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Double-beta-decay nuclear matrix elements in the QRPA framework

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Abstract

We review the recent work done by the Jyväskylä–La Plata collaboration on the calculation of nuclear matrix elements for various modes of double-beta decays. Whenever helpful, we connect our recent work to the historical background in order to highlight the progress achieved in the field of double-beta decay. At the same time, we introduce some new concepts and ideas to treat e.g. possible backgrounds in Gamow–Teller strength functions generated in (p,n) and (n,p) reactions.

1. Introduction

Thanks to the neutrino-oscillation experiments much has been learned about the basic properties of the neutrino in the last decade [1]: their relative masses and mixing properties are now known quite accurately. The underground experiments are complementary to the oscillation experiments and they can determine the fundamental (Majorana) nature and absolute mass scale of the neutrinos via detection of the neutrinoless double-beta ($0\nu\beta\beta$) decay of atomic nuclei. To extract this information from the prospective data, we need to know the details of nuclear structure for the decaying nuclei. The structure information can be crystallized in the form of nuclear matrix elements (NMEs) intimately intertwined with the lepton aspects of the various forms of double-beta decays [2, 3].

In the present work, we aim to provide a glimpse of the recent progress achieved by the Jyväskylä–La Plata collaboration in calculating values of the NMEs for various double-beta and related processes. We address both the electron- and positron-emitting modes of the two-neutrino double-beta ($2\nu\beta\beta$) and $0\nu\beta\beta$ decays. We also shed light on the double-electron capture (ECEC) modes that are presently under vigorous experimental scrutiny. Furthermore, we explore various independent and complementary ways to extract direct or indirect information on the $2\nu\beta\beta$ and $0\nu\beta\beta$ NMEs. These methods include, among others, the various charge-exchange reactions and our recent work on their possible isovector spin monopole (IVSM) backgrounds.

All the above-described decays and processes are handled in the general framework of the quasiparticle random-phase approximation (QRPA) by combining its charge-conserving mode (ccQRPA) and proton–neutron mode (pnQRPA). In particular, we address the long-standing 'g_{pp} problem' of the pnQRPA and present various ways to deal with this problem. We also highlight our recent work on the single-particle aspects of the calculations related to the spin–orbit-partner orbitals and orbital occupancies. It should be stressed that the topics are discussed in quite a compact manner due to space limitation and details are given in the many cited references.

2. Lepton aspects of double-beta decay

Here we present briefly the formalism that we use to compute the double-beta NMEs as well as the Gamow–Teller β^+ and β^- decay amplitudes and the associated strength functions.

2.1. The $2\nu\beta\beta$-decay amplitude

The $2\nu\beta\beta$-decay half-life, $t_{1/2}^{(2\nu)}$, for a transition from the initial ground state, 0_i^+, to the final J^+ state, J_f^+ (here either the ground state or some excited 0^+ or 2^+ state), can be compactly written in the form

$$\left[t_{1/2}^{(2\nu)}(0_i^+ \rightarrow J_f^+)_\alpha\right]^{-1} = G_\alpha^{(2\nu)}(J)\left|M_\alpha^{(2\nu)}(J)\right|^2 , \tag{1}$$

where $\alpha = \beta^-\beta^-, \beta^+\beta^+, \beta^+\mathrm{EC}, \mathrm{ECEC}$ is the mode of beta decay. Here $G_\alpha^{(2\nu)}(J)$ is the leptonic phase-space factor for the different double-beta channels: double-electron emission ($\beta^-\beta^-$), double-positron emission ($\beta^+\beta^+$), positron emission combined with electron capture (β^+EC) and double-electron capture (ECEC) [2, 4]. For the $\beta^-\beta^-$ and $\beta^+\beta^+$ modes, the NME is written as

$$M_\alpha^{(2\nu)}(J) = \sum_{mn} \frac{M_\mathrm{F}^J(1_m^+)\langle 1_m^+|1_n^+\rangle M_\mathrm{I}(1_n^+)}{D_m} , \quad \alpha = \beta^-\beta^-, \beta^+\beta^+, \tag{2}$$

and for the rest of the channels, the expressions are compactly written in [4]. The amplitudes connecting the initial ground state (the initial amplitude I) and the final ground or excited state (the final amplitude F) are

$$M_\mathrm{I}(1_n^+) = \left(1_n^+ \left\| \sum_k t_k^\pm \boldsymbol{\sigma}_k \right\| 0_i^+\right) , \quad M_\mathrm{F}^J(1_m^+) = \left(J_f^+ \left\| \sum_k t_k^\pm \boldsymbol{\sigma}_k \right\| 1_m^+\right) , \tag{3}$$

where $t_k^\pm$ is the flavor-changing operator for the kth nucleon in the β^+ or β^- direction. The quantity D_m is the energy denominator containing the average energy of the 1^+ states emerging from the two pnQRPA calculations, one for the initial nucleus and the other for the final nucleus. The denominator can thus be written as

$$D_m = \left(\tfrac{1}{2}\Delta + \tfrac{1}{2}[E(1_m^+) + \tilde{E}(1_m^+)] - M_i c^2\right)/m_\mathrm{e}c^2 , \tag{4}$$

where Δ is the nuclear mass difference of the $\beta\beta$ initial and final states, $\tilde{E}(1_m^+)$ is the energy of the mth 1^+ state in a pnQRPA calculation based on the initial ground state, $E(1_m^+)$ is the same for a calculation based on the final ground state and $M_i c^2$ is the mass energy of the initial nucleus. The quantity $\langle 1_m^+|1_n^+\rangle$ is the overlap between the two sets of 1^+ states [2].

2.2. Gamow–Teller strength functions

The Gamow–Teller strength functions in the (p,n) (GT^-) and (n,p) (GT^+) directions consist of the following quantities:

$$\mathrm{GT}_m^- = \left|\left(1_m^+ \left\| \sum_k t_k^- \boldsymbol{\sigma}_k \right\| 0_i^+\right)\right|^2 , \quad \mathrm{GT}_m^+ = \left|\left(1_m^+ \left\| \sum_k t_k^+ \boldsymbol{\sigma}_k \right\| 0_f^+\right)\right|^2 , \tag{5}$$

where 0_i^+ is the ground state of an (N, Z) nucleus and 0_f^+ is the ground state of an $(N-2, Z+2)$ nucleus. The states 1_m^+ are members of the set of 1^+ states in the intermediate nucleus of double-beta decay.

