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Canonicity in subvarieties of BL-algebras

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Abstract. We prove that every subvariety of BL-algebras which is not finitely generated is not σ-canonical. We also prove π-canonicity for an infinite family of subvarieties of BL-algebras that are not finitely generated. To do so we study the behavior of canonical extensions of ordered sums of posets.

1. Introduction

Canonical extensions were introduced by Jónsson and Tarski for Boolean algebras with operators (see [20] and [21]) and generalized for distributive lattices, lattices, and posets with different internal operations in [14], [15], [13] and [11]. They provide an algebraic formulation of what is otherwise treated via topological duality or relational methods.

If $\mathbf{A} = (A, \{f_i, i \in I\})$ is a distributive lattice with operations, the canonical extension A^σ of the lattice $(A, \wedge, \vee)$ is a doubly algebraic distributive lattice that contains A as a separating and compact sublattice. The main problem is to extend the extra operations $\{f_i, i \in I\}$ to A^σ and check if this new structure is an algebra in the same class as $\mathbf{A}$. There are two natural ways to extend an operation f: one is the canonical extension f^σ and the other is the dual canonical extension f^π (see [14] or Lemma 2.1). Then there are two possible candidates for the canonical extension of $\mathbf{A}$, namely the canonical extension $\mathbf{A}^\sigma$ and the dual canonical extension $\mathbf{A}^\pi$. A class of algebras is called σ-canonical or π-canonical if it is closed under canonical or dual canonical extensions, respectively.

BL-algebras were introduced by Hájek (see [17]) as the algebraic counterpart of basic logic. BL-algebras can be viewed as distributive lattices with additional operations; therefore, one can analyze canonicity for different subvarieties of these algebras. The failures of σ-canonicity and π-canonicity for the variety of BL-algebras were proved in [7], together with the non-canonicity for some well known subvarieties of BL-algebras. There are also some results in [22] that imply non-canonicity for BL-algebras.

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An interesting approach to the study of BL-algebras is the one developed in [1]. The results obtained there rely on the following two facts: the variety of BL-algebras is generated by BL-chains (totally ordered BL-algebras) and each BL-chain can be uniquely decomposed as the ordinal sum of a totally ordered family of totally ordered Wajsberg hoops. Based on these two facts, we study σ- and π-canonicity for subvarieties of BL-algebras.

The main idea of our study is characterizing the behavior of canonicity with respect to the operation of ordinal sum. This requires a careful investigation of canonical extensions of ordered sum of posets. Such an investigation is carried out in Section 2. Once this is accomplished, in Section 3 we completely characterize π- and σ-canonical extensions for ordinal sum of hoops in terms of canonical extensions for the summands.

In the fourth section, after giving some preliminaries on BL-algebras, we analyze σ-canonicity for each subvariety of BL-algebras. We conclude that the only subvarieties of BL-algebras that enjoy σ-canonicity are those that are finitely generated. Although the negative results about σ-canonicity for some subvarieties of BL-algebras still hold in the case of π-canonicity, we give some non-trivial positive results on π-canonicity for special subvarieties of BL-algebras.

In the first section we collect all the preliminary results and definitions about canonicity necessary to achieve our aim. For details, see [14] and [11].

Notation. Throughout the paper, we will denote algebras by boldface letters $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc., and their corresponding universes by the ordinary type of the same letter A, B, C, etc. We denote a poset by $\langle X, \leq \rangle$. When there is no danger of confusion, we denote the poset $\langle X, \leq \rangle$ simply by the universe X.

2. Preliminaries on canonical extensions

An *extension* e of a poset $\langle X, \leq \rangle$ is an order embedding $e \colon X \to Y$. To simplify notation, we suppress the embedding e and call $\langle Y, \leq \rangle$ an extension of $\langle X, \leq \rangle$, assuming that X is a subposet of Y. An element of Y is called *closed* if it is the infimum in Y of some non-empty downdirected subset F of X. Dually, if an element of Y is the supremum of some non-empty updirected subset F of X, it is called *open*.

An extension Y of a poset X is called a *completion* if Y is a complete lattice. In this case, Y is called *dense* if each element of Y is both the supremum of the closed elements below it and the infimum of the open elements above it. The extension Y is called *compact* if given $D, U \subseteq X$, non-empty downdirected and updirected sets, respectively, such that $\bigwedge_Y D \leq \bigvee_Y U$, then there exist $x \in D$ and $y \in U$ such that $x \leq y$. If a completion Y of X is dense and compact it is called a *canonical extension of* X.

Every poset $\langle X, \leq \rangle$ has a canonical extension which is unique up to an isomorphism that fixes $\langle X, \leq \rangle$ (see [11, Theorems 2.5 and 2.6]). We will denote it by $\langle X, \leq\rangle^\sigma$ or X^σ. From now on, we denote by $K(X^\sigma)$ and $O(X^\sigma)$ the sets of closed and open elements of X^σ, respectively. We recall some easy facts that will be used in the course of the proofs without explicit mention:

- $X = K(X^\sigma) \cap O(X^\sigma)$,
- If $x \in K(X^\sigma)$, then $x = \bigwedge_{X^\sigma}\{y \in X : x \leq y\}$,
- If $x \in O(X^\sigma)$, then $x = \bigvee_{X^\sigma}\{y \in X : x \geq y\}$,
- $\bigwedge_{X^\sigma} X^\sigma = \bigwedge_{X^\sigma} X$ and $\bigvee_{X^\sigma} X^\sigma = \bigvee_{X^\sigma} X$.
- $X = X^\sigma$ if and only if X is a finite lattice.

Given an order preserving map $f\colon X \to Y$, we consider two extensions $f^\sigma, f^\pi\colon X^\sigma \to Y^\sigma$, that can be computed according to the next lemma.

Lemma 2.1 (Lemma 3.4, [11]). *For every order preserving map $f\colon X \to Y$ we have:*

(1) $f^\sigma(c) = \bigwedge\{f(x) : c \leq x \in X\}$ *for every* $c \in K(X^\sigma)$*;*
(2) $f^\sigma(a) = \bigvee\{f^\sigma(c) : a \geq c \in K(X^\sigma)\}$*;*
(3) $f^\pi(o) = \bigvee\{f(x) : o \geq x \in X\}$ *for every* $o \in O(X^\sigma)$*;*
(4) $f^\pi(a) = \bigwedge\{f^\pi(o) : a \leq o \in O(X^\sigma)\}$*.*

In case $f\colon X \to Y$ is an order reversing map, the canonical and dual canonical extensions of f are defined by the following procedure:

- Consider the function $g\colon X^d \to Y$, where X^d denotes the poset whose order is obtained by reversing the order of X and $g(x) = f(x)$ for every $x \in X$.
- Compute $g^\sigma, g^\pi\colon (X^d)^\sigma \to Y$ according to Lemma 2.1, using the fact that $K\left(\left(X^d\right)^\sigma\right) = O(X^\sigma)$ and $O\left(\left(X^d\right)^\sigma\right) = K(X^\sigma)$.
- Since $\left(X^d\right)^\sigma = (X^\sigma)^d$, let $f^\sigma, f^\pi\colon X^\sigma \to Y$ be such that $f^\sigma(x) = g^\sigma(x)$ and $f^\pi(x) = g^\pi(x)$ for each $x \in X^\sigma$.

Let $f\colon \prod_{i=1}^n X_i \to Y$ be a map that preserves the order in some coordinates and reverses it in others. The extensions f^σ, $f^\pi\colon \left(\prod_{i=1}^n X_i\right)^\sigma \to Y^\sigma$ can be computed following the previous procedure coordinatewise and recalling that

- $(X \times Y)^\sigma = X^\sigma \times Y^\sigma$,
- $K\left((X \times Y)^\sigma\right) = K(X^\sigma) \times K(Y^\sigma)$,
- $O\left((X \times Y)^\sigma\right) = O(X^\sigma) \times O(Y^\sigma)$.

Let $\mathbf{A} = \left\langle A, \{f_i\}_{i\in I}\right\rangle$ be an algebra and $\leq$ an order over the set A. If every operation f_i preserves or reverses the order in each coordinate, we define two candidates to extend the algebra $\mathbf{A}$: the *canonical extension* $\mathbf{A}^\sigma = \left\langle A^\sigma, \{f_i^\sigma\}_{i\in I}\right\rangle$ and the *dual canonical extension* $\mathbf{A}^\pi = \left\langle A^\sigma, \{f_i^\pi\}_{i\in I}\right\rangle$ of $\mathbf{A}$.

A class of algebras is σ-*canonical* or π-*canonical* if it is closed under canonical or dual canonical extensions, respectively.

There are many results, positive and negative, about canonicity of classes of algebras. Two of the most important are the following theorem and its corollary.

Theorem 2.2 (see [15]). *If a class $\mathfrak{K}$ of similar algebras with distributive lattice reduct is closed under ultraproducts and σ-canonical (π-canonical) extensions, then the variety generated by $\mathfrak{K}$ is σ-canonical (π-canonical).*

Corollary 2.3. *If $\mathcal{V}$ is a finitely generated variety of algebras with a distributive lattice reduct, then $\mathcal{V}$ is σ-canonical and π-canonical.*

3. Canonical extensions of sums of posets

In this section, we will describe the behavior of canonical extensions of ordered sums of posets.

Definition 3.1. Let I be a poset, and let $\langle X_i, \leq_i \rangle$ be a family of pairwise disjoint non-empty posets indexed by I. The *ordered sum* $\biguplus_I \langle X_i, \leq_i \rangle$ is a poset whose universe is $\bigcup_{i \in I} X_i$ and the order is defined by

$$a \leq_{\uplus} b \text{ iff } \begin{cases} a, b \in X_i \text{ and } a \leq_i b \text{ for some } i \in I, \text{ or} \\ a \in X_i \text{ and } b \in X_j \text{ for some } i < j \text{ in } I. \end{cases}$$

Note that ordered sums preserve existing joins and meets of non-empty sets. Given an ordered sum $X = \biguplus_I \langle X_i, \leq_i \rangle$, we define $\chi \colon X \to I$ by $\chi(a) = i$ if and only if $a \in X_i$.

Hence, χ is an order preserving function that preserves arbitrary existing joins and meets. This function will help us deal with canonical extensions of ordered sums of posets.

