PHYSICAL REVIEW D **86,** 076009 (2012)

Resonance Chiral Lagrangian analysis of $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$

D. Gómez Dumm[1] and P. Roig[2]

[1] *IFLP, CONICET — Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina*

[2] *Grup de Física Teòrica, Institut de Física d'Altes Energies, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain*

(Received 10 August 2012; published 8 October 2012)

The hadronization structure of $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ decays is analyzed using chiral perturbation theory with resonances, considering only the contribution of the lightest meson resonances at leading order in the $1/N_C$ expansion. After imposing the asymptotic behavior of vector spectral functions ruled by QCD, unknown effective couplings are determined by fitting the $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ branching ratio and decay spectrum to recent data. Predictions for the partner decay $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ and the low-energy behavior of the cross section $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ are also discussed.

DOI: 10.1103/PhysRevD.86.076009 PACS numbers: 11.15.Pg, 12.38.−t, 12.39.Fe

I. INTRODUCTION

Tau decays represent an ideal benchmark where one can analyze diverse topics in elementary particle physics [1]. In particular, semileptonic decay channels $\tau^- \to H^-\nu_\tau$, where H is some hadronic state, allow a rather clean theoretical analysis of the hadronization of the $V - A$ currents in the presence of QCD interactions, since there is no hadron pollution to the leptonic current. Thus, these processes provide a suitable tool to find out intrinsic properties of the involved hadron resonances [2–6]. In this work, we concentrate on the analysis of $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays. For these channels, the contributions of scalar and pseudoscalar resonances are expected to be negligible, since they turn out to be forbidden at tree level by symmetry arguments, such as G-parity conservation. In the limit of isospin symmetry, the corresponding amplitudes are driven by the vector current, allowing a precise study of the couplings in the odd-intrinsic parity sector.

Concerning the theoretical description, it is well-known that in the very low-energy domain [$E \ll M_\rho$, where M_ρ is the mass of the $\rho(770)$ meson], chiral perturbation theory (χPT) [7] is the adequate tool to describe hadronic τ decays [8]. However, this approach fails when the invariant mass of the hadronic state becomes comparable with the mass of the lightest vector and/or axial-vector resonances; therefore, a new strategy is needed in order to enlarge the domain of applicability of χPT to higher energies. One way out in this sense is to abandon the Lagrangian approach: one can model τ decay amplitudes by taking the lowest-order χPT results to fix the normalization of the form factors at low energies, incorporating the dominant vector and axial–vector meson resonance exchanges by modulating the amplitudes with *ad hoc* Breit-Wigner functions [2–5,9]. However, it can be seen that in the low-energy limit, this approach is, in general, not consistent with next-to-leading-order χPT [7]; hence, the usage of this procedure to reproduce QCD-ruled amplitudes is questionable [10,11]. An alternative approach is to include the lightest resonances as active degrees of freedom in the theory. This can be done by adding resonance fields to the χPT Lagrangian, without any dynamical assumption [12–15]. The inclusion of these fields can be carried out together with an expansion in the inverse of the number of colors (N_C) [16–19]: at the lowest order in the $1/N_C$ expansion, one gets from QCD an effective theory which includes a spectrum of infinite zero-width states. However, we know from phenomenology that resonance widths are relevant, and that the underlying dynamics is dominated by the lightest resonances. Hence, we consider here a model in which resonance widths are incorporated, taking into account—in a way consistent with QCD symmetry requirements—only the lightest resonant states which dominate the processes under study.[1]

A basic assumption of our approach is that the lightest resonant states are the dominant ones in low-energy phenomenology. In this way, for a given process, it should be sufficient to introduce only the lightest resonance multiplet carrying the appropriate quantum numbers, while the inclusion of higher states can be carried out as a correction [21,22]. On the other hand, the Lagrangian is built upon some fundamental QCD-based features: the effective interactions have to satisfy QCD symmetries, the low-energy behavior has to be consistent with χPT, and the asymptotic behavior of Green functions and associated form factors has to satisfy QCD constraints. These requirements imply several relations among the effective couplings which render the theory predictive. The aim of this work is to study within this framework the decays $\tau^- \to \eta\pi^-\pi^0\nu_\tau$, $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$, and the low-energy limit of the cross section $\sigma(e^+e^- \to \eta\pi^+\pi^-)$.

[1] The idea of considering a minimal number of hadronic states, which, for a given Green function, satisfy QCD short and long distance constraints within the large N_C limit, has been also considered in the context of the so-called minimal hadronic approximation to large-N_C QCD [20].

The article is organized as follows: In Sec. II, we recall how the χPT Lagrangian with resonances is built (see, e.g., Ref. [23]). The relevant hadronic form factors for the decays under study are given in Sec. III. In Sec. IV, we derive the QCD-ruled high-energy constraints on the couplings, which reduce the number of unknowns to only four. In Sec. V, we show that two of these unknowns can be bounded from other phenomenological studies performed within the same framework. In this way, we end up with two unknown couplings, which appear to be highly correlated [24,25]. The possible values of these couplings are analyzed by fitting experimental data on the differential decay distribution of $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ and taking into account the present upper limit on the branching ratio (BR) for $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$. The low-energy behavior of the cross section $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ is also discussed. Our conclusions are presented in Sec. VI. Finally, in Appendices A and B, we analyze other possible contributions to the decay amplitudes and quote some useful isospin relations.

II. THEORETICAL FRAMEWORK

Our effective Lagrangian is basically ruled by the approximate chiral symmetry of light-flavored QCD—which drives the interaction of light pseudoscalar mesons—and the $SU(3)_V$ assignments of resonance multiplets [12,14]. As we will see, for the processes under consideration, it is possible to achieve a good agreement with present experimental data without the inclusion of excited multiplets. Moreover, it is seen that vector meson dominance turns out to be a good approximation [12], since spin-zero resonance contributions vanish at tree level in the very accurate isospin symmetry limit (see Appendix A). In the case of τ decays, owing to the relatively large τ mass, it occurs that several resonances reach their on-shell condition when the amplitudes are integrated over the full phase space. The corresponding pole singularities can be regularized by including finite (energy-dependent) resonance widths, thus departing from the lowest order in the $1/N_C$ expansion. Here, we adopt the prescription in Ref. [26], where energy-dependent resonance widths have been calculated in a well-defined way using our Lagrangian formalism.

