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Inversion of Umarov–Tsallis–Steinberg's q-Fourier transform and the complex-plane generalization

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ABSTRACT

We introduce a complex q-Fourier transform as a generalization of the (real) one analyzed in [S. Umarov, C. Tsallis, S. Steinberg, Milan J. Math. 307 (2008)]. By recourse to tempered ultradistributions we show that this complex-plane generalization overcomes all the troubles that afflict its real counterpart.

1. Introduction

Nonextensive statistical mechanics (NEXT) [1–3], a current generalization of the Boltzmann–Gibbs (BG) case, is actively studied in diverse areas of science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index q (with $q = 1$ recovering the standard BG entropy). It has been applied to a variety of systems, such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], and Fokker–Planck equation applications [16].

An idiosyncratic feature of NEXT is that it can be advantageously expressed via q-generalizations of standard mathematical concepts [17]. Included are, for instance, the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [18]. The q-Fourier transform F_q exhibits the nice property of transforming q-Gaussians into q-Gaussians [18]. Recently, plane waves and the representation of the Dirac delta into plane waves have also been generalized [20–23].

We will be concerned here with the fact that a generic analytical expression for the inverse q-FT for arbitrary functions and any value of q does not exist [24]. Investigations revolving around this fact and related questions might be relevant for field theory, condensed matter physics, engineering (e.g., image and signal processing) and mathematical areas for which the standard FT and its inverse play important roles. It has been recently shown [25] that, in the $1 \leq q < 2$ particular case, and for non-negative functions (e.g., probability distributions), it is possible, by using special kinds of information, to obtain a bi-univocal relation between a function and its q-FT.

In this work we focus attention on the fact that, for fixed q, the q-Fourier transform is *not* one-to-one. The F_q—scenario can be vastly improved, however, by recourse to tempered ultradistributions (TUD) [19], which help generalizing such a transform to the complex plane. The generalization ameliorates the troubles that (see, for example [25]) afflict the real F_q.

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Why TUD? Because they solve characterization problems for analytic functions whose boundary values are elements of the spaces of distributions, or, conversely, of finding representations of elements of the quoted spaces of generalized functions by analytic functions. Many papers concern themselves with the ultradistribution spaces of Sebastiao e Silva [28]. Such spaces are related to the solvability and the regularity problems of partial differential equations. Because of such relations, the study of the structural problems as well as problems of various operations and integral transformations in this setting is interesting in itself. Thus, an analysis of spaces of distributions considered as boundary values of analytic functions having appropriate growth estimates is of great value. One wishes to deal, in particular, with the Dirac's integral representation in ultradistribution spaces, with the convolution of tempered ultradistributions and ultradistributions of exponential type (in Quantum Field Theory), and with the integral transforms of tempered ultradistributions, of which the best known is the complex Fourier transformation [19].

2. The complex q-Fourier transform and its inverse

So-called q-exponentials

$$e_q(x) = [1 + (1-q)x]_+^{1/(1-q)}, \tag{2.1}$$

are the hallmark of Tsallis's statistics [1]. They are generalizations of the ordinary exponential functions and coincide with them for $q = 1$. We start our considerations by appealing to a *complex* q-exponential. More precisely, we discuss $e_q(ikx)$ for $1 \le q < 2$ with k a real number (see Ref. ([23])

$$e_q(ikx) = [1 + i(1-q)kx]^{\frac{1}{1-q}}. \tag{2.2}$$

We remind the reader that *distributions* are a class of linear functionals that map a set of conventional and well-behaved functions, called test functions, onto the set of real (complex) numbers. In physics, the most celebrated example is Dirac's delta, which, in turn, is the derivative of the Heaviside step function H

$$H(x) = \begin{cases} 1 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{2.3}$$

Tempered distributions are a subset of the set of distributions. For them, the test functions are members of a special space called a Schwartz space $\mathcal{S}$. This is the function space of functions all of whose derivatives are rapidly decreasing. $\mathcal{S}$ has the important property that the Fourier transform is an automorphism on this space. This property enables one, by duality, to define the Fourier transform for elements in the dual space of $\mathcal{S}$, which is, precisely, that of tempered distributions. In physics, it is sometimes necessary to work with functions that grow exponentially in space or time. For those cases the Schwartz space of tempered distributions is too restrictive. So called ultradistributions satisfy that need [19]. A tempered ultradistribution is a continuous linear functional defined on the space of entire functions rapidly decreasing on straight lines parallel to the real axis [19].

