J. Math. Anal. Appl. 363 (2010) 310–326

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Maximal operators, Riesz transforms and Littlewood–Paley functions associated with Bessel operators on *BMO* ☆

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ARTICLE INFO

Article history:
Received 22 January 2009
Available online 13 August 2009
Submitted by L. Grafakos

Dedicated to our friend and teacher José Manuel Méndez Pérez to celebrate his 60th birthday

Keywords:
BMO
Heat and Poisson semigroups
Bessel operator
Riesz transform
g-function

ABSTRACT

In this paper we study boundedness properties of certain harmonic analysis operators (maximal operators for heat and Poisson semigroups, Riesz transforms and Littlewood–Paley g-functions) associated with Bessel operators, on the space $BMO_o(\mathbb{R})$ that consists of the odd functions with bounded mean oscillation on $\mathbb{R}$.

1. Introduction

By $BMO_o(\mathbb{R})$ we denote the space constituted by all those odd functions with bounded mean oscillation on $\mathbb{R}$. This space can be characterized as follows. An odd function $f \in L^1_{loc}(\mathbb{R})$ is in $BMO(\mathbb{R})$, that is, f has bounded mean oscillation on $\mathbb{R}$, if and only if, for all $1 \leqslant p < \infty$ (equivalently, for some $1 \leqslant p < \infty$) there exists $C_p > 0$ such that, for every interval $I = (a, b)$

$$\left\{\frac{1}{|I|}\int_I \left|f(x) - f_I\right|^p dx\right\}^{1/p} \leqslant C_p, \quad 0 < a < b < \infty, \tag{1}$$

and also

$$\left\{\frac{1}{|I|}\int_I \left|f(x)\right|^p dx\right\}^{1/p} \leqslant C_p, \quad 0 = a < b < \infty. \tag{2}$$

Here, as usual, $|I|$ denotes the length of I and $f_I = \frac{1}{|I|}\int_I f(x)\,dx$. Moreover, for every $1 \leqslant p < \infty$, $\inf\{C_p > 0\colon (1) \text{ and } (2) \text{ hold}\}$ is equivalent to the usual $\|f\|_{BMO(\mathbb{R})}$ (see, for instance, [14, Chapter 1] definitions and properties concerning $BMO(\mathbb{R})$).

☆ This paper is partially supported by MTM2007/65609.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.08.006

$BMO_o(\mathbb{R})$ coincides with the dual $H^1_o(\mathbb{R})'$ of the subspace $H^1_o(\mathbb{R})$ of $H^1(\mathbb{R})$ that consists of all the odd functions in the Hardy space $H^1(\mathbb{R})$. The space $H^1_o(\mathbb{R})$ was studied in [4] and [10], where several characterizations of $H^1_o(\mathbb{R})$ are obtained. In the sequel we denote by BMO_+ the space that consists of all those $f \in L^1_{loc}([0,\infty))$ such that the odd extension f_o of f to $\mathbb{R}$ is in $BMO(\mathbb{R})$. On BMO_+ we consider the natural norm. Our objective in this paper is to study the behavior on BMO_+ of maximal operator, Riesz transform and Littlewood–Paley g-functions associated with Bessel operators.

Muckenhoupt and Stein [12] began the development of harmonic analysis related to Bessel operators. They considered the Bessel operator B_λ, $\lambda > 0$, defined by $B_\lambda = -x^{-2\lambda} D x^{2\lambda} D$, with $D = \frac{d}{dx}$. In [12] Poisson integrals and conjugate of Poisson integrals associated with B_λ were introduced. Recently, L^p-boundedness properties for the higher order Riesz transform [5] and for the Littlewood–Paley g-functions [6] in the B_λ context have been established.

Here we consider the Bessel operator $\Delta_\lambda = -x^{-\lambda} D x^{2\lambda} D x^{-\lambda}$, with $\lambda > 0$. If J_ν denotes the Bessel function of the first kind and order ν, for every $y > 0$, the function $\varphi_y(x) = \sqrt{xy}\, J_{\lambda-\frac{1}{2}}(xy)$, $x \in (0,\infty)$, is an eigenfunction of Δ_λ and

$$\Delta_\lambda\big(\sqrt{xy}\, J_{\lambda-\frac{1}{2}}(xy)\big) = y^2 \sqrt{xy}\, J_{\lambda-\frac{1}{2}}(xy), \quad x, y \in (0,\infty).$$

The Poisson kernel associated with the operator Δ_λ is given by

$$P^\lambda(t,x,y) = \int_0^\infty e^{-tz} \varphi_x(z) \varphi_y(z)\, dz, \quad t, x, y \in (0,\infty).$$

According to [12, (16.4)] (see also [19]) we have that

$$P^\lambda(t,x,y) = \frac{2\lambda t (xy)^\lambda}{\pi} \int_0^\pi \frac{(\sin\theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1-\cos\theta))^{\lambda+1}}\, d\theta, \quad t, x, y \in (0,\infty).$$

The Poisson integral $P^\lambda_t(f)$ is defined by

$$P^\lambda_t(f)(x) = \int_0^\infty P^\lambda(t,x,y) f(y)\, dy, \quad t, x > 0.$$

The family $\{P^\lambda_t\}_{t>0}$ constitutes a semigroup of linear and bounded operators in $L^p(0,\infty)$, $1 \leqslant p \leqslant \infty$. L^p-boundedness properties of the maximal operator

$$P^\lambda_*(f) = \sup_{t>0} \big|P^\lambda_t(f)\big|$$

were established in [7] and [8].

The heat kernel associated with the operator Δ_λ is

$$W^\lambda(t,x,y) = \int_0^\infty e^{-tz^2} \varphi_x(z) \varphi_y(z)\, dz, \quad t, x, y \in (0,\infty).$$

According to [18, 13.31(1)], we can write

$$W^\lambda(t,x,y) = \frac{1}{\sqrt{2t}} \left(\frac{xy}{2t}\right)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}\left(\frac{xy}{2t}\right) e^{-\frac{x^2+y^2}{4t}}, \quad t, x, y \in (0,\infty),$$

where I_ν denotes the modified Bessel function of the first kind and order ν. The heat integral $W^\lambda_t(f)$ of f is defined by

$$W^\lambda_t(f)(x) = \int_0^\infty W^\lambda(t,x,y) f(y)\, dy, \quad t, x > 0.$$

Then, $\{W^\lambda_t\}_{t>0}$ is a semigroup of bounded and linear operators in $L^p(0,\infty)$, $1 \leqslant p \leqslant \infty$. The maximal operator associated with $\{W^\lambda_t\}_{t>0}$ is given by

$$W^\lambda_*(f) = \sup_{t>0} \big|W^\lambda_t(f)\big|$$

and it was investigated on L^p-spaces in [7].

Bennett, DeVore and Sharpley [2, Th. 4.2(b)] proved that if $\mathcal{M}$ denotes the (uncentered) Hardy–Littlewood maximal operator on $\mathbb{R}^n$, then, for every $f \in BMO(\mathbb{R}^n)$, either $\mathcal{M}f \in BMO(\mathbb{R}^n)$ or $\mathcal{M}f \equiv \infty$. The function $f(x) = \log_+ |x|$, $x \in \mathbb{R}^n$, is an example of the second situation. In [9] it was introduced a BMO type space on $\mathbb{R}^n$ associated with Schrödinger operators where the maximal operator $\mathcal{M}$ is bounded. This is the case for the maximal operators, W^λ_*, P^λ_* and the Hardy–Littlewood maximal operator $\mathcal{M}_0$ on $(0,\infty)$, on BMO_+ as we state in the following proposition.

