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On packing and covering polyhedra of consecutive ones circulant clutters[☆]

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ABSTRACT

Building on work by G. Cornuéjols and B. Novick and by L. Trotter, we give different characterizations of contractions of consecutive ones circulant clutters that give back consecutive ones circulant clutters. Based on a recent result by G. Argiroffo and S. Bianchi, we then arrive at characterizations of the vertices of the fractional set covering polyhedron of these clutters. We obtain similar characterizations for the fractional set packing polyhedron using a result by F.B. Shepherd, and relate our findings with similar ones obtained by A. Wagler for the clique relaxation of the stable set polytope of webs. Finally, we show how our results can be used to obtain some old and new results on the corresponding fractional set covering polyhedron using properties of Farey series. Our results do not depend on Lehman's work or blocker/antiblocker duality, as is traditional in the field.

1. Introduction

Two fundamental problems in polyhedral combinatorics are the study of the fractional set covering polyhedron,

$$P_{\mathrm{c}}(M) = \{x \in \mathbb{R}^n : Mx \geq \mathbf{1}, x \geq \mathbf{0}\},$$

and the fractional set packing polyhedron,

$$P_{\mathrm{p}}(M) = \{x \in \mathbb{R}^n : Mx \leq \mathbf{1}, x \geq \mathbf{0}\},$$

where $M \in \mathbb{R}^{r\times n}$ is a 0–1 matrix with no zero columns, the inequalities indicate componentwise comparison, and $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the vectors in $\mathbb{R}^n$ of all zeros and all ones (respectively), with the subindex n omitted if no confusion arises.

Recall that a clutter, $\mathscr{C} = (V, E)$, is a collection of subsets of (a finite set) V with the property $A \not\subset B$ for all $A, B \in E$, $A \neq B$. $\mathscr{C}$ is *trivial* if either $E = \emptyset$ or $E = \{\emptyset\}$. We can associate to any nontrivial clutter $\mathscr{C}$ the matrix $M(\mathscr{C})$, whose columns are indexed by V and whose rows are the incidence vectors of the members of E.

Dominating (resp. dominated) rows of M are redundant in the definition of $P_{\mathrm{c}}(M)$ (resp. $P_{\mathrm{p}}(M)$), and we may assume that M does not have them. Thus, M is naturally associated with a clutter $\mathscr{C}$ such that $M = M(\mathscr{C})$, and in this case we set $P_{\mathrm{c}}(\mathscr{C}) = P_{\mathrm{c}}(M)$ and $P_{\mathrm{p}}(\mathscr{C}) = P_{\mathrm{p}}(M)$.

Of particular interest are packing polyhedra related to graphs. When $\mathscr{C}$ is the clutter of maximal cliques of the graph G, $P_{\mathrm{p}}(\mathscr{C})$ is often denoted by QSTAB(G), the clique relaxation of the stable set polyhedron STAB(G).

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If $J \in \mathbb{R}^{r\times n}$ is the all ones matrix having the same dimensions as M, eliminating dominated rows of M is equivalent to eliminating dominant rows of $J-M$, which is also a 0–1 matrix. Moreover, for $\mathbf{1}_n \cdot x > 1$ the inequality $Mx \le \mathbf{1}_r$ is equivalent to the inequality $(J-M)\widetilde{x} \ge \mathbf{1}_r$, where $\widetilde{x} = (\mathbf{1}_n \cdot x - 1)^{-1} x$. Hence, we may relate packing and covering polyhedra by studying the nonlinear involution Φ defined on $\{x \in \mathbb{R}^n : \mathbf{1}_n \cdot x \neq 1\}$ by

$$\Phi(x) = \frac{1}{\mathbf{1}_n \cdot x - 1}\, x. \tag{1.1}$$

This was done by Shepherd [14] who showed:

Theorem 1.1. *Given the clutter $\mathscr{C}$, with n vertices and r edges, and corresponding matrix $A = M(\mathscr{C})$, let $J \in \mathbb{R}^{r\times n}$ be the matrix of all ones, $\widetilde{A} = J - A$, and $\widetilde{\mathscr{C}}$ such that $M(\widetilde{\mathscr{C}}) = \widetilde{A}$.*

Then, the only vertices of $P_{\mathrm{p}}(A)$ in $\{x : \mathbf{1} \cdot x \le 1\}$ are $\mathbf{0}$ and the canonical basis (called trivial vertices), and if A has no column of zeros, Φ gives a 1–1 correspondence between the nontrivial vertices of $P_{\mathrm{p}}(\mathscr{C})$ and the vertices of $P_{\mathrm{c}}(\widetilde{\mathscr{C}})$.

The transformation Φ given in (1.1) is remarkably well suited to the consecutive ones circulant clutters $\mathscr{C}_n^k$, which are defined for positive integers n and k with $1 \le k \le n-1$, by considering $V(\mathscr{C}_n^k) = \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ and $E(\mathscr{C}_n^k) = \{C_i : i \in \mathbb{Z}_n\}$, where $C_i = \{i, i+1, \dots, i+(k-1)\}$ (additions modulo n), and we write $M_n^k = M(\mathscr{C}_n^k)$. (Since no other circulant clutters are considered in this paper, from now on "circulant" will be a synonym for "consecutive ones circulant".)

With the notations of Theorem 1.1, if $\mathscr{C} = \mathscr{C}_n^k$, then $\widetilde{\mathscr{C}} = \mathscr{C}_n^{n-k}$, and therefore properties of $P_{\mathrm{p}}(\mathscr{C}_n^k)$ may be translated to properties of $P_{\mathrm{c}}(\mathscr{C}_n^{n-k})$, and conversely.

A web W_n^k is the graph having vertices $\mathbb{Z}_n$ and edges $\{ij : 0 < |i-j| \le k\}$. Although W_n^k makes sense as long as $2k < n$, it is traditional to require that $2(k+1) \le n$, and in particular, $n \ge 2$. Thus W_n^0 is the graph having n vertices and no edges, but the complete graph is not considered to be a web. This implies that its stability number, $\alpha(W_n^k)$, is always greater than 2. An antiweb $\overline{W}_n^{k-1}$ is prime if $\gcd(n, k) = 1$.

The clutters $\mathscr{C}_n^k$ are closely related to the webs W_n^k and their graph complements, the antiwebs $\overline{W}_n^k$. It turns out that, for $3k \le n$, $P_{\mathrm{p}}(\mathscr{C}_n^k)$ may be looked at as $\mathrm{QSTAB}(W_n^{k-1})$. Trotter [15] was one of the first to study the graph $\overline{W}_n^{k-1}$, which he denoted by $W(n, k)$, giving several characterizations for having $\overline{W}_{n'}^{k'-1}$ as an induced subgraph of $\overline{W}_n^{k-1}$. For instance, he showed:

Theorem 1.2. *If $k \le n/2$, $1 \le k' \le k$, $n' \le n$, and $k' \le n'/2$, then $\overline{W}_{n'}^{k'-1}$ is a (node) induced subgraph of $\overline{W}_n^{k-1}$ if and only if*

$$kn' \le k'n \quad \text{and} \quad (k'-1)n \le (k-1)n'. \tag{1.2}$$

Of course, the same characterization holds for the webs W_n^{k-1}, and it has been used to study properties of the packing polyhedra associated with webs and antiwebs (see, e.g., [18] for a summary). For example, Wagler [16, Theorem 2] proved:

Theorem 1.3. *If $k \ge 1$, $\mathrm{STAB}(\overline{W}_n^k)$ has as only nontrivial facets rank constraints associated with prime antiwebs.*[1]

Recall that, if $G = (V, E)$ is a graph, any subset of nodes V' induces a subgraph G' and the *rank constraint* $\sum_{i\in V'} x_i \le \alpha(G')$, which is valid in $\mathrm{STAB}(G)$ (but need not be a facet, and not all facets are always rank constraints).

On another track, Cornuéjols and Novick [7] studied several classes of ideal and minimally nonideal clutters.

A clutter $\mathscr{C} = (V, E)$ is *ideal* if all the vertices of $P_{\mathrm{c}}(\mathscr{C})$ are 0–1. Given $N \subset E$, the *deletion minor* $\mathscr{C} \setminus N$ and the *contraction minor* $\mathscr{C}/N$ are defined by taking $V(\mathscr{C}\setminus N) = V(\mathscr{C}/N) = V - N$ (the set difference of V and N), $E(\mathscr{C}\setminus N) = \{A \in E : A \cap N = \emptyset\}$, and $E(\mathscr{C} - N)$ are the minimal members of $\{A - N : A \in E\}$. If V_1 and V_2 are disjoint subsets of V, $\mathscr{C}/V_1 \setminus V_2$ is a *minor* of $\mathscr{C}$, which is *proper* if either $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$. A *minimally nonideal* clutter (mni clutter for short) is a clutter which is not ideal and all its proper minors are ideal. The clutters $\mathscr{C} = (V, E)$ and $\mathscr{C}' = (V', E')$ are *isomorphic*, denoted by $\mathscr{C} \sim \mathscr{C}'$, if there is a bijection $\varphi : V \to V'$ such that $A \in E$ if and only if $\varphi(A) \in E'$.

As part of their work, Cornuéjols and Novick gave a classification of the clutters $\mathscr{C}_n^k$ which are either ideal or mni by studying contractions of $\mathscr{C}_n^k$ which are isomorphic to another circulant clutter. The main tools in their proof are results by Lehman [11,12], and the following lemma, which relates arithmetic conditions, contractions, and the existence of a (simple directed) cycle in the graph G_n^k, the directed graph having vertex set $V(\mathscr{C}_n^k) = \mathbb{Z}_n$, and (i, i') is an arc of G_n^k if and only if $i' - i \pmod{n}$ is either k or $k+1$.