2.3. Neutrinoless double-beta decay

Along the lines described in section 2.1, the $0\nu\beta\beta$-decay half-life can be written as [2, 5, 6]

$$\left[t_{1/2}^{(0\nu)}(0_i^+ \to 0_f^+)_\alpha\right]^{-1} = G_\alpha^{(0\nu)}(M^{(0\nu)'})^2|\langle m_\nu\rangle|^2 , \quad \alpha = \beta^-\beta^-, \beta^+\beta^+, \beta^+\mathrm{EC} , \tag{6}$$

where $\langle m_\nu\rangle$ is the effective neutrino mass [2]. The NME of (6) can be written as a linear combination of the Gamow–Teller, Fermi and tensor terms [7, 8], i.e.

$$M^{(0\nu)'} = \left(\frac{g_\mathrm{A}}{g_\mathrm{A}^\mathrm{b}}\right)^2\left[M_\mathrm{GT}^{(0\nu)} - \left(\frac{g_\mathrm{V}}{g_\mathrm{A}}\right)^2 M_\mathrm{F}^{(0\nu)} + M_\mathrm{T}^{(0\nu)}\right], \tag{7}$$

where $g_\mathrm{A}^\mathrm{b} = 1.25$ is the bare-nucleon value of the axial-vector coupling constant. Here we consider only the final ground state or excited 0^+ states since $0\nu\beta\beta$ decays to 2^+ final states are strongly suppressed [9]. Values for the phase-space factors $G_\alpha^{(0\nu)}$ are given in [2, 6, 10, 11] for the value $g_\mathrm{A} = 1.25$ as is required by the definition of the NME $M_\alpha^{(0\nu)'}$ in the above equations. Details of the associated NMEs are given e.g. in [2, 12].

2.4. Resonant neutrinoless double-electron capture

The resonant neutrinoless double-electron capture (R0νECEC) was studied in [13, 14] from the lepton aspect points of view. There it was suggested that the fulfillment of a resonance condition in this decay could enhance the decay rates up to a factor of a million. The R0νECEC decay proceeds between two atomic states in the form

$$e^- + e^- + (A, Z) \to (A, Z-2)^* \to (A, Z-2) + \gamma + 2X, \tag{8}$$

where the capture of two atomic electrons leaves the final nucleus in an excited state that decays by one or more gamma rays and the atomic vacancies are filled by outer electrons with emission of x-rays. The corresponding half-life can be written as

$$\left[T_{1/2}^{\mathrm{R}0\nu\mathrm{ECEC}}(J^+)\right]^{-1} = G_{0\nu}^{\mathrm{ECEC}}(J^+)\left|M_{0\nu}^{\mathrm{ECEC}}(J^+)\right|^2 \frac{|\langle m_\nu\rangle|^2\Gamma}{(Q-E)^2 + \Gamma^2/4} , \tag{9}$$

where $J = 0, 2$ is the angular momentum of the nuclear final state. The difference $Q - E$ is the degeneracy of the initial and final states, Q being the difference between the masses of the initial and final atoms (decay Q value) and E is the total energy of the excited state in the final atom (consisting of the nuclear excitation energy and the excitation energy of the two holes in the electronic shells). The quantity Γ is the decay width of the two holes in the atomic shells [13].

The phase-space integral for the R0νECEC mode can be written as

$$G_{0\nu}^{\mathrm{ECEC}}(J^+) = \left(\frac{G_\mathrm{F}\cos\theta_\mathrm{C}}{\sqrt{2}}\right)^4 \frac{g_\mathrm{A}^4}{4\pi^2\ln 2} m_\mathrm{e}^6 \mathcal{N}(J)_x^2 , \tag{10}$$

where $\mathcal{N}(J)_x$ is the normalization of the relativistic Dirac wavefunction for a uniformly charged spherical nucleus [6] for the mode x relating to the atomic orbitals where the capture is from. For the 0^+ final states, the corresponding NME is related to the one of (7) by

$$M_{0\nu}^{\mathrm{ECEC}}(0^+) = \frac{1}{R_A} M^{(0\nu)'} , \quad R_A = 1.2A^{1/3}\,\mathrm{fm}. \tag{11}$$

The involved NMEs for the 2^+ final states have been detailed in [15].

2.5. Beta transition amplitudes

Here we discuss only the Gamow–Teller type of allowed beta decays. For the Fermi and first-forbidden transitions, involved in the lateral beta feeding of many nuclei participating in double-beta processes, the reader is referred to the more detailed accounts in [8, 15, 16]. The allowed Gamow–Teller beta-decay transitions of interest in this work are of the type $1^+ \rightarrow 0^+, 2^+$. For them, the $\log ft$ value is defined as [16]

$$\log ft = \log(f_0 t_{1/2}) = \log\left[\frac{6147}{B_{\rm GT}}\right], \quad B_{\rm GT} = \frac{g_{\rm A}^2}{2J_i + 1}\left(J^+ \| \sum_k t_k^{\pm} \boldsymbol{\sigma}_k \| 1^+\right)^2 \tag{12}$$

for the initial 1^+ and final $J^+ = 0^+, 2^+$ states, and for the β^+ or β^- type of transitions. Here f_0 is the leptonic phase-space factor for the allowed β^- or β^+/EC decays as defined in [16].

3. Ingredients of the QRPA formalism

Here we summarize briefly the nuclear-structure ingredients that are used to compute the NMEs encountered in the different processes listed in section 2.

3.1. The pnQRPA and the '$g_{\rm pp}$ problem'

A thorough account of the formalism used in the pnQRPA model is given in [2, 16, 17]. Instead of giving the details of the formalism here, we concentrate on the implications of using such a formalism in calculations of decay rates of various modes of double-beta decays. The pnQRPA was successfully used to explain the long $2\nu\beta\beta$ half-lives in [18] by suppression of the associated NME relative to its simple single-particle estimate. The explanation called for the use of the strength parameter $g_{\rm pp}$ of the particle–particle part of the proton–neutron interaction in the 1^+ channel. This parameter governs the relative magnitudes of the particle–particle and particle–hole terms in the nuclear Hamiltonian and the wavefunctions obtained thereby. While the couplings of the particle–hole channel were fixed by fitting the observed energy of the 1^+ GTGR, there was no direct way of fixing the couplings of the pp channels acting on the same states. This ambiguity was coined the '$g_{\rm pp}$ problem' and ever since the determination of the exact value of this parameter has caused vivid discussion (see sections 7.1 and 7.2).

