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The Shapley value for arbitrary families of coalitions

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A B S T R A C T

We address the problem of finding a suitable definition of a value similar to that of Shapley's, when the games are defined on a subfamily of coalitions with no structure. We present two frameworks: one based on the familiar efficiency, linearity and null player axioms, and the other on linearity and the behavior on unanimity games. We give several properties and examples in each case, and give necessary and sufficient conditions on the family of coalitions for the approaches to coincide.

1. Introduction

In cooperative games in characteristic function form we are given a finite set $N = \{1, 2, \ldots, n\}$ of players and a real valued function v—which we will refer to as the *game*—defined on all the subsets of $N, \mathscr{P}(N)$, with $v(\emptyset) = 0$. A *value* is a mapping ϕ which assigns to each game v and $i \in N$ a real number $\phi_i(v)$.

Perhaps the most celebrated value is the one given by Shapley (1953), which has been extensively studied and generalized.

When n is large, the exponentially large data involved make it certainly unrealistic to have a complete knowledge of v in practice. Even if n is small, some coalitions may not actually form, either because the players cannot meet or communicate with each other, or because of incompatibilities among them (for instance, "parties" in voting). Thus, the study of the so called restricted games was initiated by the works of Thrall and Lucas (1963), Aumann and Dréze (1974), Myerson (1977a,b), and Owen (1977), to cite a few. Some of these studies consider the way in which coalitions are formed, such as Myerson's communication graph, while others focus on the axioms and underlying structures, such as lattices, antimatroids, or convex geometries. The book by Bilbao (2000) presents a general treatment and references in this direction.

There have been many other types of generalizations of the Shapley value, such as that by Shapley and Shubik (1954), where subfamilies of games are considered, but we do not study them here, nor do we consider the relation with the core. For recent treatments of the Shapley value and its generalizations, we refer the reader to the corresponding chapters in the book by Aumann and Hart (2002), and to the paper by Moretti and Patrone (2008).

This work concerns the problem of finding a suitable definition of a value ϕ satisfying properties similar to those of Shapley's, when the underlying family of coalitions has no structure. Recent papers related to our work are those by Honda and Grabisch (2006), Lange and Grabisch (2009) and Faigle and Peis (2008). It is worth mentioning also the work by Castro et al. (2009), where the calculation of the Shapley value is based on sampling, alleviating the problem of huge data for large n.

Indicating by $\subset$ the inclusion between two sets, and by $\subsetneq$ the strict inclusion, we assume there is a family of *coalitions* $\mathscr{K}, \mathscr{K} \subset \mathscr{P}(N)$, with $\emptyset$ and N in $\mathscr{K}$, and consider the associated family of games defined on it, $\mathscr{V} = \{v : \mathscr{K} \to \mathbb{R}, v(\emptyset) = 0\}$. Notice that we do not ask for *superadditivity* of games, i.e., the condition $v(A \cup B) \geqslant v(A) + v(B)$ for all disjoint A and B is not required for v to be in $\mathscr{V}$ ($\mathscr{K}$ need not be closed under disjoint unions).

Note. From now on, ϕ will denote a value function defined on the set of games $\mathscr{V}$ of a family of coalitions $\mathscr{K}$.

Shapley (1953) introduced his value, denoted here by ϕ^S, by means of three axioms. Although a number of alternative set of axioms have been proposed, we discuss mostly those originally given by Shapley, adapting them to our setting.

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The first axiom, *symmetry*, is defined by taking a permutation π on N and considering for $v \in \mathscr{V}$ the game $v \circ \pi$ (the composition), and the symmetry axiom asks that $\phi_i(v \circ \pi) = \phi_{\pi(i)}(v)$ for all π, v and i. Since in general we do not have $\pi(K) \in \mathscr{K}$ for all $K \in \mathscr{K}$, this axiom makes sense only partially in our context (that is, only for permutations leaving invariant $\mathscr{K}$). For example, if $N = \{1,2,3\}$ and $\mathscr{K} = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, N\}$, we may say that the players 2 and 3 play a symmetrical role. But, if $\mathscr{K} = \{\emptyset, \{1\}, \{2\}, \{1,3\}, N\}$, there are no symmetric players.

A related and much weaker concept is that of *equivalent players*, a property of the family $\mathscr{K}$ which does not depend on the value ϕ:

Definition 1.1. The players i and j in N are *equivalent* (with respect to the family $\mathscr{K}$) if $\{K \in \mathscr{K} : i \in K\} = \{K \in \mathscr{K} : j \in K\}$.

If there is no way to distinguish between two players, we may ask that they obtain the same payoff:

Property 1.2 (*EE*). *ϕ is egalitarian on equivalent players if whenever i and j are equivalent players, $\phi_i(v) = \phi_j(v)$ for all games v.*

Notice that if $\mathscr{K} = \mathscr{P}(N)$, there are no players which are different and equivalent, and **EE** is trivially satisfied. We call it property rather than axiom, since it depends on the family of coalitions being considered.

Shapley's second axiom is currently called the *carrier* axiom, though he denominated it *efficiency*. Recall that a set R is a carrier of v if $v(S) = v(S \cap R)$ for all $S \subset N$, and the axiom states that $v(R) = \sum_{i \in R} \phi_i(v)$ for every carrier R of v. Although N is a carrier of any game, for an arbitrary family of coalitions $\mathscr{K}$ we may not have $S \cap R \in \mathscr{K}$ for all $S, R \in \mathscr{K}$ (the *intersection property*). So again, this axiom does not make sense in a general context.

Here we use the term efficiency following current practice:

Axiom 1.1 (*E*). ϕ is *efficient* if $\sum_{i \in N} \phi_i(v) = v(N)$ for all $v \in \mathscr{V}$.

Recall that for $\mathscr{K} = \mathscr{P}(N)$, $i \in N$ is a null player for v if $v(S) = v(S \cup \{i\})$ whenever $S \subset N$ and $i \notin S$, and that the carrier axiom is equivalent to **E** and the *null player axiom*. In general we may not have S and $S \cup \{i\}$ in $\mathscr{K}$, but this time we may adapt the definition.

Let $G_{\mathscr{K}} = (\mathscr{K}, \mathscr{A})$ be the directed graph whose nodes are the coalitions in $\mathscr{K}$ and $(K, K') \in \mathscr{A}$ if $K \subsetneq K'$ and there is no $K'' \in \mathscr{K}$ such that $K \subsetneq K'' \subsetneq K'$. Formally, $G_{\mathscr{K}}$ is the transitive reduction of the graph induced by the inclusion on $\mathscr{K}$. For $i \in N$, let $\mathscr{A}_i = \{(K, K') \in \mathscr{A} : i \in K' \setminus K\}$.

The following definition coincides with the usual one when $\mathscr{K} = \mathscr{P}(N)$:

Definition 1.3. $i \in N$ is a *null player* for $v \in \mathscr{V}$ if $v(K) = v(K')$ for all $(K, K') \in \mathscr{A}_i$.

Axiom 1.2 (*N*). ϕ satisfies the *null player* axiom if for every $v \in \mathscr{V}$, $\phi_i(v) = 0$ for every null player i of v.

The third and final axiom by Shapley, which he called *aggregation*, asks for the equality $\phi(v + w) = \phi(v) + \phi(w)$ for all $v, w \in \mathscr{V}$, but it is more convenient to ask for linearity, i.e., adding the condition $\phi(\alpha v) = \alpha \phi(v)$ for any constant $\alpha > 0$, so that we have independence from scale. Let us observe that, since we are not assuming superadditivity of games, if $v \in \mathscr{V}$ then $-v \in \mathscr{V}$, and, actually, $\mathscr{V}$ is a linear space on $\mathbb{R}$ of dimension $|\mathscr{K}| - 1$ ($|A|$ denotes the cardinal of the finite set A).

Axiom 1.3 (*L*). ϕ is *linear* if $\phi(\alpha v + \beta w) = \alpha \phi(v) + \beta \phi(w)$ whenever $v, w \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{R}$.

The linearity of a value immediately leads to the study of its behavior on linear bases of $\mathscr{V}$. Usually two such bases are considered, both parameterized by $\mathscr{K}^* = \{R \in \mathscr{K} : R \neq \emptyset\}$. The first one consists of the *elementary* games E_R, defined for $R \in \mathscr{K}^*$ by

$$E_R(K) = \begin{cases} 1 & \text{if } R = K, \\ 0 & \text{otherwise,} \end{cases}$$

and the other consists of the *unanimity* games U_R ($R \in \mathscr{K}^*$), defined by

$$U_R(K) = \begin{cases} 1 & \text{if } R \subset K, \\ 0 & \text{otherwise.} \end{cases}$$

Unlike elementary games, unanimity games are superadditive and therefore better suited to cooperative game theory. On the other hand, elementary games are easier to work with because of the simple representation

$$v = \sum_{K \in \mathscr{K}^*} v(K) E_K \quad \text{for all } v \in \mathscr{V}. \tag{1.1}$$

Shapley showed that, when $\mathscr{K} = \mathscr{P}(N)$, the carrier and symmetry axioms imply the following property, which we take as axiom (cf. Aumann, 1990):

Axiom 1.4 (*EU*). ϕ is *egalitarian on unanimity games* if for all $T \in \mathscr{K}^*$,

$$\phi_i(U_T) = \begin{cases} 1/|T| & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that neither the carrier nor the symmetry axioms can be used for general families, but **EU** makes perfect sense in our setting.

