap A_2}. \tag{3.3}$$

According to (3.3), we have $\chi_{A_1\setminus A_2} = \chi_{A_1} - \chi_{A_1\cap A_2} = \chi_{A_1} - T(\chi_{A_1\cap A_2})$, hence (3.1) implies

$$T(\chi_{A_1\setminus A_2}) = T(\chi_{A_1} - T(\chi_{A_1\cap A_2})) = T(\chi_{A_1}) - T(\chi_{A_1\cap A_2}) = \chi_{A_1} - \chi_{A_1\cap A_2} = \chi_{A_1\setminus A_2}. \tag{3.4}$$

By (3.4), we get

$$T(\chi_{A_1\cup A_2}) = T(\chi_{A_1} + \chi_{A_2\setminus A_1}) = T(\chi_{A_1} + T(\chi_{A_2\setminus A_1})) = T(\chi_{A_1}) + T(\chi_{A_1\setminus A_2}) = \chi_{A_1} + \chi_{A_1\setminus A_2} = \chi_{A_1\cup A_2}. \tag{3.5}$$

Now, let $\{A_k\}_{k\in\mathbb{N}} \subset \mathcal{L}$. From (3.5) we obtain that $g_N := \chi_{\bigcup_{k=1}^N A_k} \in \mathcal{L}$, $N \in \mathbb{N}$, and $\{g_N\}_{N\in\mathbb{N}}$ is a nondecreasing sequence such that $\lim_{N\to\infty} d_\varphi(g_N, g) = 0$, where $g = \chi_{\bigcup_{k=1}^\infty A_k}$. Since $T(g_N) = g_N$, $N \in \mathbb{N}$, and T is φ-closed, it follows that $T(g) = g$. □

Theorem 3.12. *Let $\varphi \in \widetilde{\mathcal{F}}$ and let $T\colon L^\varphi \to L^\varphi$ be an operator. Assume that $\mu_\varphi(f/R_T)$ is a singleton for all $f \in L^\varphi$. If T is quasiadditive, quasialgebraic, φ-closed, φ-expectation invariant and φ-bounded, then R_T is a bounded measurable linear subspace of L^φ and $Tf = \mu_\varphi(f/R_T)$ for all $f \in L^\varphi$.*

Proof. Since T is quasialgebraic, it is quasihomogeneous and $T0 = 0$. By Lemma 3.3, we have

$$R_T = \{f \in L^\varphi : Tf = f\}.$$

Let $\mathcal{L}$ be the sub σ-algebra given in Lemma 3.11, and let $A_1, A_2 \in \mathcal{L}$, $\alpha_1, \alpha_2 \in \mathbb{R}$. By hypothesis, we have

$$\begin{aligned} T(\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2}) &= T(\alpha_1\chi_{A_1} + \alpha_2 T(\chi_{A_2})) = T(\alpha_1\chi_{A_1} + T(\alpha_2 T(\chi_{A_2}))) \\ &= T(\alpha_1\chi_{A_1}) + T(\alpha_2 T(\chi_{A_2})) = T(\alpha_1 T(\chi_{A_1})) + T(\alpha_2 T(\chi_{A_2})) \\ &= \alpha_1 T(\chi_{A_1}) + \alpha_2 T(\chi_{A_2}) = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2}. \end{aligned}$$

Therefore, $\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} \in R_T$. So, linear combinations of characteristic functions of sets of $\mathcal{L}$ are in R_T. The facts that the totality of such functions is dense in $L^\varphi(B, \mathcal{L}, m)$ with respect to d_φ, and T is φ-closed, imply $T(f) = f$ for all $f \in L^\varphi(B, \mathcal{L}, m)$. Hence,

$$L^\varphi(B, \mathcal{L}, m) \subset R_T. \tag{3.6}$$

We claim that

$$R_T \cap L^\infty(B) \subset L^\varphi(B, \mathcal{L}, m). \tag{3.7}$$

Indeed, let $g \in R_T \cap L^\infty(B)$. Then there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \le g \le \beta$. Hence,

$$\int_B \varphi(|g|)\, dm \le \varphi(|\max\{|\alpha|, |\beta|\}|)m(B) < \infty,$$

and so $g \in L^\varphi$. Set $I = [\alpha, \beta]$, and let $\mathcal{M}$ be the Borel σ-algebra on I. Now, we will prove that if $D \in \mathcal{M}$, then $g^{-1}(D) \in \mathcal{L}$, and consequently $g \in L^\varphi(B, \mathcal{L}, m)$.

