Leibniz in Paris: a discussion concerning the infinite number of all units[1]

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Abstract

In this paper, we analyze the arguments that Leibniz develops against the concept of infinite number in his first Parisian text on the mathematics of the infinite, the *Accessio ad arithmeticam infinitorum*. With this goal, we approach this problem from two angles. The first, rather philosophical or axiomatic, argues against the number of all numbers appealing to a *reductio ad absurdum*, showing that the acceptance of the infinite number goes against the principle of the whole and the part, which is analytically demonstrated. So, discussing the ideas of Galileo, Leibniz concludes that the infinite number equals 0. Moreover, Leibniz seems to arrive at the same conclusion through his rule for adding the infinite series resulting from the harmonic triangle. Although he acknowledges the conjectural character of this conclusion, he seems to consider it to be a reinforcement of his first argument. Moreover, in reconstructing the justification of the given rule, we try to show that Leibniz does not appeal to the application of infinitesimal quantities, but rather to a treatment of the infinite series in terms of totalities.

Keywords

Leibniz? Galileo? mathematics, infinite number, infinite series, infinitesimal calculus, mathematical conjecture

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Introduction

In 1672 Leibniz arrived in the French capital, an important cultural and intellectual center of the seventeenth century, where he resided for four years. As soon as he settled down, he met the renowned Dutch scientist C. Huygens, thanks to whom the philosopher from Leipzig began his studies in mathematics. Huygens immediately decided to test the mathematical skills of the young legate from Mainz, proposing that he undertake the sum of the reciprocal series of the triangular numbers (that is, $\frac{1}{1}, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, etc.$).[2] Leibniz started working on this immediately and he obtained amazing results. He also became acquainted with all the philosophical and scientific developments to which he had not had access previously. He read, for instance, Galileo's *Discorsi e dimostrazioni matematiche, intorno a due nuove scienze* (1638), a text which impacted him deeply. On the first part of this work, the Italian scientist thoroughly tried to justify a composition of the continuum of an infinite number of indivisibles, which obliged him to specify the nature of the indivisible and the infinite. One of the conclusions drawn by Galileo was that in the infinite number the whole would not be greater than the part. This consequence of the Galilean exam will be a constant point of discussion in Leibnizian thought.[3]

Briefly, according to Galileo, the finite and the infinite differ from the point of view of the scope of human knowledge, in the sense that both infinities and indivisibles transcend our finite understanding, namely, because of their magnitude or size (infinities) or their smallness (indivisibles).[4] Galileo considered that the incomprehension of the infinite due to the limitation of human understanding must lead us to recognize that there are properties which can be assigned to the finite that cannot be extrapolated to the infinite. In this sense, the properties of 'being greater than', 'being smaller than' and 'being equal to' are particularly important. These attributes, comprehensible in the finite, are inconveniently assigned to the infinite.[5] The distinction between the scope of the finite and of the infinite entails significant consequences. For instance, it is evident that the axiom of Euclid according to which the whole is greater than the part[6] possesses a narrow scope of application if we consider the presuppositions of Galileo. This implies some important

[2] A II 1, 344. For the philosophical significance of Leibniz's mathematical investigations on the infinite in the Parisian period, see Philop Beeley, "Approaching Infinity. Philosophical Consequences of Leibniz's Mathematical Investigations in Paris and Thereafter," in *The Philosophy of Young Leibniz*, edited by Mark Kulstad, Mogens Laerke and David Snyeder (Stuttgart: Franz Steiner Verlag, 2009) p. 29-48, and also for the time before the Parisian period, Richard T. W. Arthur, "Actual Infinitesimals in Leibniz's Early Thought," in *The Philosophy of Young Leibniz*, edited by Mark Kulstad, Mogens Laerke and David Snyeder (Stuttgart: Franz Steiner Verlag, 2009), p. 11-28.

[3] Regarding Galileo's theory of the infinite and its relation with the continuum problem, among others, see Eberhard Knobloch, "Galileo and Leibniz: Different Approaches to Infinity," *Archive for History of Exact Sciences*, 54 (1999), 87-99; Manuel Sellés García, "La paradoja de Galileo," *Asclepio. Revista de Historia de la Medicina y de la Ciencia*, 58, 1 (2006): p. 113-148; Carlos Solis Santos, "El atomismo inane de Galileo", *Theoria*, 59 (2007): p. 213-231, and Samuel Levey, "Comparability of Infinities and Infinite Multitude", in *G. W. Leibniz, interrelations between Mathematics and Philosophy*, edited by Norma Goethe, Philip Beeley and David Rabouin (Dordrecht; Heidelberg; New York; London: Springer, 2015), p. 157-187.

[4] EN, VIII, 73.

[5] EN, VIII, 77-78.

[6] Euclid, *Elements*, I, Com. not. n°. 8.

conclusions regarding the infinite number, supposing we can conceive it:[7] let us take two sets such that one of them is at the same time a subset of the other, for example, the set of the natural numbers and the set of the square numbers (see *image 1*).

Natural numbers:	1	2	3	4	5	6	7	8	9	etc.
	|	|	|	|	|	|	|	|	|	|
Square numbers:	1	4	9	16	25	36	49	64	81	etc.

Image 1

As we can see, there are three square numbers contained in the first nine natural numbers (that is, 1, 4 and 9). Hence, in any finite number, regardless how large it is, the whole (in this case, the set of natural numbers) is greater than the part (the set of square numbers in the first nine natural numbers). Moreover, the proportion of contained square numbers decreases in larger numbers (that is: while in the first nine natural numbers 1 out of 3 is a square number, in the first hundred numbers 1 out of 10 is a square number, and so on). If the aforementioned properties (that is, being greater than, smaller than or equal to) cannot be extrapolated to the infinite, in an alleged infinite number the whole would not be greater than the part. Thus, the infinite numbers would be *beyond the scope of the whole-part axiom*. Hence, if there is a number which fulfills the requisites of the infinite number (as we have said, supposing we can conceive it), it would not be a large number. Thus, an alleged infinite number must rather be compared with the number one, since only in it are contained as many squares number as natural numbers (or more generally, any power of this number is equal to itself).[8] Leibniz seems to understand that Galileo's view implies that the infinite number represents a whole.