In his 1953 paper, Shapley presented also a bargaining model in which, starting with a single player, players are added one at a time until everyone has been admitted, and each player is given her marginal contribution (according to the game v) when incorporated. If every order is possible and all orders are equally probable, then $\phi_i^S(v)$ is the corresponding expectation.

These marginal contributions are quite apparent in the usual expression for ϕ^S,

$$\phi_i^S(v) = \sum_{t=1}^{n} \frac{(t-1)!(n-t)!}{n!} \sum_{i \in T : |T| = t} (v(T) - v(T \setminus \{i\})), \tag{1.2}$$

which leads to the following definition:

Definition 1.4. ϕ is *marginalist* if there exist coefficients $\lambda_i(K,K') \in \mathbb{R}$, defined for each $i \in N$ and $(K,K') \in \mathscr{A}_i$, such that

$$\phi_i(v) = \sum_{(K,K') \in \mathscr{A}_i} \lambda_i(K,K')\,(v(K') - v(K)) \quad \textit{for all } i \in N \quad \textit{and} \quad v \in \mathscr{V}. \tag{1.3}$$

When convenient, we set $\lambda_i(K,K') = 0$ if $i \notin K' \setminus K$, so that the sum in (1.3) may be taken over all $(K,K') \in \mathscr{A}$, and not just $\mathscr{A}_i$.

If $\mathscr{K} = \mathscr{P}(N)$, there is only one player in $K' \setminus K$ if $(K,K') \in \mathscr{A}$, and precisely $\mathscr{K}$ is *regular* (Honda and Grabisch, 2006) if $|K' \setminus K| = 1$ for all $(K,K') \in \mathscr{A}$. In general, we may have several players in $K' \setminus K$, and the following property states that in every arc the marginal gain is equally distributed among its members.

Property 1.5 (IS). *A marginalist value ϕ has internal symmetry if it may represented in the form* (1.3) *with*

$$\lambda_i(K,K') = \lambda_j(K,K') \quad \textit{for all } (K,K') \in \mathscr{A} \quad \textit{and} \quad i,j \in K' \setminus K. \tag{1.4}$$

As is the case of **EE**, **IS** is satisfied by every value when $\mathscr{K} = \mathscr{P}(N)$. Actually, **IS** implies **EE**, since i and j are equivalent players if and only if $\mathscr{A}_i = \mathscr{A}_j$. As the representation as a marginalist value need not be unique, it may be the case that in one representation (1.4) is satisfied but not in another (see Example 2.5).

1.1. Summary of results

In this paper we present two frameworks for defining a Shapley value on general subfamilies, exhibiting differences and similarities.
In Section 2 we discuss the first one, based on **N**, **L** and **E**, and related to Shapley's bargaining model. We show that **N** and **L** are equivalent to marginalism (Theorem 2.4), though the representation may not be unique (Example 2.5), and we characterize those $\mathscr{K}$ for which it is (Theorem 2.6). By adding **E**, we show that a marginalist value determines a flow in $G_{\mathscr{K}}$ (Theorem 2.7), and by adding **IS**, we obtain the converse (Theorem 2.8).
N, **L** and **E** are not enough to uniquely determine the Shapley value when $\mathscr{K} = \mathscr{P}(N)$, and another axiom, such as symmetry in the classical setting, is needed. This remains true for general $\mathscr{K}$, even if **IS** is required, and in the last part of Section 2 we discuss several possibilities. In Section 2.1, closely related to the probabilistic study by Weber (1988), we use the decomposition of a flow along $\emptyset - N$ paths in $G_{\mathscr{K}}$ (Theorem 2.10 and Example 2.11). We present different models according to the weights given to the paths, and show their differences in Example 2.12. In Section 2.2, we present a generalization of the potential flow model of Lange and Grabisch (2009), obtaining uniqueness (Theorem 2.18) when the *regularity axiom*, **R** (defined in Section 2), is satisfied.
In Section 3 we study the second framework, which is inspired in the Hart and Mas-Colell (1989) treatment of the potential. We use **L** and **EU** to find existence and uniqueness of a value, ϕ^U, which also satisfies **E** and **EE** (Theorem 3.1). We devote the remainder of this section to study when ϕ^U is marginalist, giving first some examples and then some necessary conditions under sensible hypotheses. Theorem 3.4 reveals a quantity related to the coefficients of ϕ^S in (1.2) and the potential models. As a consequence, we obtain that for regular families, ϕ^U is marginalist if and only if $\mathscr{K} = \mathscr{P}(N)$ (Corollary 3.7). Corollary 3.8 exposes a property, P-1, on coalitions and Theorem 3.9 shows that P-1 implies that ϕ^U is marginalist. In Corollary 3.10 we completely characterize those families for which ϕ^U is marginalist with nonnegative marginal coefficients. Finally, in Example 3.12 we show that P-1 may not be simplified easily.
In the last section we make a few concluding comments.

2. Marginalist and efficient values

For $i \in N$, let us define

$$\mathscr{K}_i^+ = \{K \in \mathscr{K} : (K,K') \in \mathscr{A}_i \text{ for some } K' \in \mathscr{K}\},$$
$$\mathscr{K}_i^- = \{K \in \mathscr{K} : (K',K) \in \mathscr{A}_i \text{ for some } K' \in \mathscr{K}\},$$
$$\mathscr{K}_i = \mathscr{K}_i^+ \cup \mathscr{K}_i^-,$$

so that $\mathscr{K}_i$ is the set of coalitions which are endpoints of an arc in $\mathscr{A}_i$. For a given $i \in N$, let us denote by $c(i)$ the number of (weak) components induced by $\mathscr{A}_i$, and denote each of them by $\mathscr{C}_{i,k} = (\mathscr{K}_{i,k}, \mathscr{A}_{i,k}),\ k = 1,\dots,c(i)$. Notice that $\mathscr{C}_{i,k}$ is directed and bipartite: if $(K,K') \in \mathscr{A}_i$, then $K \in \mathscr{K}_i^+$ if and only if $K' \in \mathscr{K}_i^-$, and we cannot have $K \in \mathscr{K}_i^- \cap \mathscr{K}_i^+$.

Note. If a graph G is directed, we will denote by G° the underlying undirected version.

Example 2.1. Let $n = 5$ and

$$\mathscr{K} = \{\emptyset,\ K_1 = \{1\},\ K_2 = \{2\},\ K_3 = \{1,2,3,4\},\ K_4 = \{1,2,3,5\}, N\}.$$

There are 8 arcs in $\mathscr{A}$,

$$e_1 = (\emptyset,K_1), \quad e_2 = (\emptyset,K_2), \quad e_3 = (K_1,K_3), \quad e_4 = (K_1,K_4), \quad e_5 = (K_2,K_3), \quad e_6 = (K_2,K_4), \quad e_7 = (K_3,N), \quad e_8 = (K_4,N),$$

so that

$$\mathscr{A}_1 = \{e_1,e_5,e_6\}, \quad \mathscr{A}_2 = \{e_2,e_3,e_4\}, \quad \mathscr{A}_3 = \{e_3,e_4,e_5,e_6\}, \quad \mathscr{A}_4 = \{e_3,e_5,e_8\}, \quad \mathscr{A}_5 = \{e_4,e_6,e_7\}.$$

Hence, we have

$$c(1) = c(2) = c(4) = c(5) = 2, \quad c(3) = 1,$$

and numbering arbitrarily on k the components $\mathscr{C}_{i,k} = (\mathscr{K}_{i,k}, \mathscr{A}_{i,k})$, we may write, for example (others are symmetrical):

$$\mathscr{K}_{1,1} = \{\emptyset, K_1\}, \quad \mathscr{A}_{1,1} = \{e_1\}, \quad \mathscr{K}_{1,2} = \{K_2,K_3,K_4\}, \quad \mathscr{A}_{1,2} = \{e_5,e_6\}, \quad \mathscr{K}_{3,1} = \{K_1,K_2,K_3,K_4\}, \quad \mathscr{A}_{3,1} = \{e_3,e_4,e_5,e_6\}.$$

Notice that $\mathscr{C}^{\circ}_{i,k}$ are actually trees for $i \neq 3$ and $k = 1, 2$, but $\mathscr{C}^{\circ}_{3,1}$ is a cycle. □

The following results are easy to prove.

Lemma 2.2. *ϕ satisfies* **N** *if and only if $\phi_i(v) = 0$ whenever $v \in \mathscr{V}$ and $i \in N$ are such that for all $k = 1, \ldots, c(i)$ and $(R, R') \in \mathscr{A}_{i,k}$ there holds $v(R) = v(R')$.*

Lemma 2.3. *If ϕ satisfies* **N**, *$i \in N$, and $S \notin \mathscr{K}_i$, then $\phi_i(E_S) = 0$.*

It is clear that a marginalist value satisfies **N** and **L**, and we now show that there are no others (cf. Bilbao, 2000, Chapter 7, Lange and Grabisch, 2009, Property 3).

Theorem 2.4. *ϕ is marginalist if and only if it satisfies* **N** *and* **L**.

