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Constructive Logic with Strong Negation as a Substructural Logic

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Abstract

Spinks and Veroff have shown that constructive logic with strong negation (CLSN for short), can be considered as a substructural logic. We use algebraic tools developed to study substructural logics to investigate some axiomatic extensions of CLSN. For instance, we prove that Nilpotent minimum logic is the extension of CLSN by the prelinearity axiom. This generalizes the well-known result by Monteiro and Vakarelov that three-valued Łukasiewicz logic is an extension of CLSN. A Glivenko-like theorem relating CLSN and three-valued Łukasiewicz logic is proved.

Keywords: Constructive logic, strong negation, nilpotent minimum logic, Nelson algebras, residuated lattices, Heyting algebras.

1 Introduction

Recently Spinks and Veroff [34, 35] have shown that constructive logic with strong negation (CLSN for short), can be considered as a substructural logic. More precisely, they showed that the algebraic models of constructive logic with strong negation, i.e. Nelson algebras, are termwise equivalent to certain involutive commutative integral residuated lattices, called *Nelson residuated lattices* or *Nelson lattices* for short. Nelson lattices form a variety $\mathcal{N}$, and this means that constructive logic with strong negation is an axiomatic extension of $\mathbf{FL_{ew}}$, the full Lambeck calculus with exchange and weakening [13, 24].

The Spinks–Veroff result paves the way for the application of algebraic techniques developed for the study of substructural logics to CLSN. Conversely, the well-known representation of Nelson algebras as pairs of disjoint elements of Heyting algebras given independently by Fidel [11] and Vakarelov [36] allows one to obtain results about Nelson lattices. The aim of this article is to contribute to this line of research.

In [34], the authors proved the equivalence between Nelson algebras and Nelson residuated lattices syntactically, by means of the automated theorem proving tool OTTER. In the first section after giving the necessary background on Nelson algebras and residuated lattices, we give an algebraic proof of the equivalence. Using a theorem of Sendlewski [33], which improves on the above-mentioned representation of Fidel and Vakarelov, we provide an algebraic proof of the fact that each Nelson algebra admits the structure of a Nelson lattice. As a byproduct we obtain a representation of Nelson lattices as pairs of disjoint elements of Heyting algebras. For the converse, we work in a more general framework: the variety $\mathcal{E}_2$, of involutive residuated lattices satisfying $x^2 = x^3$. We show that

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an algebra in $\mathcal{E}_2$ is a Nelson algebra if and only if a certain quasi equation holds (see Corollary 3.8). In this way we not only algebraically prove one of the main results of [34], but also simplify the original definition of Nelson lattices. We also express the equivalence in terms of category theory.

In Section 3, we compare the Fidel–Vakarelov construction of Nelson algebras from Heyting algebras with Jenei's [21] connected and disconnected rotations of generalized Heyting algebras. We show that both constructions coincide when applied to Heyting algebras with a meet irreducible bottom.

We begin Section 4 by considering the radical of algebras in a variety $\mathcal{K}_2$ that properly includes $\mathcal{N}$. We show that the semisimple algebras in $\mathcal{K}_2$ are the three-valued Łukasiewicz algebras. This result generalizes Monteiro's characterization of semisimple Nelson algebras [28]. In a second part, we show that in the case of Nelson residuated lattices the radical is the kernel of a homomorphism ϕ onto a three-valued Łukasiewicz algebra, extending results of [3]. The Nelson lattices $\mathbf{A}$ such that $\phi(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ form a variety. We call elements in this variety *regular Nelson lattices* and we show that they correspond to stonean Heyting algebras in Sendlewski's representation. We conclude the section establishing the already mentioned Glivenko-like theorem.

Section 5 is devoted to the study of some varieties of Nelson residuated lattices. In the first place, we show that the three-element Łukasiewicz chain is a splitting algebra in $\mathcal{N}$ and that its conjugate variety is formed by the Nelson residuated lattices $\mathbf{A}$ such that $\phi(\mathbf{A})$ is a Boolean algebra. We also show that this variety coincides with the variety of normal Nelson algebras defined by Goranko [17]. Second, we prove that the variety of regular Nelson residuated lattices is generated by the connected rotations of generalized Heyting algebras. In a third part, we prove that Nelson prelinear residuated lattices coincide with nilpotent minimum algebras (the algebras corresponding to the logic of Fodor's [12] left-continuous and not continuous t-norm on the real segment [0, 1]) [3, 10, 15, 38]. The equivalence between both varieties shows that nilpotent minimum logic is an axiomatic extension of CLSN. This equivalence relates for the first time CLSN and a logic based on a t-norm. We also show that prelinearity with respect to the residuated lattice implication is equivalent to prelinearity with respect to Nelson algebra implication. This implies that the lattice of subvarieties of the variety of prelinear Nelson algebras [25] is the same as the lattice of subvarieties of the variety of nilpotent minimum algebras [15]. In the fourth part, we obtain some results concerning the variety of Nelson lattices that satisfies the nilpotent minimum equation without prelinearity. Finally, we establish relations among subvarieties of $\mathcal{N}$ that we have considered in the article.

In the last section, we apply techniques developed in [9] and [3] to give Boolean product representations of free algebras in subvarieties of regular Nelson residuated lattices.

2 Preliminaries

2.1 Preliminaries on residuated lattices

Recall that an *integral residuated lattice-ordered commutative monoid*, or *residuated lattice* for short, is an algebra $\mathbf{A} = (A, \vee, \wedge, *, \to, \top)$ of type $(2,2,2,2,0)$ such that $\langle A, *, \top \rangle$ is a commutative monoid, $\mathbf{L}(\mathbf{A}) = \langle A, \vee, \wedge, \top \rangle$ is a lattice with greatest element $\top$, and the following residuation condition holds:

$$x * y \leq z \text{ if and only if } x \leq y \to z, \tag{1}$$

where x, y, z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure.

Although we are assuming familiarity with the theory of residuated lattices, as developed for instance in [20, 24],[1] we list some well-known properties for further reference.

Under the assumption of integrality (which means that the neutral element of the monoid reduct coincides with the greatest element of $\mathbf{L}(\mathbf{A})$) one has that

$$x \leq y \text{ if and only if } x \to y = \top, \tag{2}$$

$$x * y \leq x \wedge y, \tag{3}$$

$$\text{if } x \leq y, \text{ then } \; x*z \leq y*z, \;\; y \to z \leq x \to z \text{ and } z \to x \leq z \to y. \tag{4}$$

Residuated lattices form a variety. Indeed, the residuation condition can be replaced by the following identities [24]:

RL_1 $(x*y) \to z = x \to (y \to z)$,
RL_2 $(x*(x \to y)) \vee y = y$,
RL_3 $(x \wedge y) \to y = \top$.

A *bounded* residuated lattice (or $\mathrm{FL}_{\mathrm{ew}}$-algebra in the nomenclature of [13]) is a residuated lattice equipped with a constant $\bot$ that is the bottom of the induced lattice structure. In this case, $\bot$ turns out to be an absorbing element for $*$ and a derived unary operation $\neg$ is defined by $\neg x = x \to \bot$. As usual this operation is called the *negation operation* and an element x satisfying $x = \neg x$ is called *a negation fixpoint*.

An *implicative filter* (i-filter for short) of a bounded residuated lattice $\mathbf{A}$ is a subset $F \subseteq A$ such that $\top \in F$ and it is closed under modus ponens: $x \in F$ and $x \to y \in F$ imply $y \in F$. Implicative filters can also be characterized as subsets of A that are non-empty, upward closed and closed by $*$. If F is an implicative filter of $\mathbf{A}$ then it is the universe of a residuated sublattice of $\mathbf{A}$, i.e. F is a commutative integral residuated lattice (not necessarily bounded). Indeed, it only remains to check that F is closed by $\to$. But this is an easy consequence of integrality and residuation ($y \in F$ and $y \leq x \to y$ imply $x \to y \in F$).

For each i-filter F, the binary relation $\theta(F)$ defined by $(x,y) \in \theta(F)$ if and only if $x \to y \in F$ and $y \to x \in F$, is a congruence of the residuated lattice $\mathbf{A}$, and $F = \{x \in A : (x, \top) \in \theta(F)\}$. We shall write $\mathbf{A}/F$ to denote the quotient algebra $\mathbf{A}/\theta(F)$, and x/F to denote the equivalence class of an element $x \in A$. As a matter of fact, the correspondence $F \mapsto \theta(F)$ defines a one-one inclusion preserving correspondence between the implicative filters and the congruence relations of A. The inverse mapping is given by the correspondence $\theta \mapsto \top/\theta$.

A residuated lattice is called *involutive* if it is bounded and it satisfies the double negation equation:

$$x = \neg\neg x. \tag{5}$$

As a consequence of RL_1 in an involutive residuated lattice the operations $*$ and $\to$ are related as follows:

$$x * y = \neg(x \to \neg y), \tag{6}$$
$$x \to y = \neg(x * \neg y). \tag{7}$$

If x is an element of a residuated lattice $\mathbf{A}$, we define $x^1 = x$ and for each $n \geq 1$, $x^{n+1} = x^n * x$.

[1] Our main reference for residuated latices will be [24] because some of the results in the mentioned paper cannot be found in [13].

A *Nelson residuated lattice or simply Nelson lattice* is an involutive residuated lattice satisfying

$$((x^2 \to y) \wedge ((\neg y)^2 \to \neg x)) \to (x \to y) = \top. \tag{8}$$

REMARK 2.1
Let $\mathbf{A}$ be an involutive residuated lattice. By (4) we have that $x \to y \leq x^2 \to y$ and $(\neg y)^2 * x \leq \neg y * x$. Hence by (7), RL_1, (5) and (6), $x \to y = \neg(x * \neg y) \leq \neg((\neg y)^2 * x) = (\neg y)^2 \to \neg x$. Therefore, the equation $(x \to y) \to ((x^2 \to y) \wedge ((\neg y)^2 \to \neg x)) = \top$ holds in $\mathbf{A}$. Consequently, *Nelson lattices satisfy the equation*

$$x \to y = (x^2 \to y) \wedge ((\neg y)^2 \to \neg x).$$

THEOREM 2.2
Let $\mathbf{A}$ be a Nelson lattice. Then $\mathbf{A}$ satisfies 3-potency, i.e,

$$x^3 = x^2. \tag{9}$$

PROOF. Take $y = \neg x$ in (8). In the light of (2) we have

$$(x^2 \to \neg x) \wedge ((\neg\neg x)^2 \to \neg x) \leq x \to \neg x.$$

From (5) and the idempotency of $\wedge$ we obtain

$$x^2 \to \neg x \leq x \to \neg x,$$

and then, taking into account RL_1, we have

$$x^3 \to \bot = x^2 \to (x \to \bot) \leq x \to \neg x = x^2 \to \bot$$

which can be rewritten as $\neg x^3 \leq \neg x^2$. Therefore $\neg\neg x^3 \geq \neg\neg x^2$ and using once more (5) we obtain $x^3 \geq x^2$. Since by (4) the inequality $x^3 \leq x^2$ holds in any residuated lattice we can conclude that (9) holds in $\mathbf{A}$. ∎

2.2 *Preliminaries on Nelson algebras*

Recall that a Kleene algebra is an algebra $\mathbf{K} = (K, \wedge, \vee, \sim, \top, \bot)$ of type $(2,2,1,0,0)$ such that $(K, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice and for each $x, y \in K$ the following conditions are satisfied:

K_1 $\sim\sim x = x$,
K_2 $\sim(x \vee y) = \sim x \wedge \sim y$,
K_3 $x \wedge \sim x \leq y \vee \sim y$.

A *Nelson algebra* is an algebra $\mathbf{N} = (N, \vee, \wedge, \Rightarrow, \sim, \top, \bot)$ of type $(2,2,2,1,0,0)$ such that $(N, \vee, \wedge, \sim, \top, \bot)$ is a Kleene algebra, and the operation $\Rightarrow$ satisfies the following conditions, where x, y, z denote arbitrary elements of A:

NL_1 $x \Rightarrow x = \top$,
NL_2 $x \wedge (x \Rightarrow y) = x \wedge (\sim x \vee y)$,
NL_3 $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$,
NL_4 $x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z$.

This definition of Nelson algebras is due to Brignole and Monteiro [2], and provides an equational characterization of the N-lattices introduced by Rasiowa [31, 32].

A *generalized Heyting algebra* is a residuated lattice that satisfies $x * y = x \wedge y$. It is a *Heyting algebra* in case the lattice reduct is bounded. Clearly, i-filters in generalized Heyting algebras coincide with lattice filters.

Given a Heyting algebra $\mathbf{A}$, we shall denote by $D(\mathbf{A})$ the filter of dense elements of A, i.e. $D(\mathbf{A}) = \{x \in A : \neg x = \bot\}$. A filter F of $\mathbf{A}$ is said to be *Boolean* provided the quotient $\mathbf{A}/F$ is a Boolean algebra. It is well known and easy to check that *a filter F of the Heyting algebra $\mathbf{A}$ is Boolean if and only if $D(\mathbf{A}) \subseteq F$*. The Boolean filters of $\mathbf{A}$, ordered by inclusion, form a lattice, having the improper filter A as the greatest element and $D(\mathbf{A})$ as the smallest element.

In the course of the article, we will make use of the next representation of Nelson algebras in terms of Heyting algebras, due to Sendlewski [33] (see also [37, Capítulo 8]), which improves upon a representation that was obtained independently by Fidel [11] and Vakarelov [36].

Theorem 2.3 (Sendlewski)
Given a Heyting algebra $\mathbf{H} = (H, \vee, \wedge, \rightharpoonup, \top, \bot)$ and a Boolean filter F of $\mathbf{H}$ let

$$K(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}.$$

Then we have:

(i) $\mathbf{K}(\mathbf{H}, F) = (K(\mathbf{H}, F), \vee, \wedge, \Rightarrow, \sim, \bot, \top)$ is a Nelson algebra, when the operations are defined as follows:

$$(x, y) \vee (s, t) = (x \vee s, y \wedge t),$$

$$(x, y) \wedge (s, t) = (x \wedge s, y \vee t),$$

$$(x, y) \Rightarrow (s, t) = (x \rightharpoonup s, x \wedge t),$$

$$\sim (x, y) = (y, x),$$

$$\top = (\top, \bot), \qquad \bot = (\bot, \top).$$

(ii) Given a Nelson algebra $\mathbf{A}$, there is a Heyting algebra $\mathbf{H_A}$, unique up to isomorphisms, and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H_A}$ such that $\mathbf{A}$ is isomorphic to $\mathbf{K}(\mathbf{H_A}, F_{\mathbf{A}})$.

(iii) If F_1, F_2 are Boolean filters of $\mathbf{H}$, then $\mathbf{K}(\mathbf{H}, F_1)$ is a subalgebra of $\mathbf{K}(\mathbf{H}, F_2)$ if and only if $F_1 \subseteq F_2$.