In this paper we will analyze the discussion of Galileo's thesis on the infinite number by Leibniz at the beginning of his stay in Paris. In order to do this, we will divide this paper into four sections. In the first section we will point out that Leibniz carried out a double approach to the problem of the infinite number, a philosophical or axiomatic one (by which he shows that the infinite number implies a contradiction) and an arithmetical or analytical one (by which, through the application of rules for adding an infinite series of terms to the series of units, he shows that its sum would be 0). In the second and third sections, we will develop respectively the two approaches above mentioned. The examination of the rule for adding infinite series presented by Leibniz will be done by discussing the usual interpretation by which, in these early writings, Leibniz would already be appealing to the elimination of infinitesimal quantities. In this sense, as we will exhibit in this paper, the explanation of Leibniz's application of the rule to the case of the infinite series of units, by which he concludes that $1 + 1 + 1 + 1 + etc. = 0$, supplies a lacuna in the classical literature. After doing so, in the fourth section we will finally consider Leibniz's justification for such a rule, given a few years later.

[7] EN, VIII, 79. As Knobloch points out: "An 'infinite quantity' ("quantità infinita") would according to Galileo's conception actually be a 'contradiction in terms', because an infinite lacks precisely those properties which characterize a quantity" ("Galileo and Leibniz: Different Approaches to Infinity," *Archive for History of Exact Sciences*, 54 (1999), 94). In this sense, the conclusion pointed out by Galileo is based on the condition that an infinite number is conceivable.
[8] EN, VIII, 83.

1. The infinite number in Leibniz

When Leibniz examines Galileo's argument he immediately recognizes that, above all, the Italian scientist has failed in his understanding the requirements which define the infinite number (still regardless of the problem of its existence). Leibniz believes that, if the whole is equal to the part in the infinite number, then in this number there are not only as many powers as roots (that is, not only as many natural numbers as squares), but also as many even and odd numbers (namely, the numbers *simpliciter*) as even numbers, and as triangular, pyramidal numbers or numbers of any given progression of this kind, and as doubles, triples, etc. of the numbers *simpliciter* (that is, added to itself two, three times, etc.), and so on.[9] In other words, the set of the natural numbers must be equal to any of his subsets, not only to those of the powers. However, these requirements are not all complied with by the unit. The double of 1 (1+1), or its triple (1+1+1) is not equal to 1. Hence, when Leibniz asks with which number the infinite number would be compared or equated, he observes that the number which complies with these requirements altogether is not 1 but 0. Only in 0 are there as many even and odd numbers as even, triangular or pyramidal numbers, etc., as well as only the double or the triple of this number is equal to itself, and so on. For Leibniz, the fact that the infinite number equates to 0 means that it is nothing, or that it does not represent a whole.[10]

However, as we have said, Leibniz not only disagrees with Galileo regarding the requirements which define the infinite number but also regarding the affirmation of its existence. The philosopher from Leipzig deepens in his approach to this problem in two ways, namely, a philosophical or axiomatic and an arithmetical or analytical one. From the philosophical point of view, Leibniz carries out an exam regarding how the propositions which have to be accepted in philosophy must be so that there is a breakthrough in it. Leibniz's main concern here is to examine the scope of the whole-part axiom. As we have said, the Galilean argument assumes that this axiom, although valid for finite quantities, 'fails' in the case of the infinite. What Leibniz notes is that, if the axiom could be demonstrated, then its scope of application could not be limited to the finite. In other words, if it could be analytically demonstrated, its universal scope would be exposed. Hence, it would be demonstrated that Galileo's conclusion is wrong, or rather contradictory. From the arithmetical point of view, on the other hand, Leibniz presents arguments which show that the sum of all units, which precisely would be the infinite number, is 0. From this perspective Leibniz formally offers no arguments with which to conclude that the affirmation of the existence of the infinite number is contradictory. In any case, it is not the task of arithmetic to justify something's existence or not, but of philosophy. In this sense, the two ways by which Leibniz approaches the problem of the existence of the infinite number are complementary. Accordingly, in the next section of this work we will consider the philosophical approach, while in the subsequent one, the arithmetical approach.

2. The infinite number from the philosophical standpoint: the demonstration of the axiom

It could be consider that ultimately the difference between the proposals of the Italian scientist and the philosopher from Leipzig is based on the fact that they rely on

[9] A II 1, 348-349.
[10] A II 1, 349-350. See also A VI 3, 98 and 168.

different premises. There are indeed two premises in conflict, namely, (a) in the infinite the whole is greater than a part and (b) an infinite set, such as that of natural numbers, is a whole. While the Italian scientist holds (b) and rejects (a), the philosopher from Leipzig does the opposite. However, as E. Lison points out,[11] if were so, Leibniz could not disapprove the affirmations based on Galileo's premises through his arguments. In this sense, he could not say that the infinite number is contradictory, since the assumption of other premises makes it possible. In this section we will show that for Leibniz it was not just a matter of assumed premises, that is to say, a matter of deciding which presupposition must be held and which rejected. He indeed considered that the universal scope of the axiom can be *demonstrated*. As a consequence, for Leibniz holding the opposite would not have been a viable option.

Now, the Leibnizian intention of 'demonstrating the axiom' might seem paradoxical. Even in Leibniz's days the notion of axiom or principle was more or less understood in the terms presented by Aristotle in his *Posterior Analytics*, that is, as a first truth that is so self-evident that it is beyond any possible demonstration.[12] However, in the present discussion Leibniz notes that one of these alleged first truths is not recognized as such by Galileo:

> In fact, as this proposition: *the whole is greater than the part*, has been questioned by the highest geometers, such as Galileo and Gregory of Saint Vincent, can we continue claiming from now on that there are propositions which are evident in themselves?[13]

The fact that a first truth can easily be doubted, means for Leibniz that in philosophy no proposition must be accepted, except those which either are certain by immediate observation of the senses, or based on a clear and distinct idea or are demonstrated by a definition.[14] For the present discussion the third type of propositions, that is, those demonstrated, is particularly important. Thus, Leibniz thinks that if a proposition which is not confirmed by the senses or based on a clear and distinct idea is accepted, then it has to be demonstrated. As a result, no demonstrable proposition should be taken as self-evident. Leibniz mentions a clear example of an alleged self-evident proposition which however must be demonstrated:

> From this it is also clear that these propositions: [1] things equal to another thing are also equal to each other; [2] if the same is added to or removed from the same things, then the same things are obtained; [3] a whole is greater than a part; [4] the equimultiples are like the simples; [5] if proportional things are added to or removed from proportional things, then proportional things are obtained; etc., require a demonstration, since it is possible to doubt them, and if