Lemma 1.4 (*Lemma 4.5 in [7]*). *Suppose $2 \le k \le n-2$. If a subset N of $V(\mathscr{C}_n^k)$ induces a simple directed cycle, D, in G_n^k, then there exist $n_1, n_2, n_3 \in \mathbb{Z}_+$, $n_1 \ge 1$, such that*

(i) $nn_1 = kn_2 + (k+1)n_3$.
(ii) $\gcd(n_1, n_2, n_3) = 1$.
(iii) *If $k - n_1 \le 0$, then $\mathscr{C}_n^k/N$ is trivial, and otherwise $\mathscr{C}_n^k/N \sim \mathscr{C}_{n-n_2-n_3}^{k-n_1}$.*

[1] In this paper we allow $k = 0$ in the definition of W_n^k (so that cliques are prime antiwebs for $n \ge 2$), but Wagler defines them for $k \ge 1$.

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Based on this lemma, Argiroffo and Bianchi [3] gave a characterization of the fractional vertices of $P_c(\mathscr{C}_n^k)$: if, for given $x \in \mathbb{R}^n$, N denotes the set of indices i for which $x_i = 0$, then x is a fractional vertex of $P_c(\mathscr{C}_n^k)$ if and only if $\mathscr{C}_n^k/N$ is isomorphic to some $\mathscr{C}_{n'}^{k'}$ with $k' > 1$ and n' and k' relative primes.

The proof of Lemma 1.4 given by Cornuéjols and Novick has many points of contact with Trotter's characterizations (Theorems 1.2 and 2.6). In a way, Trotter studied properties of one set, while Cornuéjols and Novick studied properties of its complement.

This relation is exposed in Section 2, where we study contractions of $\mathscr{C}_n^k$ and find different characterizations of N so that $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$. In Theorem 2.5 we give structural conditions, following ideas by Cornuéjols and Novick. In Lemma 2.8 we show how we may shift to Trotter's point of view, and mainly repeat his arguments to obtain in Theorem 2.10 arithmetical conditions similar to those of Theorem 1.2.

In Section 3 we study how cycles are related to these contractions, extending the work by Cornuéjols and Novick. We show in Theorem 3.10 that if $m = |N|$ and $n' = n - m$, then $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$ if and only if N may be partitioned (in a unique way up to order) into $d = \gcd(m, k - k')$ disjoint simple directed cycles in G_n^k, each of length m/d. In Corollary 3.8 we show that these cycles have an interlacing property.

We start Section 4 by giving different characterizations of the vertices of $P_c(\mathscr{C}_n^k)$ (in Theorems 4.1 and 4.3), following Argiroffo and Bianchi. With the arithmetic characterizations at hand, we may establish several properties in a simple way, such as constructing or recognizing vertices in $O(n)$ steps (Remark 4.4) or a monotonicity property (Corollary 4.6). Shepherd's transformation Φ allows us to transfer the characterizations to the vertices of $P_p(\mathscr{C}_n^k)$ (Corollaries 4.7 and 4.8). We end this section by relating our findings with previous results on the extreme points of $\mathrm{QSTAB}(W_n^k)$ and Wagler's Theorem 1.3 (Lemma 4.10 and Corollaries 4.11 and 4.12).

Finally, in Section 5 we see how these characterizations may be used to obtain some old and new results on $P_c(\mathscr{C}_n^k)$. In Section 5.1 we give a proof of Cornuéjols and Novick's characterization of ideal and mni circulant clutters which does not depend on Lehman's results, using instead properties of Farey series. Near-ideal clutters were introduced by Argiroffo [2] (see also [4]) as a symmetric idea to that of near-perfect graphs and matrices studied by Shepherd [13], and in Section 5.2 we study circulant clutters with this property.

In this paper, unless otherwise indicated, operations in $\mathbb{Z}_n$, such as addition, are indicated simply by the usual symbol, such as $+$. Occasionally, the value of the modulo is emphasized by writing $a +_n b$ for $a + b \pmod n$. $(a, b)_n$ denotes the $\mathbb{Z}_n$-cyclic open interval of points strictly between a and b—so that (a, b) is an edge whereas $(a, b)_n$ is a cyclic interval—with analogous meanings for $(a, b]_n$, $[a, b)_n$ and $[a, b]_n$. For consistency, the indices of the coordinates of vectors in $\mathbb{R}^n$ are taken in $\mathbb{Z}_n$, i.e., $x = (x_0, x_1, \ldots, x_{n-1})$.

For further background on clutters, packing and covering, we refer the reader to Cornuéjols' book [6].

2. Contractions of $\mathscr{C}_n^k$

In this section we study the minor $\mathscr{C}_n^k/N$, and give necessary and sufficient conditions so that it is isomorphic to $\mathscr{C}_{n'}^{k'}$ for some n' and k'.

Throughout most of this section, we assume n, k, and $N \subset \mathbb{Z}_n$ are given, and set $\overline{N} = \mathbb{Z}_n - N$, $m = |N|$ and $n' = |\overline{N}|$.

Recalling that $C_i = [i, i + k)_n \in E(\mathscr{C}_n^k)$, we write

$$\overline{C}_i = C_i - N = C_i \cap \overline{N} \quad \text{for } i \in \mathbb{Z}_n.$$

The point of departure is the following observation (cf. [7, Remark 4.3]):

Lemma 2.1. *Let $i \in N$. Then, $\overline{C}_{i+1}$ is dominating if $2 \le k \le n - 1$.*

Let us take care first of the extreme cases for k and m.

Lemma 2.2. *If $N = \emptyset$, then $\mathscr{C}_n^k/N = \mathscr{C}_n^k$ is not a proper minor of $\mathscr{C}_n^k$.*
Otherwise, if $N \neq \emptyset$ we have:

- *If $k = 1$ then $\mathscr{C}_n^1/N$ is trivial.*
- *If $m = n - 1$ then $\mathscr{C}_n^k/N$ is trivial for $k \le n - 1$.*
- *If $k = n - 1 > 1$ then $\mathscr{C}_n^k/N \sim \mathscr{C}_{n-m}^{k-m}$ if $m < n - 1$, and otherwise $\mathscr{C}_n^k/N$ is trivial.*

Thus, it is convenient to make the assumption:

Assumption 2.3. k and m are such that $2 \le k \le n - 1$ and $0 \le m \le n - 2$.

To construct $\mathscr{C}_n^k/N$ from $\mathscr{C}_n^k$ we have to throw away any dominating redundant sets in $\{\overline{C}_i : i \in \mathbb{Z}_n\}$. In view of Lemma 2.1, we may think that we throw away first the sets $\overline{C}_{i+1}$ for $i \in N$, and then look for any other dominating set.

Lemma 2.4. *Under Assumption 2.3,*

$$E(\mathscr{C}_n^k/N) \subset \{\overline{C}_{i+1} : i \in \overline{N}\}, \tag{2.1}$$

and therefore

$$|E(\mathscr{C}_n^k/N)| \le |V(\mathscr{C}_n^k/N)|. \tag{2.2}$$

Also, equality holds in (2.1) if and only if equality holds in (2.2).

Theorem 2.5. *Suppose Assumption 2.3 holds. Then, the following are equivalent:*

(i) *There exists k' such that $1 \le k' \le \min\{n'-1, k\}$ and $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$.*
(ii) *$\mathscr{C}_n^k/N$ is not trivial and $|E(\mathscr{C}_n^k/N)| = |V(\mathscr{C}_n^k/N)|$.*
(iii) *There exists k' such that $1 \le k' \le \min\{n'-1, k\}$ and $|\overline{C}_{i+1}| = k'$ for all $i \in \overline{N}$.*

Proof. (i) ⇒ (ii): If $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$, $M = M_n^k$ and $\overline{M} = M(\mathscr{C}_n^k/N)$ are square matrices, and m columns have been removed from M to obtain $\overline{M}$, which implies $n' = n - m$. Also, exactly m rows have been deleted from M, and since $|A| = k'$ for $A \in E(\mathscr{C}_n^k/N)$, it follows that $|E(\mathscr{C}_n^k/N)| = n - m = |V(\mathscr{C}_n^k/N)|$.

(ii) ⇒ (iii): By Lemma 2.4, $E(\mathscr{C}_n^k/N) = \{\overline{C}_{i+1} : i \in \overline{N}\}$. Given $i \in \overline{N}$, let $i' \in \overline{N}$ be such that all the points in the $\mathbb{Z}_n$-cyclic interval $(i, i')_n$ are in N. Since $\overline{C}_{i+1} \in E(\mathscr{C}_n^k/N)$ and $\mathscr{C}_n^k/N$ is not trivial, then $\overline{C}_{i+1} \ne \emptyset$ and $i' \in \overline{C}_{i+1}$. Also, since between i and i' there are only elements in N, $\overline{C}_{i+1} - \{i'\} \subset \overline{C}_{i'+1}$. Moreover, $\overline{C}_{i+1}$ does not dominate $\overline{C}_{i'+1} \in E(\mathscr{C}_n^k/N)$ (Lemma 2.4), which implies that there exists i'' such that $i'' \in \overline{C}_{i'+1} - \overline{C}_{i+1}$, and therefore $|\overline{C}_{i+1}| \le |\overline{C}_{i'+1}|$. As this happens (cyclically) for all $i, i' \in \overline{N}$, we must have $|\overline{C}_{i+1}| = k'$ for all $i \in \overline{N}$, for some k', $1 \le k' \le k$.

