Phantom *k*-essence cosmologies

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We devise a method to obtain a phantom version of FRW *k*-essence cosmologies with homogeneous *k* fields by applying form-invariance transformations. It can be seen that the transformation performs the maps $H \to -H$ and $\rho + p \to -(\rho + p)$, which in turn give $\gamma \to -\gamma$ and $a \to a^{-1}$. The discussion is presented in a general setup, valid for FRW *k*-essence cosmologies, and then we discuss power-law solutions for illustration purposes. First, we deal with models such that the gradient of the *k* field is not constant, and they include standard and generalized tachyon cosmologies. We concentrate on the usual tachyon and show that the phantom symmetry involves a change in the potential: and that it generates an extended superaccelerated tachyon field. Then, we turn our attention to models for which the time derivative of the *k* field is not constant, and we show the transformation can be implemented without changing the potential at all.

I. INTRODUCTION

Observational data provided by the WMAP mission [1] seem to have confirmed the existence of an epoch of accelerated expansion in the early universe. In addition, according to observations of distant supernovas [2], also the universe at present is expanding with acceleration. The idea that some kinds of scalar fields could be the agents driving those two periods of expansion is widely accepted. Those fields are described by effective theories which, in general, include in their Lagrangians noncanonical terms in field derivatives and might bring in crucial cosmological consequences like the occurrence of inflation even without a potential (purely kinetic acceleration or *k* acceleration) [3–5]. In these models, inflation is pole like; that is, the scale factor evolves like a negative power of time. An earlier theoretical framework in which (polelike) *k* acceleration arises naturally is the pre-big-bang model of string cosmology [6]. In this setup, acceleration is just due to a scalar field called the dilaton, and it will only manifest itself in the string conformal frame. Finally, for other ideas on kinetic inflation one may have a look at [7], where acceleration was put down to a dynamical Planck mass.

In the abovementioned theories with noncanonical kinetic terms in their Lagrangians one only considers terms which are functions of the square of the gradient of the scalar field (hereafter *k* field), because the equations of motion in classical theories seem to be of second order. Moreover, since *k* fields can be used for constructing dark energy models, it is common place to interpret them as some kind of matter called *k* essence [4,8,9]. Note, however, that the description of late time acceleration was not the original reason why *k* fields were introduced, but rather they were put forward as possible inflation driving agents [3,10]. Interestingly enough, as shown in [9], one can also construct tracking *k*-essence cosmologies, although there are dynamical systems arguments against their plausibility [11].

Lately, efforts in the framework of *k* essence have been directed towards model building using power-law solutions [5,12,13]. Such cosmologies may be interpreted as universes filled with barotropic fluids with a constant barotropic index. In this paper we address *k*-essence cosmologies that violate the weak energy condition $\rho > 0$, $\rho + p > 0$, but from a different perspective than earlier works [14]. The models will be dubbed phantom *k*-essence cosmologies following the terminology in [15]. Phantom matter can apparently be accommodated by current observations [16], and even though the theoretical understanding of the acceptability of phantom matter is limited, we can rely on the motivation provided by string theory [17]. Interestingly, the idea that the origin of dark energy should be searched within a fundamental theory—say, string theory—has been recently reinforced by the discovery that the holographic principle cannot be used to tell whether dark energy is present or not [18].

Theoretical cosmology with phantom models has become an active area of theoretical research. Sometimes the accent is put on exact solutions [19], whereas for some others cosmological dynamics is the key subject [20]. Related to this, at present there is no consensus as to whether a universe that violates the weak energy condition should generically possess a future singularity or big rip [21]. Now, since the idea of phantom cosmologies is pretty new, even in such a simple setting as (single field) Friedman-Robertson-Walker (FRW)

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spacetimes many questions remain open yet. Nevertheless, not long ago some authors ventured out of that basic picture and pursued generalizations such as considering an AdS geometry [22] or introducing multiple phantom fields [23].

Our approach to phantom k-essence cosmologies aims at model construction too, following the line of work successfully initiated in [25]. As customary, we will assume the k field is homogeneous. Two different ways to obtain power-law k-essence cosmologies are known, depending on whether the time derivative of the k field is constant or not. In the first case, the scalar field evolves linearly with time and the potential is necessarily of the inverse square form [8]. Power-law tachyon cosmologies [12] belong to this case. In contrast, in the second case, solutions with arbitrary potentials and nonlinear scalar fields can be found, if one imposes they have a constant barotropic index [5,13]. A nice feature of those solutions is the rich casuistics in the form of the field and its potential for a fixed power-law evolution. We apply our symmetry transformations to the two cases and highlight the differences and similarities between them.

II. FORM-INVARIANCE TRANSFORMATIONS

We assume our fluid is the source of a spatially flat homogeneous and isotropic spacetime with line element

$$ds^2 = -dt^2 + a^2(t)(dx_1^2 + dx_2^2 + dx_3^2), \qquad (1)$$

where $a(t)$ is the scale factor and the expansion or Hubble factor is defined as $H = \dot{a}/a$. Here and throughout overdoes will denote differentiation with respect to t.