[11] Elad Lison, "The Philosophical Assumptions Underlying Leibniz's Use of the Diagonal Paradox in 1672," *Studia Leibnitiana* 38 (2006): p. 197-208.
[12] Aristotle, *Posterior Analytics*, 76a30.
[13] A II 1, 351: "Cum enim ista propositio: *totum esse majus parte*, dubitationem receperit apud maximos Geometras, quales certe Galilaeus et P. Gregorius a S. Vincentio fuere, ullasne alias imposterum per se notas clamitare pergemus". Translations of this text of Leibniz are ours. With regard to Gregory of Saint Vincent, Leibniz points out that this mathematician held that the whole-part axiom fails in the angle of contact, that is, the angle between a circle and a tangent: Gregory of Saint Vincent. *Opus geometricum quadraturae circuli et sectionum coni*. Anvers, 1647, p. 871.
[14] A II 1, 351.

> they are true, they would be demonstrable, namely, from terms, that is, from definitions.[15]

Some famous axioms of Euclid, among which the one involved in the discussion with Galileo stands out, can be recognized in this quote. They are, strictly speaking, some "common notions" of *Elements*, I. Besides the fact that we can doubt the axioms, like the Italian scientist has effectively done, Leibniz recognizes another important reason why they cannot be taken as self-evident truths. In this case it is an epistemological reason: a human faculty or power which sets the same truths for all and with the same criteria cannot be established. The author recognizes that this would inevitably lead to the deepest skepticism. In this point, Leibniz is strongly critical of the Cartesian conception according to which it is possible to know propositions with evidence:[16]

> I say this [that all propositions must be demonstrated] against those who think that [truths] are known per se by I don't know what natural light, since it is known that some consider certain propositions as known per se, while others reject and differentiate them, and they do not provide a criteria of what is known per se, unless perhaps the common opinion according to which, if it were not subjected to doubts, it would lay probable foundations of demonstrations, which is shaking hands with Pyrrho.[17]

According to Leibniz, the demonstration of the axioms –as well as of any other proposition which must be taken in philosophy and is not ascertained by the senses– must be carried out from definitions. Leibniz recognizes that he is in debt to the scholastic thought regarding the thesis that truths become manifest by inspecting the terms. The next required step in Leibniz's exam consists in analyzing, on the one hand, definitions as such and, on the other, their role in demonstrations. Foremost, the fact that it can be demonstrated from definitions has in its basis the understanding that a definition is, according to Leibniz, the meaning of an idea.[18] As a linguistic expression, a definition also 'points at' the corresponding idea.[19] However, this certainly does not mean that definitions are the same for everyone. In fact, Leibniz recognizes that definitions as such are arbitrary. Therefore, the fact that a definition is the meaning of an idea does not imply that it must be accepted without further questioning. The propositions *demonstrated from* definition must be accepted, *not* the definitions themselves. The demonstrated propositions can be, as such, true or false. Regarding definitions from which demonstrations are carried out, coincident in this point with the Galilean exam, Leibniz says that "falsity must not be argued [about them] but ineptitude or obscurity".[20]

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[15]A II 1, 352: "Hinc apparet etiam propositiones istas: eidem aequalia etiam inter se esse, aequalia aequalibus addita vel demta facere aequalia, totum esse majus parte, aequimultiplicia esse ut simpla, si proportionalibus addantur demanturve proportionalia, producta esse proportionalia etc., cum dubitationis capaces sint, demonstratione indigere, et si sunt verae, demonstrabiles esse, ex terminis scilicet, seu definitionibus".

[16] For example, AT, VIII, 1, § 30.

[17]A II 1, 352: "(…) contra quam illi qui nescio quo lumine naturali per se notas putant. Cum constet quaedam ab aliquibus inter per se nota poni, quae ab aliis rejiciantur aut distinguantur, nec criterion afferri per se noti nisi forte opinionem communem, quae praeterquam quod dubitationibus obnoxia est, probabilia poneret fundamenta demonstrationum, quod est Pyrrhoniis manus dare".

[18] A II 1, 351.

[19] Oscar M. Esquisabel, "Leibniz: las bases semióticas de la *characteristica universalis*", *Representaciones*, 8, 1 (2012): p. 12.

[20] A II 1, 351: "(...) nec falsitatis, sed ineptiae obscuritatisque tantum arguendae".

The claim that every proposition which is admitted as true must be demonstrated from definitions, leads inevitably to a big problem: if definitions as such are arbitrary, then are the truths also so? That is: does this also mean that all truths depend on human will? There have been those who have claimed this interpretation. Thomas Hobbes, a decisive influence on Leibniz's early thought, was one of them.[21] The Hobbesian exposition is briefly that truths are arbitrary precisely because they depend on arbitrary definitions.[22] However, Leibniz disagrees with this interpretation. Although in the first version of the *Accessio* Leibniz held a conventionalist conception of definitions which has led him to the thesis that axioms are true by convention,[23] in the second and definite version he conceived this issue otherwise. It is true that propositions in some way depend on definitions since the first are expressed precisely by words or other signs. However, the connections of the ideas themselves, which are expressed by definitions, are, as such, non-symbolic, that is, independent of the terms of language. Such connections of ideas, as we have said, depend on sensation or on an idea, but not on definitions. Since thoughts are precisely non-symbolic, they are not arbitrary like definitions, given that the arbitrariness of them depends on the used terms. Therefore, Leibniz points out that the notations and symbols are arbitrary, be they words or characters, but the ideas are the same for all.[24]

However, there is one important feature which the use of symbols or characters admits. Leibniz indeed recognizes that on many occasions we proceed by a mere symbolic manipulation, without a direct consideration of the ideas themselves to which the characters are referred to.[25] These thoughts which proceed by means of mere symbols abstaining from the consideration of ideas are denominated by Leibniz as *blind*, for behaving precisely in this way. According to Leibniz, in blind thoughts we proceed analogously to the way we proceed in those cases in which we consider a few ideas simply and distinctly comprehended. The usefulness of the use of symbols rests ultimately in the analogical nature of their employment, mainly in cases in which very complex things are symbolically considered. Despite this advantage of the use of symbols, the philosopher from Leipzig recognizes that we lack of a system of simple, complete and ordered symbols, even in pure mathematical sciences. From this fact, Leibniz designs the project of a Language or a Philosophical Scripture.[26]

In accordance with the stipulated procedure, to demonstrate the whole-part axiom Leibniz begins by establishing the definitions of the notions which are involved, namely, whole-part, greater than and smaller than. In this regard, Leibniz says: "if (def. 1) *a* and *b* are the parts, then the whole (def. 2) will be $a + b$. Likewise, if (def. 3) *a* is *smaller than*,

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[21] On the importance of Hobbes for the main arguments in the *Accessio*, see also Ursula Goldenbaum, "Indivisibila Vera – How Leibniz Came to Love Mathematics Appendix: Leibniz's Marginalia in Hobbes' *Opera Philosophica* and *De Corpore*," in *Infinitesimal Differences: Controversies between Leibniz and his Contemporaries*, edited by Ursula Goldenbaum and David Jesseph (Berlin and New York: Walter de Gruyter), p. 67-76.