Proposition 1. *Let $\lambda > 0$. We denote by $\mathcal{N}$ the operators $\mathcal{M}_0$, W_*^λ or P_*^λ. There exists $C > 0$ such that*

$$\|\mathcal{N} f\|_{BMO_+} \leqslant C\|f\|_{BMO_+}, \quad f \in BMO_+.$$

Riesz transforms in the Δ_λ-setting were studied in [3]. The operator Δ_λ admits the factorization $\Delta_\lambda = D_\lambda^* D_\lambda$, where $D_\lambda = x^\lambda D x^{-\lambda}$ and D_λ^* represents the (formal) adjoint of D_λ in $L^2(0,\infty)$. Following the ideas developed by Stein in [13], the Riesz transform R_λ is defined by

$$R_\lambda f = D_\lambda \Delta_\lambda^{-\frac{1}{2}} f, \quad f \in C_c^\infty(0,\infty).$$

Here $C_c^\infty(0,\infty)$ denotes the space of smooth functions with compact support in $(0,\infty)$. The operator R_λ can be extended to $L^p(0,\infty)$ as a bounded operator on $L^p(0,\infty)$, for every $1 < p < \infty$, and to $L^1(0,\infty)$ as a bounded operator from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$. Moreover, for each $f \in L^p(0,\infty)$, $1 < p < \infty$,

$$R_\lambda f(x) = \lim_{\varepsilon \to 0} \int_{0, |x-y| > \varepsilon}^{\infty} R_\lambda(x,y) f(y)\, dy, \quad \text{a.e. } x \in (0,\infty), \tag{3}$$

being

$$R_\lambda(x,y) = \int_0^\infty D_{\lambda,x} P^\lambda(t,x,y)\, dt, \quad x,y \in (0,\infty),\ x \neq y.$$

According to [1, (1.6)] (also see [7]) we get

$$\left|R_\lambda(x,y)\right| \leqslant C(xy)^\lambda \begin{cases} \frac{x}{y^{2\lambda+2}}, & 2x \leqslant y, \\ \frac{1}{x^{2\lambda+1}}, & 0 < y < \frac{x}{2}, \end{cases} \tag{4}$$

and

$$\left|R_\lambda(x,y) - \frac{1}{\pi}\frac{1}{x-y}\right| \leqslant C\frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right), \quad 0 < \frac{x}{2} < y < 2x. \tag{5}$$

Then, we can prove that the Riesz transform R_λ is well defined on $L^\infty(0,\infty)$. This fact establishes a difference between the behavior of R_λ and the Hilbert transform on bounded functions [16, p. 294].

The vertical Littlewood–Paley g-function associated with the heat semigroup $\{W_t^\lambda\}_{t>0}$ for the Bessel operator Δ_λ is defined by

$$g_{h,\lambda}(f)(x) = \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} W_t^\lambda(f)(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}},$$

and the corresponding one for the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ is given by

$$g_{P,\lambda}(f)(x) = \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} P_t^\lambda(f)(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}.$$

The behavior of the Riesz transforms and g-functions on BMO_+ is established in the next proposition.

Proposition 2. *Let $\lambda > 0$. We denote by $\mathcal{N}$ the operators R_λ, $g_{h,\lambda}$ and $g_{P,\lambda}$. There exists $C > 0$ such that*

$$\|\mathcal{N} f\|_{BMO_+} \leqslant C\|f\|_{BMO_+}, \quad f \in BMO_+.$$

As for maximal operators the property stated in Proposition 2 for $g_{h,\lambda}$ and $g_{P,\lambda}$ contrasts with the corresponding one for vertical classical Littlewood–Paley g-functions (see [17]).

This paper is organized as follows. In Section 2 we prove Proposition 1 and the proof of Proposition 2 is shown in Sections 3 and 4 where we establish the estimates for the Riesz transform and for the g-functions, respectively.

Throughout this paper we always denote by C a suitable positive constant that can change from a line to the other one.

2. Maximal operators in BMO_+

In this section we present a proof of Proposition 1. We divide the proof in three parts. Each part is concerned with one of the maximal operators under considerations.

(i) By $\mathcal{M}_0$ we denote the Hardy–Littlewood maximal operator on $(0,\infty)$, that is, if $f \in L^1_{loc}([0,\infty))$,

$$\mathcal{M}_0(f)(x) = \sup_{x\in I} \frac{1}{|I|}\int_I \left|f(y)\right| dy, \quad x \in (0,\infty),$$

where the supremum is taken over all the bounded intervals I on $(0,\infty)$ such that $x \in I$.

Assume that $f \in BMO_+$, then $f_o \in BMO_o(\mathbb{R})$. Let $a > 0$, we write $f_o = f_1 + f_2$ where $f_1 = f_o\chi_{(-2a,2a)}$. Since $f_o \in L^1_{loc}(\mathbb{R})$, $\mathcal{M}f_1(x) < \infty$, a.e. $x \in \mathbb{R}$, where $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator on $\mathbb{R}$. Moreover, if $x \in (-a,a)$ and I is a bounded interval such that $x \in I$ and $I \cap (-2a,2a)^c \neq \emptyset$, by denoting $J = (-b,b)$, where $b = \max\{|y|,\ y \in I\}$, we have

$$\frac{1}{|I|}\int_I \left|f_2(y)\right| dy = \frac{1}{|I|}\int_{I\cap(-2a,2a)^c} \left|f_o(y)\right| dy \leqslant C\frac{1}{|J|}\int_J \left|f_o(y)\right| dy \leqslant C\|f\|_{BMO_+}. \tag{6}$$

Note that $|I| \leqslant |J| \leqslant 2(|I|+a) \leqslant 4|I|$. Hence $\mathcal{M}(f_2)(x) < \infty$, a.e. x, $|x| \leqslant a$. Then, we obtain that $\mathcal{M}f_o(x) < \infty$, a.e. x, $|x| \leqslant a$. Hence, we conclude that $\mathcal{M}f_o(x) < \infty$, a.e. $x \in \mathbb{R}$.

Since $f \in BMO(0,\infty)$, by proceeding as in [2, Theorem 4.2] we obtain that $\mathcal{M}_0 f \in BMO(0,\infty)$ and $\|\mathcal{M}_0 f\|_{BMO(0,\infty)} \leqslant C\|f\|_{BMO_+}$. Moreover, for every $a > 0$,

$$\frac{1}{a}\int_0^a \mathcal{M}_0(f)(x)\, dx \leqslant C\|f\|_{BMO_+}. \tag{7}$$

Indeed, let $a > 0$. As above we write $f = f_1 + f_2$, where $f_1 = f\chi_{(0,2a)}$. Then, by proceeding as in (6) we get

$$\mathcal{M}_0(f_2)(x) \leqslant 2\|f\|_{BMO_+}, \quad x \in (0,a). \tag{8}$$

Also, since $\mathcal{M}_0$ is bounded on $L^2(0,\infty)$, one has

$$\frac{1}{a}\int_0^a \left|M_0 f_1(x)\right| dx \leqslant \left(\frac{1}{a}\int_0^a \left|M_0 f_1(x)\right|^2 dx\right)^{\frac{1}{2}} \leqslant C\left(\frac{1}{a}\int_0^{2a} \left|f(x)\right|^2 dx\right)^{\frac{1}{2}} \leqslant C\|f\|_{BMO_+}. \tag{9}$$

From (8) and (9) we deduce that (7) holds.