We will work out $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays considering exact isospin symmetry. In this limit, the processes are driven only by the vector current (see Sec. III) and appear to be dominated by the contributions of the $\rho(770)$ resonance. The relevant effective Lagrangian reads

$$\mathcal{L}_{R\chi T} \doteq \mathcal{L}_{WZW} + \mathcal{L}_{\rm kin}^{\rm V} + \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle + \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + i\frac{G_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle + \sum_{i=1}^{7}\frac{c_i}{M_V}\mathcal{O}^i_{\rm VJP} + \sum_{i=1}^{4} d_i\mathcal{O}^i_{\rm VVP} + \sum_{i=1}^{5}\frac{g_i}{M_V}\mathcal{O}^i_{\rm VPPP}, \quad (1)$$

where all coupling constants are real, F and M_V being the pion decay constant and the mass of the lightest vector meson resonances, respectively. We follow here the notation in Refs. [11,12,27].[2] Accordingly, $\langle\rangle$ stands for trace in flavor space, and u^μ, χ_+ and $f_+^{\mu\nu}$ are defined by

$$u^\mu = iu^\dagger D^\mu U u^\dagger, \qquad \chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u, \qquad f_\pm^{\mu\nu} = u^\dagger F_L^{\mu\nu}u^\dagger \pm uF_R^{\mu\nu}u, \quad (2)$$

where u ($U = u^2$), χ and $F^{\mu\nu}_{L,R}$ are 3×3 matrices which contain light pseudoscalar fields, current quark masses, and external left and right currents, respectively. The matrix $V^{\mu\nu}$ includes the lightest vector meson multiplet, and $\mathcal{L}^{\rm V}_{\rm kin}$ stands for the resonance kinetic term. The first term in Eq. (1) is the Wess-Zumino-Witten interaction Lagrangian [28,29], which governs the decay amplitudes studied here in the limit of low hadron momenta. The part of this interaction which contributes to the processes considered here reads

$$\mathcal{L}_{WZW} \doteq -\frac{iN_C}{48\pi^2}\epsilon_{\mu\nu\alpha\beta}\langle\Sigma_L^\mu U^\dagger\partial^\nu r^\alpha U l^\beta + \Sigma_L^\mu l^\nu\partial^\alpha l^\beta + \Sigma_L^\mu\partial^\nu l^\alpha l^\beta - (L\leftrightarrow R)\rangle, \quad (3)$$

where $\Sigma_{L,R}$ are given by $\Sigma_L^\mu = U^\dagger\partial^\mu U$, $\Sigma_R^\mu = U\partial^\mu U^\dagger$, and l^α and r^α are left and right external currents. Finally, the operators $\mathcal{O}^i_{\rm VJP}$, $\mathcal{O}^i_{\rm VVP}$ and $\mathcal{O}^i_{\rm VPPP}$ in Eq. (1) are given by

VJP terms

$$\begin{aligned}
\mathcal{O}^1_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle,\\
\mathcal{O}^2_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\nabla_\alpha u^\nu\rangle,\\
\mathcal{O}^3_{\rm VJP} &= i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\chi_-\rangle,\\
\mathcal{O}^4_{\rm VJP} &= i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}[f_-^{\rho\sigma}, \chi_+]\rangle,\\
\mathcal{O}^5_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}u^\sigma\rangle,\\
\mathcal{O}^6_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\}u^\nu\rangle,\\
\mathcal{O}^7_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}u_\alpha\rangle;
\end{aligned} \quad (4)$$

VVP terms

$$\begin{aligned}
\mathcal{O}^1_{\rm VVP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle,\\
\mathcal{O}^2_{\rm VVP} &= i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\chi_-\rangle,\\
\mathcal{O}^3_{\rm VVP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}u^\sigma\rangle,\\
\mathcal{O}^4_{\rm VVP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}u_\alpha\rangle;
\end{aligned} \quad (5)$$

[2] In Ref. [25], two additional operators ($\tilde{\mathcal{O}}^8_{\rm VJP}$ and $\tilde{\mathcal{O}}^5_{\rm VVP}$) have been found when the singlet $\langle VVP\rangle$ Green function is considered in addition to the octet one in the $p^2 \sim m_q \sim 1/N_C$ counting. In Appendix A, we show that they do not contribute to the hadronic tau decays studied here.

$VPPP$ terms

$$\begin{aligned}
\mathcal{O}^1_{\rm VPPP} &= i\varepsilon_{\mu\nu\alpha\beta}\langle V^{\mu\nu}(h^{\alpha\gamma}u_\gamma u^\beta - u^\beta u_\gamma h^{\alpha\gamma})\rangle,\\
\mathcal{O}^2_{\rm VPPP} &= i\varepsilon_{\mu\nu\alpha\beta}\langle V^{\mu\nu}(h^{\alpha\gamma}u^\beta u_\gamma - u_\gamma u^\beta h^{\alpha\gamma})\rangle,\\
\mathcal{O}^3_{\rm VPPP} &= i\varepsilon_{\mu\nu\alpha\beta}\langle V^{\mu\nu}(u_\gamma h^{\alpha\gamma}u^\beta - u^\beta h^{\alpha\gamma}u_\gamma)\rangle, \qquad (6)\\
\mathcal{O}^4_{\rm VPPP} &= \varepsilon_{\mu\nu\alpha\beta}\langle\{V^{\mu\nu}, u^\alpha u^\beta\}\chi_-\rangle,\\
\mathcal{O}^5_{\rm VPPP} &= \varepsilon_{\mu\nu\alpha\beta}\langle u^\alpha V^{\mu\nu}u^\beta\chi_-\rangle,
\end{aligned}$$

where $h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu$. The covariant derivative ∇_μ involves pseudoscalar meson fields and l^α, r^α external currents. Its explicit expression can be found in Ref. [12].

The nonet of vector resonances V is described here using the antisymmetric tensor formulation. In the context of vector meson dominance [14], this is shown to be consistent with the usage of the χPT Lagrangian for light pseudoscalar mesons up to $\mathcal{O}(p^2)$ in the even-intrinsic parity sector and up to $\mathcal{O}(p^4)$ in the odd-intrinsic parity sector [15].