An important fact about ultradistributions is the essence of our present considerations. A tempered distribution is the cut of a tempered ultradistribution. It is *not* the "cut" of an analytic function. The interested reader can consult Refs. [28,29] in such respect. Thus, it turns out that $e_q(ikx)$ is the cut along the real k-axis of the tempered ultradistribution [28,29], a central fact for our work:

$$E_q(ikx) = \{H(x)H[\Im(k)] - H(-x)H[-\Im(k)]\}\,[1 + i(1-q)kx]^{\frac{1}{1-q}}, \tag{2.4}$$

where $H(x)$ is the Heaviside step function and $\Im(k)$ the imaginary part of the complex number k. In order to better understand the relationship between $e_q(ikx)$ and $E_q(ikx)$ the reader should take into account that, if $f(k)$ is a tempered distribution and $F(k)$ is the corresponding tempered ultradistribution, then (see Refs. [28,29] and the Appendix)

$$f(k) = F(k + i0) - F(k - i0). \tag{2.5}$$

As a consequence of (2.5) we obtain:

$$e_q(ikx) = E_q[i(k + i0)x] - E_q[i(k - i0)x]. \tag{2.6}$$

Define now the set $\Lambda_{[1,2),\infty}$ as

$$\Lambda_{[1,2),\infty} = \{f(x)/f(x) \in \Lambda^+_{[1,2),\infty} \wedge f(x) \in \Lambda^-_{[1,2),\infty}\}, \tag{2.7}$$

where

$$\Lambda^+_{[1,2),\infty} = \left\{f(x)/f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^+] \wedge [f(x) \ge 0;\ 1 \le q < 2]\right\}, \tag{2.8}$$

and

$$\Lambda^-_{[1,2),\infty} = \left\{f(x)/f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^-] \wedge [f(x) \ge 0;\ 1 \le q < 2]\right\}. \tag{2.9}$$

With the help of this set Λ and using (2.4), together with the fact that for a given $F(k, q)$

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} dq\, \delta(q-1-\epsilon)\, F(k,q) \ = F(k) \tag{2.10}$$

so that

$$f(x) = \frac{1}{2\pi} \oint_{\Gamma} dk\, F(k) e^{-ikx}, \tag{2.11}$$

we *are now in a position to define*, at this precise stage, *our* complex Umarov–Tsallis–Steinberg (UTS) q-Fourier transform (of $f(x) \in \Lambda_{[1,2),\infty}$) in the fashion

$$F(k,q) \ = [H(q-1) - H(q-2)] \times \left\{ H[\Im(k)] \int_0^{\infty} f(x) \{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}},\ dx - H[-\Im(k)] \right.$$
$$\left. \times \int_{-\infty}^{0} f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\ dx \right\}. \tag{2.12}$$

In (2.12) q is a real variable such that $1 \le q < 2$. It is of the essence that the cut along the real axis of this transform is the real UTS q-Fourier transform given in Refs. [18,23] (see next section for a simple application of this transform). Taking into account that for $q = 1$ the q-Fourier transform is the usual Fourier transform and using the formula for the inversion of the complex Fourier transform immediately yields the inversion formula for (2.12).