(iv) If $\mathcal{V}$ is a variety of Nelson algebras, then the class $\mathcal{H}^{\mathcal{V}} := \{\mathbf{H_A} : \mathbf{A} \in \mathcal{V}\}$ is a variety of Heyting algebras. ■

The subscript in $\mathbf{H_A}$ will be dropped when the context makes it clear which algebra is meant. We will write $\mathbf{K}(\mathbf{H})$ instead of $\mathbf{K}(\mathbf{H}, H)$. With this notation, the mentioned result given by Fidel in [11] and by Vakarelov in [36] can be stated as a corollary of the previous theorem:

Corollary 2.4
For every Nelson algebra $\mathbf{A}$ there is a Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{K}(\mathbf{H})$.

Since every Boolean filter of a Heyting algebra contains the filter of dense elements, another consequence of item (iii) in Theorem 2.3 that will be necessary for later results is that if $\mathbf{A} \cong \mathbf{K}(\mathbf{H}, F)$ then $\mathbf{A}$ contains a subalgebra isomorphic to $\mathbf{K}(\mathbf{H}, D(\mathbf{H}))$.

REMARK 2.5
It is worth noticing that if x,y are elements of a Heyting algebra $\mathbf{H}$ satisfying $x\wedge y=\bot$ then $x\rightharpoonup y=\neg x$. Indeed, it is always the case $\neg x=x\rightharpoonup\bot\leq x\rightharpoonup y$, and in this particular case we also have $x\wedge(x\rightharpoonup y)\leq x\wedge y=\bot$ that provides the other inequality.

3 Nelson residuated lattices and Nelson algebras

In [35, 36], the authors prove that every Nelson algebra has an underlying structure of Nelson residuated lattice and vice versa. This result is obtained with the help of the automated theorem-proving device OTTER. We aim at proving the same result in a more intuitive way.

3.1 The residuated lattice structure of a Nelson algebra

THEOREM 3.1
Given a Nelson algebra $\mathbf{A}=(A,\vee,\wedge,\Rightarrow,\sim,\bot,\top)$ define the derived binary operations $*$ and $\to$ as follows:

$$x*y:=\sim(x\Rightarrow\sim y)\vee\sim(y\Rightarrow\sim x), \tag{10}$$

$$x\to y:=(x\Rightarrow y)\wedge(\sim y\Rightarrow\sim x). \tag{11}$$

Then the system $\mathbf{R}(\mathbf{A})=(A,\vee,\wedge,*,\to,\bot,\top)$ is a Nelson residuated lattice. Moreover $\neg x=x\to\bot=\sim x$ for each $x\in A$.

PROOF. In the light of (ii) in Theorem 2.3 we can assume that $\mathbf{A}=\mathbf{K}(\mathbf{H},F)$ for some Heyting algebra $\mathbf{A}$ and some Boolean filter F of $\mathbf{A}$. Then (10) and (11) take the following forms:

$$(x,y)*(s,t)=(x\wedge s,(x\rightharpoonup t)\wedge(s\rightharpoonup y)), \tag{12}$$

$$(x,y)\to(s,t)=((x\rightharpoonup s)\wedge(t\rightharpoonup y),x\wedge t). \tag{13}$$

Considering that for $x,y\in H$ we have that $y\leq x\rightharpoonup y$ and $x\vee(x\rightharpoonup y)\geq x\vee\neg x\in D(\mathbf{H})$, we can check that the set $K(\mathbf{H},F)$ is closed under the operations $*$ and $\to$ as defined by (12) and (13).

Now we check that $K(\mathbf{H},F)$, with the lattice operations, $\bot$ and $\top$ defined as in Theorem 2.3 and $*$ and $\to$ given by (12) and (13) becomes a bounded residuated lattice.[2] Since the identities $a\rightharpoonup(b\wedge c)=(a\rightharpoonup b)\wedge(a\rightharpoonup c)$ and $(a\wedge b)\rightharpoonup c=c\rightharpoonup(b\rightharpoonup c)$ hold in any Heyting algebra, one can see that:

$$\begin{gathered}((x,y)*(s,t))*(r,z)=(x\wedge s,(x\rightharpoonup t)\wedge(s\rightharpoonup y))*(r,z)\\ =((x\wedge s)\wedge r,(x\wedge s)\rightharpoonup z\wedge(r\rightharpoonup(x\rightharpoonup t)\wedge(s\rightharpoonup y)))=\\ ((x\wedge s)\wedge r,((x\wedge s)\rightharpoonup z)\wedge((r\wedge x)\rightharpoonup t)\wedge((r\wedge s)\rightharpoonup y)),\end{gathered}$$

and

$$\begin{gathered}(x,y)*((s,t)*(r,z))=(x,y)*(s\wedge r,(s\rightharpoonup z)\wedge(r\rightharpoonup t))\\ =(x\wedge(s\wedge r),(x\rightharpoonup((s\rightharpoonup z)\wedge(r\rightharpoonup t))\wedge(s\wedge r)\rightharpoonup y))\\ ((x\wedge s)\wedge r,((x\wedge s)\rightharpoonup z)\wedge((x\wedge r)\rightharpoonup t)\wedge((s\wedge r)\rightharpoonup y))\end{gathered}$$

Thus, $*$ is an associative operation over A and $(A,*,\top)$ is a commutative monoid.

[2] For a similar construction in a more general setting see [5].

It is easy to see that $(A,\wedge,\vee,\bot,\top)$ is a bounded lattice. It remains to corroborate condition (1). This amounts to seeing that for elements $(x,y),(s,t),(r,z)\in A$ one has

$$(x\wedge s,(x\rightharpoonup t)\wedge(s\rightharpoonup y))\leq(r,z) \text{ if and only if } (x,t)\leq((s\rightharpoonup r)\wedge(z\rightharpoonup t),s\wedge z)$$

which is equivalent to

$$x\wedge s\leq r \text{ and } z\leq(x\rightharpoonup t)\wedge(s\rightharpoonup y) \text{ if and only if } x\leq(s\rightharpoonup r)\wedge(z\rightharpoonup t) \text{ and } s\wedge z\leq t.$$

Observe that if $x\wedge s\leq r$ and $z\leq(x\rightharpoonup t)\wedge(s\rightharpoonup y)$, then $x\leq s\rightharpoonup r$, and since $z\leq x\rightharpoonup t$ we have $x\leq z\rightharpoonup t$. These inequalities imply $x\leq(s\rightharpoonup r)\wedge(z\rightharpoonup t)$. Since it is also the case $z\leq(s\rightharpoonup y)$ we finally obtain $s\wedge z\leq y$. The converse implication follows in a similar manner.

The residuated lattice is involutive. As a matter of fact, since $x\wedge y=\bot$ implies $y\leq\neg x$, we have that $\neg(x,y)=(x,y)\rightarrow(\bot,\top)=(y,x)=\sim(x,y)$. To prove that (8) is satisfied, note first that by Remark 2.5 $(x,y)^2=(x,\neg x)$. Therefore

$$((x,y)^2\rightarrow(s,t))\wedge((\sim(s,t))^2\rightarrow\sim(x,y))=$$
$$((x,\neg x)\rightarrow(s,t))\wedge((t,\neg t)\rightarrow(y,x))=$$
$$(x\rightharpoonup s\wedge t\rightharpoonup\neg x\wedge t\rightharpoonup y\wedge x\rightharpoonup\neg t,x\wedge t)\leq(x,y)\rightarrow(s,t)$$

as desired. Consequently, we have shown that each Nelson algebra satisfies all the equations characterizing Nelson residuated lattices. ∎

REMARK 3.2
The above theorem can be proved in this alternative way: as done in [34] with the help of the automated prover device, check that if $\mathbf{A}=(A,\vee,\wedge,\Rightarrow,\sim,\bot,\top)$ is a Nelson algebra, after defining $x\rightarrow y:=(x\Rightarrow y)\wedge(\sim y\Rightarrow\sim x)$ the system $(A,\rightarrow,\bot,\top)$ is an involutive BCK-algebra. If the binary operation $*$ is defined on an involutive BCK-algebra by the prescription: $x*y=\neg(x\rightarrow\neg y)$, then the algebra $(A,\rightarrow,*,\top,\bot)$ is an involutive pocrim [8, Theorem 2.1]. Moreover, with the lattice operations defined as in Theorem 2.3, the mentioned system becomes an involutive residuated lattice, since the operations $\vee$ and $\wedge$ are compatible with the order $\leq$ defined by the pocrim structure by $x\leq y$ if and only if $x\rightarrow y=\top$. Obviously the resulting residuated lattice satisfies (8).

Given a Heyting algebra $\mathbf{H}$ and a Boolean filter F of $\mathbf{H}$, the Nelson lattice $\mathbf{R}(\mathbf{K}(\mathbf{H},F))$ will be denoted $\mathbf{R}(\mathbf{H},F)$. When $F=H$, we shall write simply $\mathbf{R}(\mathbf{H})$.

3.2 *From Nelson residuated lattices to Nelson algebras*

We denote by $\mathcal{E}_2$ the variety of residuated lattices characterized by (9).

LEMMA 3.3
Every residuated lattice in $\mathcal{E}_2$ satisfies the equation:

$$(z^2\rightarrow x^2)^2=(z^2\rightarrow x)^2. \tag{14}$$

PROOF. Recalling RL_1 and the inequalities (4) note that in any residuated lattice one has:

$$(z^2\rightarrow x)^2\rightarrow(z^4\rightarrow x^2)=(z^2\rightarrow x)\rightarrow[(z^2\rightarrow x)\rightarrow(z^4\rightarrow x^2)]=$$
$$(z^2\rightarrow x)\rightarrow[(z^4*(z^2\rightarrow x))\rightarrow x^2]\geq(z^2\rightarrow x)\rightarrow[(z^2*x)\rightarrow x^2]=$$
$$(z^2\rightarrow x)\rightarrow[z^2\rightarrow(x\rightarrow x^2)]=(z^2*(z^2\rightarrow x))\rightarrow(x\rightarrow x^2)\geq$$
$$x\rightarrow(x\rightarrow x^2)=x^2\rightarrow x^2=\top.$$

Hence, we have that

$$(z^2 \to x)^2 \le (z^4 \to x^2). \tag{15}$$

Moreover,

$$(z^2 \to x^2)^2 \to (z^2 \to x) = (z^2 \to x^2) \to [(z^2 \to x^2) \to (z^2 \to x)] =$$
$$(z^2 \to x^2) \to [(z^2 * (z^2 \to x^2)) \to x] \ge (z^2 \to x^2) \to (x^2 \to x) = \top.$$

Hence, we also have

$$(z^2 \to x^2)^2 \le z^2 \to x. \tag{16}$$

From (9) every element x in a lattice in $\mathcal{E}_2$ satisfies $x^n = x^2$ for each $n \ge 2$. Because of this fact and (15) we get,

$$(z^2 \to x)^2 = (z^2 \to x)^4 \le (z^4 \to x^2)^2 = (z^2 \to x^2)^2$$

while the opposite inequality follows from (16):

$$(z^2 \to x^2)^2 = (z^2 \to x^2)^4 \le (z^2 \to x)^2.$$

■

On each residuated lattice $\mathbf{A} = (\mathbf{A}, \vee, \wedge, *, \to, \top) \in \mathcal{E}_\mathbf{2}$ define the binary operation $\Rightarrow$ by the prescription[3]

$$x \Rightarrow y := x^2 \to y, \tag{17}$$

and let

$$\mathbf{A}' = (\mathbf{A}, \vee, \wedge, *, \Rightarrow, \top). \tag{18}$$

When $\mathbf{A}$ is bounded, define

$$\mathbf{N}(\mathbf{A}) = (A, \vee, \wedge, \Rightarrow, \neg, \bot, \top), \tag{19}$$

where $\neg x = x \to \bot$.

Theorem 3.4
For each $\mathbf{A} \in \mathcal{E}_\mathbf{2}$, the binary relation $\equiv$ defined on A by the prescription $x \equiv y$ if and only if $x^2 = y^2$ is a congruence of the algebra $\mathbf{A}'$, and the quotient, with the natural operations, is a generalized Heyting algebra. If $\mathbf{A}$ has a bottom $\bot$, then $\mathbf{A}'/\equiv$ is a Heyting algebra.

Proof. It is obvious that $\equiv$ is an equivalence relation. To show that it is a congruence, note first that from (9) it follows that $x \equiv y$ if and only if $x^2 \le y$ and $y^2 \le x$. Suppose that $x^2 \le s$ and $y^2 \le t$. Then we have:

$$(x \vee y)^2 = (x \vee y)^3 = x^3 \vee (x^2 * y) \vee (x * y^2) \vee y^3 \le s \vee t,$$
$$(x \wedge y)^2 \le x^2 \wedge y^2 \le s \wedge t,$$
$$(x * y)^2 \le s * t.$$

Hence if $a \equiv c$ and $b \equiv d$, then $(a \vee b) \equiv (c \vee d)$, $(a \wedge b) \equiv (c \wedge d)$ and $(a * b) \equiv (c * d)$, and it follows at once from Lemma 3.3 that $(a \Rightarrow b) \equiv (c \Rightarrow d)$. Therefore, $\equiv$ is a congruence of $\mathbf{A}'$. Moreover, it is easy to check that for all $a, b \in A$, $(a * b) \equiv (a \wedge b)$.

[3] As the referee pointed out, the relation defined by $x \preceq y$ iff $x^2 \to y = \top$ is a preorder on the algebra $\mathbf{A} \in \mathcal{E}_2$. The binary relation $\equiv$ given in Theorem 3.4 is the equivalence generated by this preorder $\preceq$.

Denote by $|a|$ the equivalence class of $a \in A$. It is clear that $\mathbf{A}'/\!\equiv$ with the natural operations $|a| \vee |b| := |a \vee b|$ and $|a| \wedge |b| := |a \wedge b|$ is a lattice, with greatest element $|\top|$. If $\mathbf{A}$ has a bottom $\perp$, then $|\perp|$ is the smallest element of $\mathbf{A}'/\!\equiv$. To show that $|a| \Rightarrow |b| := |a \Rightarrow b|$ is the Heyting implication in $\mathbf{A}'/\!\equiv$, we have to show that

$$|a| \wedge |b| \leq |c| \quad \text{if and only if} \quad |a| \leq |b| \Rightarrow |c|. \tag{20}$$

Since it follows from (9) that $|x| \leq |y|$ if and only if $x^2 \leq y$, and $|x \wedge y| = |x * y|$, (20) is equivalent to the conjunction of the following two conditions:

$$(a * (a^2 \to c))^2 \leq c, \tag{21}$$

and

$$(a * b)^2 \leq c \quad \text{implies} \quad a^2 \leq b^2 \to c. \tag{22}$$

Since (22) is obvious, to complete the proof we need to prove (21). By (9), we have

$$(a * (a^2 \to c))^2 = a^2 * (a^2 \to c)^2 = a^4 * (a^2 \to c)^2 = (a^2 * (a^2 \to c))^2 \leq c^2 \leq c.$$

■

Given a bounded residuated lattice in $\mathbf{A} \in \mathcal{E}_2$, we denote by $\mathbf{H}(\mathbf{A})$ the Heyting algebra $\mathbf{A}'/\!\equiv$.