[22] Thomas Hobbes, Hobbes, *De corpore*, I, 3, 8, in *The English Works of Thomas Hobbes,* William Molesworth ed (London: Bohn, 1839), vol. 1.

[23] Esquisabel, "Leibniz: las bases semióticas de la *characteristica universalis*", 9-14

[24] A II 1, 353.

[25] A II 1, 353-354.

[26] A II 1, 354; an examination of the way in which Leibniz understands the employment of symbols in the *Accessio* and his project of a universal language can be found in Esquisabel, Leibniz: las bases semióticas de la *characteristica universalis*, 14-26.

then $c = a + b$ (def. 4) will be *greater than*".[27] Now, the fact that he understands that every proposition which is accepted must be demonstrated from definitions, implicitly supposes a certain conception of what a demonstration is. What Leibniz has in mind here is at least similar to what Hobbes had said. The British philosopher had considered a demonstration as a syllogism or a chain of syllogisms which goes from definitions of names, to a last conclusion, in such a way that what is concluded must be in some way contained in the definitions which are at the basis of the first syllogism. If the conclusion is not found in it, it will not be found in any other syllogism which depends on the first and hence will never be reached.[28] According to Leibniz (and in this he is in explicit accordance with Pascal), this is the procedure of pure sciences, that is, those which are not empirical. As a matter of fact, in this kind of sciences a definition or a part of a definition of what is defined is enunciated.[29] The idea that everything which can be deduced from a notion must already be contained in its definition (under the assumption that there is nothing without a cause) will be repeatedly present in the Leibnizian analysis. In this case, this means that the demonstration of the whole-part axiom should be deducible from the definitions of 'whole', 'part', 'greater than' and 'smaller than'. Precisely for this reason, after establishing such definitions, Leibniz adds: "if the definitions are joined, the demonstration is composed: *the Whole* $= a + b$ (def. 2) $a + b = c$ (def. 4) $c =$ *greater than* (the same def. 4), *part* $= a$ (def. 1), $a =$ *smaller than* (def. 3)".[30]

Thus, Leibniz has an analytical demonstration of the whole-part axiom (that is, from definitions). Thereby, the universal scope of the axiom, that is, its universal validity for every quantity, either finite or infinite, is guaranteed. Therefore, it is incorrect to say that the axiom 'fails' in the infinite, as well as it is to claim that the infinite number of all units (which according to Galileo does not contradict the axiom since it is beyond its scope) does not imply any contradiction. This leads us to conclude that the infinite number of all units is impossible, precisely because it contradicts the whole-part axiom. Finally, if it is impossible, it does not exist.[31]

3. The Leibnizian rule for adding infinite series

The second kind of approach proposed by Leibniz is very important for the present discussion. One feature of this reasoning is that it does not require the whole-part axiom. In

[27] A II 1, 355: "Nam si (defin. 1) *partes* sint a, b, *totum* (defin. 2) erit $a + b$. Item si *minus* (defin. 3) sit idem a, *majus* (defin. 4) erit $c = a + b$".

[28] Hobbes, *De corpore*, I, 6, 13 and 16.

[29] A II 1, 354.

[30] A II 1, 355: "Conjunctis definitionibus connectetur demonstratio: *Totum* $= a + b$ (defin.2) $a + b = c$ (defin. 4) $c =$ *majus* (dict. defin. 4), *pars* $= a$ (def. 1), $a =$ *minus* (defin. 3)".

[31] These considerations regarding the infinite number are connected with others philosophical problems beyond the epistemological thesis of the demonstrability of axioms. For example, Rabouin points out its connection with the so-called 'ontological proof' (David Rabouin. "The Difficulty of Being Simple: On Some Interactions Between Mathematics and Philosophy in Leibniz's Analysis of Notions". In *G. W. Leibniz, interrelations between Mathematics and Philosophy*, edited by Norma Goethe, Philip Beeley and David Rabouin. Dordrecht; Heidelberg; New York; London: Springer, 2015, p. 61-62). Leibniz considered that this argument needs a 'supplement', since it supposes that the perfect or maximum being is possible, and this must be demonstrated (A VI 3, 511). This requirement is based on the fact that there are definitions of alleged greatest or 'maxima' things which imply contradictions, as, for example, a most rapid motion, a maximum number or a greatest shape (A VI 3, 520). Hence, the demonstration of the impossibility of the number of all units plays an important role in Leibniz's 'ontological proof'.

this sense, it shows the equalization of the infinite number with 0 by other means than those described above, and hence it evades the philosophical discussion. Thus, Leibniz offers a different argument, arithmetical in this case (at least in the early modern way of working in arithmetic), which complements the philosophical or axiomatic one. The convergence of different arguments was for Leibniz persuasive enough to convince him of the non-existence of the infinite number of all units.

This second argumentation is a consequence of the examination which Leibniz carries out from Huygens's proposal mentioned at the beginning of this paper. By looking for the sum of the reciprocals of the triangular numbers, Leibniz believes he has found not only a rule for adding this infinite series but for many others series too. The result of this procedure gave rise to the draft of a paper which he sent to the *Journal des Sçavans*, directed by J. Gallois, which was finally not published. This work, entitled *Accessio ad arithmeticam infinitorum*, contains Leibniz's first intervention in the field of infinite sums and it is frequently considered that the obtained result is the beginning of the Leibnizian research in the field of the mathematics of infinitesimals. As we will see, according to Leibniz, the application of this rule to certain series (that of units and of the reciprocals of natural numbers) will show the equalization of the infinite number (which is the sum of the above series) with 0.