Proof. If ϕ satisfies **N** and **L**, using (1.1) and Lemma 2.3 we obtain

$$\phi_i(v) = \sum_{K \in \mathscr{K}_i} v(K)\, \phi_i(E_K) = \sum_{k=1}^{c(i)} \sum_{K \in \mathscr{K}_{i,k}} v(K)\, \phi_i(E_K). \tag{2.1}$$

Letting

$$v_{i,k} = \sum_{K \in \mathscr{K}_{i,k}} E_K \quad \text{for } k = 1, \ldots, c(i), \tag{2.2}$$

we have

$$v_{i,k}(R) = v_{i,k}(R') \in \{0, 1\} \quad \text{for } (R, R') \in \mathscr{A}_i, \quad k = 1, \ldots, c(i),$$

since either R and R' are in the same component $\mathscr{C}_{i,k}$ and exactly one term in (2.2) is 1, or else they are in different components and all terms are 0. Hence, using **N**,

$$\phi_i(v_{i,k}) = 0 \quad \text{for } k = 1, \ldots, c(i). \tag{2.3}$$

For each component $\mathscr{C}_{i,k}$, let us fix $K_{i,k} \in \mathscr{K}_{i,k}$. From (2.2) we may write

$$E_{K_{i,k}} = v_{i,k} - \sum_{K \in \mathscr{K}_{i,k} : K \neq K_{i,k}} E_K,$$

and using **L** and (2.3),

$$\phi_i(E_{K_{i,k}}) = \phi_i(v_{i,k}) - \sum_{K \in \mathscr{K}_{i,k} : K \neq K_{i,k}} \phi_i(E_K) = - \sum_{K \in \mathscr{K}_{i,k} : K \neq K_{i,k}} \phi_i(E_K). \tag{2.4}$$

Thus, using (2.1) and (2.4),

$$\phi_i(v) = \sum_{k=1}^{c(i)} \sum_{K \in \mathscr{K}_{i,k}} v(K)\, \phi_i(E_K) = \sum_{k=1}^{c(i)} \left(v(K_{i,k})\, \phi_i(E_{K_{i,k}}) + \sum_{K \in \mathscr{K}_{i,k} : K \neq K_{i,k}} v(K)\, \phi_i(E_K) \right) = \sum_{k=1}^{c(i)} \sum_{K \in \mathscr{K}_{i,k} : K \neq K_{i,k}} (v(K) - v(K_{i,k}))\, \phi_i(E_K). \tag{2.5}$$

For each $K \in \mathscr{K}_{i,k}, K \neq K_{i,k}$, let $(K_0 = K_{i,k}, K_1, \ldots, K_s = K)$ be a path joining $K_{i,k}$ and K in $\mathscr{C}^{\circ}_{i,k}$, and let $P_{i,k}(K)$ be the subgraph of $\mathscr{C}_{i,k}$ induced by this path. We write each term of the sum in (2.5) as a telescoping sum:

$$(v(K) - v(K_{i,k}))\, \phi_i(E_K) = \sum_{j=1}^{s} (v(K_j) - v(K_{j-1}))\, \phi_i(E_K) = \sum_{(K', K'') \in \mathscr{A}_{i,k}} \lambda_{i,k,K}(K', K'')\, (v(K'') - v(K')),$$

where for $K \neq K_{i,k}$,

$$\lambda_{i,k,K}(K', K'') = \begin{cases} \phi_i(E_K) & \text{if } (K', K'') \text{ is an arc in } P_{i,k}(K), \\ -\phi_i(E_K) & \text{if } (K'', K') \text{ is an arc in } P_{i,k}(K), \\ 0 & \text{otherwise.} \end{cases}$$

and $\lambda_{i,k,K_{i,k}}(K', K'') = 0$ for all $(K', K'') \notin \mathscr{A}_i$. We may rewrite (2.5) as

$$\phi_i(v) = \sum_{k=1}^{c(i)} \sum_{K \in \mathscr{K}_{i,k}} \sum_{(K', K'') \in \mathscr{A}_{i,k}} \lambda_{i,k,K}(K', K'')\, (v(K'') - v(K')) = \sum_{(K', K'') \in \mathscr{A}} \lambda_i(K', K'')\, (v(K'') - v(K')),$$

where

$$\lambda_i(K', K'') = \sum_{k=1}^{c(i)} \sum_{K \in \mathscr{K}_{i,k}} \lambda_{i,k,K}(K', K'') \quad \text{if } (K', K'') \in \mathscr{A}_i,$$

and $\lambda_i(K', K'') = 0$ otherwise. □

Since the path in the previous proof need not be unique, it comes as no surprise that there may be different representations in the form (1.3).

Example 2.5. Consider N and $\mathscr{K}$ as in Example 2.1. The value ϕ defined by

$$\phi_i(v) = \begin{cases} v(K_1) & \text{if } i = 1, \\ \frac{1}{3}(v(K_3) - v(K_1)) & \text{if } i = 2,3,4, \\ v(N) - v(K_3) & \text{if } i = 5, \end{cases}$$

may be written also as

$$\phi_1(v) = v(K_1), \quad \phi_2(v) = \phi_4(v) = \frac{1}{3}(v(K_3) - v(K_1)), \quad \phi_3(v) = \frac{1}{3}(v(K_3) - v(K_2)) - \frac{1}{3}(v(K_4) - v(K_2)) + \frac{1}{3}(v(K_4) - v(K_1)),$$
$$\phi_5(v) = v(N) - v(K_3).$$

In one representation ϕ satisfies (1.4) (and therefore ϕ satisfies **IS**), but not in the other. Moreover, ϕ is strongly monotone in the sense of Young (1985), although in the second representation some of the coefficients are negative (see also Remark 2.9). □

Theorem 2.6. *If $\mathscr{A}_i^\circ$ is acyclic for all $i \in N$, then the representation of the marginalist value ϕ in (1.3) is unique. Conversely, if there exists a value ϕ whose representation in the form (1.3) is unique, then $\mathscr{A}_i^\circ$ is acyclic for all $i \in N$.*

Proof. If $\mathscr{A}_i^\circ$ is acyclic for all $i \in N$, by taking differences between two representations we may assume that there exist coefficients $\mu_i(K,K')$, defined for $i \in N$ and $(K,K') \in \mathscr{A}_i$, such that

$$\sum_{(K,K') \in \mathscr{A}_i} \mu_i(K,K')(v(K') - v(K)) = 0 \quad \text{for all } v \in \mathscr{V}.$$

By taking $v = E_K$, we obtain

$$\sum_{(K,K') \in \mathscr{A}_i} \mu_i(K,K') = 0 \quad \text{for all } K \in \mathscr{K}_i^+,$$
$$\sum_{(K',K) \in \mathscr{A}_i} \mu_i(K',K) = 0 \quad \text{for all } K \in \mathscr{K}_i^-.$$

If K is a leaf of the tree $\mathscr{C}_{i,k}^\circ$ and (K,K') is the only edge of $\mathscr{C}_{i,k}^\circ$, we must have $\mu_i(K,K') = 0$ or $\mu_i(K',K) = 0$, depending on whether K is in $\mathscr{K}_i^+$ or $\mathscr{K}_i^-$. Eliminating K and this edge, repeating the procedure on each remaining leaf and then on every component, we see that necessarily $\mu_i(K,K') = 0$ for all $(K,K') \in \mathscr{A}_i$.

Conversely, if for some $i \in N$ a component $\mathscr{C}_{i,k}$ contains an undirected cycle $\sigma = (K_0, K_1, \ldots, K_\ell = K_0)$, we may assume $K_0 \in \mathscr{K}_i^+$ and we may write, since ℓ is necessarily even (because $\mathscr{C}_{i,k}$ is bipartite),

$$0 = \sum_{s=1}^{\ell} (-1)^s (v(K_s) - v(K_{s-1})).$$

Hence, the null value ($\phi \equiv 0$) has at least two different representations, and therefore so does any marginalist value. □

Our next result relates marginalist and efficient values with flows in $G_{\mathscr{K}}$ (cf. Bilbao, 2000, Chapter 7; Lange and Grabisch, 2009, Property 5).

Setting $\mathscr{K}^{**} = \{K \in \mathscr{K} : K \neq \emptyset, N\}$, a *flow* in $G_{\mathscr{K}}$ is a function $f : \mathscr{A} \to \mathbb{R}$ such that

$$\sum_{K' : (K',K) \in \mathscr{A}} f(K',K) = \sum_{K' : (K,K') \in \mathscr{A}} f(K,K') \quad \text{for all } K \in \mathscr{K}^{**},$$

i.e., the conservation equations hold at interior nodes. Noticing that we do not ask for $f \geqslant 0$, the conservation equations imply

$$\sum_{K : (\emptyset,K) \in \mathscr{A}} f(\emptyset,K) = \sum_{K : (K,N) \in \mathscr{A}} f(K,N),$$

and this common number is called the *value* of the flow (regrettably, the term “value” is used for two quite different objects).