We consider the Lebesgue–Stieltjes measure $\mu_g\colon \mathcal{M} \to [0, m(B)]$ given by

$$\mu_g(C) = m(g^{-1}(C)) \quad \text{for } C \in \mathcal{M}. \tag{3.8}$$

Since $(I, \mathcal{M}, \mu_g)$ is a finite measure space, from (3.8), [4, Proposition 7.2, p. 80] and [4, Proposition 1.8, p. 43], we have

$$\int_I \varphi(|h|)\, d\mu_g = \int_B \varphi(|h \circ g|)\, dm \quad \text{whenever } h\colon I \to \mathbb{R} \text{ is a } \mathcal{M}\text{-measurable function.} \tag{3.9}$$

Let $D \in \mathcal{M}$ and $k \in \mathbb{N}$. As $\varphi(0^+) = 0$, there exists $\epsilon_k > 0$ such that

$$\varphi(\epsilon_k) < \frac{1}{2k\Lambda} \min\Big\{\frac{1}{\varphi(1)}, \frac{1}{m(I)}\Big\}.$$

Since, μ_g is a regular measure, there exist an open set U and a closed set F such that $F \subset D \subset U \subset I$ and $\mu_g(U \setminus F) < \varphi(\epsilon_k)$. By Urysohn's lemma, there exists a continuous function $f\colon I \to [0,1]$ such that $f|_F = 1$ and $f|_{(I\setminus U)} = 0$. So,

$$\int_I \varphi(|f - \chi_D|)\,d\mu_g = \int_{U\setminus F} \varphi(|f - \chi_D|)\,d\mu_g \le \varphi(1)\mu_g(U \setminus F) < \frac{1}{2k\Lambda}. \tag{3.10}$$

On the other hand, by Weierstrass' theorem there exists a polynomial P_k on I such that $|P_k(x) - f(x)| < \epsilon_k$ for all $x \in I$, and so

$$\int_I \varphi(|P_k - f|)\,d\mu_g \le \varphi(\epsilon_k)m(I) < \frac{1}{2k\Lambda}.$$

Therefore, (3.9) and (3.10) imply that

$$\begin{aligned}\int_B \varphi(|P_k(g) - \chi_{g^{-1}(D)}|)\,dm &= \int_B \varphi(|(P_k - \chi_D)\circ g|)\,dm = \int_I \varphi(|P_k - \chi_D|)\,d\mu_g \\ &\le \Lambda \int_I \varphi(|P_k - f|)\,d\mu_g + \Lambda \int_I \varphi(|f - \chi_D|)\,d\mu_g < \frac{1}{k},\end{aligned}$$

and consequently $\lim_{k\to\infty} d_\varphi(P_k(g), \chi_{g^{-1}(D)}) = 0$.

Since $g^2 = TgTg = T(TgTg) = T(g^2)$, by induction we see that $g^n = T(g^n)$, $n \in \mathbb{N}$, i.e., $g^n \in R_T$, $n \in \mathbb{N}$. By Theorem 3.6, it follows that R_T is a subspace of L^φ. Hence, $P_k(g) \in R_T$, $k \in \mathbb{N}$, and so

$$\lim_{k\to\infty} d_\varphi(T(P_k(g)), \chi_{g^{-1}(D)}) = 0.$$

As T is φ-closed, we have $T\chi_{g^{-1}(D)} = \chi_{g^{-1}(D)}$, and therefore $g^{-1}(D) \in \mathcal{L}$.

Now, from (3.6), (3.7) and Lemmas 3.3 and 3.9, we get $\overline{R_T \cap L^\infty(B)}^{\varphi} \subset L^\varphi(B, \mathcal{L}, m) \subset R_T$. Since T is φ-bounded, we have $R_T = L^\varphi(B, \mathcal{L}, m)$.

Finally, Theorem 3.6 completes the proof. □

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