(iii) ⇒ (i): Suppose $i, i' \in \overline{N}$ are such that $\overline{C}_{i+1} \subset \overline{C}_{i'+1}$. Since they have the same positive cardinality, $\overline{C}_{i+1} = \overline{C}_{i'+1}$, and there exists $i'' \in \overline{N}$ which is in both sets. Assuming, without loss of generality, that $i' - i > 0$, then $0 < i' - i < i'' - i \le k$, and therefore $i' \in \overline{C}_{i'+1} = \overline{C}_{i+1}$, which is a contradiction since $k < n$. Since there are no dominations between sets of the form $\overline{C}_{i+1}$ for $i \in \overline{N}$, and there are $n' = n - m$ of these, $|E(\mathscr{C}_n^k/N)| \ge n' \ge |V(\mathscr{C}_n^k/N)|$, and by Lemma 2.4, $E(\mathscr{C}_n^k/N) = \{\overline{C}_{i+1} : i \in \overline{N}\}$.

Now, the characteristic vectors of the sets C_i, $i \in \mathbb{Z}_n$, consist of a group of ones followed by a group of zeros (when looked at cyclically in $\mathbb{Z}_n$), and the same must hold for $\overline{C}_i = C_i \cap \overline{N}$. Hence, $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$. □

Although not explicitly stated, the last condition in Theorem 2.5 is part of the proof by Cornuéjols and Novick of Lemma 1.4, and is similar to one in the following result by Trotter [15, Theorem 3.1]:

Theorem 2.6. *Under the assumptions of Theorem 1.2, $\overline{W}_{n'}^{k'-1}$ is a (node) induced subgraph of $\overline{W}_n^{k-1}$ if and only if there exists $V' \subset \mathbb{Z}_n$ such that*

$$|[i, i + (k-1)]_n \cap V'| = k' \quad \textit{for all } i \in V'.$$

Remark 2.7. As was the case of Theorem 1.2, by taking complements we see that the result is also true for W_n^{k-1}.

Let us see the equivalence between the conditions in Theorem 2.5(iii) and that in Theorem 2.6 (without the restrictions $2k \le n$ and $2k' \le n'$ needed for webs and antiwebs):

Lemma 2.8. *Suppose $1 \le k < n$, $\overline{N} \subset \mathbb{Z}_n$, $n' = |\overline{N}|$, and $1 \le k' \le n'$. Then, the following are equivalent:*

(i) *$|[i, i + (k-1)]_n \cap \overline{N}| = k'$ for all $i \in \overline{N}$.*
(ii) *$|[i - (k-1), i]_n \cap \overline{N}| = k'$ for all $i \in \overline{N}$.*
(iii) *$|[i + 1, i + (n-k)]_n \cap \overline{N}| = n' - k'$ for all $i \in \overline{N}$.*

Proof. To show that (i) ⇒ (ii), let $\psi : \overline{N} \to \overline{N}$ be defined by $[i, \psi(i)]_n \cap \overline{N} = [i, i + (k-1)]_n \cap \overline{N}$. Since $|[i, i + (k-1)]_n \cap \overline{N}| = k'$, ψ is injective and hence bijective. Therefore, $[\psi^{-1}(i), i]_n \subset [i - (k-1), i]_n$ and $[\psi^{-1}(i), i]_n \cap \overline{N} = [i - (k-1), i]_n \cap \overline{N}$.

The implication (ii) ⇒ (i) follows similarly. (ii) ⇔ (iii) is readily seen by taking complements. □

Our next goal is to prove an arithmetic characterization for contractions of $\mathscr{C}_n^k$, analogous to that of Theorem 1.2.

The following result is a variant of [15, Corollary 3.2]:

Corollary 2.9. *Under Assumption 2.3, suppose $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$. Then*

$$k' \le |\overline{C}_{i+1}| \quad \textit{and} \quad |(i, i + (k+1)]_n \cap \overline{N}| \le k' + 1 \quad \textit{for all } i \in \mathbb{Z}_n.$$

Proof. If $i \notin N$, by Theorem 2.5 we know that $|\overline{C}_{i+1}| = k'$, and the other inequality follows by noticing that $(i, i+(k+1)]_n \cap \overline{N} \subset \overline{C}_{i+1} \cup \{i+(k+1)\}$.

If $i \in N$, consider $i', i'' \notin N$ such that $i \in (i', i'')_n \subset N$, and notice that $\overline{C}_{i'+1} \subset \overline{C}_{i+1}$ and $(i, i+(k+1)]_n \cap \overline{N} \subset \{i''\} \cup \overline{C}_{i''+1}$. □

The following result and its proof follow ideas by Trotter [15, Theorem 3.3].

Theorem 2.10. *Under Assumption 2.3, there exists N such that $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$ if and only if*

$$\frac{k'}{k} \le \frac{n'}{n} \le \frac{k'+1}{k+1}. \tag{2.3}$$

Proof. Suppose $N \subset \mathbb{Z}_n$ is such that $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$, and let $x \in \mathbb{R}^n$ denote the incidence vector of $\overline{N}$, so that

$$\sum_{i \in \mathbb{Z}_n} x_i = n'. \tag{2.4}$$

By Corollary 2.9 we have

$$k' \le \sum_{j=1}^{k} x_{i+j} \quad \text{and} \quad \sum_{j=1}^{k+1} x_{i+j} \le k'+1 \quad \text{for all } i.$$

Summing these inequalities over all i, and using (2.4), we obtain $k'n \le kn'$ and $(k+1)n' \le (k'+1)n$.

For the converse implication, for $j = 0, \dots, n'-1$ let us define $i_j = \lceil jn/n' \rceil$. Since $2 \le n' < n$, we have $(n'-1)n/n' < n-1$, and therefore $0 \le i_j < n$. Thus, $\overline{N} = \{i_0, \dots, i_{n'-1}\}$ satisfies $|\overline{N}| = n'$.

For fixed j, $0 \le j < n'$, let r be defined by $r = i_j n' - jn$, so that $0 \le r < n'$, and let $h = |\overline{C}_{i_j+1}|$, so that if $h > 0$, $\overline{C}_{i_j+1} = \{i_{j+_{n'}1}, \dots, i_{j+_{n'}h}\}$. Thus, h is the maximum integer such that, with operations in $\mathbb{R}$,

$$\frac{(j+h)n}{n'} \le i_j + k = \frac{jn}{n'} + \frac{r}{n'} + k,$$

i.e., $h = \lfloor r/n + kn'/n \rfloor$. The inequalities (2.3) and $0 \le r < n'$ imply

$$k' \le \frac{kn'}{n} \le \frac{r}{n} + \frac{kn'}{n} < \frac{(k+1)n'}{n} \le k'+1,$$

and hence, $h = k'$.

Therefore, $|\overline{C}_{i+1}| = k'$ for all $i \in \overline{N}$, and we may apply Theorem 2.5. □

3. Cycles in G_n^k

One of the fundamental ideas by Cornuéjols and Novick in their 1994 paper [7] is that we can relate contractions and cycles in G_n^k, and in this section we pursue these ideas. To do so, it is convenient to assume that N is written in *canonical form*,

$$N = \{i_0, i_1, \dots, i_{m-1}\} \quad \text{with } i_0 < i_1 < \dots < i_{m-1}, \tag{3.1}$$

with the usual order in $\mathbb{Z}$.

In this case, the function $\pi : N \to \mathbb{Z}_m$, defined by

$$\pi(i_j) = j \quad \text{for } j = 0, \dots, m-1, \tag{3.2}$$

is bijective and increasing.

For $n_1 \in \mathbb{Z}$, $0 \le n_1 \le m-1$, let us define $s : N \to N$ by

$$s(i_j) = i_{j+_m n_1} \quad \text{for } j = 0, \dots, m-1, \tag{3.3}$$

and $\delta : N \to \mathbb{Z}_n$ by

$$\delta(i) = s(i) - i \quad \text{for } i \in N, \tag{3.4}$$

so that $|(i, s(i)]_n \cap N| = |(i, i+\delta(i)]_n \cap N| = n_1 \quad \text{if } i \in N$.

Theorem 3.1. *Under the assumptions of Theorem 2.5, let $n_1 = k - k'$. Then $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$ if and only if $\delta(i) \in \{k, k+1\}$ for all $i \in N$, or, in other words, if and only if $(i, s(i)) \in G_n^k$ for all $i \in N$.*

Moreover, if either condition holds, then $|\overline{C}_{i+1}| \in \{k', k'+1\}$ for all $i \in N$, with $|\overline{C}_{i+1}| = k'+1$ if and only if $\delta(i) = k+1$.

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Proof. If $\mathscr{C}_n^k/N \sim \mathscr{C}_{n'}^{k'}$, by Corollary 2.9 for all $i \in \mathbb{Z}_n$ we have

$$|(i, i+k]_n \cap N| \le k - k' = n_1 = (k+1) - (k'+1) \le |(i, i+(k+1)]_n \cap N|,$$

which in particular implies $\delta(i) \in \{k, k+1\}$ for $i \in N$.

For the other implication, suppose $i \in N$. By the definition of s and δ, we have $|(i, i+\delta(i)]_n \cap N| = |(i, i+s(i)]_n \cap N| = n_1$. If $\delta(i) = k$, then $|C_{i+1} \cap N| = n_1$; and if $\delta(i) = k+1$, then $C_{i+1} \cap N = [i+1, s(i))_n \cap N$, and therefore $|C_{i+1} \cap N| = n_1 - 1$. In all, for $i \in N$ we have $|\overline{C}_{i+1}| \in \{k', k'+1\}$, with $|\overline{C}_{i+1}| = k'$ if and only if $\delta(i) = k$.