The schematic delta nuclear force was used in the $g_{\rm pp}$ study of [18] but similar calculations were performed also in [19] for $2\nu\beta\beta$ decays and in [17] for β^+/EC decays by the use of realistic G-matrix-based two-body interactions. These studies were later extended to the particle-number-projected pnQRPA in [20, 21] and to the renormalized pnQRPA (RQRPA) in [22, 23]. Since then the $g_{\rm pp}$-related problems have been discussed in the frameworks of exactly solvable models, group theoretical approaches, exact shell-model schemes and by the use of symmetry arguments.

3.2. The multiple-commutator model for decays to excited states

The multiple-commutator model (MCM) [24, 25] is designed to connect excited states of an even–even reference nucleus to states of the neighboring odd–odd nucleus. Earlier the MCM has been used extensively in the calculations of double-beta-decay rates e.g. in [10, 26]. The states of the odd–odd nucleus are given by the pnQRPA and the excited states of the even–even nucleus are generated by the (charge conserving) QRPA (ccQRPA) described in detail in [16].

One can take a 2_1^+ phonon (the collective first 2^+ state in an even–even nucleus) of the ccQRPA and build an ideal two-phonon J^+ state of the form

$$|J^+_{2-\mathrm{ph}}\rangle = \frac{1}{\sqrt{2}}\left[Q^\dagger(2_1^+)Q^\dagger(2_1^+)\right]_J|\mathrm{QRPA}\rangle\,, \tag{13}$$

where $|\mathrm{QRPA}\rangle$ is the vacuum of the ccQRPA and $Q^\dagger(2_1^+)$ creates the 2^+ collective state (phonon). An ideal two-phonon state consists of partner states $J^\pi = 0^+, 2^+, 4^+$ that are degenerate in energy, and exactly at an energy twice the excitation energy of the 2_1^+ state. In practice, this degeneracy is always lifted by the residual interaction between the one- and two-phonon states [27].

3.3. The proton–neutron microscopic anharmonic vibrator approach and four-quasiparticle degrees of freedom

The microscopic anharmonic vibrator approach (MAVA) was first developed for the description of the electromagnetic properties of one- and two-phonon states in even–even heavy nuclei in [27]. It was later extended to the description of beta decays in [28] where the formalism of this proton–neutron MAVA (pnMAVA) is reviewed in detail. Lately it was applied to the $2\nu\beta\beta$ decays of ^{76}Ge [29] and ^{100}Mo [30]. In [29, 30], it was found that the four-quasiparticle degrees of freedom embedded in the MAVA formalism change very little the $2\nu\beta\beta$-decay observables and gross structure of the Gamow–Teller strength functions GT$^-$ and GT$^+$ defined in (5). Hence, it can be concluded that the pnQRPA is a rather robust formalism for the description of $2\nu\beta\beta$-decay observables even beyond its basic two-quasiparticle nature.

4. Single-particle aspects of double-beta decays

The very basic ingredients of the QRPA-based double-beta calculations are the selection of the valence single-particle space and the associated single-particle energies. These in turn are connected to the occupations of the single-particle orbitals. These occupations play a key role for many observables related to beta and double-beta decays.

4.1. Size of the single-particle space

As mentioned above, a proper selection of the active single-particle space is crucial for the success of the QRPA calculations. In particular, the inclusion of all the spin–orbit partners in the single-particle basis is essential for the pnQRPA results to satisfy the Ikeda $3(N-Z)$ sum rule [16]. In [31], the aspects of including single-particle states beyond the minimal one harmonic-oscillator major shell were discussed and illustrated for the $0\nu\beta\beta$ ground-state-to-ground-state decays of ^{76}Ge, ^{82}Se, ^{128,130}Te and ^{136}Xe. A similar study was performed in [32, 33] for the $0\nu\beta\beta$ decays of ^{76}Ge, ^{82}Se and ^{136}Xe to the first excited 0^+ states in ^{76}Se, ^{82}Kr and ^{136}Ba. In these works, the final 0^+ states were assumed to be members of the two-phonon $0^+, 2^+, 4^+$ triplet discussed in section 3.2 and displayed in equation (13). These triplets have been observed experimentally in numerous nuclei [34].

In all these studies, the importance of including all spin–orbit partners beyond the simple one valence harmonic-oscillator shell used in the shell-model calculations [35] of double-beta processes was stressed. Omission of these spin–orbit partners and the other single-particle states around the valence shell could cause a serious underestimation of the magnitudes of the $0\nu\beta\beta$ NMEs. The effects turned out to be somewhat different for the ground-state-to-ground-state and ground-state-to-excited-state decays [31, 33].

4.2. Role of the orbital occupancies

The experimental work of [36, 37] in measuring the single-particle-orbital occupancies in the valence $1p{-}0f_{5/2}{-}0g_{9/2}$ space in ^{76}Ge and ^{76}Se has been an incentive to many theoretical investigations about the role of these occupancies in the $0\nu\beta\beta$-decay rates of various nuclei. In the first of these investigations [38], the measured occupancies in ^{76}Ge [36] were used to modify the mean-field single-particle energies such that the experimental occupancies were reproduced. This led the NME of the ^{76}Ge decay to come closer in value to the corresponding shell-model-computed [39] NME, which was an interesting result by its own right. Later the effect was verified in [40]. Also within the shell model, the replacement of the original computed orbital occupancies with the occupations simulating the experimental ones caused the computed NME to approach the one obtained in the pnQRPA [35].

An extensive study of these effects was performed in [31] for the ground-state-to-ground-state $0\nu\beta\beta$ decays of the ^{76}Ge, ^{82}Se, ^{128,130}Te and ^{136}Xe nuclei and for the ground-state-to-excited-state decays of the ^{76}Ge, ^{82}Se and ^{136}Xe nuclei [32, 33]. In these studies, the BCS-computed single-particle occupancies were replaced by the shell-model-computed [39] single-particle occupancies. It was found that no drastic effects upon the $0\nu\beta\beta$ decay rates emerged although the effects were different in the case of the ground-state-to-ground-state and ground-state-to-excited-state transitions. For more details, the reader is referred to the mentioned original articles.