Consider now

$$F(k) = \lim_{\epsilon\to 0+} \int_1^2 \delta(q-1-\epsilon) F(k,q) dq,$$

together with

$$f(x) = \frac{1}{2\pi} \int_{\Gamma} dk\, F(k) e^{-ikx}.$$

Since for $q = 1$ our Eq. (2.12) is the complex Fourier transform

$$F(k) = H[\Im(k)] \int_0^{\infty} dx f(x) e^{ikx} - H[-\Im(k)] \int_{-\infty}^{0} dx f(x) e^{ikx},$$

from (2.12) one straightforwardly obtains

$$f(x) = \frac{1}{2\pi} \oint_{\Gamma} \left[\lim_{\epsilon\to 0^+} \int_1^2 F(k,q) \delta(q-1-\epsilon)\ dq \right] e^{-ikx}\ dk. \tag{2.13}$$

Eqs. (2.12) and (2.13) solve the problem of inversion of the q-Fourier transform, *which is now of the desired one-to-one character* (see Ref. [25] for fixed q). Of course, from (2.5) and (2.6), on the real axis, we obtain for (2.12) and (2.13)

$$F(k,q) = [H(q-1) - H(q-2)] \times \int_{-\infty}^{\infty} f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\ dx, \tag{2.14}$$

for the real transform, and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\lim_{\epsilon\to 0^+} \int_1^2 F(k,q)\delta(q-1-\epsilon)\ dq \right] e^{-ikx}\ dk, \tag{2.15}$$

for its inverse. The preceding considerations may become friendlier if we look at the example developed in the next section.

3. Important example

As an instructive illustration we evaluate now the q-Fourier transform of the Heaviside function

$$f(x) = H(x). \tag{3.1}$$

In this case,

$$F(k,q) = [H(q-1) - H(q-2)]H[\Im(k)] \int_0^{\infty} [1 + (1-q)ikx]^{\frac{1}{1-q}}\ dx. \tag{3.2}$$

Using the result given in Ref. [26] formula 3.194 3, page 285, we have

$$F(k,q) = [H(q-1) - H(q-2)]H[\Im(k)] \frac{\Gamma\left(\frac{2-q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}\right)} [(1-q)ik]^{-1}, \tag{3.3}$$

and, finally,

$$F(k,q)=[H(q-1)-H(q-2)]\frac{i}{2-q}\frac{H[\Im(k)]}{k}. \tag{3.4}$$

In the same way, if we select:

$$f(x)=H(-x),$$

we obtain the result:

$$F(k,q)=[H(q-1)-H(q-2)]\frac{i}{2-q}\frac{H[-\Im(k)]}{k}. \tag{3.5}$$

Taking into account that $H(x)+H(-x)=1$ we have, for

$$f(x)=1,$$

the expression

$$F(k,q)=[H(q-1)-H(q-2)]\frac{i}{2-q}\frac{1}{k}=[H(q-1)-H(q-2)]\frac{2\pi}{2-q}\delta(k), \tag{3.6}$$

which is the formula obtained by us in Ref. ([23]). If we insert the right hand side of (3.6) into (2.13) we obtain

$$f(x)=\frac{1}{2\pi}\oint_{\Gamma}\left[\lim_{\epsilon\to 0^+}\int_1^2\frac{2\pi}{2-q}\delta(k)\delta(q-1-\epsilon)\,dq\right]e^{-ikx}\,dk=-\frac{i}{2\pi}\oint_{\Gamma}\frac{e^{-ikx}}{k}\,dk=1 \tag{3.7}$$

which is an example of using the inversion formula (2.13).

4. A second example

As another illustrative example let us consider the q-Fourier transform of

$$f(x)=qH(x).$$

In this case, noting that we have to deal with two Tsallis indexes (q and q') we have

$$F(k,q')=[H(q'-1)-H(q'-2)]\times H[\Im(k)]\int_0^{\infty}q\{1+i(1-q')kxq^{(q'-1)}\}^{\frac{1}{1-q'}}\,dx, \tag{4.1}$$

so that

$$F(k,q')=[H(q'-1)-H(q'-2)]\frac{iq^{(2-q')}}{2-q'}\frac{H[\Im(k)]}{k}. \tag{4.2}$$

If we use now the inversion formula, we are led to

$$f(x)=\frac{1}{2\pi}\oint_{\Gamma}\lim_{\epsilon\to 0^+}\left\{\int_1^2\frac{iq^{(2-q')}}{2-q'}\frac{H[\Im(k)]}{k}\delta(q'-1-\epsilon)\,dq'\right\}e^{-ikx}\,dx=\frac{iq}{2\pi}\oint_{\Gamma}\frac{H[\Im(k)]}{k}e^{-ikx}\,dx=qH(x). \tag{4.3}$$

This example clearly illustrates the fact that the q-dependence of $f(x)$ is preserved by the transformation.