The *Accessio*, which was reviewed several times, only introduces the rule of procedure for adding both the reciprocals of the triangular numbers and the successions generated by the triangular numbers (that is, of the reciprocals of the complete series of Pascal's triangle), but not its demonstration, which is the subject of later works. In this section we will first analyze the Leibnizian rule, in order to consider its demonstration, and finally present the problems which arise both from the rule and its justification, and from its application to the sum from which the infinite number would result.

According to the usual reconstruction of the Leibnizian procedure for the sum of infinite series, at some point the philosopher from Leipzig appeals to the elimination of infinitesimal quantities. J. Hofmann and more recently S. Levey explain the procedure in this way:[32]

1. Consider two infinite series, a) and b), such that the elements of b) are the differences between the consecutive elements of the a) series, namely:

a) $a_1, a_2, a_3, a_4 \ \dots, a_n, a_{n+1}$, etc.

b) $b_1 = a_1 - a_2;\ b_2 = a_2 - a_3;\ b_3 = a_3 - a_4, \dots, b_n = a_n - a_{n+1}$, etc.

2. Given these infinite series, then [Rule]: $b_1 + b_2 + b_3 + \cdots + b_n = a_1 - a_{n+1}$

3. For example, let us try to obtain the sum of the series of the reciprocals of the triangular numbers (b).

4. First of all we must determine the generating rule of the reciprocals of the triangular numbers: $b_n = \frac{2}{n\,(n+1)}$. Thus, for example:

[32] Joseph Hofmann, *Leibniz in Paris, 1672-1676. His growth to mathematical maturity* (Cambridge & New York: Cambridge University Press, 1974), p. 12-22, and Samuel Levey, "Leibniz on Mathematics and the Actually Infinite Division of Matter," *The Philosophical Review* 107, 1 (1998): p. 49-96.

(b_1) $\frac{2}{1\,(1+1)} = \frac{2}{2} = \frac{1}{1}$

(b_2) $\frac{2}{2\,(2+1)} = \frac{2}{6} = \frac{1}{3}$

(b_3) $\frac{2}{3\,(3+1)} = \frac{2}{12} = \frac{1}{6}$

(b_4) $\frac{2}{4\,(4+1)} = \frac{2}{20} = \frac{1}{10}$

And so on.

5. According to [Rule], for obtaining the sum of (b) we must find a series (a) such that the sum of $(b) = a_1 - a_{n+1}$.

6. The generating rule of such sequence (a) is: $a_n = \frac{2}{n}$. Thus, for example: $(a_1) = \frac{2}{1}$; $(a_2) = \frac{2}{2}$; $(a_3) = \frac{2}{3}$; $(a_4) = \frac{2}{4}$; and so on.

7. We can observe what was indicated in the step 1 point b), since: $(b_1)\,\frac{1}{1} = \frac{2}{1} - \frac{2}{2}$; $(b_2)\,\frac{1}{3} = \frac{2}{2} - \frac{2}{3}$; $(b_3)\,\frac{1}{6} = \frac{2}{3} - \frac{2}{4}$; etc.

8. According to [Rule]: $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10}\,etc. = \frac{2}{1} - \frac{2}{n+1}$

9. Now, supposing that the series (b) and (a) have a last term, that is, making *n* infinitely big, such last term will be infinitely small and, hence, it can be eliminated, so (Levey, 1998: 72):

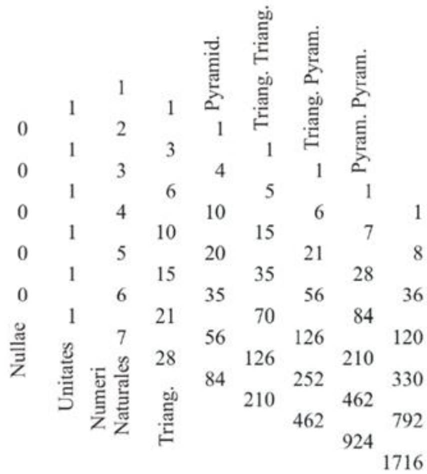
$$sum\ of\ (b) = \frac{2}{1} - \frac{2}{n+1} = \frac{2}{1} = 2$$

However, the concept of the infinitesimal is not presented either in the rule presented by Leibniz in the *Accessio*, nor in the foundation presented later. He rather supposes that the infinite sums can be treated as wholes, that is to say, it always makes sense to speak of a result of the sum, regardless of its nature.[33]

1. The rule of the *Accessio* is proposed not for adding the reciprocals of the triangular numbers only (from now on, 'triangular fractions'; Leibniz himself names them in this way when he justifies the rule: A VII 3, 366), but as a universal rule, that is, a rule that also allows us to add the pyramidal fractions (the differences between the pyramidal numbers are the triangular ones), the triangulo-triangular (the differences are the pyramidals), the triangulo-pyramidal (the differences are the triangulo-triangular), and so on. Leibniz

[33] For another approach to Leibniz's method applied in the *Accessio*, see Michel Serfati, "Order in Descartes, Harmony in Leibniz: Two Regulative Principles of Mathematical Analysis," *Studia Leibnitiana* 45, 1 (2013): p. 75-78; Michel Serfati,"'On the Sum of All Differences' and the Origin of Mathematics According to Leibniz: Mathematical and Philosophical Aspects," in *Perspectives on Theory of Controversies and the Ethics of Communication,* edited by Dana Riesenfeld and Giovanni Scarafile (Dordrecht; Heidelberg; New York; London: Springer, 2014), p. 69-79 and David Rabouin. "The Difficulty of Being Simple: On Some Interactions Between Mathematics and Philosophy in Leibniz's Analysis of Notions". In *G. W. Leibniz, interrelations between Mathematics and Philosophy,* edited by Norma Goethe, Philip Beeley and David Rabouin. Dordrecht; Heidelberg; New York; London: Springer, 2015, p. 57-63.

presents a table, which he calls *replicated arithmetical progression*, and which is the equivalent of Pascal's triangle:[34]



Nullae	Unitates	Numeri Naturales	Triang.	Pyramid.	Triang. Triang.	Triang. Pyram.	Pyram. Pyram.	
0	1	1	1	1	1	1	1	1
0	1	2	3	4	5	6	7	8
0	1	3	6	10	15	21	28	36
0	1	4	10	20	35	56	84	120
0	1	5	15	35	70	126	210	330
	1	6	21	56	126	252	462	792
		7	28	84	210	462	924	1716