Consider now $i \in \overline{N}$. We observe that $|\overline{C}_{i+1}| \ge k'$, since otherwise there would exist $c \in C_{i+1} \cap N$ with $\delta(c) < k$. Suppose $a = i - 1 \in N$. If $\delta(a) = k$, then $i + k \in \overline{N}$, and therefore $k' = |\overline{C}_{a+1}| = |\overline{C}_{i+1}|$. Else, if $\delta(a) = k+1$, then $i + k = s(a)$ and $k' + 1 = |\overline{C}_{a+1}| = |\overline{C}_{i+1}| + 1$. Finally, if $i \in \overline{N}$, let t be the largest positive integer such that $[i - t, i]_n \cap N = \emptyset$. Then $k' \le |\overline{C}_{i+1}| \le |\overline{C}_i| \le \dots \le |\overline{C}_{i-t}| = k'$. $\square$

The last part of the previous proof also shows:

Lemma 3.2. *Under the assumptions of Theorem 2.5, suppose n_1 is such that $1 \le n_1 < \min\{k, m\}$ and $\delta(a) \in \{k, k+1\}$ for all $a \in N$, and $i, i' \in \overline{N}$ are such that $i' - i \in \{k, k+1\}$. Then, $|(i, i')_n \cap N| = n_1$.*

Proof. In fact, since $i' \in \overline{N}$, the statement is equivalent to $|C_{i+1} \cap N| = n_1$, which in turn is equivalent to $|\overline{C}_{i+1}| = k'$. $\square$

Once we have $\delta(i) \in \{k, k+1\}$, it is natural to think of cycles in G_n^k.

Lemma 3.3. *Suppose $1 \le k \le n-1$, $1 \le m \le n-2$, $1 \le n_1 < m$, and $\delta(i) \in \{k, k+1\}$ for all $i \in N$.*

Then, there exist $D_1, \dots, D_d$ disjoint simple directed cycles in G_n^k, having a common length, such that $N = \cup_r V(D_r)$.

Furthermore, $d = \gcd(m, n_1)$ and the decomposition is unique (up to the order).

Proof. Since the function s, defined in (3.3), is bijective (and N is finite), it induces a decomposition of G_n^k consisting of simple directed cycles of the form $(i, s(i), s^2(i), \dots, s^{o(i)})$, where $o(i)$ is the order of $i \in N$ with respect to s; that is, $o(i) = \min\{j > 0 : s^j(i) = i\}$.

Using the function $\pi : N \to \mathbb{Z}_m$, defined in (3.2), which is an order preserving bijection, we see that the dicycles induced by s are mapped to dicycles of the directed graph $\widetilde{G}$ with vertex set $\mathbb{Z}_m$ and arcs $(z, z +_m n_1)$ for all $z \in \mathbb{Z}_m$. Thus, for $i \in N$, the order $o(i)$ is the least positive integer that multiplied by n_1 is a multiple of m. In other words, if $d = \gcd(m, n_1)$,

$$o(i) = m/d \quad \text{for all } i \in N,$$

and all the dicycles in $\widetilde{G}$ or G_n^k have the same length, namely, m/d. $\square$

Let us study the case of just one cycle. Given a simple directed cycle D in G_n^k, let us set $N = V(D)$ and define n_2 to be the number of arcs of length k in D, and n_3 to be the number of arcs of length $k+1$. Since D is a simple directed cycle, then $n_2 k + n_3(k+1)$ is a multiple of n, and therefore there exists a unique n_1 such that

$$n_1 n = n_2 k + n_3\,(k+1). \tag{3.5}$$

For fixed n_1, the general solution for the unknowns n_2 and n_3 of the diophantine Eq. (3.5) is given by

$$n_2 = -n_1 n + z(k+1) \quad \text{and} \quad n_3 = n_1 n - zk \qquad \text{for any } z \in \mathbb{Z},$$

and adding these equations we obtain $n_2 + n_3 = z$. On the other hand, if $m = |N|$, since D is simple, we have

$$m = n_2 + n_3, \tag{3.6}$$

and therefore,

$$n_2 = -n_1 n + m(k+1), \qquad n_3 = n_1 n - mk. \tag{3.7}$$

Thus, given n_2 and n_3 we may obtain m and n_1 by means of Eqs. (3.6) and (3.5), and, conversely, given m and n_1 we may obtain n_2 and n_3 by means of Eq. (3.7). If these hold, then the relations

$$\frac{km}{n} \le n_1 \le \frac{(k+1)m}{n} \tag{3.8}$$

also hold, and conversely:

Lemma 3.4. *Let n, k, m be given, with $1 \le k \le n-1$ and $0 \le m \le n-1$. Then there exist $n_1, n_2, n_3 \ge 0$ satisfying Eqs. (3.5) and (3.6) if and only if*

$$\left\lceil \frac{km}{n} \right\rceil = \left\lfloor \frac{(k+1)m}{n} \right\rfloor. \tag{3.9}$$

Moreover,

(i) *n_1 is uniquely determined by*

$$n_1 = \left\lceil \frac{km}{n} \right\rceil, \tag{3.10}$$

and therefore n_2 and n_3 are uniquely determined by Eqs. (3.7),

(ii) *if $0 < m$ and $k < n - 1$, then $0 < n_1 \le \min\{m - 1, k\}$,*

(iii) $\gcd(n_1, n_2, n_3) = \gcd(n_1, m)$.

Proof. We just observe that for $1 \le k \le n - 1$ and $0 \le m \le n - 1$ we have

$$\frac{km}{n} \le \frac{(k+1)m}{n} < \frac{km}{n} + 1,$$

and therefore, in general, $\lceil km/n \rceil \ge \lfloor (k+1)m/n \rfloor$. □

Lemma 3.5. *Suppose the following conditions are satisfied:*

(1) $1 \le k \le n - 1$,
(2) *D is a simple directed cycle in G_n^k, having n_2 arcs of length k and n_3 arcs of length $k + 1$,*
(3) *n_1 is defined by Eq. (3.5),*
(4) *$N = V(D)$ is written in the canonical form (3.1) ($m = |N|$),*
(5) *$\widetilde{s} : N \to N$ is defined by*

$$\widetilde{s}(i) = i' \quad \text{if } (i, i') \text{ is an arc of } D. \tag{3.11}$$

Then, $|(i, \widetilde{s}(i)]_n \cap N| = n_1$, i.e., $\widetilde{s} = s$, where s is defined in Eq. (3.3).

Proof. We first observe that $\widetilde{s}$ is well defined and is a bijection since D is simple.

If D has an arc of length n, then it is actually a loop, and therefore $N = \{i_0\}$, $m = n_1 = 1$, $\widetilde{s}(i_0) = i_0$, and the result follows.

Otherwise, it is enough to show the result for the case $j = 0$ and $i_0 = 0$, by considering the directed cycle $D - i_j$. For $i \in N$, let $\widetilde{\delta}(i) = \widetilde{s}(i) - i$, and consider the auxiliary integer sequence defined by

$$a_0 = 0\ (= i_0), \qquad a_j = a_{j-1} + \widetilde{\delta}(i_{j-1}) \quad \text{for } j = 1, \dots, m - 1,$$

so that $a_j = i_j \pmod{n}$.

For each $t = 0, \dots, n_1 - 1$, there is exactly one value $r(t) = j$ such that $tn \le a_j < tn + \widetilde{s}(i_0)$, since there are no arcs of length less than k or more than $k + 1$. On the other hand, since D is simple, the function r is injective. Thus there are exactly n_1 points in the cyclic interval $[i_0, \widetilde{s}(i_0))_n$. □

Applying Lemma 3.3 to $N = V(D)$, since the decomposition is unique, yields an alternative proof of Lemma 1.4(ii) (see also [1] for yet another proof):

Lemma 3.6. *If the assumptions of Lemma 3.5 hold, then* $\gcd(m, n_1) = 1$.

Suppose now $D_1, D_2, \dots, D_d$ are d disjoint simple directed cycles of G_n^k, having a common length $\overline{m}$ ($<n$), and let $N = \cup_r D_r$. Using Lemma 3.4 for the values n, k and $\overline{m}$, we see that the corresponding parameters of the cycles, say $\overline{n}_1$, $\overline{n}_2$, and $\overline{n}_3$, coincide. Therefore, all the cycles have the same number of arcs of length k, and the same number of arcs of length $k + 1$. Thus, in the graph $\cup_r D_r$ there are $n_2 = d\overline{n}_2$ arcs of length k, $n_3 = d\overline{n}_3$ arcs of length $k + 1$, and if n_1 is defined by Eq. (3.5), then $n_1 = d\overline{n}_1$. Also, the total number of vertices is $m = d\overline{m}$. Moreover, by Lemma 3.6, we must have $\gcd(\overline{m}, \overline{n}_1) = 1$, so that $\gcd(m, n_1) = d$.