5. Short-range correlations and other refinements of the NME calculations

The traditional way to account for the short-range correlations between the two decaying nucleons in the $0\nu\beta\beta$ processes has been to use the Jastrow correlation function [41]. These correlations cause an effective repulsion between the two decaying nucleons so as to prevent their overlap in the 1 fm region of their relative distance. This distance corresponds to the large average momentum exchange ($\sim$ 200 MeV) occurring in the propagation of the virtual Majorana neutrino between the two decay vertices. The use of the Jastrow correlator causes an unrealistically large suppression of the $0\nu\beta\beta$ NMEs as noticed in [42]. In this article, it was proposed that the Jastrow short-range correlations should be replaced by the correlations affected by the unitary correlation operator method (UCOM) [43].

The UCOM is a softer way to account for the short-range correlation effects in the $0\nu\beta\beta$ decays than the Jastrow method, as pointed out in [42] for the $0\nu\beta\beta$ decay of ^{48}Ca, treated in the shell-model framework. The related effects were further studied for the pnQRPA in [12, 44] and the obtained NMEs were reviewed in [45]. Results of these studies were verified in [46] where also a self-consistent approach was devised with compatible results with the UCOM method. In all these studies, the magnitudes of the UCOM-correlated NMEs were larger than the magnitudes of the Jastrow-correlated NMEs.

Further refinements in the computation of the $0\nu\beta\beta$ decays were introduced in [7] in the form of finite-size dipole form factors of nucleons and higher order nucleon currents, including the interference between the vector, axial-vector and induced pseudo-scalar contributions. All the latest calculations include these corrections to the $0\nu\beta\beta$ NMEs. The short-range correlations and the finite-size and higher-order-current refinements were studied in detail in [44] and later in [39, 47].

6. Competition between beta decays and double-beta decays

The nuclei ^{48}Ca and ^{96}Zr share the same interesting feature: they both beta decay and double-beta decay. In both cases, the initial nuclei ^{48}Ca and ^{96}Zr β^- decay extremely slowly to the lowest 6^+ (6th forbidden non-unique decay), 5^+ (4th forbidden unique decay) and 4^+ (4th forbidden non-unique decay) states in ^{48}Sc and ^{96}Nb. After this, they decay fast to the excited states in ^{48}Ti and ^{96}Mo. In the unique beta decays, only one NME is involved, whereas in the non-unique decays, several matrix elements are involved (for review of the underlying lepton and nuclear-structure aspects, see [48, 49]).

In [50], it was found by using the shell-model approach that the computed single-beta-decay half-life of ^{48}Ca was some 25 times longer than the measured $2\nu\beta\beta$ decay half-life of ^{48}Ca. The beta-decay half-life was shown to be dominated by the 4th forbidden unique decay to the 5^+ state in ^{48}Sc with a computed half-life of $1.1^{+0.8}_{-0.6} \times 10^{21}$ years. In [51], the nuclear model that was used to describe the nuclear wavefunctions involved in the $A = 96$ decays was the pnQRPA. As in the case of ^{48}Ca also here the beta-decay half-life turned out to be dominated by the 4th forbidden unique decay to the 5^+ state in ^{96}Nb with a computed half-life of 2.4×10^{20} years. This half-life is an order of magnitude longer than the measured $2\nu\beta\beta$ decay half-life of ^{96}Zr. This finding leads to the important conclusion that the single-beta-decay channel does not contaminate the geochemical measurements of the double-beta half-life of ^{96}Zr.

7. Theoretical and experimental probes of the double-beta-decay NMEs

General investigations of the important ingredients of the NMEs related to $0\nu\beta\beta$ decays have been performed in [39, 44, 47]. These surveys into the guts of the NMEs have to do with the 'g_{pp} problem', multipole decompositions, radial dependences and short-range correlations. Below we address a number of important aspects of the $0\nu\beta\beta$ NMEs with respect to these analyses and with respect to the complementary experimental information accumulating in the recently performed experiments on various observables related to the $2\nu\beta\beta$ and $0\nu\beta\beta$ decays.

7.1. The parameter g_{pp} from two-neutrino double-beta decays

The idea of fixing the value of the strength parameter g_{pp} (particle–particle channel of the proton–neutron two-body interaction) by the data on half-lives of $2\nu\beta\beta$ decays (see the recent compilation [52]) was advocated in [53] and adopted in many subsequent works dealing with the $0\nu\beta\beta$ decays. The method is based on the idea that the $2\nu\beta\beta$ NME represents a bulk property of the GT^- and GT^+ amplitudes which pnQRPA can describe reliably. Fixing the value of g_{pp} by the experimental value of the $2\nu\beta\beta$ NME improves the stability of the $0\nu\beta\beta$ NME with respect to the varying sizes of the single-particle valence spaces used in the pnQRPA calculations [53]. A shortcoming of the method is that it can be used only for those nuclei for which the $2\nu\beta\beta$ decay half-life is known experimentally. Also in the cases of a (near) single-state dominance (SSD) (see section 7.5), this method could lead to a contradiction between the computed and measured β^- and EC decay rates of the intermediate 1^+ ground state, as discussed in the following section.

7.2. The parameter g_{pp} from beta decays

In the previous section, the experimental value of the $2\nu\beta\beta$ NME was mentioned as a way to fix the value of the strength parameter g_{pp}. An alternative to it would be to use the measured $\log ft$ value (see equation (12)) of the β^- decay of the lowest 1^+ state in the intermediate nucleus of the double-beta decay. This method is the only alternative for those nuclei where the $2\nu\beta\beta$ decay half-life is not known, as demonstrated in the recent work [8] where a set of seldom studied $2\nu\beta\beta$ and $0\nu\beta\beta$ decays was investigated for the half-life estimates. The use of beta decays in determining the value of g_{pp} was also advocated in [54] in cases where the $2\nu\beta\beta$ decay rate is to a large extent determined by the virtual transition through the 1^+ ground state of the intermediate nucleus (SSD, see section 7.5). In these cases, it could be misleading to determine the value of g_{pp} by the experimental $2\nu\beta\beta$ decay half-life since this can lead to quite wrong computed log ft values for the β^- and EC decays of the 1^+ ground state of the intermediate nucleus, as demonstrated for the ^{116}Cd and ^{128}Te decays in [54].