By combining the above arguments we conclude that $(\mathcal{M}_0 f)_o \in BMO_o(\mathbb{R})$ and $\|\mathcal{M}_0 f\|_{BMO_+} \leqslant C\|f\|_{BMO_+}$.

(ii) We now analyze the maximal operator W^λ_* associated with the heat semigroup $\{W^\lambda_t\}_{t>0}$. Assume that $f \in BMO_+$.

According to [11, (5.16.5)] we have that

$$0 \leqslant W^\lambda(t,x,y) \leqslant C\frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}, \quad t,x,y \in (0,\infty), \tag{10}$$

and also,

$$W^\lambda(t,x,y) \leqslant C\begin{cases} \frac{1}{\sqrt{t}}(\frac{xy}{t})^\lambda e^{-\frac{y^2+x^2}{4t}}, & \frac{xy}{2t} \leqslant 1, \\ \frac{1}{\sqrt{t}}(\frac{xy}{t})^\lambda e^{-\frac{(x-y)^2}{4t}}, & \frac{xy}{2t} \geqslant 1. \end{cases} \tag{11}$$

It is well known that

$$\sup_{t>0}\int_0^\infty \frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}\left|f(y)\right| dy \leqslant C\mathcal{M}_0(f)(x), \quad x \in (0,\infty). \tag{12}$$

Then

$$W^\lambda_*(f)(x) \leqslant C\mathcal{M}_0(f)(x), \quad x \in (0,\infty).$$

Hence, by (7), for every $a > 0$, we have

$$\frac{1}{a}\int_0^a W^\lambda_*(f)(x)\, dx \leqslant C\|f\|_{BMO_+}. \tag{13}$$

On the other hand, we can write

$$\left|\sup_{t>0}\left|W_t^\lambda(f)(x)\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right|\right|$$
$$\leqslant \sup_{t>0}\int_0^{\frac{x}{2}} W^\lambda(t,x,y)\left|f(y)\right|dy + \sup_{t>0}\int_{2x}^{\infty} W^\lambda(t,x,y)\left|f(y)\right|dy, \quad x\in(0,\infty).$$

From (10) it follows that

$$\int_0^{\frac{x}{2}} W^\lambda(t,x,y)\left|f(y)\right|dy \leqslant C\int_0^{\frac{x}{2}} \frac{1}{\sqrt{t}}e^{-\frac{(x-y)^2}{4t}}\left|f(y)\right|dy \leqslant C\int_0^{\frac{x}{2}} \frac{1}{\sqrt{t}}e^{-\frac{x^2}{16t}}\left|f(y)\right|dy$$
$$\leqslant \frac{C}{x}\int_0^{\frac{x}{2}}\left|f(y)\right|dy \leqslant C\|f\|_{BMO_+}, \quad t,x\in(0,\infty).$$

By using (11) we get

$$\int_{2x}^{\infty} W^\lambda(t,x,y)\left|f(y)\right|dy \leqslant C\left(\int_{2x,\ \frac{xy}{2t}\leqslant 1}^{\infty}\left(\frac{xy}{t}\right)^\lambda \frac{e^{-\frac{y^2}{4t}}}{\sqrt{t}}\left|f(y)\right|dy + \int_{2x,\ \frac{xy}{2t}>1}^{\infty}\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{t}}\left|f(y)\right|dy\right)$$
$$\leqslant C\int_{2x}^{\infty}\left(\frac{xy}{t}\right)^\lambda \frac{e^{-\frac{y^2}{16t}}}{\sqrt{t}}\left|f(y)\right|dy \leqslant Cx^\lambda\int_{2x}^{\infty}\frac{|f(y)|}{y^{\lambda+1}}\,dy = Cx^\lambda\sum_{k=1}^{\infty}\int_{2k^{2/\lambda}x}^{2(k+1)^{2/\lambda}x}\frac{|f(y)|}{y^{\lambda+1}}\,dy$$
$$\leqslant Cx^\lambda\sum_{k=1}^{\infty}\frac{1}{(2k^{2/\lambda}x)^{\lambda+1}}\int_0^{2(k+1)^{2/\lambda}x}\left|f(y)\right|dy \leqslant C\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{2(k+1)^{2/\lambda}x}\int_0^{2(k+1)^{2/\lambda}x}\left|f(y)\right|dy$$
$$\leqslant C\|f\|_{BMO_+}, \quad t,x\in(0,\infty).$$

Hence, we have proved that

$$\sup_{t>0}\left|W_t^\lambda(f)(x)\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right| \in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t>0}\left|W_t^\lambda(f)(x)\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right|\right\|_\infty \leqslant C\|f\|_{BMO_+}. \tag{14}$$

Moreover,

$$\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right| \in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right\|_\infty \leqslant C\|f\|_{BMO_+}. \tag{15}$$

Indeed, we can write

$$\left|\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x} W^{\lambda}(t,x,y)f(y)\,dy\right| - \sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right|$$
$$\leqslant \sup_{t>0}\int_{\frac{x}{2}}^{2x}\left|W^{\lambda}(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right|\left|f(y)\right|dy, \quad x\in(0,\infty).$$

According to (10), it follows that

$$\int_{\frac{x}{2},\,\frac{xy}{2t}\leqslant 1}^{2x}\left|W^{\lambda}(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right|\left|f(y)\right|dy$$
$$\leqslant C\int_{\frac{x}{2},\,\frac{xy}{2t}\leqslant 1}^{2x}\frac{1}{\sqrt{t}}\left(\left(\frac{xy}{t}\right)^{\lambda}+1\right)e^{-\frac{x^2+y^2}{4t}}\left|f(y)\right|dy$$
$$\leqslant C\int_{\frac{x}{2}}^{2x}\frac{|f(y)|}{\sqrt{x^2+y^2}}\,dy\leqslant \frac{C}{x}\int_0^{2x}\left|f(y)\right|dy\leqslant C\|f\|_{BMO_+}, \quad t,x\in(0,\infty).$$

Also, by using [11, 5.16.5], we get

$$\int_{\frac{x}{2},\,\frac{xy}{2t}\geqslant 1}^{2x}\left|W^{\lambda}(t,x,y)-\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right|\left|f(y)\right|dy\leqslant C\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{t}}\left(\frac{t}{xy}\right)^{\frac{1}{2}}e^{-\frac{(x-y)^2}{4t}}\left|f(y)\right|dy$$
$$\leqslant \frac{C}{x}\int_0^{2x}\left|f(y)\right|dy\leqslant C\|f\|_{BMO_+}, \quad t,x\in(0,\infty).$$

Hence (15) is established.