Consider any two different FRW perfect fluid solutions to the Einstein field equations, a and $\bar{a}$, each one generated by energy density and pressure ρ, p and $\bar{\rho}, \bar{p}$, respectively. The sets of differential equations that have been solved to obtain those solutions are, in fact, different. Now, in the framework of a long-term project [24,25] it has been shown that a link between those cosmological models can be established using a form-invariance transformation which uses as only input the relation between the energy densities of the two fluids. Ours is, therefore, an uncommon equivalence concept. The seed and transformed cosmological models will be characterized by the set of quantities $\{H, \rho, p\}$ and $\{\bar{H}, \bar{\rho}, \bar{p}\}$ which, as usual, represent the Hubble factor, energy density, and pressure. Each set of those quantities will satisfy the customary Einstein equations. We will say the second set corresponds to a cosmological model obtained from the seed one through a form-invariance transformation generated by $\bar{\rho}(\rho)$. Interestingly, there is one form-invariance transformation [25] which preserves the energy density of the fluid and corresponds to

$$\bar{H} = -H, \qquad (2)$$

$$\bar{\rho} + \bar{p} = -(\rho + p). \qquad (3)$$

Clearly, it flips the sign of the barotropic index $\gamma \equiv -2\dot{H}/3H^2$, so in what follows we will we referring to it as to the "phantom transformation." Noticeably, from Eq. (2) it

follows that $\bar{a} \propto a^{-1}$, and this duality in the scale factor, which here allows one to generate phantom k-essence cosmologies, is identical to the one appearing in the pre-big-bang models [6]. Finally, expanding models can be constructed if we trade the initial singularity of the a solution for the final big rip of the $\bar{a}$ one.

We turn now to the specific setting of k-essence cosmologies with an homogeneous k field ϕ derived from the factorizable Lagrangian

$$\mathcal{L} = -V(\phi)F(x), \qquad (4)$$

where $x = -\dot{\phi}^2$. This form is suggested by the Born-Infeld Lagrangian

$$\mathcal{L} = -V(\phi)\sqrt{1+x}, \qquad (5)$$

which was associated with the tachyon by computations in boundary string field theory [26]. Such a Lagrangian also arises in open bosonic string theory [27] and is a key ingredient in the effective theory of D-branes [28].

Under the assumptions made, k essence can be interpreted in terms of a barotropic perfect fluid with equation of state $p = (\gamma - 1)\rho$. The Einstein equations reduce, then, to

$$3H^2 = \frac{VF}{1-\gamma}, \qquad (6)$$

$$\dot{H} = xVF_x. \qquad (7)$$

It can also be seen that

$$\gamma = -\frac{2xF_x}{F - 2xF_x}. \qquad (8)$$

A consequence of Eqs. (6) and (7) is the conservation equation

$$(F_x + 2xF_{xx})\ddot{\phi} + 3HF_x\dot{\phi} + \frac{V'}{2V}(F - 2xF_x) = 0, \qquad (9)$$

where $V' = dV/d\phi$.

Finally, we get

$$\bar{V}\bar{F} = \frac{1+\gamma}{1-\gamma}VF \qquad (10)$$

by applying the phantom transformation defined by Eqs. (2) and (3).

III. PHANTOM k-ESSENCE COSMOLOGIES ARISING FROM SYMMETRIES

In this section we will discuss the application of the phantom transformation to power-law k-essence cosmologies.

A. Models with $x = $ const

It is well known that for power-law solutions with $x = $ const the potential is necessary of the form $V = V_0 \phi^{-2}$,

with V_0 a constant. F, of course, is also constant. This is the case in power-law tachyon cosmologies, for instance. In order to avoid unnecessary complications, we will illustrate the phantom symmetry for the usual tachyon case, but, of course, it could be equally applied to other $x=$const models, like the generalized tachyon cosmologies in [5].

For the usual tachyon, one has

$$\rho = \frac{V}{\sqrt{1-\dot{\phi}^2}}, \tag{11}$$

$$p = -V\sqrt{1-\dot{\phi}^2}, \tag{12}$$

$$\gamma = \frac{\rho+p}{\rho} = \dot{\phi}^2, \tag{13}$$

and the Friedmann equation can be cast as

$$3H^2 = \frac{V_0}{\phi^2\sqrt{1-\dot{\phi}^2}}. \tag{14}$$

Applying the phantom symmetry to Eq. (13) we get

$$\bar{\gamma} = -\dot{\phi}^2, \tag{15}$$

$$3\bar{H}^2 = -\frac{\bar{V}_0}{\phi^2\sqrt{1+\dot{\phi}^2}}. \tag{16}$$

Thus, actually, Eqs. (14) and (16) arise from two different Lagrangians. Note also that the sign of the square of the time derivative of the k field gets reversed in the phantom transformation.