If $\chi(A,x)$ is the indicator function of the set A (that is, $\chi(A,x) = 1$ if $x \in A$ and is 0 otherwise), and ϕ is marginalist and written in the form (1.3),

$$\sum_{i \in N} \phi_i(v) = \sum_{i \in N} \sum_{(K,K') \in \mathscr{A}_i} \lambda_i(K,K')(v(K') - v(K)) = \sum_{(K,K') \in \mathscr{A}} (v(K') - v(K)) \left(\sum_{i \in N} \lambda_i(K,K') \chi(\mathscr{A}_i, (K,K')) \right)$$
$$= \sum_{(K,K') \in \mathscr{A}} \Lambda(K,K')(v(K') - v(K)), \tag{2.6}$$

where

$$\Lambda(K,K') = \sum_{i \in N} \lambda_i(K,K') \chi(\mathscr{A}_i, (K,K')) = \sum_{i : i \in K' \setminus K} \lambda_i(K,K').$$

If $R \in \mathscr{K}^{**}$, $v = E_R$, and ϕ is efficient, from (2.6) we obtain, successively,

$$\sum_{i \in N} \phi_i(E_R) = -\sum_{(R,K') \in \mathscr{A}} \Lambda(R,K') + \sum_{(K,R) \in \mathscr{A}} \Lambda(K,R) = E_R(N) = 0, \quad \sum_{(R,K') \in \mathscr{A}} \Lambda(R,K') = \sum_{(K,R) \in \mathscr{A}} \Lambda(K,R).$$

On the other hand, if $v = E_N$, (2.6) turns into

$$\sum_{i\in N}\phi_i(E_N) = \sum_{(K,N)\in\mathscr{A}} \Lambda(K,N) = E_N(N) = 1.$$

Thus we have:

Theorem 2.7. *If ϕ is marginalist (written in the form (1.3)) and efficient, then the quantities*

$$\Lambda(K,K') = \sum_{i:i\in K'\setminus K} \lambda_i(K,K'), \quad (K,K')\in\mathscr{A}, \tag{2.7}$$

define a flow of value 1 in $G_{\mathscr{K}}$.

We may always find a converse by forcing internal symmetry:

Theorem 2.8. *Suppose Λ is a flow of value 1 in $G_{\mathscr{K}}$. Then, if*

$$\lambda_i(K,K') = \frac{\Lambda(K,K')}{|K'\setminus K|} \quad \textit{for all } i\in N \textit{ and all } (K,K')\in\mathscr{A}_i,$$

the value ϕ defined by (1.3) is a marginalist and efficient value satisfying **IS**.

Proof. Only efficiency needs to be checked, and this is easily done. □

Remark 2.9. In our setting, the strong monotonicity of Young (1985) may be expressed as: if $v\in\mathscr{V}$ satisfies $v(K')\geqslant v(K)$ for all $(K,K')\in\mathscr{A}_i$, then $\phi_i(v)\geqslant 0$. Lange and Grabisch (2009) show that for marginalist values on regular families, it is equivalent to having non-negative coefficients λ in (1.3). This does not hold in general, as it is possible to construct a marginalist value satisfying **E**, **IS**, strong monotonicity, unique representation, but such that some of the coefficients in (1.3) are negative.

Notice that the value ϕ in Example 2.5 satisfies strong monotonicity and has a representation with some negative coefficients, but the representation is not unique. □

2.1. Maximal chains

In a network with source s and sink t, the simplest $s-t$ flow we can think of is a flow along a $s-t$ path, having value, say, 1 for arcs of the path and 0 for other arcs. Since $G_{\mathscr{K}}$ is a $\emptyset - N$ network, where the (simple, directed) $\emptyset - N$ paths are maximal chains for inclusion, this suggests that for a maximal chain $W=(K_0=\emptyset, K_1,\ldots,K_{\ell(W)}=N)$ of length $\ell(W)$, we consider the flow of value 1 associated with W,

$$\Lambda_W(K,K') = \begin{cases} 1 & \text{if } (K,K')=(K_{j-1},K_j) \text{ for some } j=1,\ldots,\ell(W),\\ 0 & \text{otherwise,}\end{cases}$$

and the corresponding marginalist, efficient and satisfying **IS** value given by Theorem 2.8,

$$\psi_{W,i}(v) = \sum_{j=1}^{\ell(W)} \frac{v(K_j)-v(K_{j-1})}{|K_j\setminus K_{j-1}|}\chi(K_j\setminus K_{j-1},i) \quad \text{for } i\in N, \quad v\in\mathscr{V}. \tag{2.8}$$

Writing $\psi_{W,}(v)$ in the form (1.3), we obtain

$$\lambda_i(K,K') = \frac{1}{|K'\setminus K|}\chi(W,(K,K'))\chi(K'\setminus K,i), \tag{2.9}$$

where we interpret that the arc $(K,K')\in W$ if and only if $(K,K')=(K_{j-1},K_j)$ for some $j=1,\ldots,\ell(W)$.

Let us denote with $\mathscr{W}$ the family of all maximal chains, and for a given $\gamma:\mathscr{W}\to\mathbb{R}$ satisfying

$$\sum_{W\in\mathscr{W}}\gamma(W)=1, \tag{2.10}$$

let us consider the sum of the flows Λ_W weighted by γ,

$$\overline{\Lambda}(K,K') = \sum_{W\in\mathscr{W}}\gamma(W)\Lambda_W(K,K') = \sum_{\substack{W\in\mathscr{W}\\(K,K')\in W}}\gamma(W) \quad \text{for } (K,K')\in\mathscr{A}, \tag{2.11}$$

we see that $\overline{\Lambda}$ is a flow on $G_{\mathscr{K}}$ of value 1.

The corresponding value,

$$\bar\phi(v) = \sum_{W\in\mathscr{W}}\gamma(W)\psi_{W,}(v), \tag{2.12}$$

is marginalist, efficient and satisfies **IS**, and writing it in the form (1.3), from (2.9) and (2.11) we see that the corresponding coefficients are

$$\bar\lambda_i(K,K') = \frac{\chi(K'\setminus K,i)}{|K'\setminus K|}\sum_{\substack{W\in\mathscr{W}\\(K,K')\in W}}\gamma(W) = \frac{\overline{\Lambda}(K,K')}{|K'\setminus K|}\chi(K'\setminus K,i). \tag{2.13}$$

We have:

Theorem 2.10. *For a value ϕ, the following are equivalent:*

(i) *There exists $\gamma : \mathscr{W} \to \mathbb{R}$ satisfying (2.10) such that ϕ may be written as in (2.12).*
(ii) *There exists $\Lambda : \mathscr{A} \to \mathbb{R}$ defining a flow in $G_{\mathscr{K}}$ of value 1 such that ϕ may be written as*

$$\phi_i(v) = \sum_{(K,K') \in \mathscr{A}_i} \frac{\Lambda(K,K')}{|K' \setminus K|}(v(K') - v(K)). \tag{2.14}$$

(iii) *ϕ is marginalist, efficient and satisfies* **IS**.

Proof. That (i) implies (ii) follows from the previous discussion: the coefficients γ in (2.10) define the flow of value 1 in (2.11), which in turn defines the value $\phi = \bar{\phi}$ in (2.12) with marginal coefficients given by (2.13). Conversely, if (ii) is satisfied, we may decompose the flow Λ along $\emptyset - N$ paths (there are no cycles in $G_{\mathscr{K}}$), and find coefficients γ so that (2.10) holds and (2.11) holds with $\Lambda = \overline{\Lambda}$. By (2.12) we obtain the value $\bar{\phi}$, whose marginal coefficients coincide with those of ϕ, by (2.13) and (2.14), and therefore $\phi = \bar{\phi}$. In all, (ii) implies (i), and they are equivalent.

(ii) implies (iii) by Theorem 2.8. On the other hand, if (iii) is satisfied, by Theorem 2.7 we may find a flow Λ in $G_{\mathscr{K}}$, of value 1, satisfying (2.7), and the internal symmetry implies

$$\lambda_i(K,K') = \frac{\Lambda(K,K')}{|K' \setminus K|} \quad \text{for all } i \in N \quad \text{and} \quad (K,K') \in \mathscr{A},$$

so that (ii) is satisfied. Therefore, (ii) and (iii) are equivalent. □

The coefficients γ in Theorem 2.10 need not be unique or nonnegative, as the following example shows.

Example 2.11. Let $\mathscr{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2\}, K_3 = \{1,2\}, K_4 = \{1,2,3\}, K_5 = \{1,2,4\}, N = \{1,2,3,4\}\}$. The maximal chains or $\emptyset - N$ paths are

$$W_1 = (\emptyset, K_1, K_3, K_4, N), \quad W_2 = (\emptyset, K_1, K_3, K_5, N), \quad W_3 = (\emptyset, K_2, K_3, K_4, N), \quad W_4 = (\emptyset, K_2, K_3, K_5, N),$$

so that if $\Lambda(K,K') = 1/2$ for all $(K,K') \in \mathscr{A}$, and ϕ is defined by (2.14), then

$$\phi = \psi_{W_1} + \psi_{W_3} = \psi_{W_2} + \psi_{W_4} = \alpha(\psi_{W_1} + \psi_{W_3}) + \beta(\psi_{W_2} + \psi_{W_4}),$$

as long as $\alpha + \beta = 1$ (so that we may take, say, $\alpha < 0$). □

If γ satisfies (2.10) and $\gamma(W) \geqslant 0$ for all $W \in \mathscr{W}$, we may think either that ϕ is a convex combination of the ψ_{W_i}'s, or, in the spirit of Shapley's bargaining model, that the coefficient $\gamma(W)$ is the probability of the chain W being chosen. At any rate, when varying γ (keeping nonnegativity), we obtain what might be called the Weber set (Weber, 1988).