Now we denote by $\{W_t\}_{t>0}$ the classical heat semigroup, that is, we write

$$W_t(f_o)(x)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{(x-y)^2}{4t}}f_o(y)\,dy, \quad t\in(0,\infty)\text{ and }x\in\mathbb{R}.$$

Since f_o is an odd function we can write

$$W_t(f_o)(x)=\frac{1}{\sqrt{4\pi t}}\int_0^{\infty}\left(e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right)f(y)\,dy, \quad t\in(0,\infty)\text{ and }x\in\mathbb{R}. \tag{16}$$

Moreover $W_t(f_o)$ is odd, for every $t>0$. By splitting the integral one gets

$$\left|W_t(f_o)(x)-\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|$$
$$\leqslant\frac{1}{\sqrt{4\pi t}}\int_0^{\frac{x}{2}}\left|e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right|\left|f(y)\right|dy$$
$$+\frac{1}{\sqrt{4\pi t}}\int_{2x}^{\infty}\left|e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right|\left|f(y)\right|dy+\frac{1}{\sqrt{4\pi t}}\int_{\frac{x}{2}}^{2x}e^{-\frac{(x+y)^2}{4t}}\left|f(y)\right|dy$$
$$\leqslant\frac{C}{\sqrt{t}}\left(\int_0^{\frac{x}{2}}\left|\frac{(x-y)^2-(x+y)^2}{4t}\right|e^{-\frac{(x-y)^2}{4t}}\left|f(y)\right|dy\right.$$

$$+\int_{2x}^{\infty}\left|\frac{(x-y)^2-(x+y)^2}{4t}\right|e^{-\frac{(x-y)^2}{4t}}\left|f(y)\right|dy+\int_{\frac{x}{2}}^{2x}e^{-\frac{(x+y)^2}{4t}}\left|f(y)\right|dy\Bigg)$$

$$\leqslant C\left(\int_0^{\frac{x}{2}}\frac{xy}{t^{\frac{3}{2}}}e^{-\frac{x^2}{16t}}\left|f(y)\right|dy+\int_{2x}^{\infty}\frac{xy}{t^{\frac{3}{2}}}e^{-\frac{y^2}{16t}}\left|f(y)\right|dy+\int_{\frac{x}{2}}^{2x}\frac{|f(y)|}{x+y}\,dy\right)$$

$$\leqslant C\left(\frac{1}{x}\int_0^{\frac{x}{2}}\left|f(y)\right|dy+x\int_{2x}^{\infty}\frac{1}{y^2}\left|f(y)\right|dy+\frac{1}{x}\int_{\frac{x}{2}}^{2x}\left|f(y)\right|dy\right)$$

$$\leqslant C\left(\frac{1}{x}\int_0^{2x}\left|f(y)\right|dy+x\sum_{k=1}^{\infty}\int_{2xk^2}^{2x(k+1)^2}\frac{1}{y^2}\left|f(y)\right|dy\right)$$

$$\leqslant C\left(\frac{1}{x}\int_0^{2x}\left|f(y)\right|dy+\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{1}{2x(k+1)^2}\int_0^{2x(k+1)^2}\left|f(y)\right|dy\right)\leqslant C\|f\|_{BMO_+},$$

for every $t, x \in (0,\infty)$. Hence,

$$\sup_{t\in(0,\infty)}\left|W_t(f_o)(x)\right|-\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t\in(0,\infty)}\left|W_t(f_o)(x)\right|-\sup_{t>0}\left|\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|\right\|_\infty\leqslant C\|f\|_{BMO_+}. \tag{17}$$

We deduce from (14), (15) and (17) that

$$\sup_{t>0}\left|W_t^\lambda(f)(x)\right|-\sup_{t>0}\left|W_t(f_o)(x)\right|\in L^\infty(0,\infty)$$

and

$$\left\|\sup_{t>0}\left|W_t^\lambda(f)\right|-\sup_{t>0}\left|W_t(f_o)\right|\right\|_\infty\leqslant C\|f\|_{BMO_+}. \tag{18}$$

According to (13) and (18), in order to see that

$$\sup_{t>0}\left|W_t^\lambda(f)\right|\in BMO_+\quad\text{and}\quad\left\|\sup_{t>0}\left|W_t^\lambda(f)\right|\right\|_{BMO_+}\leqslant C\|f\|_{BMO_+},$$

it is sufficient to establish that $\sup_{t>0}|W_t(f_o)|\in BMO(\mathbb{R})$ and that

$$\left\|\sup_{t>0}\left|W_t(f_o)\right|\right\|_{BMO(\mathbb{R})}\leqslant C\|f_o\|_{BMO(\mathbb{R})}.$$

We have to show that $\sup_{t>0}|W_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$ (see [15]). From (7) and (12) we get

$$\frac{1}{a}\int_0^a\sup_{t>0}\left|W_t(f_o)(x)\right|dx\leqslant C\|f\|_{BMO_+},\quad a>0.$$

Then, since $\sup_{t>0}|W_t(f_o)|$ is even, $\sup_{t>0}|W_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$. Thus we prove that $W_*^\lambda(f)\in BMO_+$ and $\|W_*^\lambda(f)\|_{BMO_+}\leqslant C\|f\|_{BMO_+}$.

(iii) Let $f\in BMO_+$. By using subordination formula we can write

$$P^\lambda(t,x,y)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}W^\lambda\left(\frac{t^2}{4u},x,y\right)du,\quad t,x,y\in(0,\infty). \tag{19}$$

Then,

$$\sup_{t>0}\left|P_t^\lambda(f)\right| \leqslant \sup_{t>0}\left|W_t^\lambda(f)\right|.$$

Hence, from (13) we deduce that

$$\frac{1}{a}\int_0^a P_*^\lambda(f)(x)\,dx \leqslant C\|f\|_{BMO_+}, \quad a>0. \tag{20}$$

Moreover, by taking into account the proof of (18), it follows

$$\sup_{t>0}\left|P_t^\lambda(f)(x)\right| - \sup_{t>0}\left|P_t(f_o)(x)\right| \in L^\infty(0,\infty),$$

and

$$\left\|\sup_{t>0}\left|P_t^\lambda(f)(x)\right| - \sup_{t>0}\left|P_t(f_o)(x)\right|\right\|_\infty \leqslant C\|f\|_{BMO_+}, \tag{21}$$

where

$$P_t(f_o)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{t}{(x-y)^2+t^2} f_o(y)\,dy, \quad t>0,\ x\in\mathbb{R}. \tag{22}$$

From (12) one infers that $\sup_{t>0}|P_t(f_o)(x)| \leqslant C\mathcal{M}_0(f)(x)$, $x\in(0,\infty)$. Then, by (7),

$$\frac{1}{a}\int_0^a \sup_{t>0}\left|P_t(f_o)(x)\right| dx \leqslant C\|f\|_{BMO_+}, \quad a>0.$$

Hence, since $\sup_{t>0}|P_t(f_o)|$ is even, $\sup_{t>0}|P_t(f_o)(x)|<\infty$, a.e. $x\in\mathbb{R}$. It deduces that $\sup_{t>0}|P_t(f_o)|\in BMO(\mathbb{R})$ (see [15]). (20) and (21) allow us to conclude that $P_*^\lambda f\in BMO_+$, and $\|P_*^\lambda f\|_{BMO_+} \leqslant C\|f\|_{BMO_+}$.