III. FORM FACTORS IN $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$

In the Standard Model, $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decay amplitudes can be written as

$$\mathcal{M} = -\frac{G_F}{\sqrt{2}}V_{ud}\bar{u}_{\nu_\tau}\gamma^\mu(1-\gamma_5)u_\tau\mathcal{H}_\mu, \qquad (7)$$

where $V_{ud} \simeq \cos\theta_C$ is the relevant Cabibbo-Kobayashi-Maskawa mixing and $\mathcal{H}_\mu$ is the hadron matrix element of the left-handed QCD current $(V-A)_\mu$. In general, for a decay of a τ lepton into three pseudoscalar mesons, the hadronic tensor $\mathcal{H}_\mu$ can be written as [6]

$$\begin{aligned}
&\langle h_1(p_1)h_2(p_2)h_3(p_3)|(V-A)^\mu|0\rangle\\
&= F_1^A(Q^2,s_1,s_2)V_1^\mu + F_2^A(Q^2,s_1,s_2)V_2^\mu\\
&\quad + iF_3^V(Q^2,s_1,s_2)V_3^\mu + F_4^A(Q^2,s_1,s_2)Q^\mu, \qquad (8)
\end{aligned}$$

where

$$\begin{aligned}
V_1^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)(p_1-p_3)_\nu,\\
V_2^\mu &= \left(g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right)(p_2-p_3)_\nu,\\
V_3^\mu &= \varepsilon^{\mu\alpha\beta\gamma}p_{1\alpha}p_{2\beta}p_{3\gamma}, \qquad (9)\\
Q^\mu &= (p_1+p_2+p_3)^\mu,\\
s_i &= (Q-p_i)^2.
\end{aligned}$$

The upper indices in the form factors indicate the participating currents, either the axial vector (A), or the vector one (V). The form factors F_1^A and F_2^A drive a transition to hadronic states with quantum numbers $J^P = 1^+$, while F_3^V and F_4^A correspond to outgoing states with $J^P = 1^-$ and $J^P = 0^-$, respectively. Let us focus on the amplitude for the transition $\tau^- \to \eta_8(p_1)\pi^-(p_2)\pi^0(p_3)\nu_\tau$, considering the limit of exact isospin symmetry. First of all, it is easy to see that for this process, the axial-vector form factors vanish from G-parity conservation; therefore, the dynamics will be essentially determined by the form factor F_3^V.

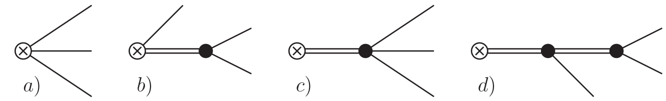

FIG. 1. Topologies contributing to the final hadron state in $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays in the $N_C \to \infty$ limit. Crossed circles indicate QCD vector current insertions. Single lines represent pseudoscalar mesons (η, π) while double lines stand for ρ-resonance intermediate states.

From the effective Lagrangian in Eq. (1), the diagrams which contribute to F_3^V are those represented in Fig. 1, where single solid lines correspond to π and η mesons and double lines to the $\rho(770)$ resonance. The corresponding contributions to the vector form factor read

$$F_3^{V(a)}{}_{(\eta_8\pi\pi)} = \frac{N_C}{6\sqrt{6}\pi^2F^3}, \qquad (10)$$

$$\begin{aligned}
F_3^{V(b)}{}_{(\eta_8\pi\pi)} = {}& \frac{8G_V}{\sqrt{3}F^3M_V}\frac{1}{M_\rho^2-s_1}[c_{125}Q^2 - c_{1256}s_1\\
&+ c_{1235}m_\eta^2 + 8c_3(m_\pi^2-m_\eta^2)], \qquad (11)
\end{aligned}$$

$$\begin{aligned}
F_3^{V(c)}{}_{(\eta_8\pi\pi)} = {}& -\frac{16F_V}{\sqrt{3}M_VF^3}\frac{1}{M_\rho^2-Q^2}\\
&\times[g_{123}s_1 - g_2(Q^2+2m_\pi^2-m_\eta^2)\\
&- (g_1-g_3)2m_\pi^2 + g_{45}m_\pi^2], \qquad (12)
\end{aligned}$$

$$\begin{aligned}
F_3^{V(d)}{}_{(\eta_8\pi\pi)} = {}& -\frac{8\sqrt{2}}{\sqrt{3}}\frac{F_VG_V}{F^3}\frac{1}{M_\rho^2-Q^2}\frac{1}{M_\rho^2-s_1}\\
&\times[d_3(Q^2+s_1) + (d_{12}-d_3)m_\eta^2\\
&+ 8d_2(m_\pi^2-m_\eta^2)], \qquad (13)
\end{aligned}$$

where we have defined

$$\begin{aligned}
&c_{125} = c_1 - c_2 + c_5, \quad c_{1256} = c_1 - c_2 - c_5 + 2c_6,\\
&c_{1235} = c_1 + c_2 + 8c_3 - c_5, \quad g_{123} = g_1 + 2g_2 - g_3, \qquad (14)\\
&g_{45} = 2g_4 + g_5, \quad d_{12} = d_1 + 8d_2.
\end{aligned}$$

The amplitude for the τ decay into the $\eta_0\pi^-\pi^0$ hadronic state can be read from Eqs. (10) to (13) by simply multiplying $F_3^{V(a,b,c,d)}{}_{(\eta_8\pi\pi)}$ by $\sqrt{2}$. Then, the matrix elements for the decays into the physical hadronic states $\eta\pi^-\pi^0$ and $\eta'\pi^-\pi^0$ can be obtained by considering $\eta_8-\eta_0$ mixing. Here, we will consider a double angle mixing scheme [30], which is consistent with the large-N_C expansion [31]. Using a notation similar to that in Ref. [32], the SU(3) octet and singlet fields are collected in a doublet $\eta_B^T \equiv (\eta_8, \eta_0)$, while the physical fields are included in $\eta_P^T \equiv (\eta, \eta')$. These doublets are related by the transformation $\eta_B = (\mathcal{M})^T\eta_P$, where [32]

$$\mathcal{M} = \begin{pmatrix} \cos\theta_P(1-\delta_8/2)+\sin\theta_P\delta_{80}/2 & -\sin\theta_P(1-\delta_0/2)-\cos\theta_P\delta_{80}/2 \\ \sin\theta_P(1-\delta_8/2)-\cos\theta_P\delta_{80}/2 & \cos\theta_P(1-\delta_0/2)-\sin\theta_P\delta_{80}/2 \end{pmatrix}. \quad (15)$$

In the framework of RχT, the parameters δ_8, δ_0, and δ_{80} can be in fact derived from an effective Lagrangian which involves scalar resonances [12]. If the latter are organized in a U(3) matrix S, from the lowest order Lagrangian

$$\mathcal{L}^S = c_d\langle S u_\mu u^\mu\rangle + c_m\langle S\chi_+\rangle, \quad (16)$$

one gets

$$\delta_8 = \frac{8c_dc_m}{M_S^2}\frac{M_8^2}{F^2}, \quad \delta_0 = \frac{8c_dc_m}{M_S^2}\frac{M_0^2}{F^2}, \quad \delta_{80} = \frac{8c_dc_m}{M_S^2}\frac{M_{80}^2}{F^2}, \quad (17)$$

where[3]

$$M_8^2 = \frac{1}{3}(4M_K^2 - M_\pi^2), \qquad M_0^2 = \frac{1}{3}(2M_K^2 + M_\pi^2),$$
$$M_{80}^2 = -\frac{2\sqrt{2}}{3}(M_K^2 - M_\pi^2). \quad (18)$$

Here, we take for M_π and M_K the isospin averaged values of the pion and kaon masses, neglecting higher-order corrections in the combined chiral and $1/N_C$ expansion. In addition, we assume $c_dc_m = F^2/4$ [34,35], which is required by high-energy QCD in the $N_C \to \infty$ limit. Finally, from $\eta - \eta'$ phenomenology we take $M_S \simeq 0.980$ GeV and $\theta_P = (-13.3 \pm 0.5)°$ [36].