As already mentioned, **N**, **L** and **E** jointly do not guarantee uniqueness of the value (and much less of the coefficients γ), even if **IS** is required. Let us give some examples of axioms or properties we may require of the coefficients γ so as to obtain uniqueness.

1. All chains have equal weight or probability, $\gamma(W) = 1/|\mathscr{W}|$. This is the case of the Shapley value, when $\mathscr{K} = \mathscr{P}(N)$, and corresponds to an indifference principle, an idea which goes back to Laplace.
2. A chain has more weight if it is longer. For instance, if proportional to the length, we would set

$$\gamma(W) = \frac{\ell(W)}{\sum_{W' \in \mathscr{W}} \ell(W')}.$$

3. Conversely, we may ask it to have less weight if it is longer. For instance, we may take the harmonic mean of the cardinals,

$$\gamma(W) = \frac{1/\ell(W)}{\sum_{W' \in \mathscr{W}} (1/\ell(W'))}.$$

4. Borrowing from Information Theory, we could define the *entropy* of a chain $W = (K_0 = \emptyset, K_1, \ldots, K_\ell = N)$ by

$$H(W) = -\sum_{k=1}^{\ell} p_k \log_2 p_k,$$

where $p_k = |K_k \setminus K_{k-1}|/n$, and then weigh according to this index. For instance, substituting $H(W)$ for $\ell(W)$ in models 2 or 3.

Notice that in all these models we have $\gamma(W) \geqslant 0$, and that the corresponding values coincide if $\ell(W) = n$ for all $W \in \mathscr{W}$. In particular, in all of them we get back the Shapley value, ϕ^S, if $\mathscr{K} = \mathscr{P}(N)$.

The next example illustrates the differences between these models.

Example 2.12. Consider $\mathscr{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{1,2\}, K_3 = \{3\}, N = \{1,2,3\}\}$, having just two maximal chains, $W_1 = (\emptyset, K_1, K_2, N)$ and $W_2 = (\emptyset, K_3, N)$. Then:

$$\gamma(W_1) = \gamma(W_2) = 0.5, \quad \text{in model 1},$$
$$\gamma(W_1) = 0.6, \quad \gamma(W_2) = 0.4, \quad \text{in model 2},$$
$$\gamma(W_1) = 0.4, \quad \gamma(W_2) = 0.6, \quad \text{in model 3},$$

and, assuming γ proportional to the entropies, in model 4 we have

$$H(W_1) \simeq 1.58496, \quad H(W_2) \simeq 0.918296,$$
$$\gamma(W_1) \simeq 0.63316, \quad \gamma(W_2) \simeq 0.36684. \quad \square$$

Example 3.2 shows another possibility when $\mathscr{K}$ has a particular structure, in which we still have $\gamma \geqslant 0$, but is not included in any of the models above.

2.2. The potential model of Lange and Grabisch

Taking a somewhat different route from the previous subsection, we follow now Lange and Grabisch (2009), extending their potential flow model. They obtained uniqueness of the value for regular families by adding the following:

Axiom 2.13 (*R*). A value $\phi : \mathscr{V} \to \mathbb{R}^n$ represented in the form (2.14) satisfies the *regularity axiom* if

$$\sum_{(K,K')\in W} \Lambda(K,K'), \tag{2.15}$$

is constant for all $W \in \mathscr{W}$, i.e, it is independent of $W \in \mathscr{W}$.

Recall that a flow $f : \mathscr{A} \to \mathbb{R}$ is *potential* if there exists $P : \mathscr{K} \to \mathbb{R}$, the *potential of the flow*, such that (in our setting)

$$f(K,K') = P(K') - P(K) \quad \text{for all } (K,K') \in \mathscr{A}. \tag{2.16}$$

The potential P is not unique, since $P + c$ is also a potential for any constant c, so it is usual to fix the potential at a point, for instance, by asking $P(\emptyset) = 0$. For $K \in \mathscr{K}$, let us set

$$\delta^+(K) = \{K' \in \mathscr{K} : (K,K') \in \mathscr{A}\},$$
$$\delta^-(K) = \{K' \in \mathscr{K} : (K',K) \in \mathscr{A}\},$$
$$\delta(K) = \delta^+(K) \cup \delta^-(K),$$

that is, $\delta(K)$ is the set of neighbors of K in $G^\circ_{\mathscr{K}}$, and let

$$\mathrm{d}^-(K) = |\delta^-(K)|, \quad \mathrm{d}^+(K) = |\delta^+(K)|, \quad \mathrm{d}(K) = |\delta(K)|,$$

be the corresponding degrees.

The following result shows that the value of the potential at an inner node is the average of the potentials of its neighbors, reminiscent of the behavior of harmonic functions of continuous variables.

Lemma 2.14. *If f is a flow in $G_{\mathscr{K}}$ with potential P, then*

$$P(K) = \frac{1}{\mathrm{d}(K)} \sum_{K'\in\delta(K)} P(K') \quad \text{for all } K \neq \emptyset,\ N. \tag{2.17}$$

Proof. Since f is a flow, $\sum_{(K',K)\in\mathscr{A}} f(K',K) = \sum_{(K,K')\in\mathscr{A}} f(K,K')$ for all $K \in \mathscr{K}^{**}$, or, using (2.16),

$$\sum_{(K',K)\in\mathscr{A}} (P(K) - P(K')) = \sum_{(K,K')\in\mathscr{A}} (P(K') - P(K)) \quad \text{for all } K \in \mathscr{K}^{**}.$$

In turn, these equations are equivalent to

$$\mathrm{d}(K)P(K) - \sum_{K'\in\delta(K)} P(K') = 0 \quad \text{for all } K \in \mathscr{K}^{**}. \quad \square \tag{2.18}$$

Lemma 2.15. *Given a family $\mathscr{K}$ and $\alpha \in \mathbb{R}$, there exists a unique potential flow whose potential P satisfies $P(\emptyset) = 0$ and $P(N) = \alpha$.*

Proof. The equations (2.18) and the conditions $P(\emptyset) = 0$ and $P(N) = \alpha$, determine a linear system of the form

$$(D - A)P = b, \tag{2.19}$$

where A is the adjacency matrix of the graph $\widehat{G}$ induced by $\mathscr{K}^{**}$ in $G^\circ_{\mathscr{K}}$, and D is a diagonal matrix with $D_{KK} = \mathrm{d}(K)$. $D - A$ is diagonally dominant, since

$$\sum_{K'\in\mathscr{K}^{**}} A_{KK'} = |\delta(K) \cap \mathscr{K}^{**}| \leqslant \mathrm{d}(K) \quad \text{for all } K \in \mathscr{K}^{**},$$

with strict inequality if $\emptyset \in \delta(K)$ or $N \in \delta(K)$. Hence, if $\widehat{G}$ is connected, the matrix $A - D$ is irreducible and therefore invertible (see e.g., Ortega, 1990), and (2.19) has a unique solution P. If $\widehat{G}$ is not connected (as in Example 2.12), we repeat this procedure for each component. The equations (2.18), equivalent to those in (2.17), ensure that (2.16) defines a flow. $\square$

By choosing a suitable constant α and using linearity, we obtain:

Corollory 2.16. *Given a family $\mathscr{K}$, there exists a unique potential flow of value 1 (and $P(\emptyset) = 0$).*

Lemma 2.17. *Let ϕ be representable in the form (2.14). Then, ϕ satisfies* **R** *if and only if Λ is a potential flow, i.e., there exists $P : \mathscr{K} \to \mathbb{R}$ such that $P(\emptyset) = 0$ and*

$$\Lambda(K, K') = P(K') - P(K) \quad \textit{for all } (K, K') \in \mathscr{A}. \tag{2.20}$$

Proof. Assume ϕ is in the form (2.14) and satisfies **R**. For $K \in \mathscr{K}$, let us consider a chain W such that $K \in W$, and define

$$P(K) = \sum_{\substack{(K',K'')\in W \\ K''\subset K}} \Lambda(K', K'').$$

$P(K)$ does not depend on the chosen chain, since if W' is another chain with $K \in W'$, and we consider the chain W'' which coincides with W' up to K and then continues as in W, using **R** we would have

$$\sum_{(K',K'')\in W} \Lambda(K', K'') = \sum_{(K',K'')\in W'} \Lambda(K', K'') = \sum_{(K',K'')\in W''} \Lambda(K', K''),$$

and since W and W'' coincide from K on, and W' and W'' up to K,

$$\sum_{\substack{(K',K'')\in W \\ K''\subset K}} \Lambda(K', K'') = \sum_{\substack{(K',K'')\in W'' \\ K''\subset K}} \Lambda(K', K'') = \sum_{\substack{(K',K'')\in W' \\ K''\subset K}} \Lambda(K', K'').$$

So, if $(K, K') \in \mathscr{A}$, we may choose a chain containing K and K', obtaining $P(K') = P(K) + \Lambda(K, K')$. Conversely, if ϕ may be written in the form (2.14) and there exists a function P such that (2.20) holds, the sum in (2.15) is telescopic and reduces to $P(N) - P(\emptyset) = P(N)$ for every $W \in \mathscr{W}$. □

Theorem 2.10, Corollary 2.16 and Lemma 2.17 imply the following result (cf. Lange and Grabisch, 2009, Theorem 6):

Theorem 2.18. *Given a family $\mathscr{K}$, there exists a unique ϕ satisfying* **N**, **L**, **E**, **IS** *and* **R**. *Moreover, ϕ may be written in the form (2.14), with Λ defined by (2.20), where P is a potential satisfying $P(\emptyset) = 0$ and $P(N) = 1$.*

Proof. To see the existence, Corollary 2.16 shows that there exists a potential flow of value 1 on $G_{\mathscr{K}}$, Theorem 2.10 then shows that this flow induces a value ϕ satisfying **N**, **L**, **E** and **IS**, and representable in the form (2.14), so that we may apply Lemma 2.17 to show that it also satisfies **R** and satisfies (2.20).