3. Riesz transform in BMO_+

Our objective is to prove Proposition 2 for the Riesz transforms R_λ. Firstly, note that, by (4) and (5), the Riesz transform R_λ is defined on $L^\infty(0,\infty)$. Indeed, let $f\in L^\infty(0,\infty)$. It is known (see, for instance, [16, p. 294]) that the limit

$$\lim_{\varepsilon\to 0}\int_{0,|x-y|>\varepsilon}^{\infty} f(y)\left(\frac{1}{x-y}+\chi_{(1,\infty)}(y)\frac{1}{y}\right)dy \tag{23}$$

exists for almost every $x\in(0,\infty)$. We now prove that

$$R_\lambda(f)(x) = \int_0^{\frac{x}{2}} R_\lambda(x,y)f(y)\,dy + \lim_{\varepsilon\to 0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} R_\lambda(x,y)f(y)\,dy + \int_{2x}^\infty R_\lambda(x,y)f(y)\,dy,$$

for almost all $x\in(0,\infty)$. According to (4), we get

$$\int_0^{\frac{x}{2}} \left|R_\lambda(x,y)\right|\left|f(y)\right| dy \leqslant C\frac{1}{x^{\lambda+1}}\int_0^{\frac{x}{2}} y^\lambda\left|f(y)\right| dy \leqslant C\|f\|_\infty, \quad x\in(0,\infty),$$

and

$$\int_{2x}^\infty \left|R_\lambda(x,y)\right|\left|f(y)\right| dy \leqslant Cx^{\lambda+1}\int_{2x}^\infty \frac{|f(y)|}{y^{\lambda+2}}\,dy \leqslant C\|f\|_\infty, \quad x\in(0,\infty).$$

On the other hand, one has

$$\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} R_\lambda(x,y) f(y)\,dy = \int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(R_\lambda(x,y) - \frac{1}{\pi}\frac{1}{x-y}\right) f(y)\,dy + \frac{1}{\pi}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \frac{1}{x-y} f(y)\,dy, \quad \varepsilon, x \in (0,\infty). \tag{24}$$

From (5) one deduces that, for every $x \in (0,\infty)$,

$$\int_{\frac{x}{2}}^{2x} \left|R_\lambda(x,y) - \frac{1}{\pi}\frac{1}{x-y}\right| \left|f(y)\right| dy \leqslant C\|f\|_\infty \int_{\frac{x}{2}}^{2x} \frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right) dy \leqslant C\|f\|_\infty,$$

because $\int_{\frac{x}{2}}^{2x} \frac{1}{y}(1 + \log_+ \frac{\sqrt{xy}}{|x-y|})\,dy = \int_{\frac{1}{2}}^{2} \frac{1}{u}(1 + \log_+ \frac{\sqrt{u}}{|1-u|})\,du$. Moreover, we write

$$\begin{aligned}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \frac{1}{x-y} f(y)\,dy = {} & \int_{0,|x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \\ & - \int_{\frac{x}{2},|x-y|>\varepsilon}^{2x} f(y)\frac{\chi_{(1,\infty)}(y)}{y}\,dy - \int_{2x,|x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \\ & - \int_{0,|x-y|>\varepsilon}^{\frac{x}{2}} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy, \quad \varepsilon, x \in (0,\infty).\end{aligned}$$

Note that, for each $x \in (0,\infty)$,

$$\int_{\frac{x}{2}}^{2x} \left|f(y)\chi_{(1,\infty)}(y)\right| \frac{dy}{y} \leqslant C\|f\|_\infty,$$

that

$$\begin{aligned}&\int_{2x}^{\infty} \left|\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right| \left|f(y)\right| dy \\ &\quad \leqslant C\|f\|_\infty \left(\int_{2x}^{2x+1} \left(\frac{1}{|x-y|} + \frac{1}{y}\right) dy + \int_{2x+1}^{\infty} \frac{x}{|x-y|y}\,dy\right) \leqslant C\left(\frac{1}{x} + 1\right)\|f\|_\infty,\end{aligned}$$

and

$$\int_0^{\frac{x}{2}} \left|\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right| \left|f(y)\right| dy \leqslant C(1+x)\|f\|_\infty.$$

Then, by (23) and (24) we conclude that the limit

$$R_\lambda(f)(x) = \lim_{\varepsilon\to 0} \int_{0,|x-y|>\varepsilon}^{\infty} R_\lambda(x,y) f(y)\,dy \tag{25}$$

exists for almost every $x \in (0,\infty)$. This property shows different behavior of Hilbert transform (see (23)) and R_λ-transform on $L^\infty(0,\infty)$.

We now prove Proposition 2 for Riesz transform R_λ.

Assume that $f \in BMO_+$. If we consider f_e the even extension of f to $\mathbb{R}$, according to [16, p. 294],

$$H(f_e)(x) = \lim_{\varepsilon\to 0} \int\limits_{-\infty,|x-y|>\varepsilon}^{+\infty} \left(\frac{1}{x-y} + \frac{\chi_{(-1,1)^c}(y)}{y}\right) f_e(y)\,dy \in BMO(\mathbb{R}),$$

because $f_e \in BMO(\mathbb{R})$. Since f_e is even we can write

$$\begin{aligned} H(f_e)(x) &= \lim_{\varepsilon\to 0} \int\limits_{0,|x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy + \int\limits_{-\infty,|x-y|>\varepsilon}^{0} \left(\frac{1}{x-y} + \frac{\chi_{(-\infty,-1)}(y)}{y}\right) f(-y)\,dy \\ &= \lim_{\varepsilon\to 0} \int\limits_{0,|x-y|>\varepsilon}^{\infty} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy + \int\limits_{0,|x+y|>\varepsilon}^{\infty} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \\ &= \lim_{\varepsilon\to 0} \int\limits_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy + \int_0^{\frac{x}{2}} \frac{2x}{x^2-y^2} f(y)\,dy + \int_{2x}^{\infty} \frac{2x}{x^2-y^2} f(y)\,dy \\ &\quad + \int_{\frac{x}{2}}^{2x} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy, \quad x \in (0,\infty). \end{aligned}$$

Note that $H(f_e)$ is odd. We are going to see that

$$H(f_e)(x) - \lim_{\varepsilon\to 0} \int\limits_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \in L^\infty(0,\infty) \tag{26}$$

and

$$\left\| H(f_e)(x) - \lim_{\varepsilon\to 0} \int\limits_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \right\|_\infty \leqslant C\|f\|_{BMO_+}. \tag{27}$$

We have to analyze three terms. One gets, as in the proof of (17) in the previous section,

$$\left| \int_0^{\frac{x}{2}} \frac{2x}{x^2-y^2} f(y)\,dy \right| \leqslant C\frac{1}{x} \int_0^x \left|f(y)\right| dy \leqslant C\|f\|_{BMO_+}, \quad x \in (0,\infty),$$

and

$$\left| \int_{2x}^{\infty} \frac{2x}{x^2-y^2} f(y)\,dy \right| \leqslant Cx \int_{2x}^{\infty} \left|f(y)\right| \frac{dy}{y^2} \leqslant C\|f\|_{BMO_+}, \quad x \in (0,\infty).$$

Also, we obtain

$$\left| \int_{\frac{x}{2}}^{2x} \left(\frac{1}{x+y} - \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \right| \leqslant \frac{C}{x} \int_0^{2x} \left|f(y)\right| dy \leqslant C\|f\|_{BMO_+}, \quad x \in (0,\infty).$$