Recently, the measurements of beta decays related to double-beta processes have gained new impetus. An example constitutes the experiments in the TITAN ion trap at the TRIUMF radioactive beam facility [55].

7.3. Charge-exchange reactions

Charge-exchange reactions are one of the major tools for the scrutiny of the strength functions of various multipole transitions in nuclei [56]. The most familiar of these reactions are the (p,n) and (n,p) reactions (see e.g. [57]) that correspond to the GT$^-$ and GT$^+$ strength functions (see section 2.2), respectively. Recently, the other type of charge-exchange reactions (d,^{2}He), corresponding to the GT$^+$ branch, have attracted attention [58–62]. The GT$^-$ type of reaction (^{3}He,t) has also raised interest as discussed in [63].

In the (d,^{2}He) and (^{3}He,t) works, the measured GT$^-$ and GT$^+$ strength functions are used to reconstruct the NMEs of the $2\nu\beta\beta$ decays in several nuclei (see e.g. [64] and the previously mentioned works). Such reconstruction, if successful, can help in constraining the parameter spaces of different nuclear models aiming at calculations of the $2\nu\beta\beta$ NME. In these works, the focus has been to learn something about the decomposition of the $2\nu\beta\beta$ NMEs (the various 1^+ contributions in the intermediate nucleus) and even go beyond and try to map e.g. the 2^- contributions in order to learn something about the $0\nu\beta\beta$ NME.

7.4. Muon capture

The charge-exchange reactions can be used directly or indirectly to access the virtual transitions occurring in the $2\nu\beta\beta$ and $0\nu\beta\beta$ decays. As proposed in [65, 66], these virtual transitions can also be probed indirectly via the ordinary muon-capture (OMC) reaction. This can be achieved by capturing an atomic K-shell muon in the final nucleus of double-beta decay thus reaching states of the intermediate nucleus of double-beta decay. The advantage of this method is that due to the large mass of the captured muon (some 200 times the mass of an electron), the capture transitions can reach highly excited states in the intermediate nucleus. The disadvantage is that theoretically the description of the OMC process is complicated by the interference of the induced pseudoscalar current with the vector and axial-vector currents of the nucleons. This is because the importance of the pseudoscalar current is magnified by the large momentum exchange between the muon and the initial nucleus.

The usefulness of the OMC in probing the virtual transitions depends on how well the strong OMC transitions correspond to the strong virtual transitions in the $2\nu\beta\beta$ and $0\nu\beta\beta$ decays. In [67], it was found by using the nuclear shell model with well-established two-body interactions that the leading OMC and virtual transitions correlated strongly in the $2\nu\beta\beta$ decays of ^{36}Ar, ^{46}Ca and ^{48}Ca. This feature suggests that the OMC has the potential to be a powerful tool to probe double-beta decays at least in light nuclei.

7.5. SSD in two-neutrino double-beta decays

Lately, the issue about the experimental reconstruction of the matrix elements for two-neutrino double-beta decays by means of charge-exchange reactions (see section 7.3) and β^- and electron-capture decays (see section 7.2) has attracted a lot of attention among experimentalists and theoreticians. The literature on the subject is very rich and we shall focus here on the essentials of the so-called single-state dominance (SSD). The SSD hypothesis states that if the ground state of the intermediate odd–odd nucleus, mediating the $2\nu\beta\beta$ transition, is a 1^+ state, then, because in that case the energy denominator would be the smallest possible, the value of the NME will be dominated by the product of the EC and β^- matrix elements connecting the initial and final ground states with the ground state of the intermediate nucleus. Because both quantities can be determined experimentally, one may then obtain a model-independent value for the $2\nu\beta\beta$ NME.

We have explored the possibility for SSD in a series of papers [68, 69] and concluded that the SSD hypothesis may be good for some nuclei but for some other nuclei, the apparent SSD is only an artifact. In the latter case, the value of the NME may indeed be close to the one predicted by the SSD hypothesis but only via cancellations between low-lying and high-lying transitions, not via the single virtual transition through the lowest 1^+ state. The mechanisms to produce the apparent SSD can be nicely tracked down by following the accumulation of the $2\nu\beta\beta$ NME as a function of the energy of the intermediate 1^+ states. This question has been addressed experimentally in the charge-exchange reactions and is still a matter of vigorous investigation (see section 7.3).

7.6. Multipole decompositions of the $0\nu\beta\beta$ NMEs

The $0\nu\beta\beta$ NMEs $M^{(0\nu)'}$ can be decomposed into contributions of different intermediate multipoles. This decomposition can be made in two ways, either through the different multipole states J^π of the intermediate nucleus or through different couplings J' of the two decaying nucleons [47, 70]. For the Gamow–Teller NME, these decompositions can be schematically written as

$$M_{\mathrm{GT}}^{(0\nu)} = \sum_{J^\pi} \sum_{J'} M_{\mathrm{GT}}^{(0\nu)}(J^\pi, J'), \tag{14}$$

where $M_{\mathrm{GT}}^{(0\nu)}(J^\pi, J')$ is given explicitly in [12, 47]. The J^π decompositions were studied recently for the $0\nu\beta^-\beta^-$ decays to the ground states in [31] and to excited two-phonon states of (13) in [33]. Both types of decomposition were studied for both the ground-state and excited-state positron-emitting decays in [71].

In [71], interesting qualitative differences in the decompositions of the NMEs corresponding to the decays of ^{106}Cd to different final states in ^{106}Pd were found. For the ground-state NME, the decomposition in terms of J^π was the typical one of the pnQRPA calculations [31, 47] and the decomposition in terms of J' was typical of the shell-model [70] and pnQRPA [47] calculations. Here typical means that in the J^π decomposition, the 1^+ and 2^- multipoles are the most prominent ones and in the J' decomposition, there is a large positive monopole contribution, a smaller negative quadrupole contribution and much smaller, mostly negative, higher multipole contributions. For the one-ccQRPA-phonon states (see section 3.2), the pattern resembles that of the ground state for the J^π decomposition but in the case of the J' decomposition, the majority of the higher multipole contributions are positive instead of negative. The behavior of the two-ccQRPA-phonon (see section 3.2) NME is qualitatively totally different: both decompositions have both large positive and large negative contributions. In the J' decomposition, the monopole component is no more the dominant one. The alternating structures of these decompositions conspire to produce larger Jastrow than UCOM NMEs which deviates from the main stream of results (see section 5).