To see the uniqueness, if ϕ satisfies **N**, **L**, **E** and **IS**, Theorem 2.10 implies that it is representable in the form (2.14), and since ϕ satisfies **R**, Lemma 2.17 shows that the corresponding flow Λ is a potential flow. The uniqueness now follows from Corollary 2.16. □

Let us make a few comments on this model.

- Although there are some similarities between this model and that of Hart and Mas-Colell (1989), the latter is a potential on the value for a given game, and the former is a potential on the coefficients (the flow).
- Lange and Grabisch (2009) give an example of their model on regular families in which some of the coefficients Λ are negative, and hence some of the coefficients γ (by (2.11)), unlike the models we presented in Section 2.1. Of course, the corresponding value in their example is not strongly monotone in the sense of Young (1985) (see Remark 2.9).
- Furthermore, the potential model applied to the family $\mathscr{K}$ in Example 2.11, yields $\Lambda(K, K') = 1/2$ for all $(K, K') \in \mathscr{A}$. Therefore, the coefficients γ of Section 2.1 are not uniquely defined in general.
- Comparing with the models in Section 2.1, in the potential flow model some arcs and chains with fewer information could receive a larger weight. For instance, when applied to Example 2.12, it coincides with the harmonic means model 3.

3. Unanimity games

If ϕ satisfies **L** and **EU**, then

$$\phi_i\left(\sum_{T\in\mathscr{K}^*} a_T U_T\right) = \sum_{T\in\mathscr{K}^*} a_T \phi_i(U_T) = \sum_{T\in\mathscr{K}^*} \frac{a_T}{|T|}\chi(T, i) \quad \text{for all } i \in N,$$

and since unanimity games form a basis of $\mathscr{V}$, we have:

Theorem 3.1. *There exists a unique value, ϕ^U, satisfying* **L** *and* **EU**, *and it is given by*

$$\phi_i^U\left(\sum_{T\in\mathscr{K}^*} a_T U_T\right) = \sum_{T\in\mathscr{K}^*} \frac{a_T}{|T|}\chi(T, i) \quad \textit{for all } i \in N. \tag{3.1}$$

Moreover, ϕ^U satisfies **E** *and* **EE**.

Notice that:

- The expression (3.1) does not involve explicitly any arcs of $G_{\mathscr{K}}$ (it is not in marginalist form), and it is well defined as long as there is a nonempty coalition in the family. In particular, $N \in \mathscr{K}$ is not needed (but then **E** does not make much sense).
- ϕ^U is the restriction of the Shapley value ϕ^S to the subspace generated by $\{U_T : T \in \mathscr{K}, T \neq \emptyset\}$. That is, if for $T \in \mathscr{K}^*$ and $K \in \mathscr{P}(N)$ we let

$$\overline{U}_T(K) = \begin{cases} 1 & \text{if } T \subset K, \\ 0 & \text{otherwise}, \end{cases}$$

and for $v = \sum_{T \in \mathscr{K}^*} a_T U_T$ we let $\bar{v} = \sum_{T \in \mathscr{K}^*} a_T \overline{U}_T$, which is defined on $\mathscr{P}(N)$, then $\phi^U(v) = \phi^S(\bar{v})$.

- Hence, for any $v \in \mathscr{V}$, we may find a (unique) potential for $\phi^U(v)$ along the lines of Hart and Mas-Colell (1989), looking at the potential for $\bar{v}$.
- Hart and Mas-Colell (1989, p. 605) consider also a monotonicity property, which ϕ^U satisfies.
- In general, ϕ^U is not marginalist in the sense of (1.3), since it does not satisfy the null player axiom (consider $\mathscr{K} = \{\emptyset, \{1\}, N = \{1,2\}\}$ and $v = U_N$). However, through the extensions mentioned above, we do obtain a different kind of marginalism.

Let us give now some examples in which ϕ^U is marginalist.

Example 3.2. Consider $\mathscr{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2,3\}, N = \{1,2,3\}\}$, whose maximal chains are $W_1 = (\emptyset, K_1, N)$ and $W_2 = (\emptyset, K_2, N)$. ϕ^U is marginalist, since it may be checked that (recall (2.8)):

$$\phi^U(v) = \frac{2}{3}\psi_{W_1}(v) + \frac{1}{3}\psi_{W_2}(v),$$

so that, with the notations of Theorem 2.10,

$$\gamma(W_1) = \frac{2}{3} \quad \text{and} \quad \gamma(W_2) = \frac{1}{3}.$$

However, it does not fit in any of the models 1, 2, or 3 of Section 2.1, since both chains have length 2, but W_1 has weight 2/3 and W_2 has weight 1/3. It does not fit the entropy model 4 either, since both chains have an arc of length 1 and another of length 2, and hence equal entropy. Lastly, it does not fit the potential model of Section 2.2, since summing the flows along W_1 we obtain 4/3, and summing the flows along W_2 we obtain 2/3.

Since players 2 and 3 are equivalent in $\mathscr{K}$, we may consider $\overline{N} = \{\bar{1}, \bar{2}\}$, identify $\bar{1}$ with $\{1\}$, $\bar{2}$ with $\{2,3\}$, $\mathscr{P}(\bar{N})$ with $\mathscr{K}$, and any game $\bar{v}$ on $\mathscr{P}(\bar{N})$ with a game v on $\mathscr{K}$. Hence, if Φ is the Shapley value on $\overline{N}$, it is natural to define ϕ on $\mathscr{K}$ by

$$\phi_1(v) = \Phi_{\bar{1}}(\bar{v}), \quad \phi_2(v) = \phi_3(v) = \frac{1}{2}\Phi_{\bar{2}}(\bar{v}).$$

Thus, $\phi^U_2(U_N) = \phi^U_3(U_N) = 1/3$, whereas $\phi_2(U_N) = \phi_3(U_N) = 1/4$.

The previous example is easily generalized. If $\sim$ denotes the equivalence relation between players, we may consider any set N and $\mathscr{K}$ so that if $\overline{N} = N/\sim$, then $\mathscr{K}$ is isomorphic to $\mathscr{P}(\bar{N})$. In this case, ϕ^U is marginalist, and ϕ^S on $\overline{N}$ induces a value on $\mathscr{K}$ which differs from ϕ^U.

It is not the case that whenever ϕ^U is marginalist then $\mathscr{K}$ is isomorphic to $\mathscr{P}(\bar{N})$ for some $\overline{N}$, as may be readily seen by taking $n = 3$, $\mathscr{K} = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, N\}$, in which there are no equivalent players and $|\mathscr{K}| = 5$. □

Our next goal is to find conditions, necessary and/or sufficient, for the marginalism of ϕ^U. We will find it useful to refer to the following equation: if ϕ^U is marginalist and is written in the form (1.3), then

$$\phi^U_i(U_T) = \sum_{\substack{(K,K') \in \mathscr{A}_i \\ T \not\subset K, T \subset K'}} \lambda_i(K,K') = \frac{1}{|T|}\chi(T,i) \quad \text{for } T \in \mathscr{K}^* \quad \text{and} \quad i \in N, \tag{3.2}$$

with the understanding that a sum is zero if it has no terms.

Lemma 3.3. *Suppose ϕ^U is marginalist and written in the form (1.3). Then, for all $i \in N$:*

(i) *there exists a unique $K \in \mathscr{K}$ such that $(K,N) \in \mathscr{A}_i$, and,*
(ii) *$\lambda_i(K,N) = 1/n$ for such K.*

Proof. (3.2) reduces to

$$\phi^U_i(U_N) = \sum_{(K,N) \in \mathscr{A}_i} \lambda_i(K,N) = \frac{1}{n} \quad \text{for all } i \in N. \tag{3.3}$$

At least one term in the sum does not vanish, and so $N \in \mathscr{K}^-_i$ for every $i \in N$. If $(\emptyset, N) \in \mathscr{A}$ (i.e, $\mathscr{A}$ has just one arc), (3.3) implies $\lambda_i(\emptyset, N) = 1/n$, and the result is proved. Suppose now $(\emptyset, N) \notin \mathscr{A}$. For fixed $i \in N$, let $(R,N) \in \mathscr{A}_i$ (and hence, $i \notin R$). Applying (3.2) for $T = N$ and $T = R$ we obtain

$$\lambda_i(R,N) + \sum_{\substack{(R',N) \in \mathscr{A}_i \\ R' \neq R}} \lambda_i(R',N) = \frac{1}{n} \quad \text{and} \quad \sum_{\substack{(R',N) \in \mathscr{A}_i \\ R' \neq R}} \lambda_i(R',N) = 0, \tag{3.4}$$

and therefore $\lambda_i(R,N) = 1/n$. Since this holds for any $(R,N) \in \mathscr{A}_i$, then there is only one such arc (otherwise the first sum in (3.4) would exceed $1/n$), and the result follows. □

Our next aim is to try to extend Lemma 3.3 for all $T \in \mathscr{K}^*$. However, we will need some extra hypotheses.