Given the form factors, $F_3^V(Q^2, s_1, s_2)$, the spectral functions for the decays $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ are finally given by

$$\frac{d\Gamma}{dQ^2} = \frac{G_F^2|V_{ud}|^2}{128(2\pi)^5M_\tau}\left(\frac{M_\tau^2}{Q^2}-1\right)^2\frac{1}{3}\left(1+2\frac{Q^2}{M_\tau^2}\right) \times \int_{(m_\eta+m_\pi)^2}^{(\sqrt{Q^2}-m_\pi)^2} ds_2 \int_{t_-(Q^2,s_2)}^{t_+(Q^2,s_2)} ds_1 W_B(Q^2,s_1,s_2), \quad (19)$$

where the relevant structure function W_B [6] is defined by $W_B(Q^2, s_1, s_2) = V_3^2|F_3^V(Q^2, s_1, s_2)|^2$ and the limits of the integral over s_1 are

$$t_\pm(Q^2,s_2) = \frac{1}{4s_2}\{(Q^2+m_\eta^2-2m_\pi^2)^2 - [\lambda^{1/2}(Q^2,s_2,m_\pi^2) \mp \lambda^{1/2}(m_\eta^2,m_\pi^2,s_2)]^2\}, \quad (20)$$

with $\lambda(a, b, c) = (a + b - c)^2 - 4ab$. We have neglected here the neutrino mass.

[3]The fully dominant contribution to the η' mass is not due to current quark masses but to the $U(1)_A$ anomaly [33], through the topological susceptibility of gluondynamics. Hence, we keep θ_P as a free parameter, to be fitted from phenomenology.

IV. SHORT-DISTANCE CONSTRAINTS ON THE COUPLINGS

The above form factors depend on several combinations of coupling constants, besides the ρ mass and the pion decay constant. The values of these parameters are not provided by the effective theory, and their determination from the underlying QCD theory is still an open problem. However, one can get information on the effective couplings by assuming that the resonance region provides a bridge between the chiral and perturbative regimes, even when one does not include the full resonance spectrum [14]. This is implemented by matching the high-energy behavior of Green functions (or related form factors) evaluated within the resonance Lagrangian with asymptotic results obtained in perturbative QCD [11,14,15,20,22,25,27,37]. In particular, it has been shown that the analysis of the two-point Green functions $\Pi_{A,V}$ [14] and the three-point Green function VVP of QCD currents (with the inclusion of only one multiplet of vector resonances) [27] leads to the following constraints in the $N_C \to \infty$ limit:

(i) By demanding that the two-pion vector form factor vanishes at high momentum transfer, one obtains the condition $F_VG_V = F^2$ [14].

(ii) The analysis of the VVP Green function [27] leads to the following results for the couplings in Eqs. (11)–(13):

$$c_{125} = 0, \qquad c_{1235} = 0,$$
$$c_{1256} = -\frac{N_C}{32\pi^2}\frac{M_V}{\sqrt{2}F_V},$$
$$d_{12} = -\frac{N_C}{64\pi^2}\frac{M_V^2}{F_V^2} + \frac{F^2}{4F_V^2}, \quad (21)$$
$$d_3 = -\frac{N_C}{64\pi^2}\frac{M_V^2}{F_V^2} + \frac{F^2}{8F_V^2}.$$

On the other hand, it is possible to find additional constraints by requiring that the contributions of any intermediate hadronic state to the spectral function $\mathrm{Im}\Pi_V(Q^2)$ vanish in the limit $Q^2 \to \infty$. This is a reasonable assumption, since from perturbative QCD, $\mathrm{Im}\Pi_V(Q^2)$ has to go to a constant value for $Q^2 \to \infty$ [38], and the imaginary part of the two-point Green function can be understood as the sum of infinite intermediate hadronic states. Considering the intermediate $\eta^{(\prime)}\pi\pi$ hadronic states, one gets the following constraints on the coupling constants:

$$c_{125}=0, \quad c_{1256}=-\frac{N_C}{96\pi^2}\frac{M_V F_V}{\sqrt{2}F^2}, \quad d_3=-\frac{N_C}{192\pi^2}\frac{M_V^2}{F^2},$$
$$g_{123}=0, \quad g_2=\frac{N_C}{192\pi^2}\frac{M_V}{\sqrt{2}F_V}. \tag{22}$$

It is worth noticing that relations (22) are in agreement with those found in a similar analysis carried out for τ decays into $2K\pi\nu_\tau$ [39,40] and $P^-\gamma\nu_\tau$ ($P=\pi$, K) states [41]. Comparing with Eq. (21), we agree in the vanishing of c_{125}, while the constraints for c_{1256} and d_3 cannot be simultaneously satisfied keeping agreement with their values in Eq. (22). Moreover, as stated in Ref. [39], it is seen that the expected vanishing of the $\pi\gamma^\star\gamma$ form factor at high q^2 is obtained from Eq. (22) but not from Eq. (21). In any case, numerically, the differences are small, and the impact of these couplings on the observables is rather mild.[4] Thus, we choose to stick to our set of relations (22), using Eqs. (21) to fix the combinations c_{1235} and d_{12}, not obtained within our study. In this way, the analysis of short-distance constraints allows us to reduce significantly the number of unknown coupling constants in the form factors $F_3^{V(\alpha)}$ quoted in Eqs. (10)–(13). To calculate the decay amplitudes for the processes $\tau^-\to\eta^{(\prime)}\pi^-\pi^0\nu_\tau$, we end up with just four unknown parameters, namely, F_V, c_3, g_{45}, and d_2. As in the above-mentioned analysis, we will take $M_V=M_\rho$.