5. A third example

As a third case we tackle an interesting example and focus attention upon Hilhorst's work [24] to illustrate the unfortunate fact that for **fixed** q the q-Fourier transform is not one-to-one. Let $f(x)$ be given by:

$$f(x)=\begin{cases}\left(\dfrac{\lambda}{x}\right)^{\beta}; & x\in[a,b];\ 0<a<b;\ \lambda>0\\ 0; & x \text{ outside } [a,b].\end{cases} \tag{5.1}$$

The corresponding complex q-Fourier transform is:

$$F(k,q)=[H(q-1)-H(q-2)]H[\Im(k)]\times\lambda^{\beta}\int_a^b x^{-\beta}\{1+i(1-q)k\lambda^{\beta(q-1)}x^{1-\beta(q-1)}\}^{\frac{1}{1-q}}\,dx. \tag{5.2}$$

By effecting the change of variable

$$y=x^{1-\beta(q-1)}$$

we obtain for (5.2):

$$F(k,q)=[H(q-1)-H(q-2)]H[\Im(k)]\times\frac{\lambda^{\beta}}{1-\beta(q-1)}\int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}}y^{\frac{\beta(q-2)}{1-\beta(q-1)}}\{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy. \tag{5.3}$$

Eq. (5.3) can be written equivalently as:

$$\begin{aligned}F(k,q)&=[H(q-1)-H(q-2)]H[\Im(k)]\times\left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\right.\\&\quad\times\frac{\lambda^{\beta}}{1-\beta(q-1)}\int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}}y^{-\frac{\beta(2-q)}{1-\beta(q-1)}}\{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy\\&\quad+\left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\}\\&\quad\left.\times\frac{\lambda^{\beta}}{\beta(q-1)-1}\int_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}}y^{\frac{\beta(q-2)}{1-\beta(q-1)}}\{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy\right\}.\end{aligned} \tag{5.4}$$

We appeal now to the result of [26], formula 3.194 1, page 284 to evaluate (5.4)

$$\begin{aligned}&\int_{a^{1-\beta(q-1)}}^{\infty}y^{-\frac{\beta(2-q)}{1-\beta(q-1)}}\{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy\\&=\frac{(q-1)[1-\beta(q-1)]a^{\frac{q-2}{q-1}}}{(2-q)[(1-q)ik\lambda^{\beta}]^{\frac{1}{q-1}}}\\&\quad\times F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};-\frac{1}{(1-q)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right),\end{aligned} \tag{5.5}$$

and [26], formula 3.194 2 page 285

$$\begin{aligned}&\int_{0}^{a^{1-\beta(q-1)}}y^{\frac{\beta(2-q)}{\beta(q-1)-1}}\{1+i(1-q)k\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy\\&=\frac{[\beta(q-1)-1]a^{1-\beta}}{\beta-1}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};(q-1)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right),\end{aligned} \tag{5.6}$$

where $F(a,b,c;z)$is the hypergeometric function. Thus we obtain for $F(k,q)$:

$$\begin{aligned}F(k,q)&=[H(q-1)-H(q-2)]H[\Im(k)]\times\left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\times\frac{(q-1)\lambda^{\beta}}{(2-q)[(1-q)ik\lambda^{\beta}]^{\frac{1}{q-1}}}\right.\\&\quad\times\left\{a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\right.\\&\quad\left.\frac{1}{(q-1)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right)-b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\\&\quad\left.\left.\frac{1}{(q-1)ik\lambda^{\beta(q-1)}b^{1-\beta(q-1)}}\right)\right\}+\left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\}\frac{\lambda^{\beta}}{\beta-1}\\&\quad\times\left\{a^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};(q-1)ik\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right)\right.\\&\quad\left.\left.-b^{1-\beta}F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};(q-1)ik\lambda^{\beta(q-1)}b^{1-\beta(q-1)}\right)\right\}\right\}.\end{aligned} \tag{5.7}$$

