WEIGHTED A PRIORI ESTIMATES FOR SOLUTIONS OF $(-\Delta)^m u = f$ WITH HOMOGENEOUS DIRICHLET CONDITIONS .

RICARDO G. DURÁN, MARCELA SANMARTINO, AND MARISA TOSCHI

Abstract. Let u be a weak solution of $(-\Delta)^m u = f$ with Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.
Then, the main goal of this paper is to prove the following a priori estimate:

$$\|u\|_{W^{2m,p}_\omega(\Omega)} \le C \|f\|_{L^p_\omega(\Omega)},$$

where ω is a weight in the Muckenhoupt class A_p.

1. Introduction

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_+$ we denote $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} ... \partial^{\alpha_n}_{x_n}$ and

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega) \, : \, D^\alpha v \in L^p(\Omega) \quad \forall \, |\alpha| \le k\}.$$

For $u \in W^{k,p}(\Omega)$, its norm is given by

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)} .$$

We consider the homogeneous problem

$$\left\{ \begin{array}{ll} (-\Delta)^m u = f & \text{in } \Omega \\ \\ \left(\frac{\partial}{\partial \nu}\right)^j u = 0 & \text{in } \partial\Omega \quad 0 \le j \le m-1, \end{array} \right. \tag{1.1}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

In the classic paper [1], the authors obtained a priori estimates for solutions of (1.1) for smooth domain Ω given by

$$\|u\|_{W^{2m,p}(\Omega)} \le C \|f\|_{L^p(\Omega)} .$$

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A key tool to prove those estimates was the Calderón-Zygmund theory for singular integral operators.

On the other hand, after the pioneering work of Muckenhoupt [7], a lot of work on continuity in weighted norms has been developed. In particular, weighted estimates for a wide class of singular integral operators has been obtained for weights in the class of Muckenhoupt A_p. Therefore, it is a natural question whether analogous weighted a priori estimates can be proved for the derivatives of solutions of elliptic equations.

For the Laplace equation ($m = 1$), it was proved in [5] that for a weight ω belonging to the Muckenhoupt class A_p

$$\|u\|_{W^{2,p}_\omega(\Omega)} \leq C \, \|f\|_{L^p_\omega(\Omega)}$$

on a bounded domain Ω with $\partial\Omega \in C^2$.

The goal of this paper is to extend the results of [5] for powers of the Laplacian operator with homogeneous Dirichlet boundary conditions, i.e. it is to prove that

$$\|u\|_{W^{2m,p}_\omega(\Omega)} \leq C \, \|f\|_{L^p_\omega(\Omega)}, \tag{1.2}$$

for $\omega \in A_p$, where the constant C depends on Ω, m, n and the weight ω.

The main ideas for the proof of these estimates are similar to those given in [5]. However, non trivial technical modifications are needed because, for $m \geq 2$, the Green function is not positive in general and therefore, we cannot apply the maximum principle.

2. Preliminaries

In what follows we consider the problem (1.1) in a bounded domain Ω with $\partial\Omega \in C^{6m+4}$ for $n = 2$ and $\partial\Omega \in C^{5m+2}$ for $n > 2$ (the regularity on the boundary is necessary in order to use the results of the Green function given in [6]).

The solution of (1.1) is given by

$$u(x) = \int_{\Omega} G_m(x,y)\, f(y)\, dy \tag{2.1}$$

where $G_m(x,y)$ is the Green function of the operator $(-\Delta)^m$ in Ω which can be written as

$$G_m(x,y) = \Gamma(x-y) + h(x,y) \tag{2.2}$$

where $\Gamma(x-y)$ is a fundamental solution and $h(x,y)$ satisfies

$$\begin{cases} (-\Delta_x)^m h(x,y) = 0 & x \in \Omega \\ \left(\frac{\partial}{\partial\nu}\right)^j h(x,y) = -\left(\frac{\partial}{\partial\nu}\right)^j \Gamma(x-y) & x \in \partial\Omega \quad 0 \le j \le m-1 \end{cases}$$

for each fixed $y \in \Omega$.

Then

$$h(x,y) = -\sum_{j=0}^{m-1} \int_{\partial\Omega} K_j(y,P) \left(\frac{\partial}{\partial\nu}\right)^j \Gamma(P-x)\, dS \tag{2.3}$$

where $K_j(y,P)$ are the Poisson kernels and dS denotes the surface measure on $\partial\Omega$.