V. PHENOMENOLOGICAL ANALYSIS

In order to carry out a phenomenological analysis of $\tau^-\to\eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays, we take into account the available experimental information. In the case of $\tau^-\to\eta\pi^-\pi^0\nu_\tau$, this includes the measured branching fraction $\mathrm{BR}(\tau^-\to\eta\pi^-\pi^0\nu_\tau)=(1.39\pm0.10)\times10^{-3}$ [42], as well as the data on the corresponding spectral function obtained by Belle [43]. The process $\tau^-\to\eta'\pi^-\pi^0\nu_\tau$ has not been observed yet; hence, we consider only the upper bound given by the PDG [42], namely, $\mathrm{BR}(\tau^-\to\eta'\pi^-\pi^0\nu_\tau)<8.0\times10^{-5}$ at 90% confidence level.

As stated, the number of unknown parameters entering the vector form factor F_3^V in RχT can be reduced to four by means of the short-distance constraints obtained in Sec. IV. In addition, the values of F_V and g_{45} can be estimated within RχT from the phenomenological analysis of $\tau^-\to(\pi\pi\pi)^-\nu_\tau$ and $\omega\to\pi^+\pi^-\pi^0$, respectively: the best fit to the $\tau^-\to(\pi\pi\pi)^-\nu_\tau$ spectral function measured by ALEPH [44] corresponds to $F_V=0.180$ GeV [22] with an estimated error of $\sim15\%$,[5] while from the $\omega\to\pi^+\pi^-\pi^0$ branching ratio, one gets $g_{45}=-0.60\pm0.02$ [39]. In this way, we are left with only two unknowns, namely, the coupling constants c_3 and d_2. Our goal is to be able to describe the available experimental information just by fitting these two parameters.

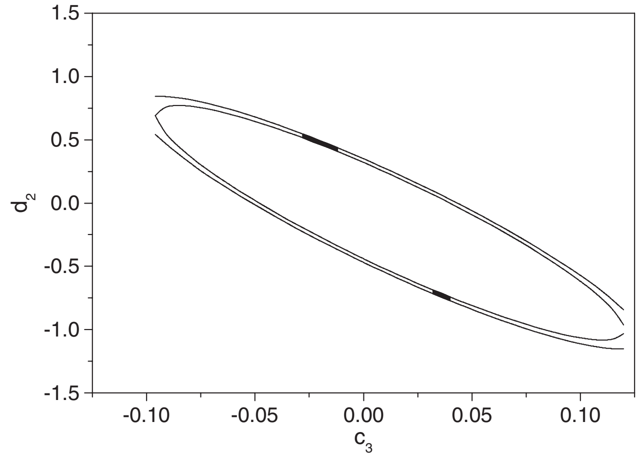

FIG. 2. Contour in the c_3-d_2 plane compatible with the branching ratio of the decay $\tau^-\to\eta\pi^-\pi^0\nu_\tau$ at the level of one standard deviation. The black strips highlight the two regions which yield lowest values of χ^2/dof (within one unit) from a fit to Belle data [43] on the corresponding spectral function.

In Fig. 2, we show the c_3-d_2 parameter region compatible with the PDG branching ratio for the mode $\tau^-\to\eta\pi^-\pi^0\nu_\tau$ at the level of one sigma. A large correlation between both couplings can be appreciated, in agreement with Refs. [24,25].[6] Then, taking into account this allowed region for c_3 and d_2, we have carried out a fit to Belle data [43] for the $\tau^-\to\eta\pi^-\pi^0\nu_\tau$ spectral function. We find two χ^2 minima, located at $(c_3,d_2)=(-0.018,0.45)$ and $(c_3,d_2)=(0.035,-0.70)$, with $\chi^2/\mathrm{dof}=3.80$ and 4.34, respectively, where the statistical error is about 10%. These values are indicated in Fig. 2, where black strips correspond to the c_3-d_2 regions which keep χ^2/dof within one unit far from the minima. The corresponding theoretical curves for the spectral function, together with experimental data, are shown in Fig. 3. We have also carried out a fit to the normalized spectral function, obtaining that the preferred values for (c_3,d_2) remain almost unchanged,

[4] The introduction of additional resonances has a different effect on the short-distance relations obtained from the VVP Green function and from the imaginary part of the vector–vector correlator. While all new contributions to the correlator are positive definite, this is not true for the VVP Green function, where cancellations are allowed. Thus, the outcome of both procedures may be different when the spectrum is restricted to the lowest–lying resonances. A convergence of both results should be recovered if the full tower of excited resonances is taken into account.

[5] Some theoretical analyses lead to the value $F_V=\sqrt{3}F\sim0.160$ GeV [41,45–50]. We have checked that a change of F_V within the range $[0.160, 0.180]$ GeV does not affect significantly the results presented in this section.

[6] In particular, as noticed in Ref. [25], there is an anticorrelation between c_3-d_2 in $\tau^-\to\eta'\pi^-\pi^0\nu_\tau$ and in associated radiative decays. Therefore, the combined study could improve the determination of these couplings.

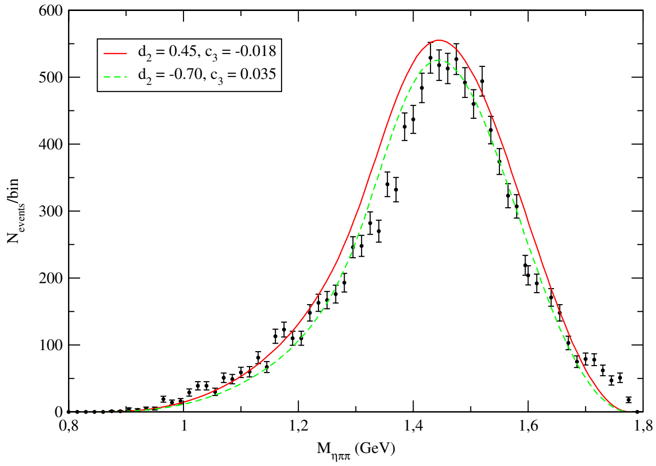

FIG. 3 (color online). Theoretical curves fitting the spectral function for $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ decay, compared to experimental data [43].