COROLLARY 3.5
Let $\mathbf{A}$ be an involutive residuated lattice in $\mathcal{E}_2$. Then the correspondence $x \mapsto \rho(x) = (|x|, |\neg x|)$ defines a homomorphism $\rho\colon \mathbf{N}(\mathbf{A}) \to \mathbf{K}(\mathbf{H}(\mathbf{A}))$. Moreover the following are equivalent conditions:

(1) ρ is injective,
(2) $x^2 = y^2$ and $(\neg x)^2 = (\neg y)^2$ imply $x = y$.

PROOF. Since $|x| \wedge |\neg x| = |x * \neg x| = |\perp|$, ρ maps A into $K(\mathbf{H}(\mathbf{A}))$, and since in an involutive residuated lattice the De Morgan laws $\neg(x \vee y) = \neg x \wedge \neg y$ and $\neg(x \wedge y) = \neg x \vee \neg y$ both hold, it follows that ρ is a lattice homomorphism. It is clear that ρ preserves bottom and top, and that $\sim \rho(x) = \rho(\neg x)$. Finally, $\rho(x) \Rightarrow \rho(y) = (|x \Rightarrow y|, |\neg(x \Rightarrow y)|)$ because in an involutive residuated lattice we have that $\neg(x^2 \to y) = x^2 * \neg y$, and $x^2 * \neg y \equiv x \wedge \neg y$. Hence, ρ is a homomorphism from $\mathbf{N}(\mathbf{A})$ into $\mathbf{K}(\mathbf{H}(\mathbf{A}))$. The last statement is obvious. ■

THEOREM 3.6
Let $\mathbf{A}$ be a Nelson lattice. Then $\mathbf{N}(\mathbf{A})$ is a Nelson algebra.

PROOF. From Theorem 2.2 we know that $\mathbf{A} \in \mathcal{E}_2$. Suppose that x, y are elements in A such that $x^2 = y^2$ and $(\neg x)^2 = (\neg y)^2$. Since $\mathbf{A} \in \mathcal{E}_2$, we can apply Lemma 3.3 to obtain $\top = (x^2 \to y^2)^2 = (x^2 \to y)^2$, and this implies that $x^2 \to y = \top$. Analogously we obtain that $(\neg y)^2 \to \neg x = \top$, and then by (8) we can conclude that $x \leq y$. Interchanging the roles of x and y we obtain that $y \leq x$. Therefore, we have seen that $\mathbf{A}$ is in $\mathcal{E}_2$ and satisfies the quasi equation $x^2 = y^2$ and $(\neg x)^2 = (\neg y)^2$ imply $x = y$. Hence by Corollary 3.5, $\mathbf{N}(\mathbf{A})$ is isomorphic to a subalgebra of the Nelson algebra $\mathbf{K}(\mathbf{H}(\mathbf{A}))$. ■

REMARK 3.7
Since $\mathbf{A}$ and $\mathbf{N}(\mathbf{A})$ have the same lattice reduct, if follows from the above theorem that *Nelson lattices are distributive*.

As an immediate consequence of the above results and the ones in the previous section, we obtain an easy characterization of Nelson residuated lattices:

COROLLARY 3.8
Let **A** be an involutive residuated lattices satisfying $x^2 = x^3$. Then **A** satisfies the quasi equation:

$$x^2 = y^2 \text{ and } (\neg x)^2 = (\neg y)^2 \quad \text{implies} \quad x = y \tag{23}$$

if and only if **A** is a Nelson lattice.

3.3 Examples of applications of Corollary 3.8

3.3.1 MV_3-algebras

Consider the bounded residuated lattice $\mathbf{L}_3$ whose universe is the set $\{0, 1/2, 1\}$ with the usual order and we require $1/2 * 1/2 = 0$ and $1/2 \to 0 = 1/2$. The operations $*$ and $\to$ are completely determined on the other elements of L_3. Since $\mathbf{L}_3$ satisfies (5), (9) and the quasi equation (23), Corollary 3.8 asserts that $\mathbf{L}_3$ is a Nelson residuated lattice. The reader can verify that $\mathbf{L}_3$ is the three-element MV-algebra [7, 18]. Call $\mathcal{MV}_3$ the variety of MV-algebras generated by $\mathbf{L}_3$. We conclude that $\mathcal{MV}_3$ is a subvariety of Nelson lattices. The algebras in $\mathcal{MV}_3$ are termwise equivalent with Moisil's three-valued Łukasiewicz algebras [1]. They are precisely the subalgebras of the Nelson algebras of the form $\mathbf{K}(\mathbf{H})$, where $\mathbf{H}$ is a Boolean algebra [6, 36]. A deeper study of the relation of $\mathcal{MV}_3$ and Nelson lattices will be carried out in Section 5.

3.3.2 The nilpotent minimum equation

Let $\mathcal{SKNM}$ be the variety of involutive residuated lattices $\mathbf{A} = (A, \vee, \wedge, *, \to, \bot, \top)$ such that their reducts $(A, \wedge, \vee, \neg, \top, \bot)$ are Kleene algebras and they satisfy the equation

$$(x^2 \to \bot) \vee (x \to x^2) = \top. \tag{24}$$

Let **A** be a subdirectly irreducible algebra in $\mathcal{SKNM}$. Since in any subdirectly irreducible residuated lattice the top element is join irreducible (see [24, Proposition 1.4]), we have that for each element $x \in A$ either $x^2 = x$ or $x^2 = \bot$. This implies that $\mathcal{SKNM}$ is a subvariety of $\mathcal{E}_2$. Moreover, $x^2 = y^2$ and $(\neg x)^2 = (\neg y)^2$ imply $x = y$. In detail, if $x^2 = x \neq \bot$ then $y^2 = y = x$. Analogously if $(\neg x)^2 = \neg x \neq \bot$ then $\neg x = \neg y$ and K_1 implies $x = y$. Since $x^2 = y^2 = \bot$ and $(\neg x)^2 = (\neg y)^2 = \bot$ imply $x = \neg x$ and $y = \neg y$ from K_3 we obtain $x = y$. From Corollary 3.8, we conclude:

THEOREM 3.9
$\mathcal{SKNM}$ is a subvariety of Nelson residuated lattices.

Equation (24) is obtained by identifying x with y in the following equation, which is known as the *nilpotent minimum equation* [10, 15]:

$$(x * y \to \bot) \vee (x \wedge y \to x * y) = \top. \tag{25}$$

Let $\mathcal{KNM}$ be the variety of involutive residuated lattices **A** such that their reducts $(A, \wedge, \vee, \neg, \top, \bot)$ are Kleene algebras and they satisfy (25).

Clearly, $\mathcal{KNM}$ is a subvariety of $\mathcal{SKNM}$. Then from the above theorem, we have:

COROLLARY 3.10
$\mathcal{KNM}$ is a subvariety of the variety $\mathcal{N}$ of Nelson residuated lattices.

We shall return to these varieties in Sections 6.4 and 6.5.

3.4 Categorical isomorphism

For the notions of category theory used in what follows we refer the reader to [26]. Let denote by $\mathbb{NA}$ the category of Nelson algebras and homomorphisms and by $\mathbb{NR}$ the category of Nelson residuated lattices and homomorphisms. We can define the following functors:

- $\mathbf{R}:\mathbb{NA}\to\mathbb{NR}$, where for each $\mathbf{A}\in\mathbb{NA}$ the lattice $\mathbf{R}(\mathbf{A})$ is the one given by Theorem 3.1 and for each morphism $h:\mathbf{A}\to\mathbf{B}$ we take $\mathbf{R}(h)=h$.
- $\mathbf{N}:\mathbb{NA}\to\mathbb{NR}$, where for each $\mathbf{N}\in\mathbb{NR}$ the lattice $\mathbf{N}(\mathbf{A})$ is the one given by Theorem 3.6 and for each morphism $h:\mathbf{A}\to\mathbf{B}$ we define $\mathbf{N}(h)=h$.

From its definition it is easy to see that the functor **N** is both, right and left adjoint of **R**. Moreover we have that both compositions **RN** and **NR** coincide with the identity in the corresponding category. Thus, we conclude:

THEOREM 3.11
The categories $\mathbb{NA}$ and $\mathbb{NR}$ are isomorphic.

4 Comparing constructions of Nelson residuated lattices

4.1 Connected and disconnected rotations of generalized Heyting algebras

The aim of this section is to compare the following two different constructions of involutive residuated lattices. The first construction is the lattice $\mathbf{R}(\mathbf{H})$ built from a Heyting algebra $\mathbf{H}$, which is a variant of the Fidel [11] and Vakarelov [36] construction of the Nelson algebra $\mathbf{K}(\mathbf{H})$. The second one is a construction given by Jenei in [21] that we next present in detail. We shall show that they are equivalent when the Heyting algebra involved has a meet irreducible bottom. The reader should compare Jenei's construction with those in [14], [27] and [39].

DEFINITION 4.1
Let $\mathbf{D}=(D,\vee,\wedge,*,\to,1)$ be a residuated lattice and let $0\notin D$. We define the disconnected rotation[4]

$$\mathbf{DR}(\mathbf{D})=((D\times\{0\})\cup(\{0\}\times D),\sqcup,\sqcap,\otimes,\hookrightarrow,\bot,\top)$$

as an algebra of type $(2,2,2,2,0,0)$ with the operations given by the following prescriptions:

$$(x,y)\sqcup(s,t)=(s,t)\sqcup(x,y)=\begin{cases}(x\vee s,0) & \text{if } y=t=0,\\ (0,y\wedge t) & \text{if } x=s=0,\\ (s,0) & \text{if } x=0 \text{ and } t=0.\end{cases}$$

$$(x,y)\sqcap(s,t)=(s,t)\sqcap(x,y)=\begin{cases}(x\wedge s,0) & \text{if } y=t=0,\\ (0,y\vee t) & \text{if } x=s=0,\\ (0,y) & \text{if } x=0 \text{ and } t=0.\end{cases}$$

[4]We have replaced Jenei's notation for one which is more useful to our purpose. Moreover, the definition of connected rotation is analogous but not exactly the same as the one given by him. The algebra $\mathbf{CR}(\mathbf{D})$ that we define is, according to [21], the connected rotation of the semigroup obtained by adding a lower bound to the residuated lattice **D**.

$$(x,y)\otimes(s,t)=(s,t)\otimes(x,y)=\begin{cases}(x*s,0) & \text{if } y=t=0,\\ (0,1) & \text{if } x=s=0,\\ (0,s\to y) & \text{if } x=0 \text{ and } t=0.\end{cases}$$

$$(x,y)\hookrightarrow(s,t)=\begin{cases}(x\to s,0) & \text{if } y=t=0,\\ (t\to y,0) & \text{if } x=s=0,\\ (0,x*t) & \text{if } y=0 \text{ and } s=0,\\ (1,0) & \text{if } x=0 \text{ and } t=0.\end{cases}$$

$$\top=(1,0) \qquad \bot=(0,1).$$

The connected rotation

$$\mathbf{CR}(\mathbf{D})=((D\times\{0\})\cup(\{0\}\times\{0\})\cup(\{0\}\times D),\sqcup,\sqcap,\otimes,\hookrightarrow,\bot,\top)$$

is an algebra of type $(2,2,2,2,0,0)$ with the operations $\sqcup,\sqcap,\otimes,\hookrightarrow$, given as in the definition of disconnected rotation over $(D\times\{0\})\cup(\{0\}\times D)$, and extended by:

$$(x,y)\sqcup(0,0)=(0,0)\sqcup(x,y)=\begin{cases}(x,y) & \text{if } y=0,\\ (0,0) & \text{otherwise,}\end{cases}$$

$$(x,y)\sqcap(0,0)=(0,0)\sqcap(x,y)=\begin{cases}(0,0) & \text{if } y=0,\\ (x,y) & \text{otherwise,}\end{cases}$$

$$(x,y)\otimes(0,0)=(0,0)\otimes(x,y)=\begin{cases}(0,0) & \text{if } y=0 \text{ and } x\neq 0,\\ (0,1) & \text{otherwise,}\end{cases}$$

$$(x,y)\hookrightarrow(0,0)=\begin{cases}(0,0) & \text{if } y=0 \text{ and } x\neq 0,\\ (1,0) & \text{otherwise,}\end{cases}$$

$$(0,0)\hookrightarrow(x,y)=\begin{cases}(0,0) & \text{if } x=0 \text{ and } y\neq 0,\\ (1,0) & \text{otherwise,}\end{cases}$$

Note that, upon defining $\neg(x,y)$ as $(x,y)\hookrightarrow\bot$, we get

$$\neg(x,y)=\begin{cases}(0,x) & \text{if } y=0,\\ (y,0) & \text{if } x=0.\end{cases}$$

THEOREM 4.2 [21]
Let **D** be a residuated lattice. The disconnected rotation **DR**(**D**) and the connected rotation **CR**(**D**) of **D** are involutive residuated lattices.

THEOREM 4.3
Let **D** be a generalized Heyting algebra. The connected and the disconnected rotations **CR**(**D**) and **DR**(**D**) of **D** are Nelson residuated lattices.

PROOF. Since the disconnected rotation of a generalized Heyting algebra is always a subalgebra of the connected rotation it is enough to prove the latter case. As a matter of fact, it is only necessary to check that $\mathbf{CR}(\mathbf{D})$ satisfies equation (8). Assume (x,y) and (s,t) are in $CR(\mathbf{D})$. We distinguish three different cases:

CASE 1
$(x,y)\in D\times\{0\}$. We have $(x,y)^2=(x,y)$. Then (8) becomes

$$((x,y)\hookrightarrow(s,t))\sqcap((\neg(s,t))^2\hookrightarrow\neg(x,y))\hookrightarrow((x,y)\hookrightarrow(s,t))=\top$$

which clearly holds.

CASE 2
$(x,y)\in\{0\}\times D$. We divide this case into two subcases: if $(s,t)\notin\{0\}\times D$, then $(x,y)\hookrightarrow(s,t)=\top$ and (8) follows. If $(s,t)\in\{0\}\times D$, we have $(x,y)^2=(s,t)^2=\bot$ and both $\neg(x,y)$ and $\neg(s,t)$ are idempotent. Then,

$$((x,y)^2\hookrightarrow(s,t))\sqcap((\neg(s,t))^2\hookrightarrow\neg(x,y))=\top\sqcap(t\to y,0)$$

while $(x,y)\hookrightarrow(s,t)=(t\to y,0)$ and (8) holds.