We recall that any fundamental solution associated to (1.1) is smooth away from the origin and it is homogeneous of degree $2m-n$ if n is odd or if $2m < n$ and the logarithmic function appears if n is even and $2m \ge n$. However, in both cases, under our assumption on the boundary domain, we have the known estimates of the Green function $G_m(x,y)$ and the Poisson kernels $K_j(x,y)$. In what follows the letter C will denote a generic constant not necessarily the same at each occurrence.

$$|D_x^\alpha G_m(x,y)| \le C \qquad \text{for } |\alpha| < 2m-n, \tag{2.4}$$

$$|D_x^\alpha G_m(x,y)| \le C \log\left(\frac{2\, diam(\Omega)}{|x-y|}\right) \qquad \text{for } |\alpha| = 2m-n, \tag{2.5}$$

$$|D_x^\alpha G_m(x,y)| \le C\, |x-y|^{2m-n-|\alpha|} \qquad \text{for } |\alpha| > 2m-n, \tag{2.6}$$

$$|D_x^\alpha G_m(x,y)| \le C\, \frac{1}{|x-y|^n} \ \min\left\{1, \frac{d(y)}{|x-y|}\right\}^m \qquad \text{for } |\alpha| = 2m, \tag{2.7}$$

$$|K_j(x,y)| \leq C \, \frac{d(x)^m}{|x-y|^{n-j+m-1}} \qquad \text{for } 0 \leq j \leq m-1, \tag{2.8}$$

where $d(x) := \mathrm{dist}(x, \partial\Omega)$ (see [6] for (2.4), (2.5) and (2.6) and [4] for (2.7) and (2.8)).

3. The estimates for the derivatives of u

In this section we state pointwise estimates for the first $2m-1$ derivatives of the function u and a weak estimate for the $2m$ derivative. These estimates will be allow to proof the main result of this work.

Lemma 3.1. *Let $u(x)$ be the solution of (1.1). Then, for $|\alpha| \leq 2m-1$ we have*

$$|D_x^\alpha u(x)| \leq C \, Mf(x),$$

where $Mf(x)$ is the usual Hardy- Littlewood maximal function of f.

Proof:

$$\begin{aligned} |D_x^\alpha u(x)| &\leq \int_\Omega |D^\alpha G_m(x,y)| \, |f(y)| \, dy \\ &\leq C \int_\Omega \frac{|f(y)|}{|x-y|^{n-1}} \, dy \leq C \, Mf(x), \end{aligned}$$

by (2.4), if $2m-n+1 \leq |\alpha| \leq 2m-1$ and by (2.5) and (2.6), if $|\alpha| \leq 2m-n$.

□

Proposition 3.2. *Given two measurable functions f and g in Ω, for $|\alpha| = 2m$ we have that*

$$\int_D |D_x^\alpha G_m(x,y) \, f(y) \, g(x)| \, dy \, dx \leq C \left(\int_\Omega Mf(x) \, |g(x)| \, dx + \int_\Omega Mg(y) \, |f(y)| \, dy \right),$$

where $D := \{(x,y) \in \Omega \times \Omega \,:\, |x-y| > d(x)\}$.

Proof: We write $D = D_1 \cup D_2$, where

$$D_1 = \{(x,y) \in D : d(y) \leq 2\, d(x)\} \quad \text{ and } \quad D_2 = \{(x,y) \in D : d(y) > 2\, d(x)\}.$$

Then, using (2.7) we have

$$\int_D |D_x^\alpha G_m(x,y)\, f(y)\, g(x)|\, dy\, dx \leq \int_D \frac{d(y)^m}{|x-y|^{n+m}}\, |f(y)|\, |g(x)|\, dy\, dx$$

$$\leq 2^m \int_{D_1} \frac{d(x)^m}{|x-y|^{n+m}}\, |f(y)|\, |g(x)|\, dy\, dx$$

$$+ \int_{D_2} \frac{d(y)^m}{|x-y|^{n+m}}\, |f(y)|\, |g(x)|\, dy\, dx = I + II. \tag{3.1}$$

Calling $\Omega_k(x) = \{z \in \Omega\, :\, 2^k d(x) \leq |x-z| < 2^{k+1} d(x)\}$,

$$\int_{D_1} \frac{d(x)^m}{|x-y|^{n+m}}\, |f(y)|\, |g(x)|\, dy\, dx \leq \int_\Omega \sum_{k=1}^{\infty} \int_{\Omega_k(x)} \frac{d(x)}{|x-y|^{n+1}}\, |f(y)|\, dy\, |g(x)|\, dx$$

$$= \int_\Omega A(x)\, |g(x)|\, dx$$

with

$$A(x) \leq \sum_{k=1}^{\infty} \int_{\{|x-y|<2^{k+1}d(x)\}} \frac{d(x)}{|x-y|^{n+1}}\, |f(y)|\, dy \leq C \sum_{k=1}^{\infty} \frac{1}{2^k}\, Mf(x) = C\, Mf(x).$$