Thus (26) and (27) are established. By using (25), (4) and (5) we have that

$$\begin{aligned} &\left| R_\lambda(f)(x) - \frac{1}{\pi} \lim_{\varepsilon\to 0} \int\limits_{\frac{x}{2},|x-y|>\varepsilon}^{2x} \left(\frac{1}{x-y} + \frac{\chi_{(1,\infty)}(y)}{y}\right) f(y)\,dy \right| \\ &\quad \leqslant C\left(\int_{\frac{x}{2}}^{2x} \frac{1}{y}\left(1 + \log_+ \frac{\sqrt{xy}}{|x-y|}\right) \left|f(y)\right| dy + \int_{\frac{x}{2}}^{2x} \frac{\chi_{(1,\infty)}(y)}{y} \left|f(y)\right| dy \right. \end{aligned}$$

$$+\frac{1}{x^{\lambda+1}}\int_0^{\frac{x}{2}} y^\lambda\big|f(y)\big|\,dy + x^{\lambda+1}\int_{2x}^{\infty}\frac{|f(y)|}{y^{\lambda+2}}\,dy\Bigg)$$

$$\leqslant C\left(\left(\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)^2 dy\right)^{\frac{1}{2}}\left(\frac{1}{x}\int_{\frac{x}{2}}^{2x}\big|f(y)\big|^2\,dy\right)^{\frac{1}{2}}\right.$$

$$\left.+\frac{1}{x}\int_0^x\big|f(y)\big|\,dy + x\int_{2x}^{\infty}\frac{|f(y)|}{y^2}\,dy\right)\leqslant C\|f\|_{BMO_+},\quad x\in(0,\infty),$$

because

$$\int_{\frac{x}{2}}^{2x}\frac{1}{y}\left(1+\log_+\frac{\sqrt{xy}}{|x-y|}\right)^2 dy=\int_{\frac{1}{2}}^{2}\frac{1}{u}\left(1+\log_+\frac{\sqrt{u}}{|1-u|}\right)^2 du<\infty,\quad x\in(0,\infty).$$

Hence,

$$R_\lambda(f)(x)-\frac{1}{\pi}\lim_{\varepsilon\to0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)\,dy\in L^\infty(0,\infty),\tag{28}$$

and

$$\left\|R_\lambda(f)(x)-\frac{1}{\pi}\lim_{\varepsilon\to0}\int_{\frac{x}{2},|x-y|>\varepsilon}^{2x}\left(\frac{1}{x-y}+\frac{\chi_{(1,\infty)}(y)}{y}\right)f(y)\,dy\right\|_{L^\infty(0,\infty)}\leqslant C\|f\|_{BMO_+}.\tag{29}$$

By combining (26), (27), (28) and (29) we conclude that

$$R_\lambda(f)-H(f_e)\in L^\infty(0,\infty)\quad\text{and}\quad\big\|R_\lambda(f)-H(f_e)\big\|_\infty\leqslant C\|f\|_{BMO_+}.\tag{30}$$

Moreover, since $H(f_e)\in BMO(\mathbb{R})$ and $H(f_e)$ is odd, for every $a\in(0,\infty)$,

$$\frac{1}{a}\int_0^a\big|H(f_e)(x)\big|\,dx=\frac{1}{2a}\int_{-a}^a\big|H(f_e)(x)\big|\,dx=\frac{1}{2a}\int_{-a}^a\left|H(f_e)(x)-\frac{1}{2a}\int_{-a}^a H(f_e)(u)\,du\right|dx$$

$$\leqslant C\big\|H(f_e)\big\|_{BMO(\mathbb{R})}\leqslant C\|f_e\|_{BMO(\mathbb{R})}\leqslant C\|f\|_{BMO_+}.$$

Then, from (30), for every $a\in(0,\infty)$,

$$\frac{1}{a}\int_0^a\big|R_\lambda(f)(x)\big|\,dx\leqslant\frac{1}{a}\int_0^a\big|R_\lambda(f)(x)-H(f_e)(x)\big|\,dx+\frac{1}{a}\int_0^a\big|H(f_e)(x)\big|\,dx\leqslant C\|f\|_{BMO_+}.$$

Hence $R_\lambda(f)\in BMO_+$ and $\|R_\lambda f\|_{BMO_+}\leqslant C\|f\|_{BMO_+}$. Thus the proof of Proposition 2 for R_λ is finished.

4. Littlewood–Paley g-functions in BMO_+

In this section we prove Proposition 2 for the Littlewood–Paley g-functions $g_{h,\lambda}$ and $g_{P,\lambda}$ associated with the heat and the Poisson semigroups for Δ_λ, respectively.

Firstly we study $g_{h,\lambda}$. Let $f\in BMO_+$. Triangle inequality for $L^2((0,\infty),\frac{dt}{t})$-norm implies that

$$\left|g_{h,\lambda}(f)(x)-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}W^\lambda(t,x,y)f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right|$$

$$=\left|\left\|t\frac{\partial}{\partial t}\int_0^\infty W^\lambda(t,x,y)f(y)\,dy\right\|_{L^2((0,\infty),\frac{dt}{t})}-\left\|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}W^\lambda(t,x,y)f(y)\,dy\right\|_{L^2((0,\infty),\frac{dt}{t})}\right|$$

$$\leqslant \left\| t\frac{\partial}{\partial t}\int_0^\infty W^\lambda(t,x,y)f(y)\,dy - t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy \right\|_{L^2((0,\infty),\frac{dt}{t})}$$

$$= \left\| t\frac{\partial}{\partial t}\left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty\right) W^\lambda(t,x,y)f(y)\,dy \right\|_{L^2((0,\infty),\frac{dt}{t})}, \quad x\in(0,\infty).$$

Then, by using the Minkowski inequality we obtain that

$$\left| g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right|$$

$$\leqslant \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty\right) \left|f(y)\right| \left\{ \int_0^\infty t\left| \frac{\partial}{\partial t} W^\lambda(t,x,y)\right|^2 dt \right\}^{\frac{1}{2}} dy, \quad x\in(0,\infty).$$

According to [7, Lemma 8] we have that

$$\left\{ \int_0^\infty t\left| \frac{\partial}{\partial t} W^\lambda(t,x,y)\right|^2 dt \right\}^{\frac{1}{2}} \leqslant C \begin{cases} \frac{y^\lambda}{x^{\lambda+1}}, & 0<y<\frac{x}{2}, \\ \frac{x^\lambda}{y^{\lambda+1}}, & 2x<y<\infty. \end{cases} \tag{31}$$

From (31) we deduce that, for every $x\in(0,\infty)$,

$$\int_0^{\frac{x}{2}} \left|f(y)\right| \left\{ \int_0^\infty t\left| \frac{\partial}{\partial t} W^\lambda(t,x,y)\right|^2 dt \right\}^{\frac{1}{2}} dy \leqslant C\int_0^{\frac{x}{2}} \frac{|f(y)|y^\lambda}{x^{\lambda+1}}\,dy \leqslant \frac{C}{x}\int_0^x \left|f(y)\right| dy \leqslant C\|f\|_{BMO_+},$$

and, as in the proof of (14) in Section 2,

$$\int_{2x}^\infty \left|f(y)\right| \left\{ \int_0^\infty t\left| \frac{\partial}{\partial t} W^\lambda(t,x,y)\right|^2 dt \right\}^{\frac{1}{2}} dy \leqslant Cx^\lambda \int_{2x}^\infty \frac{|f(y)|}{y^{\lambda+1}}\,dy \leqslant C\|f\|_{BMO_+}.$$

Hence, we conclude that

$$g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \in L^\infty(0,\infty), \tag{32}$$

and

$$\left\| g_{h,\lambda}(f)(x) - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right\|_\infty \leqslant C\|f\|_{BMO_+}. \tag{33}$$

By using again the Minkowski inequality and [7, Lemma 8] we get

$$\left| \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} W^\lambda(t,x,y)f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} - \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}} f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right|$$

$$\leqslant \left\{ \int_0^\infty \left| t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x} \left(W^\lambda(t,x,y) - \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right) f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$

$$\leqslant \int_{\frac{x}{2}}^{2x} \left|f(y)\right| \left\{ \int_0^\infty t\left| \frac{\partial}{\partial t}\left(W^\lambda(t,x,y) - \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\right)\right|^2 dt \right\}^{\frac{1}{2}} dy$$

$$\leqslant C\frac{1}{x}\int_{\frac{x}{2}}^{2x}\left|f(y)\right|dy\leqslant C\|f\|_{BMO_+},\quad x\in(0,\infty).$$