Lemma 3.2. *Let I be a complete lattice, and let $\langle X_i, \leq_i \rangle$ be a family of pairwise disjoint non-empty complete lattices indexed by I. The ordered sum $X = \biguplus_I \langle X_i, \leq_i \rangle$ is a complete lattice.*

Proof. Let $A \subseteq \bigcup_{i \in I} X_i$, and let χ be the function previously defined. Since I is a complete lattice, there exists $j \in I$ such that $j = \bigvee_I \chi(A)$.

Assume first that $j \in \chi(A)$, i.e., $A \cap X_j \neq \emptyset$. Let us check that $\bigvee_X A = \bigvee_{X_j} (A \cap X_j)$. Observe that for each $x \in A$,

$$x \leq_{\uplus} \bigvee_{X_j} (A \cap X_j).$$

Now let $c \in X$ be such that $x \leq_{\uplus} c$ for each $x \in A$. Since the ordinal sum preserves the existing joins, the assumption $A \cap X_j \neq \emptyset$ yields $\bigvee_{X_j} (A \cap X_j) = \bigvee_X (A \cap X_j) \leq_{\uplus} c$.

Assume now that $j \notin \chi(A)$. We claim that $\bigvee_X A = \bigwedge_{X_j} X_j$. This is clear if $A = \emptyset$. Otherwise, for each $x \in A$, since $\chi(x) < j$, then $x \leq_{\uplus} \bigwedge_{X_j} X_j$. Now let $y \in \bigcup_{i \in I} X_i$ be such that $x \leq_{\uplus} y$ for every $x \in A$. Hence, $\chi(x) \leq \chi(y)$ for every $x \in A$. We obtain that $j \leq \chi(y)$, and then $\bigwedge_{X_j} X_j \leq_{\uplus} y$.

The result for arbitrary meets follows similarly. □

From now on, we omit the symbol $\uplus$ in the notation $\leq_{\uplus}$. We next state the main result of this section.

Theorem 3.3. *Let I be a poset, and let $\langle X_i, \leq_i \rangle$ be a family of pairwise disjoint non-empty posets indexed by I. Assume also that each poset $\langle X_i, \leq_i \rangle$ is downdirected and updirected. Then*

$$\left(\biguplus_I \langle X_i, \leq_i \rangle\right)^\sigma \text{ is isomorphic to } \biguplus_{I^\sigma} \langle Y_j, \leq_j \rangle$$

where $\langle Y_j, \leq_j \rangle = \langle X_j, \leq_j \rangle^\sigma$ if $j \in I$ and $\langle Y_j, \leq_j \rangle = \langle \{0_j\}, =_j \rangle$ if $j \notin I$.

Proof. Let $Q = \biguplus_{I^\sigma} \langle Y_j, \leq_j \rangle$. By the previous lemma, Q is a complete lattice that extends the poset $\biguplus_I \langle X_i, \leq_i \rangle$. Observe that $\{0_j : j \in K(I^\sigma) \setminus I\} \subseteq K(Q)$ and $\{0_j : j \in O(I^\sigma) \setminus I\} \subseteq O(Q)$. Also, for each $i \in I$, we have the inclusions $K(X_i^\sigma) \subseteq K(Q)$ and $O(X_i^\sigma) \subseteq O(Q)$. We prove that Q is dense and compact.

Density. Let $x \in Q$. Then $x \in Y_j$ for some $j \in I^\sigma$. We distinguish two different possibilities:

Case 1: $j \in I$. Hence $x \in X_j^\sigma = Y_j$. Since X_j is downdirected, $\bigwedge_{X_j^\sigma} X_j$ is a closed element less than x. Therefore, the set $\{c \in K(Y_j) : c \leq x\}$ is non-empty. Thus,

$$\begin{aligned} x &= \bigvee\nolimits_{Y_j} \{c \in K(Y_j) : c \leq x\} = \bigvee\nolimits_Q \{c \in K(Y_j) : c \leq x\} \\ &\leq \bigvee\nolimits_Q \{c \in K(Q) : c \leq x\} \leq x. \end{aligned}$$

Analogously, we obtain that $x = \bigwedge_Q \{d \in O(Q) : x \leq d\}$ using that X_j is updirected.

Case 2: $j \notin I$. Then $x = 0_j$. Since $j \in I^\sigma$ and I^σ is a dense extension of I, then $j = \bigvee_I \{k \in K(I^\sigma) : k \leq j\}$. The function χ preserves the existing joins; therefore,

$$\begin{aligned} &\chi\left(\bigvee\nolimits_Q (\{c \in K(Y_k) : k \in I, k \leq j\} \cup \{0_k : k \in K(I^\sigma) \setminus I, k \leq j\})\right) \\ &\quad = \bigvee\nolimits_I (\{\chi(c) : c \in K(Y_k), k \in I, k \leq j\} \cup \{\chi(0_k) : k \in K(I^\sigma) \setminus I, k \leq j\}) \\ &\quad = \bigvee\nolimits_I (\{k \in I : k \leq j\} \cup \{k \in K(I^\sigma) \setminus I : k \leq j\}) = j. \end{aligned}$$

Since the only element in Y_j is 0_j, we conclude that

$$\bigvee\nolimits_Q (\{c \in K(Y_k) : k \in I, k \leq j\} \cup \{0_k : k \in K(I^\sigma) \setminus I, k \leq j\}) = 0_j.$$

Thus,

$$\begin{aligned} x = 0_j &= \bigvee\nolimits_Q (\{c \in K(Y_k) : k \in I, k \leq j\} \cup \{0_k : k \in K(I^\sigma) \setminus I, k \leq j\}) \\ &\leq \bigvee\nolimits_Q \{c \in K(Q) : c \leq x\} \leq x. \end{aligned}$$

Similarly, $x = \bigwedge_Q \{d \in O(Q) : x \leq d\}$. We conclude that Q is a dense extension of $\biguplus_I \langle X_i, \leq_i \rangle$.

Compactness. Let $D, U \subseteq \biguplus_I \langle X_i, \leq_i \rangle$ be down and updirected sets, respectively, such that $\bigwedge_Q D \leq \bigvee_Q U$. Since χ preserves arbitrary existing joins and meets,

$$\bigwedge\nolimits_{I^\sigma} \chi(D) = \chi\left(\bigwedge\nolimits_Q D\right) \leq \chi\left(\bigvee\nolimits_Q U\right) = \bigvee\nolimits_{I^\sigma} \chi(U).$$

From the compactness of I^σ, there exist $k \in \chi(D)$ and $l \in \chi(U)$ such that $k \leq l$.

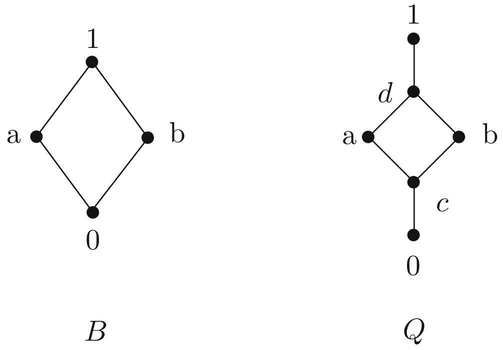

FIGURE 1

Assume first that we can choose $k < l$. This being the case, there are $x \in D$ and $y \in U$ such that $\chi(x) = k$ and $\chi(y) = l$. Hence, $x \leq y$ and the result follows.

If there is no element in $\chi(D)$ strictly smaller than an element of $\chi(U)$, we are in the case $l = k$ and $s \leq t$ for every $t \in \chi(D)$ and $s \in \chi(U)$. Hence, our assumption implies $k = \bigwedge_{I^\sigma} \chi(D) = \bigvee_{I^\sigma} \chi(U) = l$.

Therefore, it is easy to see that

$$\bigwedge_Q D = \bigwedge_{X_k^\sigma} (D \cap X_k) = \bigvee_{X_k^\sigma} (U \cap X_k) = \bigvee_Q U.$$

Clearly, $D \cap X_k$ and $U \cap X_k$ are down and updirected subsets of X_k, respectively. The compactness of X_k^σ implies that there exist $x \in D \cap X_k$ and $y \in U \cap X_k$ such that $x \leq y$. We conclude that Q is a compact extension of $\biguplus_I \langle X_i, \leq_i \rangle$. $\square$

The assumption that each X_i is down and updirected in the previous theorem is crucial. As evidence of its importance we present the next example.

Example 3.4. Consider the boolean lattice with two atoms B (see Figure 1).

Let $I = \{1, 2, 3\}$, with $1 < 2 < 3$ and let the sets X_1, X_2 and X_3 be given by $X_1 = \{0\}$, $X_2 = \{a, b\}$ and $X_3 = \{1\}$, with the order inherited from B. Then $B = \biguplus_I \langle X_i, \leq_i \rangle$. Since B is a finite lattice, $B^\sigma = B$. On the other hand, the poset $Q = \biguplus_I \langle X_i, \leq_i \rangle^\sigma$ is given by Figure 1. Hence, B^σ is not isomorphic to Q. The reason is that X_2 is neither downdirected nor updirected.

To conclude these preliminaries, we present a characterization of closed and open elements of an ordered sum of posets together with a corollary for later use.

Theorem 3.5. *Let I be a poset, and for each $i \in I$, let $\langle X_i, \leq_i \rangle$ be a down and updirected poset. If $Q = (\biguplus_I \langle X_i, \leq_i \rangle)^\sigma$, we have*

(1) $K(Q) = \{0_j : j \in K(I^\sigma) \setminus I\} \cup \left(\bigcup_{i \in I} K(\langle X_i, \leq_i \rangle^\sigma)\right)$, *and*
(2) $O(Q) = \{0_j : j \in O(I^\sigma) \setminus I\} \cup \left(\bigcup_{i \in I} O(\langle X_i, \leq_i \rangle^\sigma)\right)$.

Proof. Denote $X_i = \langle X_i, \leq_i \rangle$ and $X = \biguplus_I X_i$; thus, $Q = X^\sigma$.

For the first assertion, recall that in the proof of Theorem 3.3 it is observed that $\{0_j : j \in K(I^\sigma) \setminus I\} \cup \left(\bigcup_{i \in I} K(X_i^\sigma)\right) \subseteq K(Q)$. Let $x \in K(Q)$. There exists a downdirected set $F \subseteq X$ such that $x = \bigwedge_Q F$. Then $\chi(x) = \bigwedge_{I^\sigma} \chi(F) = j$.