7.7. Isospin symmetry breaking in double-beta-decay processes

One crucial question related to the renormalized pnQRPA (RQRPA) [22] is its reliability in the vicinity of the pnQRPA collapse point that can be reached by increasing the value of the coupling parameter g_{pp}. The RQRPA ignores this collapse and thus becomes also questionable from a more fundamental point of view. To begin with, the vanishing of the $2\nu\beta\beta$ NME is not an artifact of the pnQRPA but rather it is a manifestation of the underlying isospin symmetry, as explained in [72]. The situation is better illustrated for the case of double Fermi transitions where the matrix elements for the ground-state-to-ground-state transitions should be exactly zero if the isospin symmetry is exactly obeyed by the approximations used to calculate the involved nuclear wavefunctions. Both the pnQRPA and the RQRPA fail to reproduce the vanishing of the NME at the symmetry restoration. But while the pnQRPA collapses before reaching the symmetry point, the RQRPA passes over it and yields that way non-physical results.

As pointed out in [72], an iso-quadrupole interaction between the final ground state and the double analog state is needed to allow for a mixing between both states with a different T but the same M_T projection. The non-vanishing values of the double Fermi NMEs, obtained at the symmetry-restoration point by the pnQRPA and RQRPA approaches, are essentially due to the isospin admixture induced by the use of the BCS approximation. Thus, some caution is in order when attempting a reconstruction of the double Fermi NME by the pnQRPA or RQRPA. One should exercise similar caution when trying to construct the $2\nu\beta\beta$ NME by double Gamow–Teller transitions in these theories.

7.8. IVSM contributions

Charge-exchange reactions (see section 7.3 and the references therein) can be used to measure the GT^- and GT^+ strength functions in odd–odd nuclei. In the work of [57], the results of measurements of the (p, n) and (n, p) charge-exchange reactions on ^{116}Cd and ^{116}Sn, respectively, have been analyzed. It was shown that the dominant contribution to the cross section $\sigma_{\mathrm{GT}}(0^\circ)$ for the ^{116}Cd(p,n)^{116}In reaction is due to the $\Delta l = 0$ transitions around the energy of the Gamow–Teller giant resonance (GTGR), and that a considerable amount of strength lies at energies larger than the energy of the GTGR. Since these experiments cannot distinguish between pure GT transitions and other transitions, a possible mechanism to explain this strength is the excitation of the IVSM states by the action of the operator $\sigma r^2 t^\pm$ [73].

The interplay between the two types of modes, the $\mathrm{GT}^\pm$ and the $\mathrm{IVSM}^\pm$, was studied in the framework of the pnQRPA in [74], particularly in the energy domain $E \leqslant 30$ MeV relevant for the bulk of the GT type of strength. The position of the IVSM^- resonance, calculated at about 34 MeV, is approximately 19 MeV higher than the GTGR which lies at about 15 MeV, in good agreement with the systematics of [73]. The results of [74] do not show any significant interference between the GT^- and the IVSM^- modes. The situation was found to be different for the IVSM^+ side of excitations. Theoretically, the IVSM^+ strength is confined in the energy range $E \leqslant 30$ MeV. In this interval, the IVSM^+ intensity amounts to approximately 9.5 units of equivalent GT strength, the IVSM resonance appearing as a narrow state at energy $E \approx 23$ MeV, in agreement with [57].

It is interesting to note that in [57], the equivalent GT^+ strength measured at energies below 30 MeV was some 11 units. It was speculated that 6 units of this would be genuine GT^+ strength and the rest, 5 units, would come from the IVSM^+ mode. The 6 units of GT^+ strength would then connect strongly to the GT^- strength at energies in the GTGR region and beyond. This in turn would suggest that sizable contributions to the $2\nu\beta\beta$ NME may come from states lying at and above the energy of the GTGR. According to [74], it would seem that at the GTGR energy and beyond, the bulk of the (n,p) strength would come from the IVSM$^+$ and practically nothing from the GT$^+$ branch. This would then exclude any sizable effect on the double-beta-decay rate of ^{116}Cd from energies at and beyond the GTGR energy [74].

7.9. Transitions near closed shells and the mixing between pairing and GT excitations

A typical energy spectrum from (^{3}He,t) charge-exchange reactions on medium-mass nuclei consists of a narrow peak (IAS) and a broad distribution (GTGR). The data for $A = 48$ [75] show four states with $T = I = 1$, and a spin-dipole excitation at high energy. All these excitations participate in beta-decay and electron-capture processes, and they can bring in some very useful information about spin excitations. From the nuclear-structure point of view, the collectivity of the spin modes, based on the vibrational picture of Bohr and Mottelson [76], may be assessed at the level of the centroids of the energy spectra, but less accuracy can be obtained at the level of the strength associated with the states. Thus, the strength of the GT transitions may not be correctly reproduced unless the transition operator is renormalized. The same problems appear in the shell-model description of the same states [77].

In [78], the available experimental information for states near closed shells, with spin $S = 0, 1$ and isospin $T = 0, 1$, was analyzed in order to determine their structure as members of isoscalar-pairing multiplets or as Gamow–Teller excitations at closed shells. The study of [78] is different from the usual treatment of isoscalar pairing as an extension of the BCS formalism, the use of which may be criticized due to the vicinity of the phase transition between spherical and deformed pairing states.

The formalism of [78] was applied to calculate the response of ^{58}Cu to spin–isospin probes. The validity of the isoscalar and isovector pairing vibrational model was tested and to it Gamow–Teller excitations were added. The ground state of ^{58}Ni was described as the lowest excited isovector pairing mode of ^{56}Ni. The 1^+ states in ^{58}Cu were described as a superposition of Gamow–Teller and pairing phonons. The RPA+NFT procedure of [78] shows that the resulting effective charges are indeed energy dependent. This departure from the commonly assumed scheme of a single effective charge for all transitions (like in the shell-model case) is essential to preserve the Ikeda sum rule. The results shown in [78] demonstrate the importance of including pairing degrees of freedom together with Gamow–Teller excitations in the cases where the number of particles outside a shell closure is very small (say 2 or 4). This is relevant for the treatment of the single- and double-beta decays around closed shells, like for the ^{128,130}Te isotopes, as will be discussed in the following section.