Then

$$\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}W^\lambda(t,x,y)f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\in L^\infty(0,\infty) \tag{34}$$

and

$$\left\|\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}W^\lambda(t,x,y)f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_\infty\leqslant C\|f\|_{BMO_+}. \tag{35}$$

We denote

$$g_h(f_o)(x)=\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}W_t(f_o)(x)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}},\quad x\in\mathbb{R}.$$

We are going to see that

$$g_h(f_o)(x)-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\in L^\infty(0,\infty) \tag{36}$$

and

$$\left\|g_h(f_o)(x)-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_\infty\leqslant C\|f\|_{BMO_+}. \tag{37}$$

Note firstly that according to (16) the Minkowski inequality leads to

$$\begin{aligned}
&\left|g_h(f_o)(x)-\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right|\\
&\quad\leqslant\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_0^\infty\frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(x-y)^2}{4t}}-e^{\frac{(x+y)^2}{4t}}\right)f(y)\,dy-t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\\
&\quad\leqslant\int_{\frac{x}{2}}^{2x}\left|f(y)\right|\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x+y)^2}{4t}}\right)\right|^2dt\right\}^{\frac{1}{2}}dy\\
&\qquad+\left(\int_0^{\frac{x}{2}}+\int_{2x}^\infty\right)\left|f(y)\right|\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}\left[e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{(x+y)^2}{4t}}\right]\right)\right|^2dt\right\}^{\frac{1}{2}}dy\\
&\quad=T_1(f)(x)+T_2(f)(x),\quad x\in(0,\infty).
\end{aligned} \tag{38}$$

It is not hard to see that

$$\left|\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x+y)^2}{4t}}\right)\right|\leqslant C\frac{1}{t^{\frac{3}{2}}}e^{-\frac{(x+y)^2}{8t}},\quad t,x,y\in(0,\infty).$$

Then

$$T_1(f)(x) \leqslant C \int_{\frac{x}{2}}^{2x} |f(y)| \left\{ \int_0^\infty \frac{1}{t^2} e^{-\frac{(x+y)^2}{4t}} dt \right\}^{\frac{1}{2}} dy \leqslant C \int_{\frac{x}{2}}^{2x} \frac{|f(y)|}{x+y} dy$$
$$\leqslant C \frac{1}{x} \int_0^{2x} |f(y)| dy \leqslant C \|f\|_{BMO_+}, \quad x \in (0,\infty).$$

Also, by using the mean value theorem, we get

$$\left| \frac{\partial}{\partial t} \left(\frac{1}{\sqrt{4\pi t}} \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] \right) \right|$$
$$\leqslant C \frac{1}{t^{\frac{3}{2}}} \left(\left| e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right| + \left| \frac{(x-y)^2}{4t} e^{-\frac{(x-y)^2}{4t}} - \frac{(x+y)^2}{4t} e^{-\frac{(x+y)^2}{4t}} \right| \right)$$
$$\leqslant C \frac{xy}{t^{\frac{5}{2}}} e^{-\frac{(x-y)^2}{8t}}, \quad t \in (0,\infty) \text{ and } 0 < y < \frac{x}{2}, \text{ or } y > 2x. \tag{39}$$

Hence

$$T_2(f)(x) \leqslant C \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| xy \left\{ \int_0^\infty \frac{e^{-\frac{(x-y)^2}{4t}}}{t^4} dt \right\}^{\frac{1}{2}} dy \leqslant C \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| \frac{xy}{|x-y|^3} dy$$
$$\leqslant C \int_0^{\frac{x}{2}} |f(y)| \frac{y}{x^2} dy + x \int_{2x}^\infty \frac{|f(y)|}{y^2} dy \leqslant C \frac{1}{x} \int_0^x |f(y)| dy + x \int_{2x}^\infty \frac{|f(y)|}{y^2} dy,$$

and by proceeding as in the proof of (17) we obtain that $T_2(f)(x) \leqslant C\|f\|_{BMO_+}$, $x \in (0,\infty)$. From (38) we deduce (36) and (37). By using (33), (35) and (37) we conclude that $g_{h,\lambda}(f) \in BMO_+$ provided that $g_h(f_o) \in BMO(\mathbb{R})$, and that there exists $C > 0$ such that

$$\frac{1}{a} \int_0^a g_h(f_o)(x)\, dx \leqslant C, \quad a \in (0,\infty).$$

Since $f_o \in BMO(\mathbb{R})$, $g_h(f_o) \in BMO(\mathbb{R})$ when $g_h(f_o)(x) < \infty$, a.e. $x \in \mathbb{R}$ [17]. Let $a > 0$. We write $f_o = f_1 + f_2 + f_3$, where

$$f_1(x) = \frac{1}{2a} \int_0^{2a} f(y)\, dy := f_{(0,2a)}, \quad x \in (0,\infty),$$
$$f_2(x) = \big(f(x) - f_{(0,2a)}\big) \chi_{(0,2a)}(x), \quad x \in (0,\infty),$$
$$f_3(x) = \big(f(x) - f_{(0,2a)}\big) \chi_{(2a,\infty)}(x), \quad x \in (0,\infty),$$

and $f_i(x) = -f_i(-x)$, $x \in (-\infty,0)$, and $i = 1,2,3$.

Note that, for each $x \in (0,\infty)$,

$$g_h(f_1)(x) = |f_{(0,2a)}| \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_0^\infty \frac{1}{\sqrt{4\pi t}} \big(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \big) dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$
$$= \frac{2}{\sqrt{\pi}} |f_{(0,2a)}| \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_0^{\frac{x}{2\sqrt{t}}} e^{-u^2} du \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}$$
$$\leqslant C \|f\|_{BMO_+} \left\{ \int_0^\infty \left| t \frac{x}{4t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}} \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} = C\|f\|_{BMO_+} \left\{ \int_0^\infty \frac{x^2}{t^2} e^{-\frac{x^2}{2t}} dt \right\}^{\frac{1}{2}} \leqslant C\|f\|_{BMO_+}. \tag{40}$$

Since $g_h(f_1)$ is even, $g_h(f_1)(x) \leqslant C\|f\|_{BMO_+}$, $x \in \mathbb{R}$. It is well known that g_h is a bounded operator from $L^2(\mathbb{R})$ into itself. Then

$$\int_0^a g_h(f_2)(x)\,dx \leqslant \sqrt{a}\left\{\int_{-\infty}^{+\infty}\left|g_h(f_2)(x)\right|^2 dx\right\}^{\frac{1}{2}} \leqslant C\sqrt{a}\left\{\int_0^{2a}\left|f(x)-f_{(0,2a)}\right|^2 dx\right\}^{\frac{1}{2}}$$
$$\leqslant Ca\|f\|_{BMO_+}. \tag{41}$$

Hence, since $g_h(f_2)(x)$ is even, $g_h(f_2)(x) < \infty$, a.e. $x \in (-a,a)$.