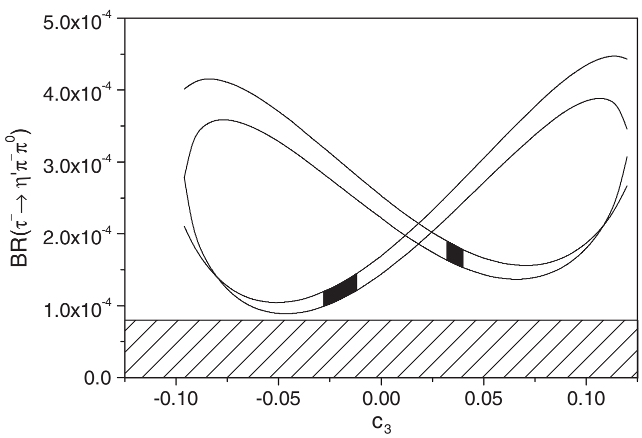

FIG. 4. Prediction for the branching ratio of the decay $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ consistent (within one sigma) with the $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ branching ratio quoted by the PDG. The horizontal line, corresponding to $\mathrm{BR}(\tau^- \to \eta'\pi^-\pi^0\nu_\tau) = 0.8 \times 10^{-4}$, represents the current PDG bound. The notation for the black strip is the same as in Fig. 2.

while χ^2/dof values get reduced to 3.0 for both minima. Finally, in order to account for the theoretical error of the high-energy predictions for the couplings c_{1256}, d_3, g_2 and d_{12}, we have also fitted the data allowing these coupling combinations to vary within $\pm 1/3$ of the values obtained from Eqs. (21) and (22).[7] It is noteworthy that the χ^2 value does not get reduced, which can be taken as an indication that our short-distance relations (obtained at leading order in $1/N_C$) lead to an appropriate effective Lagrangian to reproduce the experimental observations.

Considering the fitted values for c_3 and d_2, we can calculate the corresponding predictions for the $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ branching ratio. The results are shown in Fig. 4, where we have taken c_3 as the independent parameter. It is seen that the predictions are somewhat above the 90% confidence level upper bound quoted by the PDG, which is indicated by the shadowed region in the figure. However, the result corresponding to $c_3 = -0.018$ turns out to be rather close to the upper bound; in fact, compatibility is achieved if the width of the $c_3 - d_2$ band is enlarged considering two standard deviations in the measured value of $\mathrm{BR}(\tau^- \to \eta\pi^-\pi^0\nu_\tau)$. Future, more precise measurements of the $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ process should indicate whether our slight discrepancy arises from a weakness in the theoretical assumptions (e.g., treatment of $\eta_8 - \eta_0$ mixing, effect of excited resonances, SU(3) breaking terms in the Lagrangian [52]) or it just reflects an issue in the detection of this τ decay mode. In this regard, we emphasize the importance of making global fits with unified and consistent treatments of all hadronic currents, in order to avoid cross-contamination between different hadronic tau decay channels from misunderstood backgrounds. The improvement in the most relevant hadronic matrix elements in TAUOLA [53,54] may be a key tool in this sense.

Finally, our analysis can be used to predict $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ in the low-energy region (conversely, one could, in general, use data on e^+e^- annihilation into hadronic states to get predictions for the corresponding semileptonic tau decays [55,56][8]). The relation between this cross section and the $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ spectral function is detailed in Appendix B. One gets

$$\frac{d\Gamma(\tau^- \to \eta\pi^-\pi^0\nu_\tau)}{dQ^2} = 2f(Q^2)\sigma(e^+e^- \to \eta\pi^+\pi^-), \tag{23}$$

where $f(Q^2)$ is given by

$$f(Q^2) = \frac{G_F^2|V_{ud}|^2}{384(2\pi)^5 M_\tau}\left(\frac{M_\tau^2}{Q^2} - 1\right)^2\left(1 + 2\frac{Q^2}{M_\tau^2}\right)\left(\frac{\alpha^2}{48\pi}\right)^{-1} Q^6. \tag{24}$$

In Fig. 5, we quote our predictions for the cross section, in comparison with low-energy e^+e^- data obtained in various experiments.[9] We notice that although the η' meson decays

[7] We have also considered nonvanishing values for the coupling c_{1235}, which should be zero according to Eq. (21). Notwithstanding, have kept c_{125} and g_{123} equal to zero. Indeed, if $c_{125} \neq 0$ the Brodsky-Lepage behavior [51] of the form factor is violated, and $\mathrm{Im}\Pi_V$ goes, asymptotically, as $Q^6 \log(Q^2/M_V^2)$; if $g_{123} \neq 0$, the asymptotic growth goes as $\mathcal{O}(Q^6)$. Varying c_{1235} in the range $[-0.05, 0.05]$ does not improve the fit.

[8] A more elaborated dedicated approach, also based in RχT, has been developed for $\sigma(e^+e^- \to \eta/\pi^0\pi^+\pi^-)$ [57].

[9] We note that in this neutral current process there are additional contributions from new operators $\mathcal{O}^8_{\mathrm{VJP}}$ and $\mathcal{O}^5_{\mathrm{VVP}}$, see Ref. [25], which implies the introduction of two additional unknown couplings. However, these terms are suppressed in the large-N_C limit in the standard counting (see discussion in Appendix A).

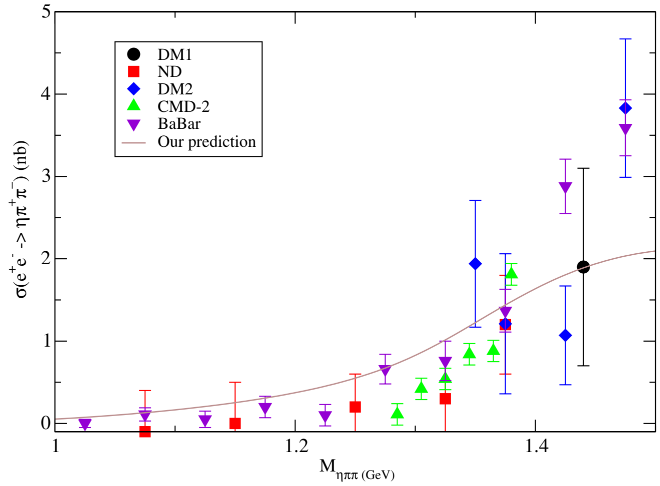

FIG. 5 (color online). Prediction for $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ low-energy behavior from our analysis of $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ decays, in comparison with DM1 [59], ND [60], DM2 [61], CMD-2 [62], and *BABAR* [63] data.

to $\eta\pi^+\pi^-$ with a fraction of about 45%, there is no significant contamination from the chain $\sigma(e^+e^- \to \eta'\gamma^\star \to \eta\pi^+\pi^-)$ since, due to C parity, this occurs at next-to-leading order in powers of the electromagnetic coupling α. From the figure, it is seen that our results are consistent with experimental data up to a center-of-mass energy of about 1.4 GeV. In fact, one should not expect our treatment to be valid beyond this energy region, where effects of excited states should be sizeable and there is no phase space suppression as in τ decay spectral functions.