Theorem 3.4. *Suppose ϕ^U may be written in the form (1.3), and either of the following is satisfied:*

H-1. $\lambda_i(K,K') \geqslant 0$ *for all* $(K,K') \in \mathscr{A}_i$,
H-2. *for each* $T \in \mathscr{K}^*$ *and* $i \in T$, *there exists at most one arc of the form* (T',T) *in* $\mathscr{A}_i$.

Then,

(i) *for all* $T \in \mathscr{K}^*$ *there exists* $\alpha(T) > 0$ *such that*

$$\sum_{K\in\mathscr{K}:T\subset K} \alpha(K) = \frac{1}{|T|},$$

(ii) *for all* $i \in T$ *there exists a unique* $K \in \mathscr{K}$ *such that* $(K,T) \in \mathscr{A}_i$, *and* $\lambda_i(K,T) = \alpha(T)$ *for such K.*

Remark 3.5. Notice that if $\mathscr{K} = \mathscr{P}(N)$, (1.2) yields

$$\alpha(T) = \frac{(|T|-1)!\,(n-|T|)!}{n!} \quad \text{for } T\neq\emptyset. \quad \square$$

Remark 3.6. Notice also that the conclusions of Theorem 3.4 imply that the representation of ϕ^U in the form (1.3) is unique, by Theorem 2.6. $\square$

Proof of Theorem 3.4. Let A be the subset of coalitions $T \in \mathscr{K}$ such that:

A-1. If $T\neq\emptyset$, then for all $K \in \mathscr{K}$ with $T \subset K, \alpha(K) > 0$ is defined, and

$$\sum_{K:T\subset K} \alpha(K) = \frac{1}{|T|} \quad \text{and} \quad \alpha(T) = \sum_{(K,T)\in\mathscr{A}_i} \lambda_i(K,T) \quad \text{for all } i \in T,$$

(in particular, $T \in \mathscr{K}_i^-$ for all $i \in T$,)
A-2. if $T\neq N$, for all $K' \in \mathscr{K}$ with $T\subsetneq K'$, K' satisfies A-1 and

$$\alpha(K') = \lambda_i(K,K') \quad \text{if } (K,K') \in \mathscr{A}_i \quad \text{and} \quad T \subset K.$$

By Lemma 3.3, we know that $N \in A$, and we will proceed in a breadth first manner, coming down from N, to show that $A = \mathscr{K}$. Let $S \in \mathscr{K}, S\neq N$, be such that $\{T \in \mathscr{K} : S\subsetneq T\} \subset A$, and let us show that necessarily $S \in A$, by considering the alternatives $S = \emptyset$ and $S\neq\emptyset$. If $S = \emptyset$, we only have to verify A-2. For $K \in \delta^+(\emptyset)$, there is only one arc ending in K and K satisfies A-1 since it is in A. Hence, $\alpha(K) = \lambda_i(\emptyset,K)$ for all $K \in \delta^+(\emptyset)$ and all $i \in K$. Thus, if $S = \emptyset$ then $S \in A$. Suppose now $S\neq\emptyset$. To see that $S \in A$, it will be enough to show that if we define

$$\alpha(S) = \frac{1}{|S|} - \sum_{K:T\subsetneq K} \alpha(K), \tag{3.5}$$

then $\alpha(S) > 0$,

$$\alpha(S) = \sum_{(K,S)\in\mathscr{A}_i} \lambda_i(K,S) \quad \text{for all } i \in S, \tag{3.6}$$

and

$$\alpha(K) = \lambda_i(S,K) \quad \text{for all } i \quad \text{and} \quad K \text{ such that } (S,K) \in \mathscr{A}_i. \tag{3.7}$$

Let us start by showing that (3.7) holds, recalling that $S\neq N$. If $i \notin S$, (3.2) simplifies to (recall that a sum is 0 if it has no terms):

$$0 = \phi_i^U(U_S) = \sum_{K':S\subsetneq K'} \sum_{\substack{(K,K')\in\mathscr{A}_i \\ S\not\subset K}} \lambda_i(K,K') = \sum_{K':(S,K')\in\mathscr{A}_i} \sum_{\substack{(K,K')\in\mathscr{A}_i \\ S\not\subset K}} \lambda_i(K,K') + \sum_{\substack{K':S\subsetneq K' \\ K'\notin\delta^+(S)}} \sum_{\substack{(K,K')\in\mathscr{A}_i \\ S\not\subset K}} \lambda_i(K,K'). \tag{3.8}$$

If K' is such that $(S,K') \in \mathscr{A}_i$, then K' satisfies A-1, and we may write

$$\sum_{\substack{(K,K')\in\mathscr{A}_i \\ S\not\subset K}} \lambda_i(K,K') = \alpha(K') - \lambda_i(S,K'),$$

which is nonnegative under either H-1 or H-2. Similarly, if $K' \in \mathscr{K}, S\subsetneq K', (S,K') \notin \mathscr{A}$, and $(K,K') \in \mathscr{A}_i$, under H-1 we must have $\lambda_i(K,K') \geqslant 0$, and under H-2, since K' satisfies A-1, we must have that $\lambda_i(K,K')$ is either 0 or $\alpha(K')$ (which is positive), so that in this case also $\lambda_i(K,K') \geqslant 0$. In all, (3.8) becomes

$$0 = \sum_{K':(S,K')\in\mathscr{A}_i} (\alpha(K') - \lambda_i(S,K')) + \sum_{\substack{K':S\subsetneq K' \\ K'\notin\delta^+(S)}} \sum_{\substack{(K,K')\in\mathscr{A}_i \\ S\not\subset K}} \lambda_i(K,K'). \tag{3.9}$$

Since the intervening terms on the right hand side are nonnegative, all the terms in the double sum vanish and $\alpha(K') = \lambda_i(S,K')$ if $(S,K') \in \mathscr{A}_i$, which implies (3.7).

To see that (3.6) is satisfied, consider $i \in S$ $(S\neq\emptyset)$.

For K' such that $S\subsetneq K'$ there are no arcs $(K,K')\in\mathscr{A}_i$ with $S\subset K$. Thus, A-1 applied to $K'\in A$ yields

$$\sum_{\substack{(K,K')\in\mathscr{A}_i\\ S\not\subset K}}\lambda_i(K,K')=\sum_{(K,K')\in\mathscr{A}_i}\lambda_i(K,K')=\alpha(K'),$$

so that (3.2) becomes

$$\phi_i^U(U_S)=\frac{1}{|S|}=\sum_{(K,S)\in\mathscr{A}_i}\lambda_i(K,S)+\sum_{T:S\subsetneq T}\alpha(T),$$

and therefore (by the definition of $\alpha(S)$ in (3.5))

$$\alpha(S)=\frac{1}{|S|}-\sum_{T:S\subsetneq T}\alpha(T)=\sum_{(K,S)\in\mathscr{A}_i}\lambda_i(K,S),\qquad(3.10)$$

which is independent of $i\in S$, and (3.6) is satisfied.

It remains to be seen that $\alpha(S)>0$.

Recall that, by Theorem 2.7, the coefficients $\lambda_i(K,K')$ associated with ϕ^U (which we are assuming marginalist), define a flow on $G_{\mathscr{K}}$. Using (3.10), the equality of incoming and outgoing flows through S $(S\neq\emptyset,N)$, and that K satisfies A-1 if $(S,K)\in\mathscr{A}$,

$$\alpha(S)\left|\{i:S\in\mathscr{K}_i^-\}\right|=\sum_{i:S\in\mathscr{K}_i^-}\alpha(S)=\sum_{i:S\in\mathscr{K}_i^-}\sum_{(K,S)\in\mathscr{A}_i}\lambda_i(K,S)=\sum_{(K,S)\in\mathscr{A}}\sum_{i:(K,S)\in\mathscr{A}_i}\lambda_i(K,S)=\sum_{(S,K)\in\mathscr{A}}\sum_{i:(S,K)\in\mathscr{A}_i}\lambda_i(S,K)=\sum_{(S,K)\in\mathscr{A}}\alpha(K)|\{i:(S,K)\in\mathscr{A}_i\}|,$$

and so $\alpha(S)>0$. Consequently, S satisfies A-1 and $S\in A$. Suppose $A\neq\mathscr{K}$, and consider $S\in\mathscr{K}\setminus A$ which is maximal for inclusion. Then, by what we have just proved, we must have $S=N$, which is a contradiction as $N\in A$. Hence, $A=\mathscr{K}$, and in particular, $\emptyset\in A$. Therefore, for every $T\in\mathscr{K}^*$:

(a) $\alpha(T)$ is defined, $\alpha(T)>0$,
(b) $\sum_{K:T\subset K}\alpha(K)=\frac{1}{|T|}$,
(c) $\alpha(T)=\sum_{(K,T)\in\mathscr{A}_i}\lambda_i(K,T)$ for all $i\in T$,
(d) $\alpha(T)=\lambda_i(K,T)$ if $(K,T)\in\mathscr{A}_i$.