In order to estimate the term II in (3.1), we first observe that for $(x,y) \in D_2$, we have that $|x-y| \geq \frac{1}{2}\, d(y)$. Then

$$\int_{D_2} \frac{d(y)^m}{|x-y|^{n+m}}\, |f(y)|\, |g(x)|\, dy\, dx \leq C \int_\Omega \sum_{k=1}^{\infty} \int_{\Omega_{k-1}(y)} \frac{d(y)}{|x-y|^{n+1}}\, |g(x)|\, dx\, |f(y)|\, dy$$

$$= \int_\Omega B(y)\, |f(y)|\, dy$$

and therefore, by the same arguments used before we have that

$$B(y) \leq C\, Mg(y)$$

and the Proposition is proved.

$\square$

In order to see how to estimate in $\Omega \setminus D$, we consider separately the function h and Γ involved in G_m.

Proposition 3.3. *If $|\alpha| \geq 2m - n + 1$, there exists a constant C such that*

$$|D^\alpha h(x,y)| \leq C\, d(x)^{2m-n-|\alpha|} \tag{3.2}$$

for $|x - y| \leq d(x)$.

Proof: In view of (2.3) we must find estimates for $D_x^\alpha(\frac{\partial}{\partial\nu})^j \Gamma(P-x)$ and $K_j(y,P)$.

From the general properties of the fundamental solution $\Gamma(x-y)$ we have that

$$\left| D_x^\alpha(\frac{\partial}{\partial\nu})^j \Gamma(P-x) \right| \leq C\,|P-x|^{2m-n-|\alpha|-j} \tag{3.3}$$

for $|\alpha| + j \geq 2m - n + 1$, and for $0 \leq j \leq m-1$, by (2.8) we have that

$$|K_j(y,P)| \leq C\, \frac{d(y)^m}{|y-P|^{n-j+m-1}} \tag{3.4}$$

for $y \in \Omega$ and $P \in \partial\Omega$.

Then by (3.3), (3.4) and the fact that if $|x-y| \leq d(x)$ then $d(y) < 2\,d(x)$, we have for $|\alpha| + j \geq 2m - n + 1$

$$\begin{aligned} |D_x^\alpha h(x,y)| &\leq C \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{d(y)^m}{|y-P|^{n-1+m-j}}\, |P-x|^{2m-n-|\alpha|-j}\, dS \\ &\leq C\, d(x)^{2m-n-|\alpha|} \sum_{j=0}^{m-1} \int_{\partial\Omega} \frac{d(y)^{m-j}}{|y-P|^{n-1+m-j}}\, dS. \end{aligned}$$

In order to see that each integral is finite we write $\partial\Omega = F_1 \cup F_2$, with

$$F_1 = \{P \in \partial\Omega : |P_0 - P| > 2\,d(y)\} \quad \textit{and} \quad F_2 = \{P \in \partial\Omega : |P_0 - P| \leq 2\,d(y)\},$$

where $P_0 \in \partial\Omega$ is that $|y - P_0| = d(y)$. And now, the convergence of these integrals follow in a standard way.

□

It follows from the previous Proposition that for each $x \in \Omega$ and $|\alpha| \geq 2m-n+1$ we have that $D_x^\alpha h(x,y)$ is bounded uniformly in a neighborhood of x and so

$$D_x^\alpha \int_\Omega h(x,y)\, f(y)\, dy = \int_\Omega D_x^\alpha h(x,y)\, f(y)\, dy. \tag{3.5}$$

On the other hand, although $D_x^\alpha \Gamma$ is a singular kernel for $|\alpha| = 2m$, taking β such that $\beta_i = \alpha_i - 1$ and $\beta_j = \alpha_j$ if $j \neq i$, we have that

$$D_{x_i} \int_\Omega D_x^\beta \Gamma(x-y)\, f(y)\, dy = Kf(x) + c(x)f(x) \tag{3.6}$$

where c is a bounded function and K is a Calderón - Zygmund operator given by

$$Kf(x) = \lim_{\epsilon \to 0} K_\epsilon f(x), \quad \text{with} \quad K_\epsilon f(x) = \int_{|x-y|>\epsilon} D_x^\alpha \Gamma(x-y)\, f(y)\, dy. \tag{3.7}$$

We will also make use of the maximal operator $\widetilde{K}f(x) = \sup_{\epsilon>0} |K_\epsilon f(x)|$. Here and in what follows we consider f defined in $\mathbb{R}^n$ extending the original f by zero.