Suppose that $j \in I$. Hence, $x \in X_j^\sigma$. This implies that $x = \bigwedge_Q F \leq \bigvee_Q X_j$. Since X_j is updirected, by the compactness of Q with respect to X, there exist $y \in X_j$ and $z \in F$ such that $x \leq z \leq y$. Therefore, $F \cap X_j \neq \emptyset$ and

$$x = \bigwedge\nolimits_Q F = \bigwedge\nolimits_{X_j^\sigma} (F \cap X_j).$$

We conclude that $x \in K\left(X_j^\sigma\right)$.

On the other hand, if $j \notin I$, then $x = 0_j$. Since F is downdirected, the set $\chi(F) \subseteq I$ is downdirected. Therefore, $j = \bigwedge_{I^\sigma} \chi(F) \in K(I^\sigma) \setminus I$; the result follows.

The proof of the second assertion can be obtained in an analogous way. □

Corollary 3.6. *Let L be a totally ordered set (a chain) with a greatest element $\top$. Then*

(1) $L^\sigma \setminus \{\top\} = (L \setminus \{\top\})^\sigma$;
(2) $K((L \setminus \{\top\})^\sigma) = K(L^\sigma) \setminus \{\top\}$;
(3) $O((L \setminus \{\top\})^\sigma) = O(L^\sigma) \setminus \{\top\}$.

Proof. Let $I = \{1, 2\}$ with the natural order. Then $X_1 = L \setminus \{\top\}$ and $X_2 = \{\top\}$ are ordered sets with the orders inherited from L. Clearly, $L = \biguplus_I X_i$. The result is a consequence of Theorems 3.3 and 3.5. □

4. Hoops and canonical extensions of ordinal sums

The theory of hoops serves as a base for the theory of BL-algebras (see [1]). As we shall see in the course of this section, basic hoops are algebras with a distributive lattice structure. Given a family of hoops indexed by a totally ordered set, a new hoop can be obtained as the ordinal sum of the members of the family. Our purpose is to investigate the canonical extensions of ordinal sums of hoops. This investigation is based on the ordered sum of posets studied in the previous section, and it will help us deal with canonical extensions of BL-algebras in subsequent ones.

Definition 4.1. A *hoop* is an algebra $\mathbf{B} = \langle B, *, \to, \top \rangle$, such that $\langle B, *, \top \rangle$ is a commutative monoid and for all $x, y, z \in B$,

$$x \to x = \top, \tag{4.1}$$

$$x * (x \to y) = y * (y \to x), \tag{4.2}$$

$$x \to (y \to z) = (x * y) \to z. \tag{4.3}$$

If $\mathbf{B} = \langle B, *, \to, \top \rangle$ is a hoop, the natural order on B is defined by $a \leq b$ iff $a \to b = \top$ and $\mathbf{B}$ satisfies the residuation law:

$$a * b \leq c \text{ iff } a \leq b \to c.$$

The partial order on any hoop is a semilattice order, where $a \wedge b = a * (a \to b)$ and $\top$ is the largest element in the order.

A hoop is called *basic* if it is isomorphic to a subdirect product of totally ordered hoops. Basic hoops form a subvariety $\mathcal{BH}$ of the variety of hoops, axiomatized by the equation

$$((x \to y) \to z) \to (((y \to x) \to z) \to z) = \top. \tag{4.4}$$

In every basic hoop, the natural order is a distributive lattice order, where $\vee$ can be defined from the hoop operations. Hence, every basic hoop $\mathbf{B}$ has the underlying structure of a distributive lattice. Moreover, the operation $*$ preserves the order in both coordinates, and $\to$ reverses the order in the first coordinate and preserves it in the second.

Next we recall the definition of ordinal sum. In the literature, the definition of ordinal sum involves only families of hoops and the resulting algebra is also a hoop (see [2]). For our purposes, we have generalized the definition for arbitrary algebras of the same similarity type as hoops.

Definition 4.2. Let $(I, \leq)$ be a totally ordered set. For each $i \in I$ let $\mathbf{B}_i = \langle B_i, *_i, \to_i, \top \rangle$ be an algebra of type $(2, 2, 0)$ such that for every $i \neq j$, $B_i \cap B_j = \{\top\}$. We define the ordinal sum as an algebra $\bigoplus_{i \in I} \mathbf{B}_i = \langle \cup_{i \in I} B_i, *, \to, \top \rangle$ of the same type with the operations $*, \to$ given by

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in B_i, \\ x & \text{if } x \in B_i \setminus \{\top\},\ y \in B_j \text{ and } i < j, \\ y & \text{if } y \in B_i \setminus \{\top\},\ x \in B_j \text{ and } i < j. \end{cases}$$

$$x \to y = \begin{cases} \top & \text{if } x \in B_i \setminus \{\top\},\ y \in B_j \text{ and } i < j, \\ x \to_i y & \text{if } x, y \in B_i, \\ y & \text{if } y \in B_i,\ x \in B_j \text{ and } i < j. \end{cases}$$

We explain the relation among the ordered sum of posets and the ordinal sum of hoops in the next remark.

Remark 4.3. Let I be a totally ordered set, and for each $i \in I$, let $\mathbf{B}_i$ be a hoop. Consider the posets $\langle B_i, \leq_i \rangle$, where $\leq_i$ is the natural order of the algebra $\mathbf{B}_i$. Then the ordinal sum $\bigoplus_{i \in I} \mathbf{B}_i$ is a hoop whose natural order $\leq$ is given by

$$a \leq b \text{ iff } \begin{cases} a, b \in B_i \text{ and } a \leq_i b \text{ for some } i \in I, \text{ or} \\ a \in B_i,\ b \in B_j \text{ and } a \neq \top \text{ for some } i < j \text{ in } I. \end{cases}$$

Therefore, the underlying poset $\langle\bigoplus_{i\in I} B_i, \leq\rangle$ of $\bigoplus_{i\in I}\mathbf{B}_i$ can be described as follows: let $\alpha\notin I$ and $I' = I\cup\{\alpha\}$, with $\alpha > i$ for each $i\in I$. Then

$$\langle\bigoplus_{i\in I} B_i, \leq\rangle = \biguplus_{I'} Z_i$$

where $Z_i = B_i\setminus\{\top\}$ with the restricted order if $i\in I$, and $Z_\alpha = \langle\{\top\}, =\rangle$.

For simplicity, we denote by $\mathbf{B}_1\oplus\mathbf{B}_2$ the ordinal sum of two summands, assuming $1 < 2$.

We are ready to describe the σ- and π-canonical extensions of sums of hoops using the canonical extensions of its components. To achieve this aim, we need to consider two different algebras of type $(2, 2, 0)$ with the same ordered universe. Let $\mathbf{L}_2 = \langle\{0, 1\}, *_{\mathbf{L}_2}, \rightarrow_{\mathbf{L}_2}, 1\rangle$, with $0 < 1$ and operations given by

$*_{\mathbf{L}_2}$	0	1
0	0	0
1	0	1

$\rightarrow_{\mathbf{L}_2}$	0	1
0	1	1
1	0	1

$\mathbf{L}_2$ is a basic hoop with two elements and $\top = 1$.

Now let $\mathbf{M} = \langle\{0, 1\}, *_{\mathbf{M}}, \rightarrow_{\mathbf{M}}, 1\rangle$, with the same order as $\mathbf{L}_2$ and binary operations defined by

$*_{\mathbf{M}}$	0	1
0	0	0
1	0	1

$\rightarrow_{\mathbf{M}}$	0	1
0	0	1
1	0	1

Clearly, $\mathbf{M}$ is not a hoop. However, it plays an important role in the following theorem.

Theorem 4.4. *Let I be a totally ordered set, and for each $i\in I$, let $\mathbf{B}_i = \langle B_i, *_i, \rightarrow_i, \top\rangle$ be a hoop such that $B_i\setminus\{\top\}$ is updirected under the natural order of $\mathbf{B}_i$. Then*

(1) $\left(\bigoplus_{i\in I}\mathbf{B}_i\right)^\pi \cong \bigoplus_{i\in I^\sigma}\mathbf{D}_i$, *where* $\mathbf{D}_i = \begin{cases}\mathbf{B}_i^\pi & \textit{if } i\in I,\\ \mathbf{L}_2 & \textit{if } i\notin I;\end{cases}$

(2) $\left(\bigoplus_{i\in I}\mathbf{B}_i\right)^\sigma \cong \bigoplus_{i\in I^\sigma}\mathbf{C}_i$, *where* $\mathbf{C}_i = \begin{cases}\mathbf{B}_i^\sigma & \textit{if } i\in I,\\ \mathbf{M} & \textit{if } i\notin I.\end{cases}$

Proof. We denote $\mathbf{B} = \bigoplus_{i\in I}\mathbf{B}_i$, $\mathbf{C} = \bigoplus_{i\in I^\sigma}\mathbf{C}_i$, and $\mathbf{D} = \bigoplus_{i\in I^\sigma}\mathbf{D}_i$. Their corresponding universes (as well as their lattice reducts) are denoted by B, C, and D, respectively. As usual, B^σ is the canonical extension of the poset B.

For each $i\in I$, since the poset $B_i\setminus\{\top\}$ is updirected and every hoop is a meet-semilattice, we conclude that $B_i\setminus\{\top\}$ is down and updirected. Following Theorem 3.3 and Remark 4.3, it is easy to check that the lattices B^σ and D are isomorphic.

Since the algebra $\mathbf{M}$ is not a hoop, if $I^\sigma\neq I$, then the ordinal sum $\mathbf{C}$ is not a hoop. Then $\mathbf{C}$ does not have a natural order. However, if we consider $\mathbf{C}$ ordered as the ordered sum $(\biguplus_{I^\sigma} C_i\setminus\{\top\})\biguplus\langle\top, =\rangle$, we easily see that the lattices C and B^σ are isomorphic.

Without danger of confusion, for each $j \in I^\sigma \setminus I$, let 0_j be the only element in $C_j \setminus \{\top\}$ as well as the only element in $D_j \setminus \{\top\}$.

From the descriptions of closed and open elements obtained in Theorem 3.5, Corollary 3.6 and from Remark 4.3, we can assert that

$$K(B^\sigma) = \{0_j : j \in K(I^\sigma) \setminus I\} \cup (\bigcup_{i\in I} K(B_i^\sigma \setminus \{\top\})) \cup \{\top\};$$
$$O(B^\sigma) = \{0_j : j \in O(I^\sigma) \setminus I\} \cup (\bigcup_{i\in I} O(B_i^\sigma \setminus \{\top\})) \cup \{\top\}.$$

To prove (1), we have to see that the canonical extensions $*^\pi$ and $\to^\pi$ of the operations of $\mathbf{B}$ that are defined according to Lemma 2.1, coincide with the operations $*_{\mathbf{D}}$ and $\to_{\mathbf{D}}$ in $\mathbf{D}$ given by the definition of ordinal sum.