The following is a generalization of Lemmas 3.5 and 3.6, and gives a converse to Lemma 3.3:

Lemma 3.7. *Suppose the following conditions are satisfied:*

(1) $1 \le k \le n - 1$, $1 \le m \le n - 2$,
(2) *$D_1, \dots, D_d$ are disjoint simple directed cycles in G_n^k, all having length m/d,*
(3) *$N = \cup_r V(D_r)$ is written in the canonical form (3.1) (so $|N| = m$).*

Then,

(i) *Eq. (3.9) holds,*
(ii) *if n_1 is the common value in (3.9), then* $\gcd(m, n_1) = d$,
(iii) *if $s : N \to N$ is defined by $s(i) = i'$ if (i, i') is an arc of D_r for some $r = 1, \dots, d$, then Eq. (3.3) holds, i.e., $s(i_j) = i_{j +_m n_1}$ for $j = 0, \dots, m - 1$.*

Proof. We observe that, if (i, i') is an arc of D_r, in the cyclic interval $(i, i']_n$ there are $\overline{n}_1$ elements of D_r (by Lemma 3.5), and also $\overline{n}_1$ elements of each D_t, $t \ne r$ (by Lemmas 3.2 and 3.5). In all, there are $d\overline{n}_1 = n_1$ elements of N in $(i, i']_n$. □

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Using the function $\pi : N \to \mathbb{Z}_m$ defined in (3.2), which is an order preserving bijection, the dicycles $D_1, \ldots, D_d$ are mapped to dicycles $\widetilde{D}_1, \ldots, \widetilde{D}_d$ of the directed graph $\widetilde{G}$ with vertex set $\mathbb{Z}_m$ and arcs $(z, z +_m n_1)$ for all $z \in \mathbb{Z}_m$. Since all simple directed cycles in $\widetilde{G}$ are translations (modulo m) of the cycle starting at $0 \in \mathbb{Z}_m$, and $\cup_r \widetilde{D}_r = \mathbb{Z}_m$, in each cyclic interval of the form $[z, z +_m d)_n$ there is exactly one point of each cycle $\widetilde{D}_r$, and this structure is carried back to N through π^{-1}. Thus, we have the following interlacing property of the cycles:

Corollary 3.8. *Suppose the assumptions of Lemma 3.7 hold. Then, in each cyclic interval of N of length d there is exactly one point of each cycle.*

Hence, if $N = \cup_r V(D_r)$ is written in the canonical form (3.1), after an eventual renaming of the cycles we have

$$V(D_r) = \{i_{r-1}, i_{(r-1)+d}, \ldots, i_{(r-1)+(\overline{m}-1)d}\}.$$

We may visualize this by splitting the elements of N to form a $d \times \overline{m}$ matrix A, where the rth row is $V(D_r)$. In other words, if $a_{r,j}$ are the entries of A, then

$$a_{r,j} = i_{r+jd}, \quad \text{for } r = 0, \ldots, d-1, \ j = 0, \ldots, \overline{m} - 1,$$

so that all the cycles have the same structure, following the columns of A.

Example 3.9. By "the same structure" we do not mean to imply that corresponding arcs have the same length. For example, if $n = 34$, $k = 7$, $m = 18$, $n_1 = 4$, we could take

$$N = \{0, 2, 4, 6, 8, 9, 12, 13, 16, 17, 19, 20, 23, 24, 26, 28, 31, 32\},$$

which decomposes into the cycles

$$D_1 = (0, 8, 16, 23, 31, 4, 12, 19, 26, 0), \qquad D_2 = (2, 9, 17, 24, 32, 6, 13, 20, 28, 2).$$

However, the first arc in D_1 has length $k + 1 = 8$, and the first arc of D_2 has length $k = 7$. Moreover, the lengths cannot be made to coincide by a cyclic rotation, since in D_1 there are three consecutive arcs of length $k + 1$, namely $(26, 0)$, $(0, 8)$ and $(8, 16)$, but this does not hold for D_2.

Using Theorem 3.1, and Lemmas 3.3 and 3.7, we can state (see also Theorem 2.5):

Theorem 3.10. *Suppose Assumption 2.3 holds, and $0 \le n_1 < \min\{k, m\}$. Then, the following are equivalent:*

(i) $\mathscr{C}_n^k / N \sim \mathscr{C}_{n-m}^{k-n_1}$.
(ii) *There exist $d = \gcd(m, n_1)$ disjoint simple dicycles in G_n^k, $D_1, \ldots, D_d$, each having length m/d, such that $N = \cup_r V(D_r)$.*

4. Vertices of related polyhedra

Given a clutter $\mathscr{C}$ and a subset $N' \subset V(\mathscr{C})$, we may interpret $P_c(\mathscr{C}/N')$ geometrically as the projection of the intersection of $P_c(\mathscr{C})$ with the subspace $\{x \in \mathbb{R}^n : x_i = 0 \text{ for all } i \in N'\}$. Thus, if for $x \in \mathbb{R}^n$ we define $N(x) = \{i \in \mathbb{Z}_n : x_i = 0\}$, we may identify $P_c(\mathscr{C}_n^k / N')$ and $\{x \in P_c(\mathscr{C}_n^k) : N' \subset N(x)\}$. Since the conditions $x_i \ge 0$, for all $i \in N'$, are some of the inequalities defining $P_c(\mathscr{C}/N')$, no new vertices are created, i.e., all the vertices of $P_c(\mathscr{C}/N')$—when regarded as a subset of $\mathbb{R}^n$—are already vertices of $P_c(\mathscr{C})$. The intersection could be empty, so there are no vertices at all, and this happens if $\mathscr{C}/N'$ is trivial. On the other hand, if $P_c(\mathscr{C})$ has a vertex x with $N(x) \ne \emptyset$ and $m = |N(x)|$, then $P_c(\mathscr{C}/N(x))$—regarded now as a subset of $\mathbb{R}^{n-m}$—has, essentially, the image of x as a vertex with no zero coordinates.

When we apply these ideas to the case $\mathscr{C} = \mathscr{C}_n^k$, we know that for some N's we get back another circulant clutter, $\mathscr{C}_{n'}^{k'}$. Thus, if x is a vertex of $P_c(\mathscr{C}_n^k)$ such that $\mathscr{C}_n^k / N(x) \sim \mathscr{C}_{n'}^{k'}$ (for appropriate n' and k'), then the vertex $x' \in P_c(\mathscr{C}_{n'}^{k'}) \subset \mathbb{R}^{n'}$, corresponding to x, has all its coordinates positive.

Argiroffo and Bianchi [3] showed the remarkable fact that the converse is true: for all fractional vertices $x \in P_c(\mathscr{C}_n^k)$, $N(x)$ is such that $\mathscr{C}_n^k / N(x)$ is a circulant clutter. Although their result is for fractional vertices, it also holds for all vertices (with minor changes in the proof, which we omit):

Theorem 4.1. *Suppose $1 \le k \le n - 1$. Then, the point x is a vertex of $P_c(\mathscr{C}_n^k)$ if and only if there exist n' and k' such that (i) $1 \le k' < n'$, (ii) $\mathscr{C}_n^k / N(x) \sim \mathscr{C}_{n'}^{k'}$, and (iii) $\gcd(n', k') = 1$.*

In this case $|N(x)| \le n - 2$, and $x_i = 1/k'$ for all $i \notin N(x)$.

Remark 4.2. More formally, Theorem 4.1 states that the vertices of $P_c(\mathscr{C}_n^k)$ have the Lehman property, although it does not depend on Lehman's results.

According to Shepherd [14], a square 0–1 matrix A is said to be Lehman if, for any $a_{ij} = 0$, the number of ones in column j is the same as the number of ones in row i, and a vertex x of $P_c(A)$ has the Lehman property if there is a square nonsingular Lehman submatrix A_x of A such that $A_x x = \mathbf{1}$ and x is zero for any column not appearing in the matrix A_x.

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Table 1
Parameters of the vertices of $P_c(\mathscr{C}_{34}^7)$.

m	n_1	d	m/d	n'	k'	Quantity
0	0	0	–	34	7	1
13	3	1	13	21	4	204
17	4	1	17	17	3	2
18	4	2	9	16	3	30634
23	5	1	23	11	2	50864
26	6	2	13	8	1	2210
27	6	3	9	7	1	26558
28	6	2	14	6	1	6137
29	6	1	29	5	1	34

Coupling Theorem 4.1 with Theorem 2.10 we have:

Theorem 4.3. *Suppose $1 \le k \le n-1$. Then, $P_c(\mathscr{C}_n^k)$ has a vertex with exactly m zero coordinates and $n' = n - m$ positive coordinates, if and only if*

(i) $m \le n-2$,
(ii)
$$\left\lceil \frac{kn'}{n} - \frac{m}{n} \right\rceil = \left\lfloor \frac{kn'}{n} \right\rfloor, \tag{4.1}$$
(iii) $\gcd(n', k') = 1$, *where k' is the common value in (4.1).*

If these conditions hold, then the value of all of the nonzero coordinates of such a vertex is $1/k'$.

Remark 4.4. Thus, we may construct a vertex using the algorithm of Theorem 2.5 in $O(n)$ steps, and conversely, we may check whether $x \in \mathbb{R}^n$ is a vertex of $P_c(\mathscr{C}_n^k)$ in $O(n)$ steps.

Example 4.5. Using Theorem 4.3, in Table 1 we have indicated the possible parameters of vertices of $P_c(\mathscr{C}_{34}^7)$.
In this table, $d = \gcd(m, n_1)$ is the number of cycles having cardinality m/d into which the set of zero coordinates decomposes. The last column indicates the quantity of vertices having those parameters, and its values were obtained using the polyhedral computational code PORTA [5].