VI. CONCLUSIONS

We have worked out the decays $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ and $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ within the framework of chiral perturbation theory with resonances. The theoretical analysis has been based on the large-N_C expansion of QCD, the low-energy limit given by χPT and the appropriate asymptotic behavior of the form factors, which helps to fix most of the initially unknown effective couplings. Indeed, after taking into account information acquired in the previous related studies, $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ amplitudes can be written in terms of only two unknown parameters.

We have carried out a phenomenological analysis taking into account the experimental data for the branching ratio and the spectrum of the decay $\tau^- \to \eta\pi^-\pi^0\nu_\tau$, as well as the present upper bound for the $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$ branching fraction. A fit to the data allows us to determine two preferred sets of values for the unknown parameters, c_3 and d_2, in the effective Lagrangian, namely, $(c_3, d_2) = (-0.018, 0.45)$ and $(0.035, -0.88)$, which lead to a reasonable overall description of the spectrum. The former set seems to be favored by the predictions for the branching ratio $\mathrm{BR}(\tau^- \to \eta'\pi^-\pi^0\nu_\tau)$, although in both cases, the theoretical values appear to be somewhat above the present experimental upper bound. Finally, using isospin symmetry, these results can be used to get a prediction for the low-energy behavior of the cross section $\sigma(e^+e^- \to \eta\pi^+\pi^-)$. The results are in good agreement with the available experimental information, and the approach can be useful [57] for the implementation of the related hadronic current in the PHOKHARA [58] Monte Carlo generator.

Our present results should be regarded as a first step in the study of the $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays in our framework. In light of higher statistics for the $\eta\pi^-\pi^0\nu_\tau$ mode, or the observation of the decay $\tau^- \to \eta'\pi^-\pi^0\nu_\tau$, this description could be improved by considering, e.g., the exchange of excited vector resonances, SU(3)-breaking terms in the Lagrangian or revising the $\eta_8 - \eta_0$ mixing scheme.

ACKNOWLEDGMENTS

We are indebted to Antonio Pich for his critical reading of our draft. We acknowledge Jorge Portolés for making useful remarks on this project. We appreciate very much that K. Hayasaka and K. Inami gave us access to Belle data for our research. We wish to thank Henryk Czyz and Simon Eidelman since their interest in the low-energy description of $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ triggered, in fact, this part of our work. We also acknowledge Simon's help in accessing the experimental data and Henryk's comments putting forward a relevant consequence of our isospin analysis. It is always encouraging and rewarding for P. R. to collaborate with T. Przedzinski, O. Shekhovtsova, and Z. Was. P. R. has benefited from discussions with R. Escribano and P. Masjuan on $\eta - \eta'$ mixing and with V. Cherepanov and I. M. Nugent on the decay modes and cross section studied in this article. This work has been partially supported by the Spanish Grants No. FPA2007-60323 and No. FPA2011-25948, and by the Spanish Consolider Ingenio 2010 Programme CPAN (CSD2007-00042). It has also been founded in part by CONICET and ANPCyT (Argentina), under Grants No. PIP02495 and No. PICT07-03-00818.

APPENDIX A: CONTRIBUTION OF SPIN-ZERO RESONANCES AND SINGLET TERMS

In this appendix, we analyze both the contribution of spin-zero (scalar and pseudoscalar) resonances and SU(3) singlet couplings to $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decays.

1. Scalar and pseudoscalar resonance exchange

Discrete symmetries of QCD constrain the possible couplings in the effective Lagrangian. One of these symmetries is G parity, which is exact in the SU(2) symmetry limit. With our conventions, the corresponding quantum numbers are $G_{A_\mu} = -1$, $G_{V_\mu} = +1$, $G_\eta = +1$, $G_{\pi^{(*)}} = -1$, $G_{f_0/\sigma} = +1$, $G_{a_0} = -1$; thus, the final state $\eta\pi^-\pi^0$ has $G = +$. As stated in Sec. III, since the axial-vector weak current has $G = -$, only the vector current can contribute to the $\tau^- \to \eta^{(\prime)}\pi^-\pi^0\nu_\tau$ decay amplitudes in this limit. The intermediate states $f_0\pi^-$, $\sigma\pi^-$ are also forbidden by G-parity conservation, which only leaves the channels $a_0^-\pi^0 \to \eta\pi^-\pi^0$ and $a_0^0\pi^- \to \eta\pi^0\pi^-$. However, the vector current is $J^P = 1^-$, while the intermediate states $a_0\pi$ have parity $P = (-1)^{J+1}$. Therefore, one can conclude that both scalar and pseudoscalar resonance contributions are strongly suppressed at tree level and can be safely neglected.

2. Contribution of double-trace terms

In Ref. [25], two additional operators have been found with respect to those in Ref. [27]. Although these operators involve two traces, hence they are suppressed in the standard counting (in powers of $p^2 \sim m_q^2$ and $1/N_C$), it is seen that they become leading when a simultaneous counting in all three expansion parameters is carried out [31]. The operators read

$$\begin{aligned}\tilde{O}^8_{VJP} &= -i\tilde{c}_8 M_V\sqrt{\frac{2}{3}}\varepsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\tilde{f}_+^{\rho\sigma}\rangle \log(\det\tilde{u})\\ \tilde{O}^5_{VVP} &= -i\tilde{d}_5 M_V^2\sqrt{\frac{2}{3}}\varepsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}V^{\rho\sigma}\rangle \log(\det\tilde{u}),\end{aligned} \tag{A1}$$

where the tildes stand for u and f matrices which include the singlet term (and would contribute to the processes considered here through the η and η' components of the η_0 meson). Once again, the contribution of these operators vanishes, since the second operator only contributes to neutral current processes, while the first one leads to the contraction of symmetric and antisymmetric tensors in the τ decay amplitudes.

APPENDIX B: ISOSPIN RELATIONS

In this appendix, we provide a derivation of Eq. (23), which allows us to relate the $\tau^- \to \eta\pi^-\pi^0\nu_\tau$ differential decay rate and the $\sigma(e^+e^- \to \eta\pi^+\pi^-)$ cross section.

We work in the limit of SU(2) isospin symmetry, and we neglect Z-exchange contributions to the hadronic e^+e^- cross section, which is a safe approximation in the considered energy range. Thus, this process will be driven by the vector current, via photon exchange. One expects to get a relation between this cross section and the vector current contribution to the decay of a tau lepton into the corresponding hadronic state.