Finally, by proceeding as in the proof of (17), we obtain

$$\int_0^a g_h(f_3)(x)\,dx = \int_0^a g_h\big((f-f_{(0,2a)})\chi_{(2a,\infty)}\big)(x)\,dx \leqslant C\int_0^a x\int_{2x}^{\infty}\left|f(y)-f_{(0,2a)}\right|\frac{dy}{y^2}\,dx$$
$$\leqslant C\left(\int_0^a x\int_{2x}^{\infty}\left|f(y)\right|\frac{dy}{y^2}\,dx + \left|f_{(0,2a)}\right|\int_0^a x\int_{2x}^{\infty}\frac{dy}{y^2}\,dx\right) \leqslant Ca\|f\|_{BMO_+}. \tag{42}$$

Then, $g_h(f_3)(x) < \infty$, a.e. $x \in (-a,a)$. We conclude that $g_h(f_o)(x) < \infty$, a.e. $x \in (-a,a)$. Hence, since $a > 0$ is arbitrary $g_h(f_0)(x) < \infty$, a.e. $x \in \mathbb{R}$, and then $g_h(f_o) \in BMO(\mathbb{R})$. Moreover, from (40), (41) and (42) we obtain that,

$$\frac{1}{a}\int_0^a \left|g_h(f_o)(x)\right|dx \leqslant C\|f\|_{BMO_+}, \quad a > 0.$$

Thus, we deduce that $g_{h,\lambda}(f) \in BMO_+$ and $\|g_{h,\lambda}(f)\|_{BMO_+} \leqslant C\|f\|_{BMO_+}$.

To analyze the Littlewood–Paley g-function $g_{P,\lambda}$ associated with the Poisson semigroup $\{P_t^\lambda\}_{t>0}$ for the Bessel operator, we can proceed as for the $g_{h,\lambda}$ case. We compare $g_{P,\lambda}$ with the g-function for the classical Poisson semigroup on $\mathbb{R}$.

According to the results established in [6, (2.8) and (2.11)] we can write

$$\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\left[P^\lambda(t,x,y)-\frac{1}{\pi}\chi_{\{\frac{x}{2}<y<2x\}}(y)\frac{t}{(x-y)^2+t^2}\right]\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}$$
$$\leqslant C(xy)^\lambda\begin{cases} x^{-2\lambda-1}, & 0<y\leqslant \frac{x}{2},\\ y^{-2\lambda-1}\big(1+\log\big(1+\frac{xy}{|x-y|^2}\big)\big), & \frac{x}{2}<y<2x,\\ y^{-2\lambda-1}, & y\geqslant 2x.\end{cases}$$

Then, by using the Minkowski inequality we get, for every $x \in (0,\infty)$,

$$\left|g_{P,\lambda}(f)(x)-\frac{1}{\pi}\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{t}{(x-y)^2+t^2}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}\right|$$
$$\leqslant\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\left(\int_0^\infty P^\lambda(t,x,y)f(y)\,dy-\frac{1}{\pi}\int_{\frac{x}{2}}^{2x}\frac{t}{(x-y)^2+t^2}f(y)\,dy\right)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}}$$
$$\leqslant\int_0^\infty\left|f(y)\right|\left\{\int_0^\infty\left|t\frac{\partial}{\partial t}\left(P^\lambda(t,x,y)-\frac{1}{\pi}\chi_{\{\frac{x}{2}<y<2x\}}(y)\frac{t}{(x-y)^2+t^2}\right)\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \leqslant C\|f\|_{BMO_+}. \tag{43}$$

On the other hand a straightforward manipulation allows us to write, for each $x \in \mathbb{R}$,

$$P_t(f_o)(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{t}{(x-y)^2+t^2}f_o(y)\,dy=\frac{1}{\pi}\int_0^\infty\left(\frac{t}{(x-y)^2+t^2}-\frac{t}{(x+y)^2+t^2}\right)f(y)\,dy.$$

We are going to see that

$$g_P(f_o)(x)-\left\{\int_0^\infty\left|\frac{t}{\pi}\frac{\partial}{\partial t}\int_{\frac{x}{2}}^{2x}\frac{t}{(x-y)^2+t^2}f(y)\,dy\right|^2\frac{dt}{t}\right\}^{\frac{1}{2}} \in L^\infty(0,\infty) \tag{44}$$

and

$$\left\| g_P(f_o)(x) - \left\{ \int_0^\infty \left| \frac{t}{\pi} \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right\|_\infty \leqslant C\|f\|_{BMO_+}, \tag{45}$$

where

$$g_P(f_o)(x) = \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} P_t(f_o)(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R},$$

being $P_t f_o$ as in (22). Indeed, the Minkowski inequality implies, for every $x \in (0,\infty)$,

$$\begin{aligned}
&\left| g_P(f_o)(x) - \left\{ \int_0^\infty \left| \frac{t}{\pi} \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right| \\
&\quad \leqslant \frac{1}{\pi} \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \left[\int_0^\infty \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) f(y)\,dy - \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)\,dy \right] \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\quad \leqslant \frac{1}{\pi} \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x+y)^2+t^2} f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\qquad + \frac{1}{\pi} \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \\
&\quad \leqslant \frac{1}{\pi} \int_{\frac{x}{2}}^{2x} |f(y)| \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \frac{t}{(x+y)^2+t^2} \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} dy \\
&\qquad + \frac{1}{\pi} \left(\int_0^{\frac{x}{2}} + \int_{2x}^\infty \right) |f(y)| \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} dy.
\end{aligned} \tag{46}$$

Moreover, we have that

$$\int_0^\infty \left| t \frac{\partial}{\partial t} \frac{t}{(x+y)^2+t^2} \right|^2 \frac{dt}{t} \leqslant C \int_0^\infty \frac{t}{((x+y)^2+t^2)^2}\,dt \leqslant \frac{C}{(x+y)^2} \leqslant \frac{C}{x^2}, \quad \frac{x}{2} < y < 2x,$$

and

$$\int_0^\infty \left| t \frac{\partial}{\partial t} \left(\frac{t}{(x-y)^2+t^2} - \frac{t}{(x+y)^2+t^2} \right) \right|^2 \frac{dt}{t} \leqslant C \begin{cases} \frac{x^2}{y^4}, & 2x < y, \\ \frac{y^2}{x^4}, & 0 < y \leqslant \frac{x}{2}. \end{cases}$$

Hence, (46) leads to

$$\begin{aligned}
&\left| g_p(f_o)(x) - \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} \int_{\frac{x}{2}}^{2x} \frac{t}{(x-y)^2+t^2} f(y)\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}} \right| \\
&\quad \leqslant C \left(\frac{1}{x} \int_{\frac{x}{2}}^{2x} |f(y)|\,dy + \frac{1}{x} \int_0^{\frac{x}{2}} |f(y)|\,dy + x \int_{2x}^\infty \frac{|f(y)|}{y^2}\,dy \right) \leqslant C\|f\|_{BMO_+}, \quad x \in (0,\infty).
\end{aligned}$$

Thus (44) and (45) are established. From (43), (44) and (45) we deduce that

$$g_{P,\lambda}(f) - g_P(f_o) \in L^\infty(0,\infty) \quad \text{and} \quad \left\| g_{P,\lambda}(f) - g_P(f_o) \right\|_\infty \leqslant C\|f\|_{BMO_+}.$$

To see that $g_{P,\lambda}(f) \in BMO_+$ and $\|g_{P,\lambda}(f)\|_{BMO_+} \leqslant C\|f\|_{BMO_+}$ we can proceed as in the proof of the corresponding property for $g_{h,\lambda}$.

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