Looking at Table 1, we notice that the 0–1 vertices may have between 26 and 29 zeros, i.e., between 5 and 8 nonzero coordinates. We can also observe that the 0–1 vertices always have fewer nonzero coordinates than fractional vertices. This is a general property which is a consequence of Theorem 4.3, since the values of $\lfloor kn'/n \rfloor$, and so of k', are increasing with n':

Corollary 4.6. *Suppose x is a 0–1 vertex of $P_c(\mathscr{C}_n^k)$, with $|N(x)| = m$, and x' is another vertex, not necessarily 0–1, with $|N(x')| = m'$. Then,*

(i) *If x' is 0–1 and $m < m'$, then, for any $m'' \in \mathbb{N}$, $m < m'' < m'$, there exists a 0–1 vertex, x'', of $P_c(\mathscr{C}_n^k)$ with $|N(x'')| = m''$.*
(ii) *If x' is a fractional vertex then $|N(x')| < m$.*

Let us see now how these characterizations are reflected when working with $P_p(\mathscr{C}_n^k)$.
Recalling the transformation Φ defined in (1.1) and Shepherd's Theorem 1.1, if $1 \le k \le n-1$, then x is a nontrivial vertex of $P_p(\mathscr{C}_n^k)$ if and only if $\Phi(x)$ is a vertex of $P_c(\mathscr{C}_n^{n-k})$. Since, by the definition of Φ, $N(x) = N(\Phi(x))$, these sets share structural properties. Moreover, all the nonzero coordinates of the nontrivial vertex x of $P_p(\mathscr{C}_n^{k-1})$ take the same value, since those of $\Phi(x)$ do. If this value is $1/k'$, then the nonzero coordinates of the vertex $\Phi(x)$ of $P_c(\mathscr{C}_n^{n-k})$ take the value

$$\frac{1}{n'/k' - 1} \frac{1}{k'} = \frac{1}{n' - k'},$$

and we may state:

Corollary 4.7. *Suppose $1 \le k \le n-1$. Then, the point x is a nontrivial vertex of $P_p(\mathscr{C}_n^k)$ if and only if there exist n' and k' such that (i) $1 \le k' < n'$, (ii) $\mathscr{C}_n^{n-k}/N(x) \sim \mathscr{C}_{n'}^{n'-k'}$, and (iii) $\gcd(n', k') = 1$.*
In this case $|N(x)| \le n-2$ and $x_i = 1/k'$ for all $i \notin N(x)$.

Replacing k by $n-k$ in the inequalities in Theorem 4.3, (4.1) now reads

$$n' - \left\lfloor \frac{kn' + m}{n} \right\rfloor = \left\lceil \frac{(n-k)n' + m}{n} \right\rceil = \left\lfloor \frac{(n-k)n'}{n} \right\rfloor = n' - \left\lceil \frac{kn'}{n} \right\rceil,$$

so that the arithmetic relations in Theorem 4.3 (and the definition of Shepherd's Φ) yield:

Corollary 4.8. *If $2 \le k \le n-1$, then $P_p(\mathscr{C}_n^k)$ has a nontrivial vertex with exactly m zero coordinates and $n' = n - m$ ($n' \ge 2$) nonzero coordinates if and only if*

(i) $\left\lceil \frac{kn'}{n} \right\rceil = \left\lfloor \frac{kn'}{n} + \frac{m}{n} \right\rfloor$,
(ii) *if k' is the common value in the previous equality, then $\gcd(n', k') = 1$.*

If these conditions hold, then the value of all of the nonzero coordinates of such a vertex is $1/k'$.

The relations

$$k' = \left\lceil \frac{kn'}{n} \right\rceil = \left\lfloor \frac{kn'}{n} + \frac{m}{n} \right\rfloor$$

in Corollary 4.8 are equivalent to (1.2), i.e., to both $kn' \le k'n$ and $(k'-1)n \le (k-1)n'$. However, in Trotter's Theorem 1.2 there is the restriction $2k' \le n'$, whereas in Corollary 4.8 we are including the condition $\gcd(n', k') = 1$. Noticing that the latter does appear in Wagler's Theorem 1.3, it seems appropriate to relate Corollaries 4.7 and 4.8 to packing polyhedra associated with webs and antiwebs.

We observe first that if $2k > n$ there are no 0–1 nontrivial vertices of $P_p(\mathscr{C}_n^k)$ by Corollary 4.8 ($kn'/n \le 1$ implies $n' = 1$ or 0), and W_n^{k-1} is not defined.

If $2k \le n$, $\mathscr{C}_n^k$ is a subset of (maximal) cliques of W_n^{k-1}, and therefore

$$\mathrm{QSTAB}(W_n^{k-1}) \subset P_p(\mathscr{C}_n^k). \tag{4.2}$$

If, furthermore, $3k \le n$, M_n^k is precisely the clique-node matrix of W_n^{k-1}, and the inclusion above turns to an equality, so that Corollaries 4.7 and 4.8 provide characterizations of the vertices of $\mathrm{QSTAB}(W_n^{k-1}) = P_p(\mathscr{C}_n^k)$ whenever the stability number of W_n^{k-1}, $\alpha(W_n^{k-1}) = \lfloor n/k \rfloor$, is 3 or more.

When $2k \le n < 3k$, there may be other maximal cliques in W_n^{k-1} besides those in $\mathscr{C}_n^k$. Wagler [17], building on work by W. Cook (1987, unpublished) and Shepherd [13], showed that in this case W_n^{k-1} is near-perfect, i.e., $\mathrm{STAB}(W_n^{k-1})$ is determined by the clique inequalities and the rank inequality $x \cdot \mathbf{1} \le 2$:

Theorem 4.9 (*Theorem 15 in [17]*). *A web is near-perfect if and only if it is perfect, an odd hole, W_{11}^2, or if it has stability number 2.*

This leads us to study the inequality $x \cdot \mathbf{1} \ge 2$ in $P_p(\mathscr{C}_n^k)$.

Lemma 4.10. *Suppose $2k \le n$ and x' is a nontrivial vertex of $P_p(\mathscr{C}_n^k)$, written in the form*

$$x' = \frac{1}{k'} \chi^{\overline{N}}, \tag{4.3}$$

where $\chi^{\overline{N}}$ is the incidence vector of $\overline{N}$, $n' = |\overline{N}| \ge 2$, $k' \ge 1$, $\mathscr{C}_n^{n-k}/N(x) \sim \mathscr{C}_{n'}^{n'-k'}$, $\gcd(n', k') = 1$, and (1.2) holds.

We have:

(i) *If $3k \le n$ then $x' \cdot \mathbf{1} \ge 2$, with strict inequality if x' is not 0–1 (i.e., if $k' > 1$).*
(ii) *If $x' \cdot \mathbf{1} \ge 2$, then x' is a nontrivial vertex of $\mathrm{QSTAB}(W_n^{k-1})$.*

Proof. Notice first that, by (4.3), the inequality $x' \cdot \mathbf{1} \ge 2$ is equivalent to $n' \ge 2k'$.

(i) Suppose $n \ge 3k$ and $n' < 2k'$. Then $n' + 1 \le 2k$, and from $(k'-1)n \le (k-1)n'$,

$$3k\,(k'-1) \le n(k'-1) \le (k-1)n' \le (k-1)(2k'-1),$$

which implies $k(k'-2) \le 1 - 2k' < 0$, and therefore we cannot have $k' \ge 2$.

Thus, either $k' = 1$ or $n' \ge 2k'$. If $k' = 1$, x is a 0–1 vertex, and since it is nontrivial, we must have $n' \ge 2$. If $k' > 1$, the condition $\gcd(n', k') = 1$ implies that we cannot have $n' = 2k'$, and hence $n' > 2k'$.

(ii) If $n' \ge 2k'$, we may use Trotter's Theorem 2.6 (actually Remark 2.7) to see that the subgraph of W_n^{k-1} induced by $\overline{N}$ is isomorphic to $W_{n'}^{k'-1}$. Hence, if Q is a clique of W_n^{k-1} (not necessarily maximal) with $Q \subset \overline{N}$, then $|Q| \le k'$. Thus, for any maximal clique Q' of W_n^{k-1} we have $\chi^{Q'} \cdot x' = |Q' \cap \overline{N}|/k' \le 1$, which implies that x' is a vertex of $\mathrm{QSTAB}(W_n^{k-1})$. □

Partitioning the vertices of $P_p(\mathscr{C}_n^k)$ into T, the set of 0–1 trivial vertices (i.e., those for which $x \cdot \mathbf{1} \le 1$), S^-, the set of fractional vertices with $x \cdot \mathbf{1} < 2$, S_2, the set of 0–1 vertices with $x \cdot \mathbf{1} = 2$, and S^+, the set of vertices with $x \cdot \mathbf{1} > 2$, we have

$$P_p(\mathscr{C}_n^k) = \mathrm{conv}(T \cup S_2 \cup S^+ \cup S^-).$$

Corollary 4.11. *If $k \le n/2$, $\mathrm{QSTAB}(W_n^{k-1}) = \mathrm{conv}(T \cup S_2 \cup S^+)$.*

In other words, x' is a nontrivial vertex of $\mathrm{QSTAB}(W_n^{k-1})$ if and only if it is a nontrivial vertex of $P_p(\mathscr{C}_n^k)$ and $\mathbf{1} \cdot x' \ge 2$ (the latter being redundant if $n \ge 3k$).

Proof. Let us consider the different possibilities for the stability number of W_n^{k-1}.

If $n \geq 3k$, we have $\mathrm{QSTAB}(W_n^{k-1}) = P_p(\mathscr{C}_n^k)$ and $S^- = \emptyset$ (by Lemma 4.10(i)), so the result follows.

By Lemma 4.10(ii) we know that $S_2 \cup S^+$ is a subset of the vertices of $\mathrm{QSTAB}(W_n^{k-1})$, and therefore $\mathrm{conv}(T \cup S_2 \cup S^+) \subset \mathrm{QSTAB}(W_n^{k-1})$.