7.10. Decays near closed shells: the case of ^{128,130}Te

A weakness of the pnQRPA method is the fragility of its results based on the use of the BCS approximation for nuclei near closed shells, like in the case of the tellurium isotopes. The results of [77] on the interplay between pairing vibrations and Gamow–Teller excitations were used to construct a model where pairs of protons are treated as pairing excitations and the neutrons described as quasiparticles [79]. In this scheme, the vacuum state is the product of the proton closed shell ($Z = 50$) and the BCS vacuum for neutrons. In both the initial and final nuclei, the neutrons are described as quasiparticles moving in a common BCS vacuum state and proton states lie above (below) the Fermi energy. The model of [79] takes into account Gamow–Teller bosons $\Gamma^+_{I=1\mathrm{M},\mathrm{T}=1\mathrm{T}_z,\mathrm{n}}$ (which include isoscalar-pairing admixtures) and the proton isovector-pairing boson $\Gamma^+_{00,11,1}$. The initial and final states are described by one proton-pairing phonon (Te isotopes) and two proton-pairing phonons (Xe isotopes), respectively. The procedure followed to extract the values of the various fields is described in [79].

The whole $2\nu\beta^-\beta^-$ process is divided into two stages. In the first decay ^{128,130}Te($Z = 52$)$\rightarrow$ ^{128,130}I($Z = 53$), the first electron–antineutrino pair is emitted. The nuclear term of the associated interaction is proportional to the Gamow–Teller operator [77]. It creates intermediate states upon which another Gamow–Teller operator acts, leading to the final state through the transition ^{128,130}I($Z = 53$)$\rightarrow$ ^{128,130}Xe($Z = 54$). These intermediate states must be treated as final (initial) states in the NFT sense [79], for the first (second) stage of the process, respectively. The couplings of the pairing sector are determined from the observed mass differences that yield the energy of the isovector-pairing excitations. The isoscalar-pairing coupling is taken from [78], scaled to the actual masses $A = 128, 130$. One particular feature of the formalism is the energy dependence of the renormalization factors which appear in the expression of the matrix elements between initial (final) and intermediate states [79]. In the standard pnQRPA, these factors are just BCS occupation factors. The results shown in table II of [79], for $A = 128$ and $A = 130$, show the suppression of the NMEs which was obtained without any additional renormalization of the couplings of the pnQRPA. We believe that this is a clear improvement in the theoretical description of $2\nu\beta^-\beta^-$ observables in this mass region.

8. Positron-emitting and ECEC processes

A lot of work has been done in experimental [52] and theoretical [1, 2] investigations of the double-β^- decays of nuclei due to their favorable decay Q values. The positron-emitting modes of decays, $\beta^+\beta^+$, β^+EC and ECEC, are much less studied. Theoretical studies of these modes include reference [4] for the general, nuclear-model-independent frameworks of two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays and reference [6] for the general frameworks of the neutrinoless $\beta^+\beta^+$ and β^+EC decays. The formalism for the resonant neutrinoless double-electron capture (R0νECEC) was first developed in [13] and later discussed and extended to its radiative variant ($0\nu\gamma$ECEC) in [14]. Experimental search for such processes was performed e.g. in [80–83].

8.1. Two-neutrino processes

The first nuclear-structure calculations of the NMEs involved in the above decays were performed for the two-neutrino and neutrinoless $\beta^+\beta^+$ decay channels of ^{78}Kr, ^{96}Ru, ^{106}Cd, ^{124}Xe, ^{130}Ba, ^{136}Ce and ^{148}Gd in [84]. After this, the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays of ^{58}Ni, ^{96}Ru, ^{106}Cd and ^{136}Ce were discussed in [85] and later the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays and neutrinoless $\beta^+\beta^+$ and β^+EC decays in ^{58}Ni, ^{78}Kr, ^{96}Ru, ^{106}Cd, ^{124}Xe and ^{136}Ce were addressed in [86]. All of these calculations considered transitions to the final ground states only. Later the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays of ^{78}Kr, ^{92}Mo, ^{96}Ru, ^{106}Cd, ^{124}Xe and ^{130}Ba were examined in [26] for both the ground states and first excited 0^+ states. This study was complemented by a joint theoretical and experimental investigation for the decay of ^{106}Cd in [87].

The RQRPA was used in [23] to calculate the NMEs of the two-neutrino $\beta^+\beta^+$ decays of ^{78}Kr and ^{106}Cd to the ground and first excited 0^+ states of the final nuclei. In [68, 69], the SSD hypothesis was examined and the NMEs related to the two-neutrino ECEC decays of ^{106}Cd and ^{136}Ce to the final ground state and two lowest excited 0^+ states were derived. More recently in [88], the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC modes of decay were discussed under the SSD hypothesis, without a quantitative nuclear-structure calculation, for several

Table 1. R0νECEC decay transitions with degeneracy parameters $Q - E$ derived from Q-value measurements of the last column. Also the involved atomic orbitals have been given. The second-last column lists the currently available half-life estimates with the reference indicated in the last column.

Transition	J_f^π	$Q - E$ (keV)	Atomic orbitals	C^{ECEC}	References
^{74}Se $\to$ ^{74}Ge	2^+	2.23	L_2L_3	$(0.2{-}100) \times 10^{43}$	[92]
^{96}Ru $\to$ ^{96}Mo	2^+	8.92(13)	L_1L_3		[93]
	0^+?	$-3.90(13)$	L_1L_1		
^{102}Pd $\to$ ^{102}Ru	2^+	75.26(36)	KL_3		[94]
^{106}Cd $\to$ ^{106}Pd	0^+?	8.39	KK	$(2.1{-}5.7) \times 10^{30}$	[71]
	$(2,3)^-$	$-0.33(41)$	KL_3		[94]
^{112}Sn $\to$ ^{112}Cd	0^+	-4.5	KK	$> 5.9 \times 10^{29}$	[95]
^{136}Ce $\to$ ^{136}Ba	0^+	-11.67	KK	$(3{-}23) \times 10^{32}$	[96]
^{144}Sm $\to$ ^{144}Nd	2^+	171.89(87)	KL_3		[94]
^{152}Gd $\to$ ^{152}Sm	0^+_{gs}	0.91(18)	KL_1	$\sim 1 \times 10^{26}$	[97]
^{156}Dy $\to$ ^{156}Gd	1^-	0.75(10)	KL_1		[98]
	0^+	0.54(24)	L_1L_1		[98]
	2^+	0.04(10)	M_1N_3		[98]
^{162}Er $\to$ ^{162}Dy	2^+	2.69(30) keV	KL_3		[93]
^{168}Yb $\to$ ^{168}Er	(2^-)	1.52(25) keV	M_1M_3		[93]

nuclei and for several 0^+ and 2^+ final states. A more refined NME and half-life calculation of the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays of ^{106}Cd to the ground state and first excited 0^+ state in ^{106}Pd was carried out in [89]. The two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays of ^{106}Cd to the final ground state were also examined within the Hartree–Fock–Bogoliubov model in [90].