Now, the requirement that the energy density gets preserved enforces

$$\bar{V}_0 = -V_0\frac{\sqrt{1+\dot{\phi}_0^2}}{\sqrt{1-\dot{\phi}_0^2}}, \tag{17}$$

where we have put $\phi=\phi_0 t$ with ϕ_0 a constant. Let us see how this corresponds to the map $a\rightarrow a^{-1}$. Using Eqs. (13) and (14) and recalling $a\propto t^{2/3\gamma}$, after some algebra we arrive at the result

$$\gamma = \frac{2}{1+\sqrt{1+9V_0^2/4}}. \tag{18}$$

For the transformed solution we must take $\bar{a}\propto t^{2/3\bar{\gamma}}$ and $\bar{\phi} = \bar{\phi}_0 t$ with $\bar{\phi}_0$ a constant. Straightforward calculations which involve Eqs. (15) and (16) give

$$\bar{\gamma} = \frac{2}{1-\sqrt{1+9\bar{V}_0^2/4}}. \tag{19}$$

Finally, if we insert Eq. (17) back into Eq. (19) and do some more algebra, we see $\bar{\gamma} = -\gamma$.

B. Models with $x\neq$const

Let us first review how power-law models are obtained in this case. The conservation equation for $\gamma=$const can be readily integrated to give

$$VF = \frac{\rho_0}{a^{3\gamma}}, \tag{20}$$

where $a\propto t^{2/3\gamma}$ and ρ_0 a constant. If Eq. (8) is viewed as a differential equation for $F(x)$, one can solve it to yield

$$F(x) = (-x)^{\gamma/2(\gamma-1)}. \tag{21}$$

Note that in this case we would not have to make any requirement on the form of V, unless we wished to obtain $\phi(t)$ explicitly on using Eqs. (20) and (21). Nevertheless, any F like Eq. (21) will give a power-law solution for every potential.

Let us assume now that V does not change ($\bar{V}=V$), because power-law solutions in this case are obtained without making any assumption on V, as we just showed. We are going to deduce now the $\bar{x}(x)$ rule necessary to implement the phantom symmetry. Since we are looking for new power-law solutions, it is admissible, with the requirement of form-invariance, to assume, from the beginning,

$$\bar{F} = (-\bar{x})^{\bar{\gamma}/2(\bar{\gamma}-1)}. \tag{22}$$

Combining Eq. (22) with Eq. (10) and using $\bar{\gamma} = -\gamma$, we get

$$\dot{\bar{\phi}}^2 = \left(\frac{1+\gamma}{1-\gamma}\right)^{2(\gamma+1)/\gamma}\dot{\phi}^{2(\gamma+1)/(\gamma-1)}. \tag{23}$$

The latter is, in fact, a very interesting result. In standard scalar field theories, the phantom transformation can be implemented just by Wick rotating the field [25], and that was also the case in tachyon cosmology, as shown in the previous section. In contrast, in the $x\neq$const case, the transformation rule for the field, as given by Eq. (23), is not so simple and does not include the Wick rotation as a particular case either.

IV. CONCLUSIONS

As part of a long-term project [24,25] we have shown here that form-invariance transformations can be used as tools for generating new exact solutions to the Einstein field equations. In particular, we have applied the method to the obtention of phantom versions of FRW k-essence cosmologies, with an accent on power-law spacetimes. The discussion has been presented in a general setup, valid for FRW k-essence cosmologies, and we have only discussed power-law models for illustration purposes.

Specifically, we have been concerned with two families of such solutions: namely, those corresponding to a scalar field

(k field) with a constant and nonconstant time derivative, respectively. In broad terms, it can be seen that the transformation flips the sign of both the barotropic index and Hubble factor. Then, if the initial singularity of the seed solution is identified with the final big rip of the transformed one, an expanding phantom universe is obtained.

Interestingly enough, we have shown that in the x = const cases implementation of the phantom transformation requires a change in the potential. In the particular case of the tachyon, a sign reversal in the square of the time derivative of the scalar field is also needed; not surprisingly, perhaps, this is exactly the same rule as for standard scalar field cosmologies. Note also that the usual tachyon is a k-essence model with $F = \sqrt{1+x}$, whereas the phantom tachyon falls into a different category within k-essence models, because it corresponds to $F = \sqrt{1-x}$. In contrast, in the $x \neq$ const case, the phantom transformation can be realized without changing the potential, but the transformation rule for the scalar field is not as simple as the previous case.

Summarizing, we have given a neat prescription for generating phantom k-essence cosmologies, and we have shown that, against what one can naively expect, in some cases a simple $\dot{\phi}^2 \to -\dot{\phi}^2$ map does not do the job. Nevertheless, we believe the subject deserves further investigation, and even though we have concentrated here on power-law spacetimes, hopefully we will widen our scope in the future to address other cases.

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