Since both η_8 and η_0 states are SU(2) singlets, we can compute isospin relations between $\eta_{0,8}\pi\pi$ channels just by taking into account the isospin of $\pi\pi$ states. Let us denote by T_{-0}, T_{0-} the amplitudes $\langle\eta\pi\pi|\bar{d}\Gamma^\mu_w u|0\rangle$ and by T_{+-}, T_{-+}, T_{00} the amplitudes $\frac{1}{\sqrt{2}}\langle\eta\pi\pi|(\bar{u}\Gamma^\mu_w u - \bar{d}\Gamma^\mu_w d)|0\rangle$, where the subscripts correspond to pion electric charges, and η can be either η_0 or η_8. We obtain the relations

$$\begin{aligned}&\frac{1}{\sqrt{2}}(T_{-0}+T_{0-}) = -\frac{1}{\sqrt{6}}(T_{+-}+T_{-+}-2T_{00}) = 0,\\ &\frac{1}{\sqrt{2}}(T_{0-}-T_{-0}) = -\frac{1}{\sqrt{2}}(T_{-+}-T_{+-}),\\ &\sqrt{3}(T_{+-}+T_{-+}-T_{00}) = 0,\end{aligned} \tag{B1}$$

which lead to

$$T_{00} = 0, \qquad T_{+-} = -T_{-+} = T_{0-} = -T_{-0}. \tag{B2}$$

Now, let us consider the electromagnetic current. One can decompose it into $I = 0$ and $I = 1$ pieces:

$$\Gamma^\mu_{em} = \frac{1}{3}(2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s) = \Gamma^\mu_{(0)} + \Gamma^\mu_{(1)}, \tag{B3}$$

where

$$\begin{aligned}\Gamma^\mu_{(0)} &= \frac{1}{6}(\bar{u}\gamma^\mu u + \bar{d}\gamma^\mu d - 2\bar{s}\gamma^\mu s),\\ \Gamma^\mu_{(1)} &= \frac{1}{2}(\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d).\end{aligned} \tag{B4}$$

One can relate the amplitudes $\langle\eta\pi\pi|\Gamma^\mu|0\rangle$ for charge and isospin (A_I) $|\eta\pi\pi\rangle$ states by

$$A_{+-} = \frac{1}{\sqrt{2}}A_1 + \frac{1}{\sqrt{3}}A_0, \qquad A_{-+} = -\frac{1}{\sqrt{2}}A_1 + \frac{1}{\sqrt{3}}A_0,$$
$$A_{00} = -\frac{1}{\sqrt{3}}A_0. \tag{B5}$$

Moreover, the vanishing of the amplitude A_2 implies

$$2A_{00} + A_{+-} + A_{-+} = 0. \tag{B6}$$

In this way, one obtains the following relations:

$$\begin{aligned}&A_{+-}+A_{00} = \frac{1}{\sqrt{2}}A_1, \quad A_{-+}+A_{00} = -\frac{1}{\sqrt{2}}A_1,\\ &A_1 = \frac{A_{+-}-A_{-+}}{\sqrt{2}},\\ &A_0 = \frac{A_{+-}+A_{-+}-A_{00}}{\sqrt{3}} = -\sqrt{3}A_{00} = \frac{\sqrt{3}}{2}(A_{+-}+A_{-+}),\end{aligned} \tag{B7}$$

which lead to

$$\begin{aligned}|A_{+-}+A_{-+}|^2+|A_{+-}-A_{-+}|^2 &= 2(|A_{+-}|^2+|A_{-+}|^2)\\ &= 4|A_{00}|^2+2|A_1|^2,\end{aligned} \tag{B8}$$

$$|A_1|^2 = |A_{+-}|^2 + |A_{-+}|^2 - 2|A_{00}|^2. \tag{B9}$$

Thus corresponding cross sections are related by

$$\begin{aligned}&\sigma(e^+e^- \to \eta\pi\pi)|_{I=1} = \sigma(e^+e^- \to \eta\pi^+\pi^-)\\ &\quad + \sigma(e^+e^- \to \eta\pi^-\pi^+) - 2\times 2\sigma(e^+e^- \to \eta\pi^0\pi^0)\\ &\simeq 2\sigma(e^+e^- \to \eta\pi^+\pi^-),\end{aligned} \tag{B10}$$

where the additional factor 2 in the first equation arises from the presence of identical particles in the final state. In the last line, we have neglected the cross section to the $\eta\pi^0\pi^0$ state, since it turns that it vanishes at the lowest order in the electromagnetic coupling α owing to C-parity conservation. For the isoscalar part, from Eq. (B7), we find

$$\sigma(e^+e^- \to \eta\pi\pi)|_{I=0} = 6\sigma(e^+e^- \to \eta\pi^0\pi^0). \quad \text{(B11)}$$

Finally, one has

$$\frac{1}{\sqrt{2}}(T^{0-} - T^{-0}) = \sqrt{2}T^{0-} = -\langle 1, 0| \frac{\bar{u}u - \bar{d}d}{\sqrt{2}} |0\rangle$$
$$= -\sqrt{2}A_1. \quad \text{(B12)}$$

Taking into account that the e^+e^- cross section into three hadrons is given by

$$\sigma_{e^+e^-\to h_1h_2h_3}(Q^2) = \frac{e^4}{768\pi^3}\frac{1}{Q^6}\int \mathrm{d}s\mathrm{d}t|F_3|^2(-V_{3\mu}V_3^{\mu*}), \quad \text{(B13)}$$

the cross sections for the different modes read ($|A_{+-}|^2 = |A_1|^2/2 = |A_{-0}|^2/2$)

$$\sigma(e^+e^- \to \eta\pi^+\pi^-) = \frac{\alpha^2}{96\pi}\frac{1}{Q^6}\int \mathrm{d}s\mathrm{d}t|T_{-0}|^2(V_{3\mu}V^{3\mu*}),$$
$$\sigma(e^+e^- \to \eta\pi^0\pi^0) = \frac{\alpha^2}{48\pi}\frac{1}{Q^6}\int \mathrm{d}s\mathrm{d}t\frac{1}{2}|T_{00}|^2(V_{3\mu}V^{3\mu*}), \quad \text{(B14)}$$

where the additional factor $1/2$ in the second equation arises from the presence of identical particles in the final state. Thus, one finally obtains

$$\frac{d\Gamma(\tau^- \to \eta\pi^-\pi^0\nu_\tau)}{\mathrm{d}Q^2} = f(Q^2)\sigma(e^+e^- \to \eta\pi\pi)|_{I=1}$$
$$= 2f(Q^2)[\sigma(e^+e^- \to \eta\pi^+\pi^-) - 2\sigma(e^+e^- \to \eta\pi^0\pi^0)]$$
$$\simeq 2f(Q^2)\sigma(e^+e^- \to \eta\pi^+\pi^-), \quad \text{(B15)}$$

where $f(Q^2)$ is the kinematical factor given in Eq. (24).

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