8.2. Neutrinoless modes

The calculation of the NMEs related to the neutrinoless positron-emission modes was started, as mentioned earlier, in [84, 86]. Later, in [10], the ground-state neutrinoless $\beta^+\beta^+$ decays of ^{124}Xe and ^{136}Ce were compared with several $\beta^-\beta^-$ decays and in [11], a systematic study of the neutrinoless $\beta^+\beta^+$ and β^+EC decays to excited 0^+ states in ^{92}Mo, ^{96}Ru, ^{106}Cd, ^{124}Xe, ^{130}Ba and ^{136}Ce was performed. In [91], the two-neutrino $\beta^+\beta^+$, β^+EC and ECEC decays as well as the neutrinoless $\beta^+\beta^+$ and β^+EC decays of ^{106}Cd to the ground state were treated within the second quasi random phase approximation framework. In [71], the positron-emitting neutrinoless double-beta decays of ^{106}Cd were revisited with up-to-date short-range correlations, nucleon form factors and higher order nucleonic currents.

8.3. Search for the resonant neutrinoless ECEC decay

Measurements of the R0νECEC decays for various nuclei were carried out in [81, 82]. Table 1 lists the best known cases of R0νECEC transitions in various nuclei where Q-value measurements have been conducted recently. These Q values have been measured exploiting the Penning-trap techniques. In the cases of ^{96}Ru and ^{106}Cd, the assignment of 0^+ spin-parity to the resonant state is uncertain.

In the table, we also list the estimated half-lives for the cases for which such estimates exist. The reference of the last column indicates the origin of the estimate. In the table, a quantity C^{ECEC} is given and it ties to the R0νECEC half-life through

$$T_{1/2}^{\mathrm{R0}\nu\mathrm{ECEC}} = \frac{C^{\mathrm{ECEC}}}{\left(\langle m_\nu\rangle[\mathrm{eV}]\right)^2}\ \mathrm{years}\ , \tag{15}$$

where the effective neutrino mass should be given in units of eV. In all the listed cases, the decay rates are suppressed by the rather sizable magnitude of the degeneracy parameter. There are much more favorable cases listed in table 1 but the associated NMEs are still waiting for their evaluation.

9. Toward QRPA theory for double-beta decay of deformed nuclei

The issue of double-beta-decay calculations for deformed nuclei was raised several years ago, in connection with the measurements of the decay of nuclei such as Mo, Nd and U. The NMEs corresponding to the DBD transitions (both the $2\nu\beta\beta$ and $0\nu\beta\beta$ modes) in some of the deformed nuclei where DBD can be measured have been calculated using deformed Hartree–Fock mean field [99], pseudo-SU_3 [100, 101] and deformed QRPA models [102, 103]. Other attempts have made use of a deformed Nilsson mean field plus pairing plus quadrupole residual interactions [104], or the nuclear energy-density-functional method with generator coordinates plus angular-momentum projection [105].

From a conventional point of view, in dealing with deformed nuclei, one may, for instance, adopt a Nilsson scheme for the single-particle levels and then solve the inverse gap equations to extract single-quasiparticle occupation numbers from the available experimental information on energy gaps. After this, one can write pnQRPA equations in the resulting quasiparticle basis [76]. The proton–neutron interaction may be fixed from phenomenology [103], though the correlated two-quasiparticle (one phonon) excitations do not actually have good angular momentum. The angular-momentum symmetry can be recovered by performing a numerical projection on angular momentum [106]. This scheme is feasible but numerically involved, because of the large number of states which should be included in the single-particle basis, and because of the dependence of the pnQRPA results upon locally defined deformation parameters.

While a lot of experience has accumulated in the past concerning two-like-particle excitations in deformed nuclei (e.g. electric and magnetic excitations in deformed nuclei based on the excitations of the same kind of particles, either neutrons or protons) [107], much less experience has been gained in treating proton–neutron excitations in deformed nuclei. Recently, the results of (deformed) Hartree–Fock mean-field treatments of double-beta emitters have been reported [99]. These results may be confronted with those produced by the Madrid group [102]. The comparison between both sets of results shows very important differences, mainly in the sector of the calculations concerning self-consistent deformations. We may then conclude this section by saying that the field of pnQRPA treatments of double-beta decays in deformed nuclei is still open for improvements. Work is in progress concerning this matter [108].

10. Calculated values of double-beta observables

In this section, we summarize our recent calculations concerning the different modes of double-beta decays. The calculated observables include the NMEs, decay half-lives and possible auxiliary information concerning the related beta-decay and other observables. These

Table 2. Our recent QRPA calculations of double-beta observables (NMEs, half-lives, etc) for the nuclei listed in the first column.

Nuclei	Mode	References
^{70}Zn, ^{86}Kr, ^{94}Zr	$\beta^-\beta^-$	[8]
^{104}Ru, ^{110}Pd, ^{124}Sn		
^{74}Se	R0νECEC	[92]
^{76}Ge, ^{82}Se	$\beta^-\beta^-$	[29, 31–33, 38, 44, 45]
^{96}Zr, ^{116}Cd	$\beta^-\beta^-$	[12, 45]
^{100}Mo	$\beta^-\beta^-$	[12, 30, 45]
^{106}Cd	β^+/EC, R0νECEC	[71, 89]
^{112}Sn	R0νECEC	[95]
^{128}Te, ^{130}Te	$\beta^-\beta^-$	[12, 31, 45, 79]
^{136}Xe	$\beta^-\beta^-$	[12, 31, 33, 45]
^{136}Ce	R0νECEC	[96]

calculations have been summarized in table 2 where the first column lists the nuclei, the second column the studied decay modes and the third column the references where these nuclei have been investigated.

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