If $2k \leq n < 3k$, by Wagler's Theorem 4.9 we have

$$\mathrm{STAB}(W_n^{k-1}) = \mathrm{conv}(T \cup S_2) = \mathrm{QSTAB}(W_n^{k-1}) \cap \{x : x \cdot \mathbf{1} \leq 2\},$$

which implies $S^- \cap \mathrm{QSTAB}(W_n^{k-1}) = \emptyset$, and by (4.2), $\mathrm{QSTAB}(W_n^{k-1}) \subset \mathrm{conv}(T \cup S_2 \cup S^+)$. $\square$

In the proof of Lemma 4.10(ii) we saw that, when $n' \geq 2k'$, Trotter's Theorem 2.6 could be used to show that $\overline{N}$, as a subset of nodes of W_n^{k-1}, induces $W_{n'}^{k'-1}$. But by the same argument we may say that $\overline{N}$, looked at now in $\overline{W}_n^{k-1}$, induces $\overline{W}_{n'}^{k'-1}$. Since by Corollary 4.11 any nontrivial vertex x' of $\mathrm{QSTAB}(W_n^{k-1})$ satisfies the inequality $n' \geq 2k'$, we may write:

Corollary 4.12. *x' is a nontrivial vertex of $\mathrm{QSTAB}(W_n^{k-1})$ if and only if $x' = \frac{1}{k'} \chi^{\overline{N}}$, where $\overline{N}$ induces the prime antiweb $\overline{W}_{n'}^{k'-1}$ in $\overline{W}_n^{k-1}$ (or, equivalently, the "prime" web $W_{n'}^{k'-1}$ in W_n^{k-1}).*

Thus, nontrivial vertices of $\mathrm{QSTAB}(W_n^{k-1})$ are in 1–1 correspondence with rank inequalities $\sum_{i \in \overline{N}} x_i \leq k'$ valid for $\mathrm{STAB}(\overline{W}_n^{k-1})$, where $\overline{N}$ induces the prime antiweb $\overline{W}_{n'}^{k'-1}$ (a clique if $k' = 1$).

By Fulkerson's antiblocking duality, we know that the undominated vertices of $\mathrm{QSTAB}(W_n^{k-1})$ are in 1–1 correspondence with the nontrivial facets of $\mathrm{STAB}(\overline{W}_n^{k-1})$ (those of the form $x \cdot a \leq 1$), and since nonzero trivial vertices of $\mathrm{QSTAB}(W_n^{k-1})$ define facets of $\mathrm{STAB}(\overline{W}_n^{k-1})$ only if $k = 1$, we see how our results are equivalent to Theorem 1.3 as long as only undominated vertices and inequalities are considered. When all vertices are considered, we may use the following characterization by Koster and Wagler [10] of the extreme points of $\mathrm{QSTAB}(G)$ for general graphs:

Theorem 4.13. *Given a graph $G = (V, E)$ and $a \in \mathbb{R}^{|V|}$, $a \geq \mathbf{0}$, let G_a be the graph induced by $\{v \in V : a_v > 0\}$, and $\mathrm{supp}(a)$ be the vector a restricted to the nonzero components only.*

Then $a \neq \mathbf{0}$ is an extreme point of $\mathrm{QSTAB}(G)$ if and only if the inequality $\mathrm{supp}(a) \cdot x \leq 1$ defines a facet of $\mathrm{STAB}(\overline{G}_a)$.

5. Applications

In this section we use the algebraic characterization of vertices of $P_c(\mathscr{C}_n^k)$ to study some families of circulant clutters.

5.1. Ideal and mni circulant clutters

Using Theorem 4.3, we may state a characterization of ideal and mni circulant clutters in algebraic terms:

Proposition 5.1. *If $n \geq 3$ and $1 \leq k \leq n-1$, then $\mathscr{C}_n^k$ is ideal or mni if and only if, for every m, $1 \leq m \leq n-2$, and n_1 such that*

$$\frac{km}{n} \leq n_1 \leq \frac{(k+1)m}{n} \quad \text{and} \quad \gcd(n-m, k-n_1) = 1,$$

there holds $n_1 = k - 1$.

In this case, $\mathscr{C}_n^k$ is mni if $\gcd(n, k) = 1$, and otherwise is ideal.

Cornuéjols and Novick [7] described many ideal and mni clutters, studying in particular the circulant clutters $\mathscr{C}_n^k$ which are ideal or mni, obtaining that, for $k \geq 2$, the only ideal circulant clutters are

$$\mathscr{C}_6^3, \quad \mathscr{C}_9^3, \quad \mathscr{C}_8^4, \quad \text{and} \quad \mathscr{C}_n^2 \text{ for even } n, \tag{5.1}$$

and the only mni circulant clutters are

$$\mathscr{C}_5^3,\ \mathscr{C}_8^3,\ \mathscr{C}_{11}^3,\ \mathscr{C}_{14}^3,\ \mathscr{C}_{17}^3,\ \mathscr{C}_7^4,\ \mathscr{C}_{11}^4,\ \mathscr{C}_9^5,\ \mathscr{C}_{11}^6,\ \mathscr{C}_{13}^7,\ \text{and } \mathscr{C}_n^2 \text{ for odd } n. \tag{5.2}$$

The first application is to show that their result may be obtained without using Lehman's results [11,12], using instead Proposition 5.1 and properties of the Farey series.

Let us recall that, for $n \in \mathbb{N}$, the Farey series of order n, $\mathfrak{F}_n$, is the set of ordered irreducible fractions of the form a/b, with $1 \leq b \leq n$ and $0 \leq a \leq b$, with the conventions $0 = 0/1$ and $1 = 1/1$. For instance, $\mathfrak{F}_4 = (0, 1/4, 1/3, 1/2, 2/3, 3/4, 1)$.

The following results may be found in the book by Hardy and Wright [9, Chapter III, Theorem 28,29, and 31], the first two being equivalent, in the sense that one may be derived from the other:

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Theorem 5.2. *If h/k and h'/k' are two successive terms of $\mathfrak{F}_n$, then $kh' - hk' = 1$.*

Theorem 5.3. *If h/k, h''/k'', and h'/k' are three successive terms of $\mathfrak{F}_n$, then*

$$\frac{h''}{k''} = \frac{h+h'}{k+k'}.$$

Theorem 5.4. *If $n > 1$, then no two successive term of $\mathfrak{F}_n$ have the same denominator.*

Proof of Proposition 5.1. In order to show the validity of (5.1) and (5.2), we consider different possibilities, eliminating from the start the case $k = 1$, for which we already know $\mathbf{1}$ is the only vertex of $P_c(\mathscr{C}_n^1)$ (and hence $\mathscr{C}_n^1$ is ideal).

Case 1. $k = 2$.

Suppose we have a vertex with m zeros. Then, the only possibilities are $n_1 = 0$ or $n_1 = 1$. If $n_1 = 0$, then $m = 0$, and $\frac{1}{k}\mathbf{1}$ is a vertex of $P_c(\mathscr{C}_n^2)$ if and only if $\gcd(n, 2) = 1$, i.e., if n is odd. If $n_1 = 1$, we may take any m with $n/3 \le m \le n/2$, but all the corresponding vertices are 0–1 since $k - n_1 = 1$.

Case 2. $3 \le k < n \le 20$.

It is easy to verify (for example, with a simple computer program), that the only pairs (n, k) satisfying both conditions of Proposition 5.1 are in agreement with the results by Cornuéjols and Novick.

We see now that in all the remaining cases $\mathscr{C}_n^k$ is neither ideal nor mni, by finding m and n_1 which verify all of the following conditions:

$$\begin{aligned} &0 < m < n-1, \quad 0 < n_1 < k, \ \ k - n_1 > 1, \\ &\frac{k}{n} \le \frac{n_1}{m} \le \frac{k+1}{n}, \quad \gcd(n-m, k-n_1) = 1. \end{aligned} \tag{5.3}$$

Case 3. $d = \gcd(n, k) > 1$, $n \ne jk$.

We take $n' = n/d$, $k' = k/d$, $m = (d-1)n'$, $n_1 = (d-1)k'$, and notice that $k' > 1$, since k does not divide n.

Case 4. $k \ge 3$, $n = jk > 20$.

Defining $m = j(k-2) - 1$ and $n_1 = k - 2$, we observe that, since $k < n = jk$, then $j \ge 2$, so that m and n_1 are positive for $k \ge 3$. We have $\gcd(n-m, k-n_1) = \gcd(2j+1, 2) = 1$, $n_1 n - km = k$, and it remains to be seen that the quantity

$$(k+1)m - n_1 n = m - (n_1 n - km) = (j-1)(k-2) - 3 \tag{5.4}$$

is nonnegative. Recalling that $j \ge 2$, we see that this is true for $k \ge 5$, whereas, using that $n = jk \ge 21$, we may check that for $k = 3, 4$ the quantity in (5.4) is always positive.

Case 5. $\gcd(n, k) = 1$, $n > 20$, $k \ge 3$.

This is where Cornuéjols and Novick used Lehman's results, and where we use Farey series.