(a) coupled with (b) imply (i), and (a), (c) and (d) imply (ii). □

Recall that $\mathscr{K}$ is regular (Honda and Grabisch, 2006) if $|K\setminus K'|=1$ for all $(K',K)\in\mathscr{A}$. Thus, if a family is regular then for every $K\in\mathscr{K}^*$ and every $(K',K)\in\mathscr{A}$, there is a unique i such that $(K',K)\in\mathscr{A}_i$. Theorem 3.4 says that (under certain conditions) if ϕ^U is marginalist then for every $K\in\mathscr{K}^*$ and every $i\in K$ there exists a unique K' such that $(K',K)\in\mathscr{A}_i$. These conditions are somewhat complementary, and the following result expresses that we cannot have them together unless $\mathscr{K}=\mathscr{P}(N)$.

Corollory 3.7. *If $\mathscr{K}$ is regular, then ϕ^U is marginalist if and only if $\mathscr{K}=\mathscr{P}(N)$.*

Proof. The classical result is that if $\mathscr{K}=\mathscr{P}(N)$ then $\phi^U=\phi^S$, and therefore marginalist.

On the other hand, if $\mathscr{K}$ is regular then it satisfies H-2, and we may use the conclusions of Theorem 3.4 and the regularity to see that for all $K\in\mathscr{K}^*$ and every $i\in K$, the set $K\setminus\{i\}$ is in $\mathscr{K}$. But this implies that $\mathscr{K}=\mathscr{P}(N)$ since $N\in\mathscr{K}$. □

The following result is an elaboration on the last sum in (3.9).

Corollory 3.8. *If ϕ^U is marginalist, and either H-1 or H-2 is satisfied, then $\mathscr{K}$ satisfies the following property:*

$$\text{If } R,T\in\mathscr{K}^* \text{ and } T\subset R, \text{then } \{R\setminus S:(S,R)\in\mathscr{A},T\not\subset S\} \text{ is a partition of } T.\qquad(\text{P}-1)$$

Proof. Using Theorem 3.4, we see that if $T\in\mathscr{K}^*,(S,R)\in\mathscr{A},T\not\subset S$, and $T\subset R$, then $R\setminus S\subset T$, since if $j\in R\setminus S$ and $j\notin T$, we would have

$$\phi_j^U(U_T)=0=\sum_{\substack{(K,K')\in\mathscr{A}_j\\ T\not\subset K,T\subset K'}}\lambda_j(K,K')=\sum_{\substack{(K,K')\in\mathscr{A}_j\\ T\subsetneq K'}}\alpha(K')\geqslant\alpha(R)>0.$$

Therefore, if $T\subset R,\bigcup_{\substack{S:T\not\subset S\\(S,R)\in\mathscr{A}}}R\setminus S\subset T$, and the union is disjoint, since if $i\in R$, there is only one arc of the form (S,R) in $\mathscr{A}_i$. Finally, by Theorem 3.4, for all $i\in T\subset R$ there exists an arc $(S,R)\in\mathscr{A}_i$. □

The following is a converse to Corollary 3.8:

Theorem 3.9. *If $\mathscr{K}$ satisfies P-1, then ϕ^U is marginalist.*

Proof. Let us define $\alpha:\mathscr{K}^*\to\mathbb{R}$ recursively by setting $\alpha(N)=1/n$, and for $S\in\mathscr{K}^{**}$,

$$\sum_{K:(S,K)\in\mathscr{A}}|K\setminus S|\alpha(K)=|S|\alpha(S),\qquad(3.11)$$

so that $\alpha(S)>0$ for all $S\in\mathscr{K}^*$. We will show that $\Lambda:\mathscr{A}\to\mathbb{R}$ defined by

$$\Lambda(K,K')=|K'\setminus K|\alpha(K'),\qquad(3.12)$$

is a flow of value 1.

The conservation of flow through $S \in \mathscr{K}^{**}$ is represented by the equality

$$\sum_{K:(S,K)\in\mathscr{A}} \Lambda(S,K) = \sum_{K:(K,S)\in\mathscr{A}} \Lambda(K,S),$$

which, given that P-1 is satisfied, written in terms of α is equivalent to (3.11):

$$\sum_{K:(S,K)\in\mathscr{A}} |K \setminus S|\alpha(K) = \sum_{K:(K,S)\in\mathscr{A}} |S \setminus K|\alpha(S) = |S|\alpha(S).$$

Also, since $\sum_{(K,N)\in\mathscr{A}} \Lambda(K,N) = |N|/n = 1$, the value of the flow Λ is 1. By Theorem 2.10 we obtain a marginalist value ϕ associated with Λ, such that

$$\lambda_i(K,K') = \frac{\Lambda(K,K')}{|K' \setminus K|} = \alpha(K') > 0 \quad \text{for all } (K,K') \in \mathscr{A}_i.$$

For $T \in \mathscr{K}^*$ and $i \in N$ we have

$$\phi_i(U_T) = \sum_{\substack{(K,K')\in\mathscr{A}_i \\ T \not\subset K, T\subset K'}} \alpha(K').$$

If $i \notin T$, the sum is 0 since the arcs (K,K') with $T \not\subset K$ and $T \subset K'$ form a partition of T, and the sum has no terms. If $i \in T$, $\phi_i(U_T) = 1/|T|$, as the flow through the cut $\{(K,K') : T \not\subset K, T \subset K'\}$ is

$$1 = \sum_{\substack{(K,K')\in\mathscr{A} \\ T \not\subset K, T\subset K'}} \Lambda(K,K') = \sum_{K':T\subset K'} \alpha(K') \sum_{\substack{(K,K')\in\mathscr{A} \\ T \not\subset K}} |K' \setminus K| = \sum_{K':T\subset K'} \alpha(K')|T| = |T|\phi_i(U_T).$$

Thus, $\phi^U = \phi$. □

Collecting the last results, we may say:

Corollary 3.10. *ϕ^U is marginalist with nonnegative marginal coefficients in (1.3) if and only if $\mathscr{K}$ satisfies P-1. In this case, the representation of ϕ^U in the form (1.3) is unique.*

The proof of Theorem 3.9 shows that if we relax the condition P-1 to

$$\text{For all } T \in \mathscr{K}^*, \{T \setminus S : (S,T) \in \mathscr{A}\} \text{ is a partition of } T, \tag{P–2}$$

then we have:

Lemma 3.11. *If $\mathscr{K}$ satisfies P-2, and α is defined as in (3.11), then the quantities Λ defined by (3.12) define a flow in $G_{\mathscr{K}}$ of value 1, and hence induce a value which is marginalist, efficient and satisfies* **IS**.

However, P-2 is not enough to enforce marginalism of ϕ^U as the following example shows:

Example 3.12. Let $n = 3$ and $\mathscr{K} = \{\emptyset, K_1 = \{1\}, K_2 = \{2\}, K_3 = \{3\}, K_4 = \{1,2\}, N\}$, which satisfies P-2. If ϕ^U were marginalist, the values of the flow Λ arriving to N would be determined, so that $\Lambda(\emptyset, K_3) = \Lambda(K_3, N) = 2/3$, and $\Lambda(K_4, N) = 1/3$. Also, by symmetry, $\Lambda(\emptyset, K_1) = \Lambda(K_1, K_4) = \Lambda(\emptyset, K_2) = \Lambda(K_2, K_4) = 1/6$. If $T = K_1$, then $\phi_2^U(U_T) = 1/3$ (even though $\phi_1^U(U_T) = 1$ and $\phi_3^U(U_T) = 0$). □

4. Concluding remarks

Our work shows that, when using classical axioms to define values similar to Shapley's on general subfamilies of coalitions, the resulting values will have quite different properties depending on the chosen set.

We presented here two broad frameworks.

The first one, based on **N**, **L** and **E**, gives rise to a marginalist value, with a rich mathematical theory behind, but an extra axiom or property is needed to obtain uniqueness (e.g., symmetry in the classical setting), for which we put forward several possibilities, all of them coinciding when $\mathscr{K} = \mathscr{P}(N)$.

In contrast, the second framework uses just **L** and **EU** and gives rise to a unique value, ϕ^U, which is efficient, but need not be marginalist.

Not surprisingly, the dissimilarities are paralleled by the choice of linear bases of $\mathscr{V}$.

ϕ^U has received little attention for general subfamilies, perhaps because typically it is not marginalist. As Corollary 3.7 shows, even for such general families as the regular ones, the marginalism of ϕ^U implies that the family contains all possible coalitions.

ϕ^U challenges the marginalist approach which has been predominant in the literature and is reflected in Section 2. More precisely, what is the "fair" value for the players in Example 3.2 when $v = U_N$?: ϕ^U assigns 1/3 to each, whereas we may think that since 2 and 3 act as a unit, player 1 should receive 1/2, and players 2 and 3 should receive 1/4 each.

We have not pursued the appropriateness or reasonability of any of the approaches we showed, as this paper is mathematically oriented and its only purpose is to exhibit some possibilities.

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