Image 2

2. Hence, the series of the corresponding reciprocals are:

Natural fractions: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, etc.$

Triangular fractions: $\frac{1}{1}, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \frac{1}{28}, etc.$

Pyramidal fractions: $\frac{1}{1}, \frac{1}{4}, \frac{1}{10}, \frac{1}{20}, \frac{1}{35}, \frac{1}{56}, \frac{1}{84}, etc.$

Triangulo-triangular fractions: $\frac{1}{1}, \frac{1}{5}, \frac{1}{15}, \frac{1}{35}, \frac{1}{70}, \frac{1}{126}, \frac{1}{210}, etc.$

Triangulo-pyramidal fractions: $\frac{1}{1}, \frac{1}{6}, \frac{1}{21}, \frac{1}{56}, \frac{1}{126}, \frac{1}{252}, \frac{1}{462}, etc.$

Pyramido-pyramidal fractions: $\frac{1}{1}, \frac{1}{7}, \frac{1}{28}, \frac{1}{84}, \frac{1}{210}, \frac{1}{462}, \frac{1}{924}, etc.$

Etc.

(In today's terms: natural fractions: $\frac{1}{n}$; triangular fractions: $\frac{2}{n(n+1)}$; pyramidal fractions: $\frac{6}{n(n+1)(n+2)}$, and so on).

3. In each of the progressions we must distinguish between the generating number and the exponent number. The generating number is that which multiplies each fraction of each progression, while the exponent number is that which indicates the order of the arithmetical progression in the table of the reciprocals of the replicated arithmetical progression. From the succession of the natural numbers on, the exponent number *coincides with the number which follows the unit immediately*. Thus, for example, the exponent number of the series of natural numbers is 2, that of the triangular numbers is 3, that of the pyramidal numbers is 4, and so on.[35]

4. With this in mind, Leibniz's rule is:

> The sum of a series of fractions whose numerator is the generating number and whose denominators are the terms of some Replicated Arithmetical

[34] A II 1, 345-346.
[35] A II 1, 346.

progression, that is, what is the same, the sum of ratios in which the antecedent is the unit and the consequent is a term of the Replicated Arithmetical progression which has the Unit as the generating number, this sum, I say, is a fraction or ratio whose numerator or antecedent is the exponent number of the immediately preceding series, that is, of the penultimate (namely, supposing a given last one), but the denominator or consequent is the exponent number of the immediately preceding series to the preceding one, that is, of the antepenultimate.[36]

5. Thus, for example:

- Sum of the pyramido-pyramidal fractions: $\frac{exponent\ number\ of\ the\ triangulo{-}pyramidal}{exponent\ number\ of\ the\ triangulo{-}triangular} = \frac{6}{5}$
- Sum of the triangulo-pyramidal fractions: $\frac{5}{4}$ (for the same reason)
- Sum of the triangulo-triangular fractions: $\frac{4}{3}$ (for the same reason)
- Sum of the pyramidal fractions: $\frac{3}{2}$ (for the same reason)
- Sum of the triangular fractions: $\frac{2}{1}$ (for the same reason)

6. The following table is therefore obtained:

0	1	2	3	4	5	6	7	etc.	exponentes
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$		
$\frac{0}{0}$	$\frac{1}{1}$	$\boxed{\frac{1}{2}}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$= \frac{1}{0}$ NB	
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{21}$	$\frac{1}{28}$	NB	
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{56}$	$\frac{1}{84}$	NB	
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{35}$	$\frac{1}{70}$	$\frac{1}{126}$	$\frac{1}{210}$		
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{6}$	$\frac{1}{21}$	$\frac{1}{56}$	$\frac{1}{126}$	$\frac{1}{252}$	$\frac{1}{462}$	NB	
$\frac{0}{0}$	$\frac{1}{1}$	$\frac{1}{7}$	$\frac{1}{28}$	$\frac{1}{84}$	$\frac{1}{210}$	$\frac{1}{462}$	$\frac{1}{924}$		
$\frac{0}{0}$	$\frac{0}{0}$	$\frac{1}{0}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	etc.	summae.

Series fractionum progressionis arithmeticae replicatae unitatis generatricis.

Image 3

7. We can observe that Leibniz, showing that the sum of the triangular fractions is 2, presents a procedure for adding an endless number of series.

[36] A II 1, 346: "Regula Universalis haec est: Summa seriei fractionum, quarum numerator est generator, nominatores sunt termini cujusdam progressionis Arithmeticae Replicatae, seu, quod idem est, summa rationum in quibus antecedens Unitas, consequens est terminus progressionis Arithmeticae Replicatae Unitatem habentis generatricem, haec summa, inquam, est fractio seu ratio cujus numerator seu antecedens est exponens seriei proximae praecedentis seu penultimae (data scilicet supposita ultima) nominator vero seu consequens est exponens seriei proxime praecedentis praecedentem, seu antepenultimae".

8. The application of the aforementioned rule to the series of natural numbers and of units constitutes Leibniz's arithmetical argument for showing that the infinite number is equal to 0.[37] Indeed, the sum of the series of units would be the infinite number of all units. As the last table shows:

- Sum of the natural fractions: $\frac{1}{0}$
- Sum of units: $\frac{0}{0}$

As we have said in section 1, the idea that the infinite number equates to 0 implies that it is nothing, or better, it does not represent a whole, in opposition to the conclusion draws by Galileo.

4. The justification of the rule

It is clear that here there is no explicit appeal to infinitesimal quantities, nor to any procedure for eliminating them. Therefore, the result of our examination seems to go against the usual interpretation. However, Leibniz seems to suppose more or less implicitly that whenever we have an infinite series there is a certain amount that corresponds with it as a totalization, either finite or infinite (regardless of whether this can make sense to us today or not). Thanks to the edition of Leibniz's work on infinite series, we have texts which show in what way he justified the rule of the *Accessio*. There are at least two texts which expose the justification of the aforementioned rule. The first one is entitled *Summa fractionum a figuratis, per aequationes* and the second one *Scheda Exigua*.[38] We will present a synthesis of the second text's argumentation. These texts, again, do not explicitly refer to infinitesimal quantities, although they appeal to the idea that the infinite series make sense even when the results are not finite quantities. We will now provide the Leibnizian justification of the aforementioned rule.