We check first that $*^\pi = *_{\mathbf{D}}$. From the fact that $*_{\mathbf{D}}$ and $*^\pi$ are commutative, we need only consider the following cases:

Suppose that $a, b \in O(B^\sigma)$; one of four cases occurs.

Case 1: $a, b \in O(B_j^\sigma \setminus \{\top\}) = O(B_j^\sigma) \setminus \{\top\}$ for some $j \in I$. Then

$$\begin{aligned} a *^\pi b &= \bigvee\nolimits_D \{c * d : c \le a, d \le b \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c * d : c \le a, d \le b \text{ and } c, d \in B_j \setminus \{\top\}\} \\ &= \bigvee\nolimits_{B_j} \{c *_j d : c \le a, d \le b \text{ and } c, d \in B_j \setminus \{\top\}\} = a *_j^\pi b = a *_{\mathbf{D}} b. \end{aligned}$$

Case 2: $a, b \in D_j \setminus \{\top\}$ and $j \in O(I^\sigma) \setminus I$. Then $a = b = 0_j$ and

$$\begin{aligned} 0_j *^\pi 0_j &= \bigvee\nolimits_D \{c * d : c, d \le 0_j \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c : c \le 0_j \text{ and } c \in B\} = 0_j = 0_j *_{\mathbf{D}} 0_j. \end{aligned}$$

Case 3: $a \in D_j \setminus \{\top\}$, $b \in D_k \setminus \{\top\}$ with $j < k$. Since $a, b \in O(B^\sigma)$, then $j, k \in O(I^\sigma)$. Thus, there exists $l \in I$ such that $j < l \le k$ and

$$\begin{aligned} a *^\pi b &= \bigvee\nolimits_D \{c * d : c \le a, d \le b \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c * d : c \le a, d \le b, \chi(d) \ge l \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c : c \le a, c \in B\} = a = a *_{\mathbf{D}} b. \end{aligned}$$

Case 4: $b = \top$. We easily obtain that $a *^\pi b = a = a *_{\mathbf{D}} b$.

Assume now $(a, b) \notin O(B^\sigma) \times O(B^\sigma)$; one of the following three cases occurs.

Case 1: $a, b \in B_j^\sigma$. Then

$$\begin{aligned} a *^\pi b &= \bigwedge\nolimits_D \{c *^\pi d : a \le c \text{ and } b \le d,\ c, d \in O(B^\sigma)\} \\ &= \bigwedge\nolimits_D \left\{c *^\pi d : a \le c \text{ and } b \le d,\ c, d \in O\left(B_j^\sigma\right)\right\} \\ &= \bigvee\nolimits_{B_j^\sigma} \left\{c *_{\mathbf{B}} d : a \le c \text{ and } b \le d,\ c, d \in O\left(B_j^\sigma\right)\right\} \\ &= \bigvee\nolimits_{B_j^\sigma} \left\{c *_j^\pi d : a \le c \text{ and } b \le d,\ c, d \in O\left(B_j^\sigma\right)\right\} \\ &= a *_j^\pi b = a *_{\mathbf{D}} b. \end{aligned}$$

Case 2: $a, b \in D_j \setminus \{\top\}$ for some $j \in I^\sigma \setminus O(I^\sigma)$. Since $j \notin I$, we have $a = b = 0_j$. Hence,

$$\begin{aligned} 0_j *^\pi 0_j &= \bigwedge\nolimits_D \{c *^\pi d : 0_j \le c, d \text{ and } c, d \in O(B^\sigma)\} \\ &= \bigwedge\nolimits_D \{c : 0_j \le c \text{ and } c \in O(B^\sigma)\} = 0_j = 0_j *_{\mathbf{D}} 0_j. \end{aligned}$$

Case 3: $a \in D_j \setminus \{\top\}, b \in D_k \setminus \{\top\}$, and $j < k$. Let $l \in O(I^\sigma)$ be such that $j \leq l < k$. Then

$$\begin{aligned} a *^\pi b &= \bigwedge\nolimits_D \{c *^\pi d : a \leq c \text{ and } b \leq d,\ c, d \in O(B^\sigma) \text{ and } \chi(c) \leq l\} \\ &= \bigwedge\nolimits_D \{c *_{\mathbf{D}} d : a \leq c < b \leq d,\ c, d \in O(B^\sigma) \text{ and } \chi(c) \leq l\} \\ &= \bigwedge\nolimits_D \{c : c \leq a,\ c \in O(B^\sigma)\} = a = a *_{\mathbf{D}} b. \end{aligned}$$

The implication of $\mathbf{B}$ is order reversing in the first coordinate and order preserving in the second coordinate. Therefore, by Lemma 2.1 and the remarks below it, to compute $\rightarrow^\pi$, we have to consider first $(a, b) \in O((B^d \times B)^\sigma)$, i.e., $a \in K(B^\sigma)$ and $b \in O(B^\sigma)$. Note that in this case, if $a, b \in D_j \setminus \{\top\}$ for some $j \in I^\sigma$, then $j \in K(I^\sigma) \cap O(I^\sigma) = I$. Therefore, we need only consider the following four cases.

Case 1: $a, b \in D_j \setminus \{\top\}$ for some $j \in I$, i.e., $a \in K(B_j^\sigma \setminus \{\top\})$ and $b \in O(B_j^\sigma \setminus \{\top\})$. Then

$$\begin{aligned} a \rightarrow^\pi b &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c, d \leq b \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c, d \leq b \text{ and } c, d \in B_j \setminus \{\top\}\} \\ &= a \rightarrow_j^\pi b = a \rightarrow_{\mathbf{D}} b. \end{aligned}$$

Case 2: $a \in D_j \setminus \{\top\}$, $b \in D_k \setminus \{\top\}$ with $j < k$. Since $a \in K(B^\sigma)$ and $b \in O(B^\sigma)$, we have $j \in K(I^\sigma)$ and $k \in O(I^\sigma)$. Thus, there exist $l, m \in I$ such that $j \leq l < m \leq k$ and

$$\begin{aligned} a \rightarrow^\pi b &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c, d \leq b \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c, d \leq b,\ \chi(c) \leq l,\ \chi(d) \geq m \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c < d \leq b \text{ and } c, d \in B\} = \top = a \rightarrow_{\mathbf{D}} b. \end{aligned}$$

Case 3: $a \in D_j \setminus \{\top\}$, $b \in D_k \setminus \{\top\}$ with $j > k$. Thus, $b \leq a$ and

$$\begin{aligned} a \rightarrow^\pi b &= \bigvee\nolimits_D \{c \rightarrow d : a \leq c, d \leq b \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{c \rightarrow d :, d \leq b \leq a \leq c \text{ and } c, d \in B\} \\ &= \bigvee\nolimits_D \{d : d \leq b, b \in B\} = b = a \rightarrow_{\mathbf{D}} b. \end{aligned}$$

Case 4: $a = \top$ or $b = \top$. It is easy to see that $a \rightarrow^\pi b = b = a \rightarrow_{\mathbf{D}} b$.

Assume now $(a, b) \notin K(B^\sigma) \times O(B^\sigma)$. Then one of the following four cases occurs:

Case 1: $a, b \in B_j^\sigma$. Then

$$\begin{aligned} a \rightarrow^\pi b &= \bigwedge\nolimits_D \{c \rightarrow^\pi d : c \leq a,\ b \leq d,\ c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \bigwedge\nolimits_D \left\{c \rightarrow^\pi d : c \leq a,\ b \leq d,\ c \in K\left(B_j^\sigma\right) \text{ and } d \in O\left(B_j^\sigma\right)\right\} \\ &= \bigwedge\nolimits_{B_j^\sigma} \left\{c \rightarrow_{\mathbf{D}} d : c \leq a,\ b \leq d,\ c \in K\left(B_j^\sigma\right) \text{ and } d \in O\left(B_j^\sigma\right)\right\} \\ &= \bigwedge\nolimits_{B_j^\sigma} \left\{c \rightarrow_j^\pi d : c \leq a,\ b \leq d,\ c \in K\left(B_j^\sigma\right) \text{ and } d \in O\left(B_j^\sigma\right)\right\} \\ &= a \rightarrow_j^\pi b = a \rightarrow_{\mathbf{D}} b. \end{aligned}$$

Case 2: $a, b \in D_j \setminus \{\top\}$ for some $j \in I^\sigma \setminus I$. Then $a = b = 0_j$ and

$$\begin{aligned} 0_j \to^\pi 0_j &= \textstyle\bigwedge_D \{c \to^\pi d : c \leq 0_j \leq d,\, c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \textstyle\bigwedge_D \{c \to_{\mathbf{D}} d : c \leq 0_j \leq d,\, c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \top = 0_j \to_{\mathbf{D}} 0_j. \end{aligned}$$

Case 3: $a \in D_j \setminus \{\top\}, b \in D_k \setminus \{\top\}$ and $j < k$. Since $a \leq b$,

$$\begin{aligned} a \to^\pi b &= \textstyle\bigwedge_D \{c \to^\pi d : c \leq a,\ b \leq d,\, c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \textstyle\bigwedge_D \{c \to_{\mathbf{D}} d : c \leq a \leq b \leq d,\, c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \top = a \to_{\mathbf{D}} b. \end{aligned}$$

Case 4: $a \in D_j \setminus \{\top\}, b \in D_k \setminus \{\top\}$ and $j > k$. By the density of I^σ, there exist $l \in O(I^\sigma)$ and $m \in K(I^\sigma)$ such that $k \leq l < m \leq j$. Then

$$\begin{aligned} a \to^\pi b &= \textstyle\bigwedge_D \{c \to^\pi d : c \leq a,\, b \leq d,\, c \in K(B^\sigma) \text{ and } d \in O(B^\sigma)\} \\ &= \textstyle\bigwedge_D \{c \to_{\mathbf{D}} d : b \leq d < c \leq a,\, c \in K(B^\sigma),\, d \in O(B^\sigma),\, \chi(d) \leq l\} \\ &= \textstyle\bigwedge_D \{d : b \leq d,\, d \in O(B^\sigma) \text{ and } \chi(d) \leq l\} = b = a \to_{\mathbf{D}} b. \end{aligned}$$

The proof of (2) is similar to the previous one. The only important detail that differs from (1) is the fact that the sum involves algebras of the form $\mathbf{M}$ when the index is in $I^\sigma \setminus I$, whereas in (1) it involves $\mathbf{L}_2$. Observe that the operations $*_{\mathbf{M}}$ and $*_{\mathbf{L}_2}$ coincide, while $\to_{\mathbf{M}}$ and $\to_{\mathbf{L}_2}$ are different only in the pair $(0, 0)$.