CASE 3
$(x,y)=(0,0)$. If $t=0$ we have $(x,y)\hookrightarrow(s,t)=\top$ and (8) follows. Otherwise $(x,y)^2=\bot$ and $(\neg(s,t))^2=\neg(s,t)$. Thus

$$((x,y)^2\hookrightarrow(s,t))\sqcap((\neg(s,t))^2\hookrightarrow\neg(x,y))=\neg(s,t)\hookrightarrow(0,0)=(0,0).$$

On the other hand

$$(0,0)\hookrightarrow(s,t)=(0,0).$$

as desired.

■

The following result is well known.

LEMMA 4.4
Let $\mathbf{D}$ be a residuated lattice such that $0\notin D$ and $D_0=D\cup\{0\}$. Then $\mathbf{D}_0=(D_0,\vee,\wedge,*,\to,0,\top)$ is a bounded residuated lattice with the operations $*,\to,\wedge,\vee$ of D extended by:

$$x*0=0*x=0 \quad x\to 0=0 \quad 0\to x=\top \quad x\wedge 0=0 \quad x\vee 0=x$$

for each $x\in D$.

Observe that if $\mathbf{D}$ is a residuated lattice with $0\notin D$ and $x,y\in D_0$ we have that $x*y=0$ if and only if $x=0$ or $y=0$. Moreover, following the notation introduced after Remark 3.2, if $\mathbf{D}$ is a generalized Heyting algebra then $\mathbf{D}_0$ is a Heyting algebra and for every pair (x,y) in the Nelson lattice $\mathbf{R}(\mathbf{D}_0)$ one and only one of the following holds:

(1) $x\in D$ and $y=0$;
(2) $x=0$ and $y\in D$;
(3) $x=y=0$.

This implies that the universes of $\mathbf{CR}(\mathbf{D})$ and $\mathbf{R}(\mathbf{D}_0)$ coincide. Now it is easy to check the next result.

THEOREM 4.5
Let $\mathbf{D}$ be a generalized Heyting algebra and $0 \notin D$. Then

$$\mathbf{CR}(\mathbf{D}) \cong \mathbf{R}(\mathbf{D}_0).$$

5 The radical

5.1 *The variety* $\mathcal{K}_2$

We denote by $\mathcal{K}_2$ the variety of involutive residuated lattices determined by (9) and

$$(x \wedge \neg x) \to (y \vee \neg y) = \top. \tag{26}$$

The variety $\mathcal{N}$ of Nelson residuated lattices is a *proper* subvariety of $\mathcal{K}_2$: the disconnected rotation of $\mathbf{L}_3$ provides an example of an algebra in $\mathcal{K}_2$ which is not a Nelson lattice.

Let $\mathbf{L}_2$ denote the two-element Boolean algebra and consider $\mathbf{R}(\mathbf{L}_2)$. The universe of $\mathbf{R}(\mathbf{L}_2)$ is the set $\{\bot = (0,1), (0,0), (1,0) = \top\}$ and $(0,0)^2 = \bot$. It is easy to see that $\mathbf{R}(\mathbf{L}_2) \cong \mathbf{L}_3$. It follows that $\mathbf{L}_3 \in \mathcal{K}_2$, and $\mathcal{MV}_3$ is the subvariety of $\mathcal{K}_2$ generated by $\mathbf{L}_3$.

A. Monteiro [28] characterized three-valued Łukasiewicz algebras as the semisimple Nelson algebras. We are going to show that they are the semisimple algebras in the variety $\mathcal{K}_2$.

Let $\mathbf{A}$ be a bounded residuated lattice. One can check in [24] that $\mathbf{A}$ is simple if and only if for all $x \in A$ such that $x \neq \top$ there exists a positive integer n such that $x^n = \bot$. Based on this characterization of simple algebras we have:

THEOREM 5.1
Every simple algebra in $\mathcal{K}_2$ is a subalgebra of $\mathbf{L}_3$.

PROOF. Assume $\mathbf{A} \in \mathcal{K}_2$ is simple and take $a \in A$ such that $a \neq \top$. Equation (9) implies that $a^2 = \bot$, and this means $a \leq \neg a$. If $\neg a < \top$ we have $(\neg a)^2 = \bot$. Hence, $\neg a \leq \neg\neg a = a$ and we obtain $\neg a = a$. Otherwise $a = \neg\neg a = \bot$. Since (26) implies the uniqueness of a negation fixpoint in $\mathbf{A}$, we have that either $A = \{\bot, \top\}$, or $A = \{\bot, a, \top\}$, with $a^2 = \bot$. In the last case $\mathbf{A}$ is isomorphic to $\mathbf{L}_3$, and in the first case $\mathbf{A}$ is isomorphic to $\mathbf{L}_2$. ■

Since each algebra in $\mathcal{MV}_3$ is a subdirect product of subalgebras of $\mathbf{L}_3$ [1, 7, 18], we have:

COROLLARY 5.2
$\mathcal{MV}_3$ is the class of semisimple algebras in $\mathcal{K}_2$.

Recall that the radical $Rad(\mathbf{A})$ of a bounded residuated lattice $\mathbf{A}$ is the intersection of all its maximal implicative filters. Since maximal implicative filters of $\mathbf{A}$ are in correspondence with maximal congruences of $\mathbf{A}$, we have:

COROLLARY 5.3
If $\mathbf{A} \in \mathcal{K}_2$, then $\mathbf{A}/Rad(\mathbf{A}) \in \mathcal{MV}_3$.

5.2 *The homomorphism* Φ

We now turn our attention to the variety $\mathcal{N}$ of Nelson residuated lattices.

For each $\mathbf{A} \in \mathcal{N}$ we shall be considering the following three unary term functions:

$$\nabla(x) = \neg(\neg x^2)^2 \qquad \Delta(x) = (\neg(\neg x)^2)^2$$
$$\phi(x) = \Delta(x) \wedge (\nabla(x \vee \neg x) \vee x).$$

Assume that $\mathbf{H}$ is a Heyting algebra and consider the corresponding Nelson lattice $\mathbf{A} = \mathbf{R}(\mathbf{H})$. Computing the operators defined above over $\mathbf{A}$ we obtain:

$$\nabla(x, y) = (\neg\neg x, \neg x) \tag{27}$$
$$\Delta(x, y) = (\neg y, \neg\neg y) \tag{28}$$
$$\phi(x, y) = (\neg\neg x, \neg\neg y). \tag{29}$$

THEOREM 5.4
Let $\mathbf{A} = (A, \vee, \wedge, *, \to, \bot, \top)$ be a Nelson lattice. Consider

$$\phi(A) = \{y \in A : y = \phi(x) \text{ for some } x \in A\}.$$

Then $\Phi(\mathbf{A}) = (\phi(A), \vee', \wedge', *, \to, \bot, \top)$ is a Nelson lattice, where for each $\star \in \{\wedge, \vee\}$ the operation $\star'$ is given by $x \star' y = \phi(x \star y)$, and ϕ is a homomorphism from $\mathbf{A}$ onto $\Phi(\mathbf{A})$.

PROOF. By Theorem 2.3, we can assume that $\mathbf{A} = \mathbf{R}(\mathbf{H}, F)$ for some Heyting algebra $\mathbf{H}$ and some Boolean filter F of $\mathbf{H}$. Then, taking into account the classical Glivenko theorem for Heyting algebras and (29), it is easy to check that the following identities holds:

$$\phi(x \to y) = \phi(x) \to \phi(y), \quad \phi(x * y) = \phi(x) * \phi(y),$$
$$\phi(x \wedge y) = \phi(\phi(x) \wedge \phi(y)), \quad \phi(x \vee y) = \phi(\phi(x) \vee \phi(y)),$$
$$\phi(\top) = \top \quad \phi(\bot) = \bot.$$

Hence for each Nelson lattice $\mathbf{A}$, ϕ is a homomorphism from $\mathbf{A}$ onto $\Phi(\mathbf{A})$, and this implies that $\Phi(\mathbf{A})$ is a Nelson lattice. ■

It is well known that in every bounded integral and commutative residuated lattice $\mathbf{A}$, $x \in Rad(\mathbf{A})$ if and only if for every positive integer n there exists a positive integer m such that $(\neg x^n)^m = \bot$. If $\mathbf{A}$ satisfies (9) we also have:

$$x \in Rad(\mathbf{A}) \text{ if and only if for every positive integer } n \text{ we have } (\neg x^n)^2 = \bot,$$

which means

$$x \in Rad(\mathbf{A}) \text{ if and only if for each positive integer } n \text{ we have } \neg x^n \leq x^n.$$

If $\mathbf{A}$ is involutive, using again by (9) we finally obtain:

$$x \in Rad(\mathbf{A}) \text{ if and only if } \neg x^2 \leq x^2. \tag{30}$$

THEOREM 5.5
Let $\mathbf{A}$ be a Nelson lattice. The following properties are satisfied for every $x \in A$:

(i) $\nabla(x) = \top$ if and only if $\neg x^2 \leq x^2$,
(ii) $\Delta(x) = \top$ if and only if $\neg x \leq x$,

(iii) $x \in Rad(\mathbf{A})$ if and only if $\nabla(x) = \top$ if and only if $\phi(x) = \top$,
(iv) $\nabla(x) \to \Delta(x) = \top$.

PROOF. (i) and (ii) are immediate consequences of the definitions of ∇ and Δ, respectively. While the first equivalence of (iii) is immediate from (i) and (30), the characterizations of ϕ and ∇ given by (29) and (27) prove the second equivalence. Lastly, to check (iv) we think of the elements of A as pairs of the form (x, y) for some x, y in a Heyting algebra $\mathbf{H}$ such that $x \wedge y = \bot_H$. Since $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$, we obtain $\neg\neg x \wedge \neg\neg y = \bot_H$. This implies

$$\neg\neg x \wedge y \leq \neg\neg x \wedge \neg\neg y = \bot_H$$

and similarly

$$\neg\neg y \wedge x \leq \neg\neg y \wedge \neg\neg x = \bot_H.$$

Hence, for each $(x, y) \in A$ we have

$$\neg\neg x \leq \neg y \quad \text{and} \quad \neg\neg y \leq \neg x.$$

We conclude from (27) and (28) that $\nabla(x, y) \to \Delta(x, y) = (\neg\neg x, \neg x) \to (\neg y, \neg\neg y) = \top$ as desired. ■

COROLLARY 5.6
If $\mathbf{A}$ is a Nelson lattice then $\mathbf{A}/Rad(\mathbf{A}) \cong \Phi(\mathbf{A})$.

PROOF. By (iii) in the above theorem, the natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\mathrm{Rad}(\mathbf{A})$ and the homomorphism ϕ from $\mathbf{A}$ onto $\Phi(\mathbf{A})$ have the same kernel. ■

COROLLARY 5.7
If $\mathbf{A}$ is a Nelson lattice then $\Phi(\mathbf{A}) \in \mathcal{MV}_3$. Moreover, if $\mathbf{M}$ is a subalgebra of $\mathbf{A}$ that belongs to $\mathcal{MV}_3$, then $\mathbf{M}$ is a subalgebra of $\Phi(\mathbf{A})$.

PROOF. The first assertion is a consequence of Corollaries 5.3 and 5.6. For the second one, simply observe that if $x \in \mathbf{L}_3$, then $\phi(x) = x$. Therefore, the equation $\phi(x) = x$, holds in $\mathbf{M}$, and this implies that $M \subseteq \phi(\mathbf{A})$. ■

REMARK 5.8
Corollary 5.7 implies that if $\Phi(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ then it is the greatest subalgebra of $\mathbf{A}$ belonging to the variety of MV-algebras.

A natural question arising from the previous results is under which conditions is the algebra $\Phi(\mathbf{A})$ a Boolean algebra. It is not hard to corroborate that in the Boolean algebra $\mathbf{L}_2$ the equation

$$\nabla(x) = \Delta(x) \tag{31}$$

is satisfied and that it is not satisfied in $\mathbf{L}_3$. Moreover, the characterizations of ϕ, ∇ and Δ given by (29), (27) and (28) imply that

$$\nabla(\phi(x)) = \nabla(x) \quad \text{and} \quad \Delta(\phi(x)) = \Delta(x) \tag{32}$$

for every x in a Nelson lattice $\mathbf{A}$. As an immediate consequence we have:

COROLLARY 5.9
A Nelson algebra $\mathbf{A}$ satisfies (31) if and only if $\Phi(\mathbf{A})$ is a Boolean algebra.

DEFINITION 5.10
A *regular Nelson lattice* is a Nelson residuated lattice **A** such that $\Phi(\mathbf{A})$ is a subalgebra of **A**, i.e. $\phi(x\vee y)=\phi(x)\vee\phi(y)$ and $\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ for all $x,y\in A$.

From the fact that ϕ is a term function it turns out that the class of regular Nelson lattices is a variety that we denote by $\mathcal{NR}$. Because of the characterization of the term function ϕ given in (29) it is easy to see that for each $x\in A$,

$$\phi(\phi(x))=\phi(x)$$

and if $x\in\phi(\mathbf{A})$, then $\phi(x)=x$. Then we have

COROLLARY 5.11
The homomorphism ϕ is a retract from the regular Nelson lattice **A** onto $\Phi(\mathbf{A})$.

The next theorem establishes a connection between the variety $\mathcal{NR}$ and a variety of Heyting algebras.

THEOREM 5.12
A Nelson lattice $\mathbf{A}\in\mathcal{NR}$ if and only if the Heyting algebra $\mathbf{H_A}$ satisfies the Stone identity $\neg x\vee\neg\neg x=\top$.