Now we are in conditions to give the following estimate:

Theorem 3.4. *Given g a measurable function and $|\alpha| = 2m$. Then there exists a constant C depending only on n, m and Ω such that, for any $x \in \Omega$,*

$$\begin{aligned}\int_\Omega |D_x^\alpha u(x)\, g(x)|\, dx \quad &\leq \quad C \left(\int_\Omega \widetilde{K}f(x)\, |g(x)|\, dx + \int_\Omega Mf(x)\, |g(x)|\, dx \right. \\ &+ \quad \left. \int_\Omega Mg(y)\, |f(y)|\, dy + \int_\Omega |f(x)|\, |g(x)|\, dx \right).\end{aligned}$$

Proof: Using the representation formula for u, by (3.5), (3.6) and (3.7) we have that

$$\begin{aligned} D_x^\alpha u(x) &= \lim_{\epsilon \to 0} \int_{\epsilon<|x-y|\leq d(x)} D_x^\alpha \Gamma(x-y)\, f(y)\, dy + c(x)f(x) \\ &+ \int_{|x-y|\leq d(x)} D_x^\alpha h(x,y)\, f(y)\, dy + \int_{|x-y|>d(x)} D_x^\alpha G(x,y)\, f(y)\, dy \\ &=: I + II + III + IV. \end{aligned} \tag{3.8}$$

By the results given above, for I, II and III we have pointwise estimates, and obtain (in the same way that in [5]) that

$$|I+II+III| \leq C\left(\widetilde{K}f(x)+|f(x)|+M\,f(x)\right).$$

However, for IV we have just a weak estimate. Indeed, for the Proposition 3.2 we have

$$\int_\Omega |IV|\,|g(x)|\,dx \quad \leq \quad C\left(\int_\Omega Mf(x)\,|g(x)|\,dx+\int_\Omega Mg(y)\,|f(y)|\,dy\right)$$

and the Theorem is proved.

□

4. Main result

We can now state and prove our main result. First we recall the definition of the A_p class for $1<p<\infty$. A non-negative locally integrable function ω belongs to A_p if there exists a constant C such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\;dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\;dx\right)^{p-1} \leq C$$

for all cube $Q\subset\mathbb{R}^n$.

For any weight ω, $L^p_\omega(\Omega)$ is the space of measurable functions f defined in Ω such that

$$\|f\|_{L^p_\omega(\Omega)}=\left(\int_\Omega |f(x)|^p\,\omega(x)\,dx\right)^{1/p}<\infty$$

and $W^{k,p}_\omega(\Omega)$ is the space of functions such that

$$\|f\|_{W^{k,p}_\omega(\Omega)}=\left(\sum_{|\alpha|\leq k}\|D^\alpha f\|^p_{L^p_\omega(\Omega)}\right)^{1/p}<\infty.$$

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain such that $\partial\Omega$ is of class C^{6m+4} for $n=2$ and $\partial\Omega$ is of class C^{5m+2} for $n\geq 2$. If $\omega\in A_p$, $f\in L^p_\omega(\Omega)$ and u a weak* *solution of (1.1), then there exists a constant C depending only on n, m, ω and Ω such that*

$$\|u\|_{W^{2m,p}_{\omega}(\Omega)} \leq C\,\|f\|_{L^p_\omega(\Omega)}.$$

Proof: Since M is a bounded operator in $L^p_\omega(\Omega)$, by Lemma 3.1 it follows that

$$\sum_{|\alpha|\leq 2m-1} \|D^\alpha_x u\|_{L^p_\omega(\Omega)} \leq C\,\|f\|_{L^p_\omega(\Omega)}.$$

Therefore, it only remains to estimate $\|D^\alpha_x u\|_{L^p_\omega(\Omega)}$ for $|\alpha| = 2m$.