Therefore, to prove (2), we need only verify that $0_j \to^\sigma 0_j = 0_j \to_{\mathbf{C}} 0_j$ when $0_j \in C_j \setminus \{\top\}$ with $j \notin I$. Thus, $j \notin K(I^\sigma) \cap O(I^\sigma)$. Since σ- and π-canonical extensions coincide on closed and open elements, we obtain that

$$\begin{aligned} 0_j \to^\sigma 0_j &= \textstyle\bigvee_C \{b \to^\sigma c : c \leq 0_j \leq b,\, b \in O \text{ and } c \in K\} \\ &= \textstyle\bigvee_C \{b \to^\pi c : c \leq 0_j \leq b,\, b \in O,\ c \in K \text{ and } c < b\} \\ &= \textstyle\bigvee_C \{b \to_{\mathbf{D}} c : c \leq 0_j \leq b,\, b \in O,\ c \in K \text{ and } c < b\} \\ &= \textstyle\bigvee_C \{c : c \leq 0_j,\ c \in K\} = 0_j = 0_j \to_{\mathbf{C}} 0_j. \end{aligned}$$ □

Corollary 4.5. *Let $I = \{1, \ldots, n\}$ be a finite set with the usual order inherited from $\mathbb{N}$. For each $i \in I$, let $\mathbf{B}_i$ be a hoop, and assume that $B_i \setminus \{\top\}$ is updirected for each $1 \leq i < n$. Then*

$$\left(\textstyle\bigoplus_{i \in I} \mathbf{B}_i\right)^\pi \cong \textstyle\bigoplus_{i \in I} (\mathbf{B}_i^\pi) \quad \textit{and} \quad \left(\textstyle\bigoplus_{i \in I} \mathbf{B}_i\right)^\sigma \cong \textstyle\bigoplus_{i \in I} (\mathbf{B}_i^\sigma).$$

Proof. Since the underlying lattice of $\bigoplus_{i \in I} \mathbf{B}_i$ is $\left(\biguplus_{I \setminus \{n\}} (B_i \setminus \{\top\})\right) \biguplus B_n$, the hypothesis that $B_n \setminus \{\top\}$ is an updirected set can be omitted from the statement of Theorem 4.4, yielding the result. □

Corollary 4.6. *Let I be a totally ordered set. For each $i \in I$, let $\mathbf{B}_i$ be a totally ordered hoop.*

(1) *If B_i is finite for every $i \in I$, then $\left(\bigoplus_{i \in I} \mathbf{B}_i\right)^\pi$ is a totally ordered bounded hoop.*
(2) *If I is infinite, then $\left(\bigoplus_{i \in I} \mathbf{B}_i\right)^\sigma$ is not a hoop.*

5. Canonical extensions of subvarieties of BL-algebras

In [1], a deep study of subvarieties of BL-algebras is developed. That study is based on a decomposition of totally ordered BL-algebras (BL-chains) into ordinal sums of some algebraic structures called totally ordered Wajsberg hoops. In this section we use this decomposition and the fact that subvarieties of BL-algebras are generated by BL-chains to investigate canonical extensions of subvarieties of BL-algebras.

5.1. Background on BL-algebras.

Definition 5.1. A *BL-algebra* is an algebra $\mathbf{A} = \langle A, \to, *, \bot, \top\rangle$ such that

- $\langle A, \to, *, \top\rangle$ is a basic hoop;
- A is bounded with lower bound $\bot$, i.e., $\bot \leq a$ for each $a \in A$.

Therefore, BL-algebras are bounded distributive lattices with monotone operators in the sense of [13]. References about BL-algebras can be found in [1], [17] and [18]. BL-algebras form a variety $\mathcal{BL}$ (see [17]). From the definition of basic hoops, we conclude:

Theorem 5.2. *Every subvariety of $\mathcal{BL}$ is generated by BL-chains.*

Remark 5.3. Readers familiar with the theory of residuated lattices (see [12]) can think of BL-algebras as commutative bounded integral residuated lattices satisfying prelinearity and divisibility.

Theorem 5.4 (see [17])**.** *The following are proper subvarieties of $\mathcal{BL}$.*

(1) *The variety $\mathcal{MV}$ of MV-algebras, see* [8].
(2) *The variety $\mathcal{G}$ of Gödel algebras (linear Heyting algebras), see* [10, 19, 23].
(3) *The variety $\mathcal{PL}$ of product algebras, see* [9].
(4) *The variety $\mathcal{B}$ of Boolean algebras.*

An *implicative filter* of a BL-algebra $\mathbf{A}$ is a subset $F \subseteq A$ satisfying that $\top \in F$, and if $x \in F$ and $x \to y \in F$, then $y \in F$. Implicative filters are in one-to-one correspondence with congruences in BL-algebras (see [17, Lemma 2.3.14]). It is worth noticing that an implicative filter F of a BL-algebra $\mathbf{A}$ is closed under $*$. Therefore F is the universe of a subhoop of the hoop reduct of $\mathbf{A}$.

A **Wajsberg hoop** is a hoop satisfying the equation

$$(x \to y) \to y = (y \to x) \to x.$$

Wajsberg hoops form a variety of hoops that we denote by $\mathcal{WH}$. Readers interested in more information about Wajsberg hoops may see [2]. The next theorem shows the strong relation between BL-chains and totally ordered Wajsberg hoops.

Theorem 5.5 (see [1] and [5]). *Every nontrivial BL-chain can be uniquely (up to isomorphism) decomposed into the ordinal sum of a family of nontrivial totally ordered Wajsberg hoops whose bottom component is bounded.*

We also have that if I is a totally ordered set with a lower bound 0, and for each $i \in I$, $\mathbf{W}_i$ is a totally ordered Wajsberg hoop with $\mathbf{W}_0$ bounded, then the algebra $\bigoplus_{i\in I} \mathbf{W}_i$ is a BL-chain. An interesting consequence of Theorem 4.4 and the previous result is the next result.

Lemma 5.6. *Let I be a totally ordered set with a lower bound. For each $i \in I$, let $\mathbf{B}_i$ be a finite totally ordered Wajsberg hoop. Then $\left(\bigoplus_{i\in I} \mathbf{B}_i\right)^\pi$ is a BL-chain.*

To conclude this section, we present some important definitions that involve finite BL-chains and ordinal sums.

The algebra $\mathbf{L}_2$ defined in the previous section is the reduct of the BL-chain $\langle\{0,1\}, *_{\mathbf{L}_2}, \rightarrow_{\mathbf{L}_2}, 0, 1\rangle$ that we will also call $\mathbf{L}_2$. Generalizing this definition, let $\mathbf{L}_m$ denote the Łukasiewicz m-element chain, i.e., the unique (up to isomorphism) MV-chain whose universe is the set

$$\left\{\frac{0}{m-1}, \frac{1}{m-1}, \ldots, \frac{m-1}{m-1}\right\},$$

with the usual order and the operations given by

$$x * y = \max\{0, x + y - 1\} \text{ and } x \rightarrow y = \min\{1 - x + y, 1\}.$$

Since bounded Wajsberg hoops are reducts of MV-algebras, with an abuse of notation we shall use $\mathbf{L}_m$ to denote both structures: the MV-chain and the Wajsberg hoop. For example, $\mathbf{L}_n$ will denote the MV-chain $\langle L_n, *, \rightarrow, 0, 1\rangle$ as well as the Wajsberg hoop $\langle L_n, *, \rightarrow, 1\rangle$. Then we shall understand $\mathbf{L}_n \oplus \mathbf{L}_m$ as the BL-algebra obtained from the ordinal sum of the MV-chain $\mathbf{L}_n$ and the Wajsberg hoop $\mathbf{L}_m$.

5.2. σ-canonicity of subvarieties of $\mathcal{BL}$. We shall prove the following theorem that extends the results in [16] for MV-algebras and those in [7] for BL-algebras:

Theorem 5.7. *Given a subvariety $\mathcal{V}$ of BL-algebras, the following statements are equivalent:*

(1) *$\mathcal{V}$ is σ-canonical.*
(2) *$\mathcal{V}$ is finitely generated.*

The implication (2) $\rightarrow$ (1) is an immediate consequence of Corollary 2.3. We devote the rest of the subsection to proving the opposite implication. The proof will be based on Theorem 5.2 and on some results of [7] that we summarize in the next theorem.

Theorem 5.8. *Let $\mathcal{V}$ be a subvariety of BL-algebras.*

(1) *If $\mathcal{PL} \subseteq \mathcal{V}$, then $\mathcal{V}$ is not σ-canonical.*

(2) *If there is a subvariety $\mathcal{S} \subseteq \mathcal{MV}$ such that $\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{S}$ is not finitely generated, then $\mathcal{V}$ is not σ-canonical.*
(3) *If $\mathcal{G} \subseteq \mathcal{V}$, then $\mathcal{V}$ is not σ-canonical.*

Before starting with the proof of Theorem 5.7, we compile some necessary results about Wajsberg hoops and ordinal sums that can be found in [1].

Theorem 5.9. *Every totally ordered Wajsberg hoop $\mathbf{W}$ satisfies one and only one of the following conditions:*

(1) *$\mathbf{W}$ is cancellative, i.e., it satisfies that if $x * y = x * z$, then $y = z$;*
(2) *$\mathbf{W}$ is bounded. In this case $\mathbf{W}$ is the bottom-free reduct of a totally ordered MV-algebra (MV-chain).*

Obviously, if W is a finite set, then $\mathbf{W}$ is the bottom-free reduct of a finite MV-chain.