PROOF. It is well known that a Heyting algebra satisfies the Stone identity if and only if it satisfies the identity $\neg\neg x\vee\neg\neg y=\neg\neg(x\vee y)$. Then if a Heyting algebra **H** satisfies the Stone identity, considering (29) we can see that $\mathbf{R}(\mathbf{H},F)$ satisfies $\phi(x\vee y)=\phi(x)\vee\phi(y)$ and $\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ for every Boolean filter F of **H**. We conclude that if $\mathbf{H_A}$ satisfies the Stone identity, then $\mathbf{A}\in\mathcal{NR}$. To prove the converse, suppose that the Stone identity does not hold in a Heyting algebra **H**. Then there is $x\in H$ such that $\neg x\vee\neg\neg x<\top$. The pairs $(\neg x,x)$ and $(x,\neg x)$ both belong to $R(\mathbf{H},D(\mathbf{H}))$, and by (29) we have that

$$\phi(\neg x,x)\vee\phi(x,\neg x)=(\neg x,\neg\neg x)\vee(\neg\neg x,\neg x)=(\neg x\vee\neg\neg x,\bot)<$$
$$(\top,\bot)=(\neg\neg(\neg x\vee x),\bot)=\phi((\neg x,x)\vee(x,\neg x)).$$

Therefore $\mathbf{R}(\mathbf{H},D(\mathbf{H}))\notin\mathcal{NR}$, and by (iii) in Theorem 2.3, we can conclude that $\mathbf{R}(\mathbf{H},F)\notin\mathcal{NR}$ for every Boolean filter F of **H**. By (ii) of the same theorem, this implies that if $\mathbf{A}\in\mathcal{NR}$, then $\mathbf{H_A}$ satisfies the Stone identity. ■

REMARK 5.13
Let **A** be a Nelson lattice. Following Monteiro [29], we denote with Γx the 'negation' of an element x in $\mathbf{N}(\mathbf{A})$, i.e. $\Gamma x=x\Rightarrow\bot=x^2\to\bot=\neg x^2$. Hence $\Gamma\Gamma x=\nabla x$. Monteiro called an element x of a Nelson algebra *regular* when $\nabla x=\Gamma\Gamma x=x$. It follows from (27) and(28) that $x=\nabla x$ if and only if $\triangle x=x$. These regular elements form a Boolean subalgebra of the MV_3-algebra $\Phi(\mathbf{A})$. By [6, Theorem 6.6] and Theorem 5.12, we can conclude that this Boolean algebra is a subalgebra of **A** if and only if $\Phi(\mathbf{A})$ is a subalgebra of **A**.

5.3 *Glivenko-like theorems*

As an application of the above results, we are going to relate provability in Nelson's constructive logic with strong negation and provability in Łukasiewicz three-valued logic, which resembles the celebrated Glivenko theorem relating classical and intuitionistic logics [16].

Let α be a propositional formula built from the propositional variables $p_1,\ldots,p_n$ with the binary connectives $\sqcup,\sqcap,\&,\Longrightarrow$ and the unary connective $\triangleleft$. Given a bounded residuated lattice **A** and

$(a_1,\ldots,a_n)\in A^n$, $\alpha^{\mathbf{A}}(a_1,\ldots,a_n)$ is the element of A obtained by replacing in α the connectives $\sqcup,\sqcap$ by the lattice join and meet, respectively, & by the monoid product, $\Longrightarrow$ by its residual and $\triangleleft$ by negation, and each p_i by a_i, $i=1,\ldots,n$. If $\triangledown\alpha$ denotes the formula $\triangleleft(\triangleleft(\alpha\&\alpha)\&\triangleleft(\alpha\&\alpha))$, it follows by induction on the complexity of α that for each bounded residuated lattice $\mathbf{A}$:

$$(\triangledown\alpha)^{\mathbf{A}}=\nabla(\alpha^{\mathbf{A}}). \tag{33}$$

The next theorem was announced by Monteiro (see the final remarks in [29]).

THEOREM 5.14
A propositional formula α is derivable in Łukasiewicz three-valued logic if and only if $\triangledown\alpha$ is derivable in Nelson constructive logic with strong negation.

PROOF. Recall from [31, 32] that a propositional formula α is derivable in Nelson constructive logic with strong negation if and only if for each Nelson lattice $\mathbf{A}$ and each $(a_1,\ldots,a_n)\in A^n$, $\alpha^{\mathbf{A}}(a_1,\ldots,a_n)=\top$, and from [7, 18] that α is derivable in Łukasiewicz three-valued logic if and only if for each MV_3-algebra $\mathbf{A}$ and each $(a_1,\ldots,a_n)\in A^n$, $\alpha^{\mathbf{A}}(a_1,\ldots,a_n)=\top$. Suppose that α is derivable in Łukasiewicz three-valued logic, and let $\mathbf{A}$ be a Nelson lattice. By Theorem 5.4 and Corollary 5.7, for $(a_1,\ldots,a_n)\in A^n$ we have $\phi(\alpha^{\mathbf{A}}(a_1,\ldots,a_n))=\alpha^{\Phi(\mathbf{A})}(\phi a_1,\ldots,\phi a_n)=\top$. By (iii) in Theorem 5.5 we also have $\nabla(\alpha^{\mathbf{A}}(a_1,\ldots,a_n)=\top$. Since $\mathbf{A}$ is an arbitrary Nelson lattice, (33) implies that $\triangledown\alpha$ is derivable in Nelson constructive logic with strong negation. Suppose now that $\triangledown\alpha$ is derivable in Nelson logic. Hence it is also derivable in three-valued Łukasiewicz logic, and it follows from (33) that $\nabla(\alpha^{\mathbf{A}}(a_1,\ldots,a_n))=\top$ for each $\mathbf{A}\in\mathcal{MV}_3$ and every $(a_1,\ldots,a_n)\in A^n$. Since the equation $\nabla x\to x=\top$ holds in $\mathbf{A}$ (because it holds in $\mathbf{L}_3$), we conclude that $\alpha^{\mathbf{A}}(a_1,\ldots,a_n)=\top$, and since $\mathbf{A}$ is an arbitrary algebra in $\mathcal{MV}_3$, α is derivable in Łukasiewicz three-valued logic. ■

According to Corollary 5.9, a similar relation between derivability in classical propositional calculus and derivability in the extension of Nelson logic by an axiom expressing (31) can be obtained.

6 Varieties of Nelson residuated lattices

In this section, we shall use all the machinery and results of the previous sections to obtain information about some subvarieties of $\mathcal{N}$. In Section 6.5, we establish the relations among the subvarieties of $\mathcal{N}$ considered in this article.

6.1 A splitting pair

A pair $(\mathcal{V}_1,\mathcal{V}_2)$ of subvarieties of a variety $\mathcal{V}$ is a *splitting pair* (of the lattice of subvarieties of $\mathcal{V}$), provided that for each subvariety $\mathcal{W}$ of $\mathcal{V}$, one has $\mathcal{W}\subseteq\mathcal{V}_1$ if and only if $\mathcal{V}_2\not\subseteq\mathcal{W}$ (or equivalently, if the lattice of subvarieties of $\mathcal{V}$ is the disjoint union of the principal ideal $(\mathcal{V}_1]$ and the principal filter $[\mathcal{V}_2)$). If $(\mathcal{V}_1,\mathcal{V}_2)$ is a splitting pair of $\mathcal{V}$, then $\mathcal{V}_2$ is generated by one subdirectly irreducible algebra $\mathbf{S}$, and $\mathcal{V}_1$ is determined (relatively to $\mathcal{V}$) by a single equation e. $\mathbf{S}$ is called a *splitting algebra* in $\mathcal{V}$ and $\mathcal{V}_1$ is its *conjugate variety*, determined by the *conjugate equation e* (see, for instance, [23, Section 2.3] and the references therein). Observe that if $\mathbf{S}$ is a splitting algebra of $\mathcal{V}$, then its conjugate variety is the largest subvariety of $\mathcal{V}$ that does not contain $\mathbf{S}$.

Let $\mathcal{S}$ be the subvariety of $\mathcal{N}$ characterized by (31).

THEOREM 6.1
$\mathbf{L}_3$ is a splitting algebra in the variety $\mathcal{N}$ of Nelson residuated lattices, with $\mathcal{S}$ as its conjugate variety and (31) as its conjugate equation.

Proof. Clearly $\mathbf{L}_3$ does not belong to $\mathcal{S}$. Suppose that $\mathcal{W}$ is a subvariety of $\mathcal{N}$ such that $\mathbf{L}_3 \notin \mathcal{W}$, and let $\mathbf{A} \in \mathcal{W}$. If $\mathbf{A}$ does not satisfy (31), then by Corollary 5.9, $\Phi(\mathbf{A}) \in \mathcal{MV}_3$ and it is not a Boolean algebra. Hence, there is a maximal implicative filter M of $\Phi(\mathbf{A})$ such that $\Phi(\mathbf{A})/M$ is isomorphic to $\mathbf{L}_3$. The composition of the homomorphism $\phi\colon \mathbf{A} \to \Phi(\mathbf{A})$ with the natural homomorphism from $\Phi(\mathbf{A})$ onto $\Phi(\mathbf{A})/M$ gives a homomorphism from $\mathbf{A}$ onto $\mathbf{L}_3$. This implies that $\mathbf{L}_3 \in \mathcal{W}$, a contradiction. Consequently $\mathcal{W} \subseteq \mathcal{S}$, and $\mathcal{S}$ is the largest subvariety of $\mathcal{N}$ not containing $\mathbf{L}_3$. ■

Sendlewski [33, Proposition 4.5] has shown that the Nelson algebras of the form $\mathbf{K}(\mathbf{H}, D(\mathbf{H}))$ for $\mathbf{H}$ a Heyting algebra form a variety, characterized by the following equation in the language of Nelson algebras:

$$(x \Rightarrow \sim x) \wedge (\sim x \Rightarrow x) = \sim x \wedge x, \tag{34}$$

and that $\mathbf{L}_3$ is not a homomorphic image of any algebra in this variety. From this results it is possible to show that the subvariety $\mathcal{S}$ of $\mathcal{N}$ considered in the above theorem is the class of the Nelson residuated lattices of the form $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$ for $\mathbf{H}$ a Heyting algebra. In Theorem 6.3, we offer a direct proof of this fact. We start by the following:

Lemma 6.2
Let $\mathbf{H}$ be a Heyting algebra. The following are equivalent conditions for all $x, y \in H$:

(i) $x \wedge y = \bot$ and $x \vee y \in D(\mathbf{H})$,
(ii) $x \wedge y = \bot$ and $\neg x \wedge \neg y = \bot$,
(iii) $\neg x = \neg\neg y$,
(iv) $\neg x \rightharpoonup x = \neg y$.

Proof. The equivalence between (i) and (ii) follows from the identity $\neg(x \vee y) = \neg x \wedge \neg y$, valid in all Heyting algebras. Since $x \wedge y = \bot$ implies $\neg\neg y \leq \neg x$, and $\neg x \wedge \neg y = \bot$ implies $\neg x \leq \neg\neg y$, we have that (ii) implies (iii), and clearly (iii) implies (ii). Suppose that (iii) holds true. Then $(\neg x \rightharpoonup x) \geq \neg\neg x = \neg y$. On the other hand, since $y \wedge x = \bot$, we have that $\neg x \rightharpoonup x \leq y \rightharpoonup x = \neg y$ (see Remark 2.5 for the last equality). Therefore (iii) implies (iv). Finally, suppose that $\neg x \rightharpoonup x = \neg y$. Since $x \leq \neg x \rightharpoonup x$, we have that $y \wedge x = \bot$. Moreover $\neg x \wedge \neg y = \neg x \wedge (\neg x \rightharpoonup x) = \neg x \wedge x = \bot$. Therefore (iv) implies (ii). ■

Theorem 6.3
A Nelson lattice $\mathbf{A} \in \mathcal{S}$ if and only if there is a Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$. That is, $\mathbf{A} \in \mathcal{S}$ if and only if $F_{\mathbf{A}} = D(\mathbf{H}_{\mathbf{A}})$.

Proof. Observe first that in the light of (6), (31) can be written as:

$$\neg x^2 \to x^2 = (\neg x \to x)^2. \tag{35}$$

By Theorem 2.3, we can assume that $\mathbf{A} = \mathbf{R}(\mathbf{H}, F)$ for some Heyting algebra $\mathbf{H}$ and some Boolean filter F of $\mathbf{H}$. It follows from (12) and (13) in the proof of Theorem 3.1 that the pair $(x, y) \in R(\mathbf{H}, F)$ satisfies (35) if and only if $(\neg x \rightharpoonup x, \neg x) = (\neg y, \neg\neg y)$, and by Lemma 6.2, this equality holds if and only if $x \vee y \in D(\mathbf{H})$. Hence $\mathbf{A} \in \mathcal{S}$ if and only if $F_{\mathbf{A}} = D(\mathbf{H}_{\mathbf{A}})$. ■

The previous result shows that $\mathcal{S}$ coincides with the subvariety of *normal Nelson algebras* in the nomenclature of [17]. Therefore, Theorem 6.1 is a reformulation of Goranko's Lemma 57 in the language of residuated lattices.

6.2 The variety $\mathcal{NR}$

Our aim is to show that $\mathcal{NR}$ is the subvariety of $\mathcal{N}$ generated by the connected rotations of generalized Heyting algebras. We start by describing directly indecomposable algebras in $\mathcal{NR}$. Recall that an algebra $\mathbf{A}$ is called *directly indecomposable* if it cannot be decomposed into the direct product of two non-trivial bounded residuated lattices. Given a bounded residuated lattice $\mathbf{A}$, $B(\mathbf{A})$ will denote the set of complemented elements of the bounded lattice $\mathbf{L}(\mathbf{A})$. The following results were proved in [24]:

LEMMA 6.4
If $\mathbf{A}$ is a residuated lattice, then $B(\mathbf{A})$ is the universe of a subalgebra of $\mathbf{A}$, which is a Boolean algebra that we shall denote by $\mathbf{B}(\mathbf{A})$.

LEMMA 6.5
A Nelson lattice $\mathbf{A}$ is directly indecomposable if and only if $\mathbf{B}(\mathbf{A})$ is the two-element Boolean algebra.

When the Nelson lattice is regular we have:

THEOREM 6.6
If $\mathbf{A} \in \mathcal{NR}$, then $\mathbf{B}(\mathbf{A}) = \mathbf{B}(\Phi(\mathbf{A}))$.

PROOF. Because of regularity we can prove that $B(\phi(\mathbf{A})) \subseteq B(\mathbf{A})$. The opposite inclusion is an immediate consequence of Corollary 5.7 and holds in any Nelson lattice. ■

From Remark 5.13 the converse of the above theorem also holds: $\mathbf{B}(\mathbf{A}) = \mathbf{B}(\Phi(\mathbf{A}))$ *implies* $\mathbf{A} \in \mathcal{NR}$.

As an easy consequence of Theorem 6.6, Lemma 6.5 and the fact that the only two algebras in $\mathcal{MV}_3$ whose Boolean skeleton is the two-element Boolean chain are $\mathbf{L}_2$ and $\mathbf{L}_3$ we conclude:

THEOREM 6.7
Let $\mathbf{A} \in \mathcal{NR}$. Then $\mathbf{A}$ is directly indecomposable if and only if $\Phi(\mathbf{A}) \cong \mathbf{L}_3$ or if $\Phi(\mathbf{A}) \cong \mathbf{L}_2$.

To characterize directly indecomposable algebras we have to study regular Nelson lattices whose images by the term function ϕ is either $\mathbf{L}_2$ or $\mathbf{L}_3$. From now on we denote by $\bot, 1/2$ and $\top$ the elements of $\mathbf{L}_3$.