Let $\omega \in A_p$ and $g(x) := (D^\alpha_x u(x))^{p-1}\,\omega(x)$. By Theorem 3.4 we see that

$$\begin{aligned}
\int_\Omega |D^\alpha_x u(x)|^p\,\omega(x)\,dx &= \int_\Omega |D^\alpha_x u(x)|\,g(x)\,dx \\
&\leq C\left(\int_\Omega \widetilde{K}f(x)\,|g(x)|\,dx + \int_\Omega Mf(x)\,|g(x)|\,dx\right. \\
&\left. + \int_\Omega Mg(y)\,|f(y)|\,dy + \int_\Omega |f(x)|\,|g(x)|\,dx\right). \qquad (4.1)
\end{aligned}$$

Since $\tilde{K}$ and M are bounded operators in $L^p_\omega(\Omega)$, applying the Hölder inequality, it follows that

$$\begin{aligned}
\int_\Omega \widetilde{K}f(x)\,|g(x)|\,dx &= \int_\Omega \widetilde{K}f(x)\,|g(x)|\,\frac{1}{\omega(x)^{1/p}}\,\omega(x)^{1/p}\,dx \\
&\leq \left(\int_\Omega \widetilde{K}f(x)^p\,\omega(x)\,dx\right)^{1/p}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q} \\
&\leq \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q}, \qquad (4.2)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the same way, we obtain that

$$\int_\Omega Mf(x)\,|g(x)|\,dx \leq \|f\|_{L^p_\omega(\Omega)}\left(\int_\Omega |g(x)|^q\,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q} \qquad (4.3)$$

and

$$\int_\Omega |f(x)|\,|g(x)|\,dx \;\leq\; \|f\|_{L^p_\omega(\Omega)} \left(\int_\Omega |g(x)|^q \,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q}. \tag{4.4}$$

For the last term in (4.1), taking into account that $\omega^{-q/p} \in A_q$, we have that

$$\begin{aligned}\int_\Omega Mg(y)\,|f(y)|\,dy \;&\leq\; \|f\|_{L^p_\omega(\Omega)} \left(\int_\Omega Mg(y)^q \,\frac{1}{\omega(y)^{q/p}}\,dy\right)^{1/q} \\ &\leq\; \|f\|_{L^p_\omega(\Omega)} \left(\int_\Omega |g(x)|^q \,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q}.\end{aligned} \tag{4.5}$$

Then, by (4.2), (4.3), (4.4) and (4.5)we have

$$\|D^\alpha_x u\|^p_{L^p_\omega(\Omega)} \;\leq\; C\,\|f\|_{L^p_\omega(\Omega)} \left(\int_\Omega |g(x)|^q \,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q}.$$

By the definition of $g(x)$,

$$\begin{aligned}\left(\int_\Omega |g(x)|^q \,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q} \;&=\; \left(\int_\Omega |D^\alpha_x u|^{(p-1)q}\,\omega(x)^q\,\frac{1}{\omega(x)^{q/p}}\,dx\right)^{1/q} \\ &=\; \left(\int_\Omega |D^\alpha_x u|^p\,\omega(x)\,dx\right)^{1/q} = \|D^\alpha_x u\|^{p/q}_{L^p_\omega(\Omega)}.\end{aligned}$$

Then we obtain

$$\|D^\alpha u\|^p_{L^p_\omega(\Omega)} \leq C\,\|f\|_{L^p_\omega(\Omega)}\,\|D^\alpha u\|^{p/q}_{L^p_\omega(\Omega)} \tag{4.6}$$

and the Theorem is proved for $u \in W^{2m,p}_\omega(\Omega)$.

Finally, we will show that the weak solutio u of (1.1) belong to $W^{2m,p}_\omega(\Omega)$:

We have that $(-\Delta)^m u = f$, with $f \in L^p_\omega(\Omega)$, then there exists a sequence $f_k \in C^\infty(\mathbb{R}^n)$ such that $\lim_{k\to\infty} f_k = f$ in $L^p_\omega(\Omega)$ [3].

For each k, there exists $u_k \in C^\infty(\Omega)$ satisfying

$$\begin{cases} (-\Delta)^m u_k = f_k & \text{in } \Omega \\ \left(\frac{\partial}{\partial\nu}\right)^j u_k = 0 & \text{in } \partial\Omega \quad 0 \leq j \leq m-1. \end{cases}$$

It is easily to see, from Lemma 3.1 that $u_k \in W^{2m-1,p}_{\omega}(\Omega)$, and obviously $u_k \in W^{2m,p}_{\omega,\,loc}(\Omega)$. Moreover for all compact set $K \subset \Omega$, we have

$$\|u_k\|_{W^{2m,p}_{\omega}(K)} \leq C(K),$$

where $C(K)$ is a constant depending on the measure of K. Indeed, taking $v_k = u_k\varphi$ with $\varphi \in C_0^{\infty}(K)$, it follows that $v_k \in W^{2m,p}_{\omega}(\Omega)$, satisfies (1.1) with $f = g_k \in L^p_{\omega}(\Omega)$, and we can use (4.6).