Lemma 5.10. *If $\mathbf{A}$ is a finite BL-chain, then $\mathbf{A} \cong \bigoplus_{i\in I} \mathbf{L}_{s_i}$ for some finite set $I \subseteq \mathbb{N}$ and $2 \leq s_i \in \mathbb{N}$.*

Lemma 5.11. *Let $\mathbf{B}$ be a BL-chain, and let $\mathbf{B} = \bigoplus_{i\in I} \mathbf{W}_i$ be its decomposition into totally ordered Wajsberg hoops given by Theorem 5.5. Let 0 be the bottom element of I. The subhoops of $\mathbf{B}$ are totally ordered hoops of the form $\mathbf{C} = \bigoplus_{i\in J} \mathbf{V}_i$ such that $J \subseteq I$ and for each $i \in J$, $\mathbf{V}_i$ is a subhoop of $\mathbf{W}_i$. The subalgebras of $\mathbf{B}$ are obtained similarly, but we must require that $0 \in J$ and that $\mathbf{V}_0$ is a subalgebra of $\mathbf{W}_0$.*

Lemma 5.12. *Every totally ordered Gödel algebra (Gödel chain) is of the form $\bigoplus_{i\in I} \mathbf{L}_2$ where I is a totally ordered set. The variety $\mathcal{G}$ of Gödel algebras is generated by an infinite family of non-isomorphic finite Gödel chains or by any infinite Gödel chain.*

Remark 5.13. The previous Lemma and Corollary 4.6 provide an alternative proof of the fact that $\mathcal{G}$, as a subvariety of BL-algebras, is not σ-canonical. Indeed, for any infinite bounded totally ordered set I, the algebra $\bigoplus_{i\in I} \mathbf{L}_2 \in \mathcal{G}$. Corollary 4.6 implies that $(\bigoplus_{i\in I} \mathbf{L}_2)^\sigma$ is not even a hoop. Therefore, any variety $\mathcal{V} \subseteq \mathcal{BL}$ that satisfies $\mathcal{G} \subseteq \mathcal{V}$ is not σ-canonical.

The following two theorems are crucial to prove Theorem 5.7. However, their proofs are long and may distract the reader's attention from the main point. Therefore, we have put them in an appendix at the end of the paper.

Theorem 5.14. *Let $\mathbf{W}$ be an infinite totally ordered Wajsberg hoop, and let $\mathbf{B} = \mathbf{L}_2 \oplus \mathbf{W}$ be a BL-chain. Then the variety of BL-algebras generated by $\mathbf{B}$ contains $\mathcal{PL}$.*

Theorem 5.15. *Let $\mathcal{V}$ be a variety of BL-algebras. If there exists an infinite set T of natural numbers such that $\{\mathbf{L}_2 \bigoplus \mathbf{L}_t : t \in T\} \subseteq \mathcal{V}$, then there exists an infinite totally ordered Wajsberg hoop $\mathbf{W}$ such that $\mathbf{L}_2 \oplus \mathbf{W} \in \mathcal{V}$.*

We fix some notation for the sequel. If $\mathbf{A}$ is a BL-algebra, we denote by $\langle \mathbf{A} \rangle_{BL}$ the subvariety of BL-algebras generated by $\mathbf{A}$. In case $\mathbf{A}$ has a reduct that is a Wajsberg hoop, $\langle \mathbf{A} \rangle_{WH}$ denotes the subvariety of Wajsberg hoops generated by the hoop reduct of $\mathbf{A}$.

We have settled all the necessary machinery to prove the implication (1) $\to$ (2) in Theorem 5.7. Let $\mathcal{V}$ be a variety of BL-algebras that is not finitely generated. By Theorem 5.2, there exists a set of BL-chains S such that $\mathcal{V}$ is generated by S. There are two possible cases for S: it contains an infinite BL-chain or it contains an infinite number of non-isomorphic finite BL-chains.

S **contains an infinite BL-chain.** Let $\mathbf{B} \in S$ be an infinite BL-chain. From Theorem 5.5, we know that $\mathbf{B}$ admits a unique decomposition $\mathbf{B} = \bigoplus_{i \in I} \mathbf{W}_i$. For each $i \in I$, $\mathbf{W}_i$ is a totally ordered Wajsberg hoop and $\mathbf{W}_0$ is a bounded Wajsberg hoop, where 0 is the bottom element of the totally ordered set I. Since $\mathbf{B}$ is infinite, at least one of the following statements is satisfied:

(1) $\mathbf{W}_0$ is an infinite MV-chain,
(2) there exists $i \in I$ with $0 < i$ such that $\mathbf{W}_i$ is an infinite totally ordered Wajsberg hoop.
(3) I is infinite and for every $i \in I$, $\mathbf{W}_i$ is a finite totally ordered Wajsberg hoop.

If case (1) holds, then by Lemma 5.11, $\mathbf{W}_0$ is a subalgebra of $\mathbf{B}$. Therefore, $\langle \mathbf{W}_0 \rangle_{BL} \subseteq \mathcal{V}$ and clearly $\langle \mathbf{W}_0 \rangle_{BL} \subseteq \mathcal{MV}$ is not finitely generated (see [8, Chapter 8]). Hence, by (ii) in Theorem 5.8, we conclude that $\mathcal{V}$ is not σ-canonical.

In the second case, from Lemma 5.11, the algebra $\mathbf{L}_2 \oplus \mathbf{W}_i$ is a subalgebra of $\mathbf{B}$. From Theorem 5.14, we conclude that $\mathcal{PL} \subseteq \langle \mathbf{L}_2 \oplus \mathbf{W}_i \rangle_{BL} \subseteq \mathcal{V}$. Therefore, $\mathcal{V}$ is not σ-canonical because of (i) in Theorem 5.8.

In case (3), observe that each $\mathbf{W}_i$ is the reduct of a finite MV-chain, $\mathbf{L}_{s_i}$. Since $\mathbf{L}_2$ is a subalgebra of $\mathbf{L}_{s_i}$ for each $i \in I$, by Lemma 5.11, $\bigoplus_{i \in I} \mathbf{L}_2$ is a subalgebra of $\mathbf{B}$. Thus, by Lemma 5.12, $\bigoplus_{i \in I} \mathbf{L}_2$ is an infinite Gödel algebra that generates $\mathcal{G}$. We conclude $\mathcal{G} \subseteq \mathcal{V}$ and from item (iii) in Theorem 5.8 (see also Remark 5.13), $\mathcal{V}$ is not σ-canonical.

S **contains an infinite number of non-isomorphic finite BL-chains.** From Lemma 5.10, let the non-isomorphic finite algebras in S be given by

$$\mathbf{B}_j \cong \bigoplus_{i \in I_j} \mathbf{L}_{n(j,i)} \qquad (j \in J)$$

where J is an infinite set of indices, and for each $j \in J$, I_j is a finite totally ordered set. Let $|I_j|$ be the cardinal of the set I_j and 0_j its bottom element. We split the proof into two different cases:

(1) There exists $k \in \mathbb{N}$ such that $|I_j| \leq k$ for all $j \in J$.
(2) $\{|I_j| : j \in J\}$ is unbounded.

If (1) happens, since $\{\mathbf{B}_j : j \in J\}$ is an infinite set of non-isomorphic BL-chains, then $T = \{n(j,i) : j \in J \text{ and } i \in I_j\}$ is infinite.

Suppose first that $\{n(j, 0_j) : j \in J\} \subseteq T$ is infinite. By Lemma 5.11,

$$\{\mathbf{L}_{n(j,0_j)} : j \in J\}$$

is an infinite set of non-isomorphic MV-chains contained in $\mathcal{V}$. In [8, Proposition 8.1.2], it is proved that $\{\mathbf{L}_{n(j,0_j)} : j \in J\}$ generates the variety $\mathcal{MV}$. Therefore $\mathcal{MV} \subseteq \mathcal{V}$ and by item (ii) in Theorem 5.8, $\mathcal{V}$ is not σ-canonical.

If $\{n(j, 0_j) : j \in J\}$ is finite, then the set $\{n(j, i) : j \in J, 0_j < i \in I_j\} \subseteq T$ is infinite. This being the case, from Lemma 5.11,

$$\{\mathbf{L}_2 \oplus \mathbf{L}_{n(j,i)} : j \in J, 0_j < i \in I_j\}$$

is an infinite set of non-isomorphic BL-chains contained in $\mathcal{V}$. Because of Theorem 5.15 and Theorem 5.14, there is an infinite totally ordered Wajsberg hoop $\mathbf{W}$ such that $\mathbf{L}_2 \oplus \mathbf{W} \in \mathcal{V}$ and $\mathcal{PL} \subseteq \mathcal{V}$. Then the hypothesis of item (i) in Theorem 5.8 is satisfied and we have the desired result.

To deal with (2), for each $j \in J$, let $\mathbf{A}_j = \bigoplus_{I_j} \mathbf{L}_2$. Lemma 5.11 implies that for each $j \in J$, the algebra $\mathbf{A}_j$ is isomorphic to a subalgebra of $\mathbf{B}_j$. Using Lemma 5.12, we conclude that $\{\mathbf{A}_j : j \in J\} \subseteq \mathcal{V}$ is an infinite set of non-isomorphic Gödel chains and $\mathcal{G} \subseteq \mathcal{V}$. Then the result follows from (iii) in Theorem 5.8.

5.3. About π-canonicity. We present some positive and some negative results about π-canonicity.

Theorem 5.16 (see [7] and [22]). *Let $\mathcal{V}$ be a subvariety of BL-algebras.*

(1) *If $\mathcal{PL} \subseteq \mathcal{V}$, then $\mathcal{V}$ is not π-canonical.*
(2) *If there is a subvariety $\mathcal{S} \subseteq \mathcal{MV}$ such that $\mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{S}$ is not finitely generated, then $\mathcal{V}$ is not π-canonical.*
(3) *If $\mathcal{V} \subseteq \mathcal{G}$ or $\mathcal{V}$ is finitely generated, then $\mathcal{V}$ is π-canonical.*

Theorem 5.17. *Let $\mathcal{V}$ be a subvariety of $\mathcal{BL}$, and let $\mathbf{A} \in \mathcal{V}$ be a BL-chain. Assume that the decomposition of $\mathbf{A}$, according to Theorem 5.5, is given by $\mathbf{A} = \bigoplus_{i \in I} \mathbf{W}_i$. If there is an $i \in I$ such that $\mathbf{W}_i$ is infinite, then $\mathcal{V}$ is not π-canonical.*

Proof. Let 0 be the least element of I. If $\mathbf{W}_0$ is infinite, then $\langle \mathbf{W}_0 \rangle_{BL}$ is a subvariety of $\mathcal{MV}$ not finitely generated. Since $\langle \mathbf{W}_0 \rangle_{BL} \subseteq \mathcal{V}$, by (2) of Theorem 5.16 we conclude that $\mathcal{V}$ is not π-canonical. Otherwise, there exists $i \in I$ with $0 < i$ such that $\mathbf{W}_i$ is infinite; by Lemma 5.11, $\mathbf{L}_2 \oplus \mathbf{W}_i$ is a subalgebra of $\mathbf{A}$. Now the result follows from Theorem 5.14 and item (1) of Theorem 5.16. $\square$

To obtain new positive results about π-canonicity, we need to consider the variety $\mathcal{GG}$ of generalized Gödel algebras. Generalized Gödel algebras can be though of as the bottom-free reducts of Gödel algebras (linear Heyting algebras). They form a variety $\mathcal{GG}$ that is the subvariety of basic hoops characterized by the equation $x * x = x$.