LEMMA 6.8
Let $\mathbf{H}$ be a Heyting algebra, F an implicative filter of $\mathbf{H}$ and let $\bot_H$ denote the bottom element in H. Let $x, y \in \mathbf{H}$ be such that $(x, y) \in \mathbf{R}(\mathbf{H}, F)$. Then $\phi(x, y)$ is a negation fixpoint if and only if $x = y = \bot_H$.

PROOF. From (29) we have that $\phi(x, y) = \neg\phi(x, y)$ if and only if $\neg\neg x = \neg\neg y$. Obviously if $x = y = \bot_H$ we have that $\phi(x, y)$ is the negation fixpoint. For the converse implication, assume $\neg\neg x = \neg\neg y$. Since $x \wedge y = \bot_H$ the inequality $x \leq \neg y$ holds. Then

$$x = x \wedge \neg\neg x \leq \neg\neg x \wedge \neg y = \neg\neg y \wedge \neg y = \bot_H.$$

Analogously one proves that $y = \bot_H$. ■

THEOREM 6.9
Let $\mathbf{A}$ be a Nelson lattice and consider $A_H = \{x \in A : \phi(x) = \top\}$. Then $\mathbf{A_H} = (A_H, \vee, \wedge, *, \to, \top)$ is a generalized Heyting algebra.

PROOF. Since $\{\top\}$ is an implicative filter of the residuated lattice $\Phi(\mathbf{A})$ and ϕ is a homomorphism, A_H is an implicative filter of $\mathbf{A}$, i.e. $\mathbf{A_H}$ is a residuated lattice. To see that it is a generalized Heyting algebra we just have to check that $x * y = x \wedge y$ for all $x, y \in A_H$. By Corollary 2.4, we can think of

A as a subalgebra of **R**(**H**) for some Heyting algebra **H**. From (29) we get $A_H = \{(x,y) \in A : \neg\neg x = \top \text{ and } \neg\neg y = \bot\}$. One easily sees that

$$A_H = \{(x,y) \in A : \neg x = \bot \text{ and } y = \bot\}.$$

Therefore for elements (x,y) and (s,t) in A_H, we get

$$(x,y) * (s,t) = (x,\bot) * (s,\bot) = (x \wedge s, \neg x \wedge \neg s) = (x \wedge s, \bot)$$

$$= (x,\bot) \wedge (s,\bot) = (x,y) \wedge (x,t),$$

as desired. ∎

THEOREM 6.10
Let $\mathbf{A} \in \mathcal{NR}$ be directly indecomposable. Assume $x \in A_H$ and $y \notin A_H$. Then $y < x$.

PROOF. Theorem 2.3 allows us to think of the elements of **A** as pairs of the form (a,b) with a,b in some Heyting algebra **H**. Then let $x = (x_1,x_2)$ and $y = (y_1,y_2)$ with $x_1,x_2,y_1,y_2 \in H$. Since **A** is directly indecomposable, from Theorem 6.7 we know that $\phi(x_1,x_2) = \top$ and either $\phi(y_1,y_2) = \bot$ or $\phi(y_1,y_2) = 1/2$. Denote by $\bot_H$ the bottom element in **H**.

As a consequence of Lemma 6.8 we have $\phi(y_1,y_2) = 1/2$ if and only if $y_1 = y_2 = \bot_H$. If $\phi(y_1,y_2) = \bot$ then (29) implies $y_1 = \bot_H$. The two previous observations yield that if $\phi(y_1,y_2) < \top$, then $y_1 = \bot_H$. Since $(x_1,x_2) \in A_H$ implies $x_1 > \bot_H$ and $x_2 = \bot_H$ we obtain the result of the theorem. ∎

THEOREM 6.11
Let $\mathbf{A} \in \mathcal{NR}$ be directly indecomposable. Then either $\mathbf{A} \cong \mathbf{DR}(\mathbf{A_H})$ or $\mathbf{A} \cong \mathbf{CR}(\mathbf{A_H})$.

PROOF. Assume that $\phi(\mathbf{A}) = L_3 = \{\bot, 1/2, \top\}$ and let $x \in A$ be such that $\phi(x) = \bot$. Then there is a unique $y_x \in A_H$ such that $x = \neg y_x$. Indeed, take $y_x = \neg x$. Then $x = \neg\neg x = \neg y_x$ and $\phi(y_x) = \phi(\neg x) = \neg\phi(x) = \top$. If there exists $z \in A_H$ such that $x = \neg z$ then $z = \neg\neg z = \neg x = y_x$. Define the map $\rho : A \to CR(A_H)$ by the prescription:

$$\rho(x) = \begin{cases} (x,0) & \text{if } \phi(x) = \top, \\ (0,0) & \text{if } \phi(x) = \frac{1}{2}, \\ (0,y_x) & \text{if } \phi(x) = \bot. \end{cases}$$

We will verify that ρ is an isomorphism. It is easy to see that ρ preserves $\bot$ and $\top$. The uniqueness of y_x and the result of Lemma 6.8 imply that ρ is one to one. To see that it is surjective take first an element **a** in $CR(A_H)$ of the form $(x,0)$ for some $x \in A_H$. Clearly $\mathbf{a} = \rho(x)$. Assume now that $\mathbf{a} = (0,y)$ for some $y \in A_H$. Let $x = \neg y$. Hence $\phi(x) = \bot$, $y = y_x$ and $\rho(x) = (0,y_x) = \mathbf{a}$. Finally, since $\phi(x) = 1/2$ for some $x \in A$, we have $\rho(x) = (0,0)$.

To see that ρ preserves $*$ we consider the following cases:

(1) $x,y \in A_H$. Then

$$\rho(x*y) = (x*y,0) = (x,0) \otimes (y,0) = \rho(x) \otimes \rho(y)$$

(2) $x \in A_H$ and $y \notin A_H$. This being the case we obtain $x*y \notin A_H$. Assume first that $\phi(y) = 1/2$. Then $\phi(x*y) = \phi(x)*\phi(y) = 1/2$. This implies that $\rho(x*y) = (0,0)$ and $\rho(x) \otimes \rho(y) = (x,0) \otimes (0,0) = (0,0)$.

If $\phi(y)=\bot$, then $\rho(x*y)=(0,z)$ where z is such that $\neg z=x*y$. On the other hand we have

$$\rho(x)\otimes\rho(y)=(x,0)\otimes(0,t)=(0,x\to t)$$

with $t=\neg y$. Note that $z=\neg(x*y)=x\to\neg y=x\to t$.

(3) $\phi(x)=\phi(y)=1/2$. This case only happens if $x=y=\neg x$. Then $x*y=x*\neg x=\bot$. Hence

$$\rho(x)\otimes\rho(y)=(0,0)\otimes(0,0)=(0,\top)=\rho(\bot).$$

(4) $\phi(x)=\phi(y)=\bot$. Then $\rho(x)\otimes\rho(y)=(0,t)\otimes(0,s)=(0,\top)$ where $\neg t=x$ and $\neg s=y$. On the other hand $\rho(x*y)=(0,z)$ with $\neg z=x*y$. To obtain the result notice that $z=\neg(x*y)=x\to\neg y$. Since $\neg y\in A_H$ from Theorem 6.10 we get $z=\top$ as desired.

(5) $\phi(x)=\bot$ and $\phi(y)=1/2$. Then $\phi(x*y)=\phi(x)=\bot$. This implies $\rho(x*y)=(0,z)$ with $\neg z=x*y$ and $\rho(x)\otimes\rho(y)=(0,t)\otimes(0,0)=(0,\top)$ where $\neg t=x$. Now we have $z=\neg\neg z=\neg(x*y)=y\to\neg x$. Another application of Theorem 6.10 yields $z=\top$.

Since $*$ is commutative we have provided a proof for all possible cases.

Now we check that ρ preserves $\neg$. For the case $\phi(x)=1/2$ the conclusion follows from the uniqueness of a fixpoint in a Nelson lattice. If $\phi(x)=\top$, then $\neg\rho(x)=(0,x)$ and $\rho(\neg x)=(0,y_{\neg x})$. Since $y_{\neg x}=\neg\neg x=x$, negation is preserved in this case. Lastly if $\phi(x)=\bot$, then $\neg\rho(x)=(y_x,0)$, with $\neg y_x=x$. On the other hand we have $\rho(\neg x)=(\neg x,0)$ and the result also follows.

Since $x\to y=\neg(x*\neg y)$ the operation $\to$ is also preserved by ϕ.

To conclude the proof observe that ρ also preserves $\wedge$ and $\vee$.

Likewise one can see that if $\phi(A)=\{\bot,\top\}$, i.e. $\Phi(\mathbf{A})=\mathbf{L}_2$ after defining the map σ from A to $DR(A_H)$ by

$$\sigma(x)=\begin{cases}(x,0) & \text{if } \phi(x)=\top,\\ (0,y_x) & \text{if } \phi(x)=\bot\end{cases}$$

we obtain an isomorphism from $\mathbf{A}$ onto $\mathbf{DR}(A_H)$.

■

COROLLARY 6.12

Let $\mathbf{A}$ be a directly indecomposable regular Nelson lattice. Then we have:

$$\nabla(x)=\begin{cases}\top & \text{if and only if } x>\neg x,\\ \bot & \text{if and only if } x\le\neg x,\end{cases}\tag{36}$$

$$\Delta(x)=\begin{cases}\top & \text{if and only if } x\ge\neg x,\\ \bot & \text{if and only if } x<\neg x,\end{cases}\tag{37}$$

$$\phi(x)=\begin{cases}\top & \text{if and only if } x>\neg x,\\ x & \text{if and only if } x=\neg x,\\ \bot & \text{if and only if } x<\neg x.\end{cases}\tag{38}$$

Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathcal{NR}$. Since $\mathbf{A}$ is directly indecomposable there exists a generalized Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{CR}(\mathbf{H})$. Then $\mathcal{NR}$ is a subvariety of the variety of Nelson residuated lattices generated by the connected rotations of generalized Heyting algebras. On the other hand, using the result of Theorem 4.5 and the characterization of ϕ given in (29) one can check that the connected rotation of a generalized Heyting algebra is a regular algebra. Therefore we get:

THEOREM 6.13
The variety $\mathcal{NR}$ is the subvariety of Nelson residuated lattices generated by the connected rotations of generalized Heyting algebras.

The above arguments also show that the disconnected rotations of generalized Heyting algebras are in $\mathcal{NR}\cap\mathcal{S}$. Observe also that if $\mathbf{A}$ is a subdirectly irreducible algebra in $\mathcal{NR}\cap\mathcal{S}$, then $\mathbf{A}$ is a disconnected rotation of a generalized Heyting algebra. Hence we have:

COROLLARY 6.14
The variety $\mathcal{NR}\cap\mathcal{S}$ is the subvariety of $\mathcal{N}$ generated by the disconnected rotations of generalized Heyting algebras.

From Theorems 5.12 and 6.3 we obtain:

COROLLARY 6.15
A Nelson lattice $\mathbf{A}$ belongs to $\mathcal{NR}\cap\mathcal{S}$ if and only if there is a stonean Heyting algebra $\mathbf{H}$ such that $\mathbf{A}\cong\mathbf{R}(\mathbf{H},D(\mathbf{H}))$.

6.3 Prelinear Nelson residuated lattices

An important class of bounded residuated lattices is the variety $\mathcal{MTL}$ determined by the *prelinearity equation:*

$$(x\to y)\vee(y\to x)=\top. \tag{39}$$

This variety was introduced in [10] as the variety generated by totally ordered bounded residuated lattices. Algebras in $\mathcal{MTL}$ are the algebraic counterpart of the logic where the conjunction is interpreted by left-continuous t-norms on the real segment $[0,1]$ (see [22]).

The involutive members of $\mathcal{MTL}$ satisfying the nilpotent minimum equation (25) are called *nilpotent minimum algebras*. The subvariety $\mathcal{NM}$ of $\mathcal{MTL}$ formed by the nilpotent minimum algebras was introduced in [10] as the variety generated by a left-continuous and not continuous t-norm, and was further investigated in [3, 15, 38]. Our next aim is to prove that $\mathcal{NM}=\mathcal{N}\cap\mathcal{MTL}$, i.e. that $\mathcal{NM}$ is the subvariety of $\mathcal{N}$ generated by totally ordered Nelson residuated lattices.

THEOREM 6.16
Every totally ordered Nelson lattice is a nilpotent minimum algebra.

PROOF. Let $\mathbf{A}$ be a totally ordered Nelson lattice. Since the homomorphism ϕ preserves the order we have $\Phi(\mathbf{A})=\mathbf{L}_2$ or $\Phi(\mathbf{A})=\mathbf{L}_3$. This means that $\mathbf{A}$ is regular and obviously it is directly indecomposable. From Theorem 6.11 there is a totally ordered generalized Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{CR}(\mathbf{H})$. From [3, Theorem 4.5] we have that $\mathbf{A}$ is a nilpotent minimum algebra. ■

COROLLARY 6.17
$\mathcal{NM}=\mathcal{N}\cap\mathcal{MTL}$.

PROOF. Let $\mathbf{A}\in\mathcal{NM}$. Since $\mathbf{A}$ satisfies (5) and (25), $\mathbf{A}$ is in the variety $\mathcal{KNM}$. It follows from Corollary 3.10 that $\mathbf{A}\in\mathcal{N}$. Therefore, we have $\mathcal{NM}\subseteq\mathcal{N}\cap\mathcal{MTL}$. The opposite inclusion is an immediate consequence of the above theorem. ■

The fact that nilpotent minimum algebras are Nelson residuated lattices has as a consequence that they are representable as $\mathbf{R}(\mathbf{H},F)$, where $\mathbf{H}$ is a Heyting algebra and F is a Boolean filter of $\mathbf{H}$. We are going to consider this representation.

Following Hájek's nomenclature [19], we call *Gödel algebra* a Heyting algebra that satisfies the prelinearity equation

$$(x \rightharpoonup y) \vee (y \rightharpoonup x) = \top,$$

and we denote by $\mathcal{G}$ the variety of Gödel algebras.