Then, it follows by the dominated convergence theorem that $u_k \in W^{2m,p}_{\omega}(\Omega)$ and applying (4.6), we have that

$$\|u_k\|_{W^{2m,p}_{\omega}(\Omega)} \leq C\, \|f_k\|_{L^p_{\omega}(\Omega)}.$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W^{2m,p}_{\omega}(\Omega)$ and there exists $v \in W^{2m,p}_{\omega}(\Omega)$ such that $\lim\limits_{k\to\infty} u_k = v$ in $W^{2m,p}_{\omega}(\Omega)$. Let see now that v solves (1.1).

Obviously, $f = \lim\limits_{k\to\infty} f_k = \lim\limits_{k\to\infty} (-\Delta)^m u_k = (-\Delta)^m v$ in $L^p_{\omega}(\Omega)$ and by the classical trace theorems in Sobolev spaces and the definition of $\omega \in A_p$, it follows that v satisfies the homogeneous boundary conditions and by uniqueness of the solution, the Theorem is proved.

□

Remark 4.2. The result of Theorem 4.1 is valid also for u a weak solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \\ \mathcal{B}_j u = 0 & \text{in } \partial\Omega \quad 0 \leq j \leq m-1 \end{cases}$$

when $\mathcal{L} := \sum_{|\alpha|\,\leq 2m} a_\alpha(x)\, D^\alpha$ is uniformly elliptic and $\mathcal{B}_j := \sum_{\,|\alpha|\,\leq\, m_j} b_{j,\alpha}(x)\, D^\alpha$, $0 \leq j \leq m-1$ are the boundary operators defined in [1].

Indeed, we define $l_1 > \max_j(2m - m_j)$ and $l_0 = \max_j(2m - m_j)$. If the coefficients $a_\alpha \in C^{l_1+1}(\overline{\Omega})$, $b_{j,\alpha} \in C^{l_1+1}(\partial\Omega)$ and $\partial\Omega \in C^{l_1+2m+1}$ we have that the Green function G_m and the Poisson kernels K_j for $0 \leq j \leq m-1$ exist whenever $l_1 > 2(l_0+1)$ for $n = 2$ and $l_1 > \frac{3}{2}\, l_0$ for $n \geq 3$.

Moreover, wherever they are defined, the Green function and the Poisson kernels of the operator $\mathcal{L}$ with these boundary conditions satisfy the estimates (2.4), (2.5), (2.6), (2.7) and (2.8) (see [4] and [6]).

Remark 4.3. Using the fact that $d(x)^\beta \in A_p$ for $-1 < \beta < p-1$ and generalizing the classical imbedding Theorems of Sobolev spaces to weighted Sobolev spaces (as we have done in [5] , Theorem 3.4) we have as a consequence of the main result: Under the hypotheses of Theorem 4.1 with $\omega = d^\gamma$, where $\gamma = k\beta$, $k \in \mathbb{N}$ and $0 \le \beta \le 1$. If $0 \le \gamma < p-1$ and $1/p - 1/q \le 2m/(n+k)$ (with $q < \infty$ when $2mp = n+k$), then there exists a constant C depending only on γ , p, q, n and Ω such that

$$\|u\|_{L^q_{d^\gamma}(\Omega)} \le C\,\|f\|_{L^p_{d^\gamma}(\Omega)}. \tag{4.7}$$

Finally, as a particular case of (4.7) taking $\gamma = m$ we have that

$$\|u\|_{L^q_{d^m}(\Omega)} \le C\,\|f\|_{L^p_{d^m}(\Omega)}$$

for $p > m+1$ and $1/p - 1/q \le 2m/(n+m)$ (with $q < \infty$ when $2mp = n+m$).

This result is proved in [4] using different arguments for the case $1/p - 1/q < 2m/(n+m)$ Our results shows that, at least in the case $p > m+1$, the estimate remains valid when $1/p - 1/q = 2m/(n+m)$

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Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
E-mail address: rduran@dm.uba.ar

Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, 1900 La Plata (Buenos Aires), Argentina
E-mail address: tatu@mate.unlp.edu.ar

Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, 1900 La Plata (Buenos Aires), Argentina
E-mail address: mtoschi@mate.unlp.edu.ar