As quoted in Theorem 5.16, item (3), the variety $\mathcal{G}$ of Gödel algebras is π-canonical and it is not σ-canonical. A proof of this fact can be deduced from the results in [15] (see [7]). A slight modification of that argument justifies the following result.

Lemma 5.18. *The variety $\mathcal{GG}$ is π-canonical and not σ-canonical.*

Let $n \in \mathbb{N}$. Consider the class $\mathfrak{K}_n = \{\mathbf{L}_n \oplus \mathbf{G} : \mathbf{G} \in \mathcal{GG}\}$. The next result implies that $\mathfrak{K}_n$ is a class of BL-algebras.

Theorem 5.19. *Let $\mathbf{A}$ be a basic hoop. For each $n \in \mathbb{N}$, $\mathbf{L}_n \oplus \mathbf{A}$ is a BL-algebra.*

Proof. Since $\mathbf{L}_n$ is bounded, we know that $\mathbf{L}_n \oplus \mathbf{A}$ is a bounded hoop. According to the definition of BL-algebra, it only remains to check that $\mathbf{L}_n \oplus \mathbf{A}$ satisfies equation (4.4) from Section 3. By the residuation law and the definition of the order in any hoop, one can see that equation (4.4) is equivalent to the inequality

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \tag{5.1}$$

Once more from the residuation law, we know that for any $x, y, z \in L_n \oplus A$, the inequality

$$z \leq ((y \to x) \to z) \to z \tag{5.2}$$

holds in $\mathbf{L}_n \oplus \mathbf{A}$. We divide the proof into three cases:

Case 1: $y \to x \in L_n \setminus \{\top\}$. Notice that $y \to x \in L_n \setminus \{\top\}$ if and only if $x < y$. Hence, $x \to y = \top$ and $(x \to y) \to z = \top \to z = z$. Inequality (5.1) follows from inequality (5.2)

Case 2: $y \to x \in A$ and $z \in L_n \setminus \{\top\}$. From the definition of ordinal sum the right hand side of inequality (5.1) is $((y \to x) \to z) \to z = z \to z = \top$, and the inequality holds.

Case 3: $y \to x \in A$ and $z \in A$. If $y \to x = \top$, then the right hand side of (5.1) is $\top$ and the inequality holds. Otherwise, $y \to x \in A$ implies $x, y \in A$. Then we are in the case $x, y, z \in A$, and inequality (5.1) holds because $\mathbf{A}$ is a basic hoop. □

Theorem 5.20. *Let $n \in \mathbb{N}$, and let $\mathcal{V}_n$ be the variety of BL-algebras generated by $\mathfrak{K}_n$. Then $\mathcal{V}_n$ is π-canonical.*

Proof. According to Theorem 2.2, we only need to see that for each $n \in \mathbb{N}$, the class $\mathfrak{K}_n$ is closed under ultraproducts and under π-canonical extensions.

Closure under ultraproducts. For an arbitrary class of algebras $\mathfrak{K}$, let $\mathbf{I}(\mathfrak{K})$ and $\mathbf{P}_u(\mathfrak{K})$ denote the classes of isomorphic images and ultraproducts of algebras from $\mathfrak{K}$, respectively. In [1, Proposition 3.3] it is proved that for a set J, if $\mathbf{A}_0^j$ and $\mathbf{A}_1^j$ for $j \in J$ are basic hoops, then $\mathbf{P}_u(\{\mathbf{A}_0^j \oplus \mathbf{A}_1^j : j \in J\}) =$

$$\{\mathbf{B}_0 \oplus \mathbf{B}_1 : \mathbf{B}_i \in \mathbf{IP}_u(\{\mathbf{A}_i^j : j \in J\}) \text{ for } i \in \{0,1\}\}.$$

Every ultrapower of a single finite algebra $\mathbf{D}$ is isomorphic to $\mathbf{D}$ (see [3, Chapter IV, Lemma 6.5]). Hence $\mathbf{P}_u(\mathbf{L}_n) \subseteq \mathbf{I}(\mathbf{L_n})$. Since $\mathcal{GG}$ is a variety, $\mathbf{P}_u(\mathcal{GG}\}) \subseteq \mathcal{GG}$. These assertions together yield $\mathbf{P}_u(\mathfrak{K}_n) \subseteq \mathfrak{K}_n$.

Closure under π-canonical extensions. Let $\mathbf{L}_n \oplus \mathbf{G} \in \mathfrak{K}_n$. From Corollary 4.5, $(\mathbf{L}_n \oplus \mathbf{G})^\pi = (\mathbf{L}_n)^\pi \oplus (\mathbf{G})^\pi$.

Since L_n is finite, $(\mathbf{L}_n)^\pi = \mathbf{L}_n$. By Lemma 5.18, we conclude that $(\mathbf{G})^\pi \in \mathcal{GG}$; thus, $(\mathbf{L}_n \oplus \mathbf{G})^\pi \in \mathfrak{K}_n$ as desired. □

6. Appendix

In this appendix, we give the proofs of Theorem 5.14 and Theorem 5.15. Both these proofs will use the fact that the totally ordered Wajsberg hoop, $[0,1]_{WH} = ([0,1], *, \to, 1)$, where $x * y = \max(0, x+y-1)$ and $x \to y = \min(1, 1-x+y)$, generates the variety of basic Wajsberg hoops, denoted $\mathcal{BWH}$. An analogous proof to that of [8, Prop. 3.5.3 and Prop. 8.11] shows that if $\mathbf{W}$ is the hoop reduct of a simple infinite MV-chain, then $\mathbf{W}$ is isomorphic to a subalgebra of $[0,1]_{WH}$ and the variety generated by $\mathbf{W}$ is $\mathcal{BWH}$ (this is because every infinite subhoop of $[0,1]_{WH}$ is dense in $[0,1]_{WH}$ and the operations $*, \to$ are continuous in $[0,1]_{WH}$).

The proof of Theorem 5.14 relies on the following two important facts:

F1 (see [1]): The variety $\mathcal{PL}$ of Product algebras is generated by any BL-chain of the form $\mathbf{L}_2 \oplus \mathbf{W}$, with $\mathbf{W}$ a cancellative totally ordered Wajsberg hoop.

F2 (see [6, Corollary 3.5]): If $\mathbf{W}$ is a basic hoop, then the variety of basic hoops generated by $\mathbf{W}$ coincides with the variety of basic hoops $\mathcal{W}$ given by $\mathcal{W} = \{\mathbf{C} : \mathbf{L}_2 \oplus \mathbf{C} \in \langle \mathbf{L}_2 \oplus \mathbf{W} \rangle_{BL}\}$.

Now we are ready to prove Theorem 5.14. To facilitate readability, we recall the statement.

Theorem 5.14. *Let $\mathbf{W}$ be an infinite totally ordered Wajsberg hoop and let $\mathbf{B} = \mathbf{L}_2 \oplus \mathbf{W}$ be a BL-chain. Then the variety of BL-algebras generated by $\mathbf{B}$ contains $\mathcal{PL}$.*

Proof. Let $\mathcal{V}$ be the variety of BL-algebras generated by $\mathbf{B}$. Following the result of Theorem 5.9, we can assert that one and only one of the following cases happens:

(1) $\mathbf{W}$ is a cancellative totally ordered Wajsberg hoop;
(2) $\mathbf{W}$ is the reduct of a simple infinite MV-chain;
(3) $\mathbf{W}$ is the reduct of a nonsimple infinite MV-chain.

In the case of item 1, we know from F1 that $\mathbf{B}$ is a product algebra that generates $\mathcal{PL}$. Thus, $\mathcal{PL} = \mathcal{V}$.

If $\mathbf{W}$ is the reduct of a simple infinite MV-chain, the variety generated by $\mathbf{W}$ is the variety $\mathcal{BWH}$. Let $\mathbf{C} \in \mathcal{BWH}$ be a cancellative totally ordered Wajsberg hoop. By F2, $\mathbf{L}_2 \oplus \mathbf{C}$ is in $\mathcal{V}$. Since $\mathbf{L}_2 \oplus \mathbf{C}$ generates the variety $\mathcal{PL}$, we conclude $\mathcal{PL} \subseteq \mathcal{V}$.

If **W** is the reduct of a nonsimple MV-chain, let **A** be the maximal proper implicative filter of **W**. The observations following the definition of implicative filters yields that **A** is a non-trivial totally ordered Wajsberg hoop. We prove that **A** is cancellative. If this were not the case (absurdum hypothesis), then from Theorem 5.9, **A** is bounded. Let z be the lower bound of A. Since $\perp \notin A$ and **A** is non-trivial, we get $z \neq \perp$ and $z \neq \top$. Note that $z * z = z^2 \in A$, and since it is always the case that $z^2 \leq z$, the fact that z is the lower bound of A yields $z^2 = z$. This means that z is a complemented element in the MV-chain reduct **W**, which is different from $\perp$ and $\top$ (see [8, Theorem 1.5.3]). But the only complemented elements in any MV-chain are $\perp$ and $\top$. This contradiction follows from the assumption that **A** has a lower bound. Therefore, we conclude that **A** is cancellative.

By Lemma 5.11, $\mathbf{L}_2 \oplus \mathbf{A}$ is a subalgebra of **B** and by F1, $\mathcal{PL} \subseteq \mathcal{V}$. □

The rest of the appendix is devoted to proving Theorem 5.15. To achieve this aim, we investigate the behavior of equations in the BL-algebra $\mathbf{L}_2 \oplus [0,1]_{WH}$, showing that if an equation ε fails in $\mathbf{L}_2 \oplus [0,1]_{WH}$, then there exists $m \in \mathbb{N}$ such that ε fails in every BL-algebra $\mathbf{L}_2 \oplus \mathbf{L}_t$ with $t \geq m$. Notice that the universe of $\mathbf{L}_2 \oplus [0,1]_{WH}$ is the set $\{\perp\} \cup [0,1]$ and $\top = 1$.