The justification starts from a property which relates the succession of natural fractions with the succession of triangular fractions. Leibniz presents a table or graphical arrangement in order to show this property:

$$\begin{array}{lccccc} A \sqcap & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ & & & & & \\ B \sqcap & \frac{1}{1} & \frac{1}{3} & \frac{1}{6} & \frac{1}{10} & \frac{1}{15} \end{array}$$

Image 4

This property, which Leibniz takes as a theorem, consists in assuming that if to each term of the natural succession the half of the term of the triangular fractions linked to it by a bar is added, then the term of the natural succession immediately prior to it (that is, that which is placed above the triangular term linked to it by the bar) is obtained.[39] As a consequence:

[37] A II 1, 349-350.
[38] A VII 3, 365-369, September of 1674, and A VII 3, 712-714, December of 1675 or February of 1676.
[39] A VII 3, 712-713.

$\frac{1}{2}$ (natural fraction) $+\frac{1}{2}$ (half of the triangular fraction linked to it by a bar) $= 1$
$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$
$\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$
$\frac{1}{5}+\frac{1}{20}=\frac{1}{4}$

On the basis of this property, Leibniz develops the proof that the infinite sum of the triangular fractions is 2, as well as that of the other progressions which are obtained from the triangular succession. To do so, Leibniz establishes, in the first place, the generating rule of the triangular numbers and, in the second, that of the reciprocals of such numbers. To obtain the triangular numbers: $\frac{y^2+ya}{2}$, where 'a' is the generating number. As in this case the generating number is the unit, $\frac{y^2+y}{2}$. Thus, $(y_1)\,\frac{1^2+1}{2}=\frac{2}{2}=\frac{1}{1}=1$; $(y_2)\,\frac{2^2+2}{2}=\frac{6}{2}=\frac{3}{1}=3$; $(y_3)\,\frac{3^2+3}{2}=\frac{12}{2}=\frac{6}{1}=6$, and so on. The triangular fractions are obtained, in turn, by this formula: $\frac{2a^2}{y^2+ya}$. Again, as 'a' is the generating number which here is the unit, $\frac{2}{y^2+y}$ is obtained. Thus, for example: $(y_1)\,\frac{2}{1^2+1}=\frac{2}{2}=\frac{1}{1}$; $(y_2)\,\frac{2}{2^2+2}=\frac{2}{6}=\frac{1}{3}$; $(y_3)\,\frac{2}{3^2+3}=\frac{2}{12}=\frac{1}{6}$; and so on (VII 3, 366). Thus, we will have the opportunity to see how Leibniz justifies the rule of the *Accessio* from a generalization of this procedure. Let us take, in the first place, the infinite sums of the successions in order from the natural succession (remember *image 3*):

(1)
$A\ \sqcap \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\,etc.$
$B\ \sqcap \frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\,etc.$
$C\ \sqcap \frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}\,etc.$
$D\ \sqcap \frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}\,etc.$
$etc. \sqcap etc.$

Before proceeding, a clarification on the sign which represents equality ($\sqcap$) should be made. This sign presents a notational ambiguity which has considerable consequences in the demonstration of the sum of the triangular fractions, since the letters A, B, C, D, etc. can be understood as a mere conventional notation used to express the series as such in an abridged form, but they can also be interpreted as quantities which result from the infinite sum of fractions. In any case, as we will see, Leibniz supposes that the algebraic operations which are applied to a finite number of terms can be equally applied if the number of terms is infinite.

Given the aforementioned ambiguity, let us now take the infinite sum A of natural fractions and let us subtract 1. This will result in:

(2) $A-1\ \sqcap \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\,etc.$

Similarly, let us now take the sum of half of each one of the terms of the infinite sum of the succession of the triangular fractions. Again, given the aforementioned ambiguity, Leibniz seems to consider that the division is distributive in relation to the sum also for infinite terms:

(3) $\frac{1}{2}B \ \sqcap \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} etc.$

Next, let us make the algebraic sum of (2) and (3). Thus, for the property exhibited in *image 4*, the following result will be obtained (which, apparently, is the restitution of the complete series of the natural fractions, that is, the series A):

(4) $A - 1 + \frac{1}{2}B \ \sqcap 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} etc.$

Thus, since by the addition (4) the series A is again obtained, Leibniz concludes that:

(5) $A - 1 + \frac{1}{2}B = A$, then,

(6) $\frac{1}{2}B = 1$, hence,

(7) $B = \frac{2}{1}$, which was what he wanted to prove.[40]

Leibniz obtains analogous conclusions by applying a similar reasoning to other series. In order to do this he recognizes, for example, that the succession of the series C (pyramidal fractions) behaves, regarding the succession of the terms of B (triangular fractions), in an analogous manner to B regarding A. The same can be said of the terms of the series D regarding C, of E regarding D, etc. For example: in analogy with the previous reasoning, let us subtract the unit from the series B ($B - 1$) and let us establish which fraction of C must be added in order to restore the original series B. We will obtain that $B - 1 + \frac{2}{3}C = B$. From here Leibniz will recognize that there is a sequence in the coefficients for the remaining series: as well as for obtaining B, $\frac{2}{3}$ of C must be added to B-1, for obtaining C, $\frac{3}{4}$ of D must be added to C-1, therefore for obtaining D $\frac{4}{5}$ of E must be added to D-1, and so on. (Let us note the sequence of the coefficients: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, etc.$). Thus, it is obtained that:

(8) $B - 1 + \frac{2}{3}C = B$; then, $\frac{2}{3}C = 1$; hence, $C = \frac{3}{2}$

(9) $C - 1 + \frac{3}{4}D = C$; then, $\frac{3}{4}D = 1$; hence, $D = \frac{4}{3}$

Consequently, the next table follows (A VII 3, 714):

[40] A VII 3, 713-714.

$$
\begin{array}{cccccc}
 & \overbrace{\quad}^{A} & \overbrace{\quad}^{B} & \overbrace{\quad}^{C} & \overbrace{\quad}^{D} & \text{etc.} \\
\frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \\
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \\
\frac{1}{1} & \frac{1}{3} & \frac{1}{6} & \frac{1}{10} & \frac{1}{15} & \\
\frac{1}{1} & \frac{1}{4} & \frac{1}{10} & \frac{1}{20} & \frac{1}{35} & \\
\frac{1}{1} & \frac{1}{5} & \frac{1}{15} & \frac{1}{35} & \frac{1}{70} & \\
\text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} & \\
\multicolumn{6}{c}{\boxed{\text{summa}}} \\
\frac{0}{\dots} & \frac{1}{0} & \frac{2}{1} & \frac{3}{2} & \frac{4}{3} & \text{etc.} \\
\multicolumn{2}{c}{\text{coniecturalia}} & \multicolumn{4}{c}{\text{certa}}
\end{array}
$$

Image 5

We see, therefore, where the rule of the *Accessio* arises. This rule (we must remember) says: the sum of each series of the reciprocals of the replicated arithmetical progressions is a ratio in which the numerator is the exponent number of the penultimate succession and the denominator is the exponent number of the antepenultimate succession. Thus, the following property is generalized: in order to obtain the sum of a given series the reciprocal of the coefficient which multiplies it must be formed so that the previous series can be obtained.