LEMMA 6.18
The following are equivalent conditions for each Heyting algebra $\mathbf{H}$ and each Boolean filter F of $\mathbf{H}$:

(i) $\mathbf{H} \in \mathcal{G}$,
(ii) $\mathbf{R}(\mathbf{H},F) \in \mathcal{NM}$,
(iii) $\mathbf{R}(\mathbf{H},F)$ satisfies the equation

$$(x^2 \to y) \vee (y^2 \to x) = \top. \tag{40}$$

PROOF. Suppose that (i) holds. Notice that since the equation $\neg(x \wedge y) = \neg x \vee \neg y$ holds in totally order Heyting algebras, it holds in $\mathbf{H}$. Hence if $x, y, s, t \in H$ and $x \wedge y = s \wedge t = \bot$, we have that

$$((x \rightharpoonup s) \wedge (t \rightharpoonup y)) \vee ((s \rightharpoonup x) \wedge (y \rightharpoonup t)) = ((x \rightharpoonup s) \vee (y \rightharpoonup t)) \wedge ((t \rightharpoonup y) \vee (s \rightharpoonup x)) \geq$$

$$(\neg x \vee \neg y) \wedge (\neg t \vee \neg s) = \neg(x \wedge y) \wedge \neg(t \wedge s) = \top.$$

Then we have that for $(x,y),(s,t) \in R(\mathbf{H},F)$, $((x,y) \to (s,t)) \vee ((s,t) \to (x,y)) = (\top, \bot)$. Therefore (i) implies (ii). Since $(x \to y) \vee (y \to x) \leq (x^2 \to y) \vee (y^2 \to x)$, (ii) implies (iii). Suppose now (iii), and let $x, y \in H$. Since $(x, \neg x)$ and $(y, \neg y)$ are in $R(\mathbf{H},F)$, we have that

$$(\top, \bot) = ((x, \neg x)^2 \to (y, \neg y)) \vee ((y, \neg y)^2 \to (x, \neg x)) \leq ((x \rightharpoonup y) \vee (y \rightharpoonup x), \bot),$$

hence $(x \rightharpoonup y) \vee (y \rightharpoonup x) = \top$, and (iii) implies (i). ∎

From the equivalence between (i) and (ii) of the above lemma we obtain:

THEOREM 6.19
A Nelson lattice $\mathbf{A}$ is a nilpotent minimum algebra if and only if $\mathbf{H_A}$ is a Gödel algebra.

From the equivalence between (ii) and (iii) in Lemma 6.18, we have that $\mathbf{A}$ is a nilpotent minimum algebra if and only if its associated Nelson algebra $\mathbf{N}(\mathbf{A})$ satisfies the equation $(x \Rightarrow y) \vee (y \Rightarrow x) = \top$. Hence, the variety $\mathcal{NM}$ coincides with the variety of Nelson algebras introduced by Monteiro [30] and independently investigated by Kracht [25].

The next Corollary is an immediate consequence of Theorems 6.3 and 6.19.

COROLLARY 6.20
A Nelson lattice $\mathbf{A}$ is in $\mathcal{NM} \cap \mathcal{S}$ if and only if there is a Gödel algebra $\mathbf{H}$ such that $A \cong \mathbf{R}(\mathbf{H}, D(\mathbf{H}))$.

Let $\mathbf{H}_n$ be the n-element totally ordered Heyting algebra, $n \geq 2$. Gispert [15] proved that the algebras $\mathbf{R}(\mathbf{H}_n)$ and $\mathbf{R}(\mathbf{H}_n, D(\mathbf{H}_n))$ are subdirectly irreducible and that they generate $\mathcal{NM}$. The arguments given in [15, Theorem 1] can be easily adapted to show that the algebras $\mathbf{R}(\mathbf{H}_n, D(\mathbf{H}_n))$ generate $\mathcal{NM} \cap \mathcal{S}$. Similar results were obtained by Kracht [25] in the language of Nelson algebras. We denote by $\mathcal{NM}_n^-$ the subvariety of $\mathcal{NM}$ generated by $\mathbf{R}(\mathbf{H}_n, D(\mathbf{H}_n))$, and by $\mathcal{NM}_n^+$ the subvariety generated by $\mathbf{R}(\mathbf{H}_n)$. Clearly $\mathcal{NM}_n^- \subseteq \mathcal{NM}_n^+$. Observe that $\mathcal{NM}_2^- = \mathcal{B}$ and $\mathcal{NM}_2^+ = \mathcal{MV}_3$.

6.4 *The variety $\mathcal{SKNM}$*

We now turn our attention to the variety $\mathcal{SKNM}$, i.e. the variety of Nelson lattices characterized by (24) (see Theorem 3.9). With the notation of (iv) in Theorem 2.3, we have:

THEOREM 6.21
$\mathcal{H}^{\mathcal{SKNM}}$ is the variety of Heyting algebras determined by the equation

$$\neg\neg x \vee (\neg\neg x \to x) = \top. \tag{41}$$

PROOF. Consider $\mathbf{H}$, a Heyting algebra. By (iii) in Lemma 6.2, we have:

$$R(\mathbf{H}, D(\mathbf{H})) = \{(x,y) \in H \times H : \neg x = \neg\neg y\}. \tag{42}$$

Hence if $x \vee y \in D(\mathbf{H})$ we obtain

$$\neg(x,y)^2 \vee ((x,y) \to (x,y)^2) = (\neg x, x) \vee (\neg x \to y, \bot) =$$
$$= (\neg x \vee (\neg x \to y), \bot) = (\neg\neg y \vee (\neg\neg y \to y), \bot).$$

This shows that $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$ satisfies (24) if and only if $\mathbf{H}$ satisfies (41).

Let $\mathcal{V}$ be the variety of Heyting algebras determined by (41) and suppose $\mathbf{H} \in \mathcal{V}$. Since $\mathbf{H}$ is isomorphic to $\mathbf{H}_{\mathbf{R}(\mathbf{H},D(\mathbf{H}))}$, we have that $\mathbf{H} \in \mathcal{H}^{\mathcal{SKNM}}$. Therefore $\mathcal{V} \subseteq \mathcal{H}^{\mathcal{SKNM}}$. To prove the opposite inclusion, suppose that $\mathbf{H}$ is a Heyting algebra not satisfying (24) and let $\mathbf{A}$ be a Nelson lattice such that $\mathbf{H}_{\mathbf{A}}$ is isomorphic to $\mathbf{H}$. Since $\mathbf{A}$ has a subalgebra isomorphic to $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$, we can conclude that $\mathbf{H} \notin \mathcal{V}$ implies $\mathbf{H} \notin \mathcal{H}^{\mathcal{SKNM}}$. ■

COROLLARY 6.22
The variety $\mathcal{SKNM}$ is a proper subvariety of the variety $\mathcal{N}$ of Nelson residuated lattices.

REMARK 6.23
$\mathcal{SKNM} \subsetneqq \{\mathbf{A} \in \mathcal{N} : \mathbf{H}_{\mathbf{A}} \in \mathcal{H}^{\mathcal{SKNM}}\}$. For instance, if $\mathbf{H}$ is the Heyting algebra obtained by adding a new top to the two-atom Boolean algebra, then $\mathbf{H} \in \mathcal{H}^{\mathcal{SKNM}}$, and $\mathbf{R}(\mathbf{H}) \notin \mathcal{SKNM}$. This result should be compared with Theorems 5.12 and 6.19.

6.5 *Some relations among subvarieties of $\mathcal{N}$*

We conclude this section summarizing in a few theorems the relations between the different subvarieties of Nelson residuated lattices studied so far.

As proved in Theorem 3.9 and in Corollary 3.10 the varieties $\mathcal{SKNM}$ and $\mathcal{KNM}$ characterized by (24) and (25), respectively are subvarieties of $\mathcal{N}$. As one would expect we have:

THEOREM 6.24
The variety $\mathcal{KNM}$ is a proper subvariety of $\mathcal{SKNM}$.

PROOF. We already noted that $\mathcal{KNM} \subseteq \mathcal{SKNM}$. Let $\mathbf{H}$ be the Heyting algebra obtained by adding a new bottom and a new top to a two-atom Boolean algebra. Then $\mathbf{R}(\mathbf{H}, D(\mathbf{H}))$ is an example of a Nelson lattice that belongs to $\mathcal{SKNM}$ and does not belong to $\mathcal{KNM}$. ■

Notice that the example given in the previous theorem is a connected rotation of the generalized Heyting algebra given by adding a top element to the two-atom Boolean algebra. This implies that there is a regular Nelson algebra which is not in $\mathcal{KNM}$. Moreover we have:

THEOREM 6.25
$\mathcal{NR} \nsubseteq \mathcal{KNM}$ and $\mathcal{KNM} \nsubseteq \mathcal{NR}$.

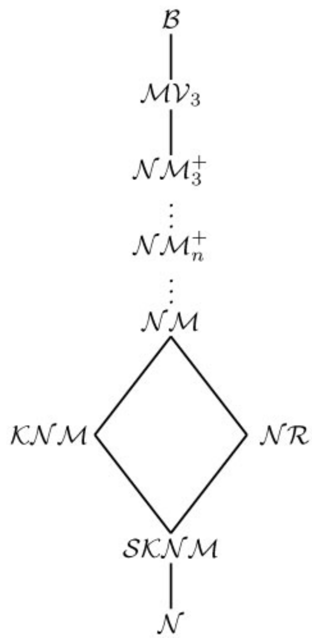

FIGURE 1. Subvarieties of $\mathcal{N}$

PROOF. Since we have already verified that $\mathcal{KNM} \nsubseteq \mathcal{NR}$ we just have to see that $\mathcal{NR} \nsubseteq \mathcal{KNM}$. To achieve this aim, let Γ be the finite distributive lattice obtained by adding a new bottom and a new top elements to the two-atom Boolean algebra. Call a and b these two atoms. It turns out that Γ possesses only one Kleene algebra structure. Since it satisfies the interpolation property [6] it is a Nelson algebra $\mathbf{A}_\Gamma$. The reader can check that $\mathbf{A}_\Gamma$ satisfies (25) and it is not regular. ■

With the previous results in mind, after calling $\mathcal{B}$ the variety of Boolean algebras we have:

THEOREM 6.26
The following chains of proper inclusions hold in the lattice of subvarieties of $\mathcal{N}$ (see Figure 1).

$$\mathcal{B} \subsetneqq \mathcal{MV}_3 \subsetneqq \mathcal{NM}_3^+ \subsetneqq \cdots \mathcal{NM}_n^+ \subsetneqq \mathcal{NM}_{n+1}^+ \subsetneqq \cdots \subsetneqq \mathcal{NM} \subsetneqq \mathcal{NR} \subsetneqq \mathcal{SKNM} \subsetneqq \mathcal{N}$$

and

$$\mathcal{NM} \subsetneqq \mathcal{KNM} \subsetneqq \mathcal{SKNM} \subsetneqq \mathcal{N}.$$

PROOF. We start by proving the first chain of inclusions. That $\mathcal{B} \subseteq \mathcal{MV}_3$ is well known. Varieties of nilpotent minimum algebras containing $\mathcal{MV}_3$ are analysed in [15].

The inclusion $\mathcal{NM} \subseteq \mathcal{NR}$ is an easy consequence of the proof of Theorem 6.16. This inclusion is also proper. Indeed, the connected rotation of a generalized Heyting algebra not satisfying the prelinearity equation provides an example of a regular Nelson lattice which is not a nilpotent minimum algebra.

Since every connected rotation of generalized Heyting algebras satisfies (24) from Theorem 6.13, we have $\mathcal{NR} \subseteq \mathcal{SKNM}$. To see that the inclusion is proper, consider the algebra $\mathbf{A}_\Gamma$ given in the proof of Theorem 6.25. We have that $\mathbf{A}_\Gamma \in \mathcal{KNM}$ and $\mathbf{A}_\Gamma \notin \mathcal{NR}$. Since by Theorem 6.24,

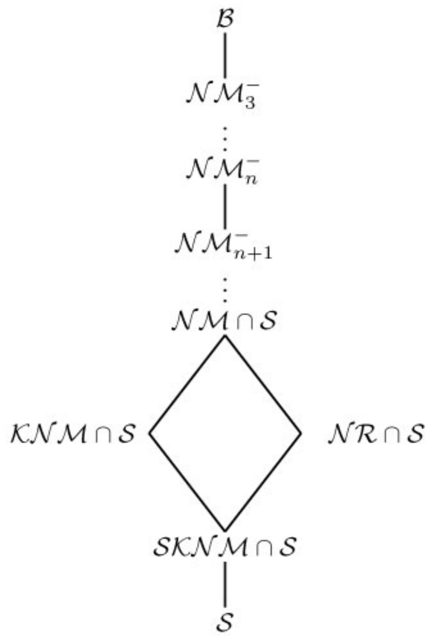

FIGURE 2. Subvarieties of $\mathcal{S}$

$\mathcal{KNM} \subseteq \mathcal{SKNM}$, $\mathbf{A}_\Gamma$ provides the needed example. Finally consider the Nelson lattice $\mathbf{R}(\mathbf{H}_{\mathbf{A}_\Gamma})$. This algebra does not satisfy (24). The counterexample is given considering the pair $(a, \bot)$.

For the second chain of inequalities it only remains to prove that $\mathcal{NM} \subsetneqq \mathcal{KNM}$. But this follows easily considering once more the algebra $\mathbf{A}_\Gamma$ which is not in $\mathcal{NM}$ since it is not regular. ■

Recall from Section 6.1 that $\mathbf{L}_3$ is a splitting algebra with $\mathcal{S}$ as its conjugate variety. Following the proof of the previous theorem, we obtain:

THEOREM 6.27
The following chains of proper inclusions hold in the lattice of subvarieties of $\mathcal{S}$ (see Figure 2)

$$\mathcal{B} \subsetneqq \mathcal{NM}_3^- \subsetneqq \cdots \subsetneqq \mathcal{NM}_n^- \subsetneqq \mathcal{NM}_{n+1}^- \subsetneqq \cdots \subsetneqq \mathcal{NM} \cap \mathcal{S} \subsetneqq$$

$$\mathcal{NR} \cap \mathcal{S} \subsetneqq \mathcal{SKNM} \cap \mathcal{S} \subsetneqq \mathcal{S} \subsetneqq \mathcal{N}$$

and

$$\mathcal{NM} \cap \mathcal{S} \subsetneqq$$

$$\mathcal{KNM} \cap \mathcal{S} \subsetneqq \mathcal{SKNM} \cap \mathcal{S} \subsetneqq \mathcal{S} \subsetneqq \mathcal{N}.$$

7 Free regular Nelson lattices

For $\mathcal{K}$ a variety of algebras we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in $\mathcal{K}$ over an arbitrary set of generators X. Our aim is to generalize the results of [3] to describe free algebras in certain subvarieties of $\mathcal{NR}$. We shall refer the reader to that paper repeatedly. In what follows we assume familiarity of the reader with weak Boolean product decompositions of residuated lattices.

7.1 Representation of regular Nelson lattices as weak Boolean products of directly indecomposable algebras

It easily follows from Corollary 5.7 that for every Nelson residuated lattice $\mathbf{A}$, $\mathbf{B}(\mathbf{A})$ is a subalgebra of $\Phi(\mathbf{A})$.

If U is a filter of the Boolean algebra $\mathbf{B}(\mathbf{A})$, we shall denote by $\langle U\rangle$ the i-filter of $\mathbf{A}$ generated by U. As usual, given a Boolean algebra $\mathbf{B}$ one can provide the set of its ultrafilters with the Stone topology to obtain the corresponding Boolean space $Sp(\mathbf{B})$.