Lemma 6.1 (see [4], Lemma 3.6)**.** *Assume that $\tau(x_1, \ldots, x_n)$ is a term function in the language of BL-algebras. Let $(a_1, \ldots, a_n) \in (\mathbf{L}_2 \oplus [0,1]_{WH})^n$ be such that $a_j \in [0,1]_{WH}$ for some $j \in \{1, 2, \ldots, n\}$. Then*

$$\perp = \tau(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n)$$

if and only if for any $y \in [0,1]_{WH}$,

$$\perp = \tau(a_1, \ldots, a_{j-1}, y, a_{j+1}, \ldots, a_n).$$

Lemma 6.2. *Let $\tau(x_1, \ldots, x_n)$ be a term function in the language of BL-algebras, and let $(a_1, \ldots, a_n) \in (\mathbf{L}_2 \oplus [0,1]_{WH})^n$ be such that $\perp < \tau(a_1, \ldots, a_n)$. Assume that $a_j \neq \perp$ for some $1 \leq j \leq n$. Consider*

$$\tau'(y) = \tau(a_1, \ldots, a_{j-1}, y, a_{j+1}, \ldots a_n)$$

as a function defined in the real interval $[0,1]$. Then the image of τ' is included in $[0,1]$ and τ' is continuous.

Proof. We present a proof by induction on the complexity of τ.

If τ is a term function of complexity 0, then $\tau(x_1, \ldots, x_n) = x_i$ for some $1 \leq i \leq n$, or $\tau(x_1, \ldots, x_n) \in \{\perp, \top\}$ for every $(x_1, \ldots, x_n) \in (\mathbf{L}_2 \oplus [0,1]_{WH})^n$. If $\tau(x_1, \ldots, x_n) = x_j$, then $\tau'(y) = y$ is clearly continuous. If $\tau(x_1, \ldots, x_n) = x_i$ for some $i \neq j$, then $\tau'(y) = a_i \in [0,1]$ is constant and the result holds. If $\tau(x_1, \ldots, x_n) = \top$ for every $(x_1, \ldots, x_n) \in (\mathbf{L}_2 \oplus [0,1]_{WH})^n$, the result also holds. Lastly, the hypothesis $\perp < \tau(a_1, \ldots, a_n)$ implies that it can not be the case that $\tau(x_1, \ldots, x_n) = \perp$ for every $(x_1, \ldots, x_n) \in (\mathbf{L}_2 \oplus [0,1]_{WH})^n$.

Assume that we have proved the statement for all term functions τ satisfying the hypothesis whose complexity is less than $k > 0$. Let τ be a term function

of complexity k. Since $\tau(a_1,\ldots,a_n)\neq\bot$, Lemma 6.1 implies that the image of $\tau'(y)$ is included in $[0,1]$. To check continuity, we consider the following two cases.
Case 1: $\tau=\phi*\varphi$ for some term functions ϕ,φ of complexity less than k. Recall that $\bot$ is an absorbing element for the operation $*$, i.e, $x*y=\bot$ if and only if $x=\bot$ or $y=\bot$. Hence, from the hypothesis $\bot<\tau(a_1,\ldots,a_n)$, we conclude $\bot<\phi(a_1,\ldots,a_n)$ and $\bot<\varphi(a_1,\ldots,a_n)$. The induction hypothesis yields that $\phi'(y)$ and $\varphi'(y)$ are continuous functions that arise by fixing $x_i=a_i$ for all the variables except x_j. Therefore, $\tau'(y)=\max(0,\phi'(y)+\varphi'(y)-1)$ is a continuous function from $[0,1]$ into $[0,1]$.
Case 2: $\tau=\phi\to\varphi$ for some term functions ϕ,φ of complexity less than k. From the hypothesis $\bot<\tau(a_1,\ldots,a_n)$, one possible case is that $\bot=\phi(a_1,\ldots,a_n)$. If this happens, then the result of Lemma 6.1 yields that $\phi(a_1,\ldots,a_{j-1},y,a_{j+1},\ldots,a_n)=\bot$ for any $y\in[0,1]$. Therefore,

$$\phi(a_1,\ldots,a_{j-1},y,a_{j+1},\ldots,a_n)\leq\psi(a_1,\ldots,a_{j-1},y,a_{j+1},\ldots,a_n)$$

for every $y\in[0,1]$, and the definition of the order yields $\top=1=\tau'(y)$ for all $y\in[0,1]$. If it were the case that $\bot=\varphi(a_1,\ldots,a_n)$ and $\bot<\phi(a_1,\ldots,a_n)$, then the definition of ordinal sum would yield $\bot=\tau(a_1,\ldots,a_n)$, contradicting our hypothesis. Therefore, the only remaining possibility is that both $\varphi(a_1,\ldots,a_n)$ and $\phi(a_1,\ldots,a_n)$ are greater than $\bot$. By the induction hypothesis, $\phi'(y)$ and $\varphi'(y)$ are continuous functions from $[0,1]$ into $[0,1]$ and $\tau'(y)=\min(1,1-\phi'(y)+\varphi'(y))$. We conclude that τ' is a continuous function from $[0,1]$ into $[0,1]$. □

As an easy generalization of the previous result we obtain

Lemma 6.3. *Let $\tau(x_1,\ldots,x_n)$ be a term function in the language of BL-algebras, and let $(a_1,\ldots,a_n)\in(\mathbf{L}_2\oplus[0,1]_{WH})^n$ be such that $\bot<\tau(a_1,\ldots,a_n)$. Assume that there are $j_1,j_2,\ldots,j_k\in\{1,\ldots n\}$ with $0<k\leq n$ such that $a_{j_i}\neq\bot$. Then the function $\tau'(x_{j_1},\ldots,x_{j_k})$ that arises by fixing $x_i=a_i$ in τ, for all variables x_i such that $i\notin\{j_1,\ldots j_k\}$, is a continuous function from $[0,1]^k$ into $[0,1]$.*

Lemma 6.4. *If an equation ε in the language of BL-algebras does not hold in $\mathbf{L}_2\oplus[0,1]_{WH}$, then there is an $m\in\mathbb{N}$ such that ε fails in $\mathbf{L}_2\oplus\mathbf{L}_t$ for every $t\geq m$.*

Proof. Every equation $\alpha=\beta$ in the language of BL-algebras can be written equivalently as $(\alpha\to\beta)*(\beta\to\alpha)=\top$. Let ε be an equation in the language of BL-algebras given by

$$\tau(x_1,\ldots,x_n)=\top. \tag{6.1}$$

Without danger of confusion, let $\tau(x_1,\ldots,x_n)$ denote the term function associated with ε. Assume that ε fails in $\mathbf{L}_2\oplus[0,1]_{WH}$. Then there is an n-tuple $(a_1,\ldots,a_n)\in(\mathbf{L}_2\oplus[0,1]_{WH})^n$ such that $\top>\tau(a_1,\ldots,a_n)$.

If $\tau(a_1, \dots, a_n) = \perp$, we can apply Lemma 6.1 and conclude that for each $s > 2$ and for each n-tuple $(b_1, \dots, b_n) \in (\mathbf{L}_2 \oplus \mathbf{L}_s)^n$ that satisfies $b_i = a_i$, if $a_i = \perp$, then $\tau(b_1, \dots, b_n) = \perp$. Therefore, the equation (6.1) fails in any algebra of the form $\mathbf{L}_2 \oplus \mathbf{L}_s$ with $s \geq 2$.

Now assume that $\top > \tau(a_1, \dots, a_n) > \perp$. Since $\mathbf{L}_2$ is a subalgebra of $\mathbf{L}_2 \oplus [0,1]_{WH}$, it cannot be the case that $a_i = \perp$ for each $i = 1, \dots, n$. Let $\tau'(x_{j_1}, \dots x_{j_k})$ be the function obtained from $\tau(x_1, \dots, x_n)$ by fixing $x_i = a_i$ when $a_i = \perp$. By Lemma 6.3, τ' is a continuous function from $[0,1]^k$ into $[0,1]$. Since $\tau'(a_{j_1}, \dots, a_{j_k}) < \top$, the continuity of τ' implies that there is an $m \in \mathbb{N}$ such that for all $t \geq m$ there is a k-tuple $(c_{j_1}, \dots, c_{j_k}) \in (\mathbf{L}_t)^k$ that satisfies $\tau'(c_{j_1}, \dots, c_{j_k}) < \top$. Therefore, the n-tuple $(d_1, \dots, d_n) \in (\mathbf{L}_2 \oplus \mathbf{L}_t)^n$ given by

$$d_i = \begin{cases} c_i & \text{if } a_i \neq \perp, \\ \perp & \text{otherwise,} \end{cases}$$

satisfies $\tau(d_1, \dots, d_n) = \tau'(c_{j_1}, \dots, c_{j_k}) < \top$. This last assertion implies that equation (6.1) fails in $\mathbf{L}_2 \oplus \mathbf{L}_t$ for all $t \geq m$. □

Now we are ready to prove the promised theorem:

Theorem 5.15. *Let $\mathcal{V}$ be a variety of BL-algebras. If there exists an infinite set T of natural numbers such that $\{\mathbf{L}_2 \bigoplus \mathbf{L}_t : t \in T\} \subseteq \mathcal{V}$, then there exists an infinite totally ordered Wajsberg hoop $\mathbf{W}$, such that $\mathbf{L}_2 \oplus \mathbf{W} \in \mathcal{V}$.*

Proof. We shall check that $\mathbf{L}_2 \oplus [0,1]_{WH} \in \mathcal{V}$. Since $\mathcal{V}$ is a variety, it is enough to see that for every equation ε, ε holds in $\mathbf{L}_2 \oplus [0,1]_{WH}$ if and only if it holds in $\mathbf{L}_2 \oplus \mathbf{L}_t$ for each $t \in T$. One implication is a consequence of the fact that for each $t \in T$, $\mathbf{L}_2 \oplus \mathbf{L}_t$ is a subalgebra of $\mathbf{L}_2 \oplus [0,1]_{WH}$. The opposite one follows from Lemma 6.4. □

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