Conclusions: some problems of the Leibnizian exposition

In short: Leibniz deals with Galileo's conclusions regarding the infinite number in two complementary ways. On the one hand, he demonstrates the whole-part axiom by means of definitions, in order to justify its universal scope. Hence, it would be inconvenient to say, as Galileo has done, that the properties of being greater than, being smaller than, and being equal to, cannot be applied to infinites. Thus, the infinite number implies a contradiction. This is what we have called the 'philosophical' or 'axiomatic' approach of Leibniz. On the other hand, Leibniz shows that we must not say that the infinite number equates to 1, but to 0, all of which means that it is nothing, or better, that it does not represent a whole. Leibniz applied the rule for adding the infinite series of the harmonic triangle to the infinite series of units to show the astonishing conclusion that the infinite sum of units would be 0. This is what we have called the 'arithmetic' or 'analytical' approach of Leibniz. It seems that Leibniz considered that the results of these two approaches are complementary, that is, they are in solidarity with each other: the infinite number is impossible, that is, it is nothing, or better, it is not a whole.

Beyond this, it should be note that Leibniz's rule and its justification raise questions regarding both the presuppositions which are in the basis of the proof and the validity of the application of the rule to the series of natural fractions and units. Let us return, in the first place, to the problem of the ambiguity of the equalities previously formulated (namely, A, B, C, D). It seems clear that Leibniz interprets the corresponding letters not only as abbreviations of the series but also as expressions of unknown quantities. If this interpretation is correct, then it follows that Leibniz deals with all the series as if we could consider them as wholes, that is, such that each one of them amounts to a certain quantity, finite or infinite, which can be algebraically operated with.

Implicitly, however, the course of Leibniz's argumentation seems to suggest that the operations with an infinite number of terms do not maintain the same characteristics as the operations with a finite number of terms. Thus, for example, the step (4) ($A - 1 + \frac{1}{2}B \sqcap 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} etc.$) does not authorize the step (5) ($A - 1 + \frac{1}{2}B = A$) in the case of a finite series, since if A and B have n terms in a one to one correspondence, $A - 1 + \frac{1}{2}B = A$ will have *n-1* terms, so the result of the operation in the finite case does not restitute the original series. In other words, if the series A is the sum of the reciprocals of the k first natural numbers, we will have that $A - 1 + \frac{1}{2}B = A - \frac{1}{K}$. However, Leibniz seems to suppose that in an infinitary case, both series, A and $A - 1 + \frac{1}{2}B$, have the same cardinality as formulas, that is, the same number of terms. This seems to suggest that the algebraic operations in the field of the finite are not conserved in the field of the infinite.

There is, however, another possible answer, which consists in appealing to infinitesimal quantities. If we introduce these quantities and consider a series as a totality gifted with a last infinitesimal term, this term will become negligible and therefore dispensable. This would be the case of the missing term in the series $A - 1 + \frac{1}{2}B$, which would become infinitely small since the denominator is infinitely big and, in that case, both sums would be equal. Although this interpretation is closely related to the usual, which is correct if what we are considering is Leibniz's mature thought, it does not seem to be applicable to the method of argumentation which he uses in these texts, since, as we have seen, there is no explicit reference to the concept of an infinitesimal quantity, neither to the elimination procedure of such quantities. In short, the strategy followed by Leibniz for obtaining the sum of the series of the triangular fractions does not consist in the introduction of infinitesimal quantities, but in the possibility of treating the infinite series as wholes which always give a certain quantity as the result, either finite or infinite. From this point of view, the usual interpretations, which claim that Leibniz already in 1672 obtained the sum of the triangular fractions by means of operating with infinitesimal quantities, are not correct. In other words, in the Leibnizian argumentation what matters is that the series can be treated as a whole which always gives a quantity as the result, but not that it must be considered as a whole ended by a last dispensable infinitesimal.

A second problem arises precisely in relation with the infinite number. While Leibniz elaborates the justification of the rule of the *Accessio*, he recognizes that the results of its application to the first two series are "conjectural", not "certain" like the others. In this sense, he recognizes that the rule is useful for the sum of the series of the triangular numbers and the following series, but not for the sum of the preceding ones. From a contemporary point of view, we can recognize that the first two series are not convergent, while that of the triangular fractions (as well as those that follow it) is. Although in Leibniz's time there were no convergence criteria, he recognized the difference between convergent and non-convergent series. Notwithstanding, he applied a rule for adding convergent series even to non-convergent series, such as the series of natural numbers and of units. In other words, the infinite series were treated by Leibniz as summable wholes, whether they are convergent or not.

In third place, if what has been said is correct, namely, that Leibniz takes the sums as wholes, then some problems arise in relation to the strategy designed by Leibniz to find the sums of the triangular fractions and of the remaining fractions of the harmonic triangle. Indeed, the series of the reciprocals of the natural numbers (series A) is essential to the proof developed by Leibniz, and he knows that this series does not give a finite sum as the result (that is, it is not a convergent series). From Leibniz's point of view, this is to say that its sum gives as a result an infinite quantity, which according to our interpretation follows from having considered the possibility of totalizing any kind of infinite sum. Thus, a conflict with his explicit rejection of the existence of the infinite number arises. In other words, the Leibnizian foundation seems to appeal to entities which his very conception rejects. Perhaps at this point we should appeal to something said by Leibniz in 1676, that is, that it corresponds to the metaphysicians to examine whether the nature of things accepts infinite bounded quantities or not, since for the geometer is enough to demonstrate what follow from their supposition.[41]

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