The next theorem follows from [24] (see also [3, 9]).

THEOREM 7.1
Let $\mathcal{V}\subseteq\mathcal{NR}$. The free regular *Nelson lattice* $\mathbf{Free}_{\mathcal{V}}(X)$ can be represented as a weak Boolean product of the family

$$(\mathbf{Free}_{\mathcal{V}}(X)/\langle U\rangle): U\in Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$$

over the boolean space $Sp\ \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$.

7.2 Boolean skeleton of $\mathrm{Free}_{\mathcal{V}}(X)$

If $\mathcal{V}$ is a variety of regular Nelson lattices one can always consider the set

$$\mathcal{MV}_{\mathcal{V}}=\{\mathbf{C}\in\mathcal{MV}_3:\mathbf{C}=\Phi(\mathbf{A})\text{ for some }\mathbf{A}\in\mathcal{V}\}. \tag{43}$$

For each variety $\mathcal{V}\subseteq\mathcal{NR}$, $\mathcal{MV}_{\mathcal{V}}=\mathcal{V}\cap\mathcal{MV}_3$. Indeed, since for every $\mathbf{A}\in\mathcal{NR}$, $\Phi(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ that belongs to $\mathcal{MV}_3$, $\mathcal{MV}_{\mathcal{V}}\subseteq\mathcal{V}\cap\mathcal{MV}_3$. The opposite inclusion follows from the fact that ϕ is the identity on each $\mathbf{A}\in\mathcal{MV}_3$. From Theorem 6.1 it follows that

$$\mathcal{MV}_{\mathcal{V}}=\mathcal{MV}_3\quad\text{if and only if}\quad\mathcal{V}\nsubseteq\mathcal{S}\quad\text{if and only if}\quad\mathbf{L}_3\in\mathcal{V}.$$

Otherwise $\mathcal{MV}_{\mathcal{V}}$ is the variety of Boolean algebras $\mathcal{B}$. The proof of [3, Theorem 5.2] can be easily adapted to obtain:

THEOREM 7.2
Let X be a set of free generators of the free algebra $\mathbf{Free}_{\mathcal{V}}(X)$ and let $Z=\{\phi(x):x\in X\}$. Then

$$\Phi(\mathbf{Free}_{\mathcal{V}}(X))=\mathbf{Free}_{\mathcal{MV}_{\mathcal{V}}}(Z).$$

In [4] it is proved that $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_3}(Z))$ is the free Boolean algebra over the poset $Y=\{\nabla(z),\Delta(z): z\in Z\}$. Considering $Z=\{\phi(x):x\in X\}$ as in Theorem 7.2, from (32) the set Y can be rewritten as $Y=\{\nabla(x),\Delta(x):x\in X\}$. From Corollary 5.9 if $\mathcal{MV}_{\mathcal{V}}$ is $\mathcal{B}$, then $\Delta(x)=\nabla(x)$ holds in $\mathcal{V}$. An application of Theorems 7.2 and 6.6 yields:

THEOREM 7.3
Let $\mathcal{V}\subseteq\mathcal{NR}$. Then

(a) If $\mathcal{MV}_{\mathcal{V}}=\mathcal{B}$, then $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))=\mathbf{Free}_{\mathcal{B}}(Y)$, with $Y=\{\nabla(x):x\in X\}$.
(b) If $\mathcal{MV}_{\mathcal{V}}=\mathcal{MV}_3$, then $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra over the poset $Y=\{\nabla(x),\Delta(x): x\in X\}$.

This theorem together with the fact that ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ are bijective correspondence with monotone maps from $Y=\{\nabla(x),\Delta(x):x\in X\}$ into the two-element Boolean algebra, give a complete description of the spectrum of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ (cf [3]).

Consider the poset $Y=\{\nabla(x),\Delta(x):x\in X\}$. For each upward closed subset $S\subseteq Y$ consider the set G_S given by the join of the following four sets:

$$\{\nabla(x):\nabla(x)\in S\},\quad \{\neg\nabla(x):\nabla(x)\notin S\},$$

$$\{\Delta(x):\Delta(x)\in S\},\quad \{\neg\Delta(x):\Delta(x)\notin S\}.$$

Then the correspondence that assigns to each upward closed subset $S\subseteq Y$ the Boolean filter U_S generated by G_S defines a bijection from the set of upward closed subsets of Y onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_{\mathcal{NR}}(X))$. Notice that in case $\nabla(x)=\Delta(x)$ holds in $\mathcal{V}$ for each subset $S\subseteq Y=\{\Delta(x):x\in X\}$ the set G_S is simply $\{\Delta(x):\Delta(x)\in S\}\cup\{\neg\Delta(x):\Delta(x)\notin S\}$.

We refer to each ultrafilter of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ by U_S making explicit reference to the subset S that corresponds to it.

7.3 *Directly indecomposable stalks of* $\mathrm{Free}_{\mathcal{V}}(X)$

Based on the description of the spectrum of $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ we will characterize the stalk $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ for subvarieties $\mathcal{V}\subseteq\mathcal{NR}$ satisfying a special property.

If $\mathcal{GH}$ denotes the variety of generalized Heyting algebras, consider the set

$$\mathcal{GH}_{\mathcal{V}}=\{\mathbf{G}\in\mathcal{GH}:\mathbf{A}=\mathbf{DR}(\mathbf{G}) \text{ for some } \mathbf{A}\in\mathcal{V}\} \tag{44}$$

LEMMA 7.4
For each variety $\mathcal{V}\subseteq\mathcal{NR}$ the set $\mathcal{GH}_{\mathcal{V}}$ is a subvariety of generalized Heyting algebras.

PROOF. It is easy to corroborate that $\mathcal{GH}_{\mathcal{V}}$ is closed under subalgebras and homomorphic images. To see that if $\mathbf{G}$ and $\mathbf{H}$ are in $\mathcal{GH}_{\mathcal{V}}$ their product is also in the variety simply observe that the map $f:\mathbf{DR}(\mathbf{G}\times\mathbf{H})\to\mathbf{DR}(\mathbf{G})\times\mathbf{DR}(\mathbf{H})$ given by $f((x,y),0)=((x,0),(y,0))$ and $f(0,(x,y))=((0,x),(0,y))$ is a monomorphism. This argument can be generalized for the product of an arbitrary family of algebras. ■

We shall say that a variety $\mathcal{V}\subseteq\mathcal{NR}$ is *good* if one of the following conditions is satisfied:

(1) $\mathcal{V}\subseteq\mathcal{S}$ (or equivalently $\mathcal{MV}_{\mathcal{V}}=\mathcal{B}$) or
(2) $\mathbf{G}\in\mathcal{GH}_{\mathcal{V}}$ if and only if $\mathbf{CR}(\mathbf{G})\in\mathcal{V}$.

The varieties $\mathcal{NR}$, $\mathcal{NM}$ and $\mathcal{MV}_3$ are examples of good varieties. In [15, Theorem 3] one can see that there are subvarieties of nilpotent minimum algebras which are not good.

From now on we assume that $\mathcal{V}$ is a good variety. We fix $\langle U_S\rangle$ for some upward closed set $S\subseteq\{\nabla(x),\Delta(x):x\in X\}$. The set

$$X/\langle U_S\rangle=\{x/\langle U_S\rangle:x\in X\}$$

generates $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$.

THEOREM 7.5
The directly indecomposable Nelson lattice $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ has a negation fixpoint if and only if there exists $x\in X$ such that $\nabla(x)\notin S$ and $\Delta(x)\in S$.

PROOF. From Corollary 6.12 one can assert that $\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ has a negation fixpoint if and only if $\Delta(y)\neq\nabla(y)$ for some $y\in\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle$ and this happens if and only if $\Delta(x/\langle U_S\rangle)\neq\nabla(x/\langle U_S\rangle)$ for some $x\in X/\langle U_S\rangle$. Based on the result of Theorem 5.5 (iv), the remaining of the proof is analogous to that of [3, Theorem 5.6]. ■

Recall from Theorem 6.9 that $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$ is the generalized Heyting algebra whose universe is the preimage by $\top$ of ϕ. Let X_S be given by

$$X_S=\{x/\langle U_S\rangle:\phi(x/\langle U_S\rangle)=\top\}\cup\{\neg x/\langle U_S\rangle:\phi(x/\langle U_S\rangle)=\bot\}. \tag{45}$$

THEOREM 7.6
Assume that $\mathcal{V}$ is a good subvariety of $\mathcal{NR}$. Then $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$ is the free generalized Heyting algebra in $\mathcal{GH}_{\mathcal{V}}$ generated by X_S, i.e.

$$(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}\cong\mathbf{Free}_{\mathcal{GH}_{\mathcal{V}}}(X_S).$$

PROOF. Assume first that $\mathcal{MV}_{\mathcal{V}}=\mathcal{MV}_3$. Let $\mathbf{G}\in\mathcal{GH}_{\mathcal{V}}$ be a generalized Heyting algebra and let $f:X_S\to G$ be an arbitrary function. From the definition of $\mathcal{GH}_{\mathcal{V}}$ and the fact that $\mathcal{V}$ is good we know that $\mathbf{CR}(\mathbf{G})\in\mathcal{V}$. Define $f':X\to\mathbf{CR}(\mathbf{G})$ by

$$f'(x)=\begin{cases}(f(x/\langle U_S\rangle),0) & \text{if } \phi(x/\langle U_S\rangle)=\top,\\ (0,0) & \text{if } \phi(x/\langle U_S\rangle)=\frac{1}{2},\\ (0,f(\neg x/\langle U_S\rangle)) & \text{if } \phi(x/\langle U_S\rangle)=\bot,\end{cases}$$

By the definition of free algebra, there exists a homomorphism $g':\mathbf{Free}_{\mathcal{V}}(X)\to\mathbf{CR}(\mathbf{G})$ such that $g'(x)=f'(x)$ for all $x\in X$.

Following the proof of [3, Theorem 5.8] and based on Corollary 6.12 we have $g'(\langle U_S\rangle)\subseteq\{\top\}$. Then there exists a unique homomorphism $g:\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle\to\mathbf{CR}(\mathbf{G})$ such that $g(y/\langle U_S\rangle)=g'(y)$ for all $y\in\mathbf{Free}_{\mathcal{V}}(X)$.

An analogous proof to that of [3, Theorem 4.15] shows that the generalized Heyting algebra generated by X_S is $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$. Consider h, the restriction of g to $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$. Observe that

$$h((\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}})\subseteq G\times\{0\}$$

and $G\times\{0\}$ is the universe of the generalized Heyting algebra $(\mathbf{CR}(\mathbf{G}))_{\mathbf{H}}$.

Let $\gamma:(\mathbf{CR}(\mathbf{G}))_{\mathbf{H}}\to\mathbf{G}$ be the isomorphism given by $\gamma(x,0)=x$. Then $\gamma\circ h:(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}\to\mathbf{G}$. If $x/\langle U_S\rangle\in X_S$ then $\phi(x/\langle U_S\rangle)=\top$, thus

$$\gamma\circ h(x/\langle U_S\rangle)=\gamma(g'(x))=\gamma(f'(x))=\gamma(f(x/\langle U_S\rangle),0)=f(x/\langle U_S\rangle).$$

If $\neg(x/\langle U_S\rangle)\in X_S$, then $\phi(x/\langle U_S\rangle)=\bot$. Hence

$$\gamma\circ h(\neg(x/\langle U_S\rangle))=\gamma(g'(\neg x))=\gamma(\neg(g'(x)))=\gamma(\neg(f'(x)))=$$

$$=\gamma(\neg(0,f(\neg x/\langle U_S\rangle)))=f(\neg x/\langle U_S\rangle).$$

We have found a homomorphism $\gamma\circ h$ from $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$ into the generalized Heyting algebra $\mathbf{G}$ that extends the map $f:X_S\to\mathbf{G}$. We conclude that $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$ is the free generalized Heyting algebra in $\mathcal{GH}_{\mathcal{V}}$ over X_S.

Assume now that $\mathcal{MV}_{\mathcal{V}}=\mathcal{B}$. This being the case, take $\mathbf{G}\in\mathcal{GH}_{\mathcal{V}}$, let $f:X_S\to G$ be an arbitrary function and define $f':X\to\mathbf{DR}(\mathbf{G})$ by

$$f'(x)=\begin{cases}(f(x/\langle U_S\rangle),0) & \text{if } \phi(x/\langle U_S\rangle)=\top,\\ (0,f(\neg x/\langle U_S\rangle)) & \text{if } \phi(x/\langle U_S\rangle)=\bot.\end{cases}$$

Following step by step the proof of the previous case we can conclude that the algebra $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S\rangle)_{\mathbf{H}}$ is the free generalized Heyting algebra in $\mathcal{GH}_{\mathcal{V}}$ over X_S.

■

Based on the results of this section and on the characterization of directly indecomposable regular Nelson residuated lattices given in Theorem 6.11 we conclude:

THEOREM 7.7
Let $\mathcal{V}\subseteq\mathcal{NR}$ be a good variety and let $\mathcal{MV}_{\mathcal{V}}$ and $\mathcal{GH}_{\mathcal{V}}$ be given as in (43) and (44).

(1) If $\mathcal{V}\subseteq\mathcal{S}$, then $\mathbf{Free}_{\mathcal{V}}(X)$ is a weak Boolean product of algebras of the form

$$\mathbf{DR}(\mathbf{Free}_{\mathcal{GH}_{\mathcal{V}}}(X_S))$$

over the Boolean space $Sp\,\mathbf{Free}_{\mathcal{B}}(Y)$, where $Y=\{\nabla(x):x\in X\}$ and for each $U_S\in Sp\,\mathbf{Free}_{\mathcal{B}}(Y)$ and X_S is given by (45).

(2) If $\mathbf{L}_3\in\mathcal{V}$, then $\mathbf{Free}_{\mathcal{V}}(X)$ is a weak Boolean product of algebras of the form

$$\mathbf{CR}(\mathbf{Free}_{\mathcal{GH}_{\mathcal{V}}}(X_S)) \quad\text{or}\quad \mathbf{DR}(\mathbf{Free}_{\mathcal{GH}_{\mathcal{V}}}(X_S))$$

over the Boolean space corresponding to the free Boolean algebra over the poset $Y=\{\nabla(x),\Delta(x):x\in X\}$ and X_S is given by (45).

As the reader would have already noticed, this description of free algebras generalizes the one in [3] for free nilpotent minimum algebras. Another case of description of free algebras in subvarieties of $\mathcal{NR}$ can be found in [9]. In that paper the authors consider the subvariety of MTL-algebras satisfying $\Delta(x)=\nabla(x)$ and $\neg\neg(\neg\neg x\to x)=\top$. Some of the results in that paper are also particular cases of Theorem 7.7.

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