Since k/n is irreducible, then:

$$\frac{k}{n} \in \mathfrak{F}_n. \tag{5.5}$$

By Theorem 5.3,

$$\frac{k}{n} = \frac{\alpha+\gamma}{\beta+\delta}, \tag{5.6}$$

where α/β and γ/δ are the terms surrounding k/n in $\mathfrak{F}_n$, i.e.,

$$\frac{k-1}{n} \le \frac{\alpha}{\beta} < \frac{k}{n} < \frac{\gamma}{\delta} \le \frac{k+1}{n}, \tag{5.7}$$

$$\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1. \tag{5.8}$$

Furthermore, by Theorem 5.4, consecutive terms in $\mathfrak{F}_n$ cannot have the same denominator, and therefore,

$$0 < \beta < n, \quad 0 < \delta < n, \tag{5.9}$$

which together with (5.6) and (5.5), imply

$$\beta + \delta = n, \quad \alpha + \gamma = k. \tag{5.10}$$

If $\alpha > 1$, we set $m = \delta$ and $n_1 = \gamma$, and using Eq. (5.5) through (5.10), we see that $k - n_1 = k - \gamma = \alpha > 1$, and the conditions in (5.3) are satisfied.

Let us consider the case $\alpha = 1$. Theorem 5.2 tells us that $1 = \beta k - \alpha n = \beta k - n$; that is, $n = \beta k - 1$. We copy the techniques used in Case 4, where we had $n = jk$, by defining $m = \beta(k-2) - 2$ and $n_1 = k - 2$, and observe that,

Table 2
Near-ideal $\mathscr{C}_n^k$ for $3 \le k < n$, $31 \le n \le 50$.

n	k	Quantity	n	k	Quantity
31	10,13,14,15,18,...	17	41	13,17,18,19,20,24,...	22
32	10,13,14,15,19,...	17	42	18,19,20,25,...	20
33	14,15,16,19,...	17	43	14,18,19,20,21,25,...	23
34	11,14,15,16,20,...	18	44	14,18,19,20,21,26,...	23
35	11,15,16,17,21,...	18	45	19,20,21,22,27,...	22
36	15,16,17,21,...	18	46	15,19,20,21,22,27,...	24
37	12,15,16,17,18,22,...	20	47	15,20,21,22,23,28,...	24
38	12,16,17,18,22,...	20	48	20,21,22,23,28,...	24
39	16,17,18,19,23,...	20	49	16,21,22,23,24,29,...	25
40	13,17,18,19,24,...	20	50	16,21,22,23,24,30,...	25

since $k < n$, then $\beta \ge 2$, so that $m > 0$ for $k \ge 4$, and also for $k = 3$, since $21 \le n < \beta k$ and so $\beta \ge 8$. We have $n_1 n - km = n_1(\beta k - 1) - k(\beta n_1 - 2) = k + 2$.

It remains to be seen that

$$(k+1)m - n_1 n = m - (n_1 n - km) = (\beta - 1)(k - 2) - 6 \tag{5.11}$$

is a nonnegative number. Recalling that $\beta \ge 2$, we see that this is true for $k \ge 8$, whereas, using that $n = \beta k - 1 \ge 21$, we see that for $k = 3, 4, 5, 6, 7$ the quantity in (5.11) is always positive. □

5.2. Near-ideal circulant clutters

Following Argiroffo [2], a circulant clutter $\mathscr{C}_n^k$ is *near-ideal* if the convex hull of the 0–1 vertices of $P_c(\mathscr{C}_n^k)$ is

$$P_c(\mathscr{C}_n^k) \cap \{x \in \mathbb{R}^n : \mathbf{1} \cdot x \ge \lceil n/k \rceil\}.$$

Therefore, the family of near-ideal circulant clutters includes those of ideal and mni circulant clutters.

Turning the definition around, we may say that a circulant clutter is *not* near-ideal, if there exists a fractional vertex x of $P_c(\mathscr{C}_n^k)$ for which

$$\mathbf{1} \cdot x \ge \left\lceil \frac{n}{k} \right\rceil. \tag{5.12}$$

In what follows, we restrict our attention to the case $k \ge 3$, since for $k = 2$ we already know that $\mathscr{C}_n^2$ is either ideal or mni.

We first observe that, since $n > k$, we must have $\lceil n/k \rceil \ge 2$. Using the notations of Theorem 4.3, we see that if there is a vertex of $P_c(\mathscr{C}_n^k)$ with exactly n' coordinates taking the value $1/k' < 1$, then k'/n' and $1/\lceil n/k \rceil$ cannot coincide (and both are members of the Farey series $\mathfrak{F}_n$). Thus, we may state:

Proposition 5.5. *Suppose n and k are given, $n > k \ge 3$. Then $\mathscr{C}_n^k$ is not near ideal if and only if there exist n' and k' such that*

$$k > k' > 1, \quad \gcd(k', n') = 1, \quad \frac{n'}{k'} > \left\lceil \frac{n}{k} \right\rceil, \quad \frac{n}{k+1} \ge \frac{n'}{k'+1}. \tag{5.13}$$

It is rather simple to construct—with a computer program—a table such as Table 2, where, for each value of n, $31 \le n \le 50$, we have indicated the values of k, $3 \le k < n$, for which $\mathscr{C}_n^k$ is near-ideal. The ellipses indicate that all subsequent values of k, up to $n - 1$, make $\mathscr{C}_n^k$ near-ideal, and the last column is the quantity of near-ideal clutters ($k \ge 3$) for a given n. Let us recall—once more—that $\mathscr{C}_n^2$ is always near-ideal.

We observe in Table 2 that, for fixed n, higher values of k yield $\mathscr{C}_n^k$ near-ideal, whereas if $k \ge 3$ but is small compared to n, then $\mathscr{C}_n^k$ is not near-ideal. We show these results in Propositions 5.7 and 5.8 later, but let us consider a special case first.

If n is multiple of k, i.e., if $\lceil n/k \rceil = n/k = \mu$, we see that, for $n > 20$, as we did in Case 4 of the previous subsection, we may take $m = \mu(k-2) - 1$, $n_1 = k - 2$, and obtain $n'/k' > \lceil n/k \rceil$, provided

$$(k+1)m - n_1 n = (\mu - 1)(k - 2) - 3 \tag{5.14}$$

is a nonnegative number.

This is certainly true for $k \ge 5$ (since $n/k = \lceil n/k \rceil \ge 2$). For $k = 3$ the only values of $\lceil n/k \rceil$ making the quantity in (5.14) negative are $n/k = 2, 3$, but we already know that $\mathscr{C}_6^3$ and $\mathscr{C}_9^3$ are ideal. Similarly, for $k = 4$ the only value of n/k making the quantity in (5.14) negative is $n/k = 2$, and we know that $\mathscr{C}_8^4$ is ideal. Thus, we have the following property, already observed by Argiroffo and Bianchi [3]:

Proposition 5.6. *If $k \ge 3$, then for any $\mu \ge 2$, $\mathscr{C}_{\mu k}^k$ is not near-ideal except for $\mathscr{C}_6^3$, $\mathscr{C}_9^3$ and $\mathscr{C}_8^4$, which are ideal.*

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Suppose $\lceil n/k \rceil > n/k$, and with the previous notations, suppose $k' = 2$. Then n' is odd, and since $n' > \lceil n/k \rceil\ k' \geq 2k' = 4$, we must have $n' \geq 5$, and therefore

$$\frac{n'}{k'+1} = \frac{n'}{3} \geq \frac{5}{3} > \frac{3}{2}.$$

On the other hand, if $k' \geq 3$,

$$\frac{n'}{k'+1} > 2 \times \frac{k'}{k'+1} \geq 2 \times \frac{3}{4} = \frac{3}{2},$$

so that all fractional vertices of $P_c(\mathscr{C}_n^k)$ are eliminated by the inequality (5.12) if, for instance,

$$\frac{n}{k+1} \leq \frac{3}{2}.$$

Proposition 5.7. *If $k \geq \frac{2}{3}\, n - 1$, then $\mathscr{C}_n^k$ is near-ideal.*

A very similar result, with the bound $k \geq \lfloor 2n/3 \rfloor$, was obtained by Argiroffo [2], using very different techniques, involving blockers.

Finally, we show that if $k \geq 3$ is small compared to n, then $\mathscr{C}_n^k$ is not near-ideal:

Proposition 5.8. *If $k \geq 3$ and $n \geq 13k$, then $\mathscr{C}_n^k$ is not near-ideal.*

Proof. Let us take $k' = 2$ and $n' = 2\ \lceil n/k \rceil + 1$, so that

$$\gcd(n', k') = 1,$$
$$\frac{n'}{k'} = \frac{2\ \lceil n/k \rceil + 1}{2} > \lceil n/k \rceil\,,$$
$$\frac{n}{k+1} = \frac{n}{k} \times \frac{k}{k+1} > (\lceil n/k \rceil - 1) \times \frac{3}{4} \geq \frac{n'}{3} = \frac{n'}{k'+1},$$

where in the last line we have used that $k/(k+1) \geq 3/4$ if $k \geq 3$ in the first inequality and $n \geq 13k$ in the last. Thus the conditions (5.13) are satisfied and $\mathscr{C}_n^k$ is not near-ideal. □

As a final remark on near-ideal circulant clutters, looking at Table 2 we observe a "chaotic" behavior for intermediate values of k. For example, if $\mathscr{C}_n^k$ is near-ideal, then not necessarily $\mathscr{C}_{n+1}^k$ or $\mathscr{C}_n^{k+1}$ are near-ideal, or the number of near-ideal circulant clutters (the third column in the table) is not monotone. Although the algebraic characterizations in Propositions 5.1 and 5.5 are similar, it seems that we cannot hope for a classification of near-ideal circulant clutters, in the spirit of that given by Cornuéjols and Novick for ideal and mni circulant clutters, or Wagler's Theorem 4.9 on near-perfect webs.

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