From (5.7) we appreciate a crucial fact: our $F(k,q)$ does *depend* on a and b (remember that we have shown in Section 2 that $F(k,q)$ is one to one). That the mentioned dependence is of the essence will become evident right now. If we fix q and select $\beta=1/(q-1)$ we immediately reproduce the result obtained by Hilhorst. In fact, in this case we obtain for (5.7):

$$\begin{aligned}F(k,q)&=\lambda^{\frac{1}{q-1}}\frac{q-1}{2-q}H[\Im(k)]\,[H(q-1)-H(q-2)]\times\left[a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{q-1},\frac{2-q}{q-1};(q-1)ik\lambda\right)\right.\\&\quad\left.-\ b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{q-1},\frac{2-q}{q-1};(q-1)ik\lambda\right)\right].\end{aligned} \tag{5.8}$$

According to Ref. [27]:

$$F(-a, b, b, -z) = (1+z)^a.$$

Then, (5.8) simplifies to:

$$F(k,q) = \lambda^{\frac{1}{q-1}} \frac{q-1}{2-q} H[\Im(k)]\,[H(q-1) - H(q-2)]\,.\left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right)[1+(1-q)ik\lambda]^{\frac{1}{1-q}}\,. \tag{5.9}$$

If we use the value of λ given by Hilhorst, then

$$\lambda = \left[\left(\frac{q-1}{2-q}\right)\left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right)\right]^{1-q}.$$

We now obtain

$$F(k,q) = H[\Im(k)]\,[H(q-1) - H(q-2)]\,[1+(1-q)ik\lambda]^{\frac{1}{1-q}}\,. \tag{5.10}$$

We see according to (5.10) that in this case $F(k,q)$ is *independent of a and b*, i.e., the same for all legitimate pairs a–b, and, as consequence, not one-to-one for fixed q. On the real axis (5.10) takes the form

$$F(k,q) = [H(q-1) - H(q-2)]\,[1+(1-q)i(k+i0)\lambda]^{\frac{1}{1-q}} = [H(q-1) - H(q-2)]\,[1+(1-q)ik\lambda]^{\frac{1}{1-q}}\,, \tag{5.11}$$

which is the result obtained in Ref. ([24]).

Conclusions

Using tempered ultradistributions we have introduced a complex q-Fourier transform $F(k,q)$ that exhibits nice properties and is one-to-one.

This solves a serious flaw of the original F_q-definition, i.e., not being of the essential one-to-one nature, as illustrated in detail in Section 3.

In this work we have shown that if we eliminate the requirement that q be fixed, and let it float in its proper interval $[1,2)$, the complex generalization of the F_q-definition restores the one-to-one character.

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Appendix. Tempered ultradistributions and distributions of exponential type

For the benefit of the reader we give a brief summary of the main properties of distributions of exponential type and tempered ultradistributions.

Notations. The notations are almost textually taken from Ref. [29]. Let $\mathbb{R}^n$ (res. $\mathbb{C}^n$) be the real (resp. complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, \ldots, x_n)$ (resp $z = (z_1, z_2, \ldots, z_n)$). We shall use the notations:

(a) $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

(b) $x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, \ldots, x_n \geqq 0$

(c) $x \cdot y = \sum_{j=1}^{n} x_j y_j$

(d) $|\, x \,| = \sum_{j=1}^{n} |\, x_j \,|$.

Let $\mathbb{N}^n$ be the set of n-tuples of natural numbers. If $p \in \mathbb{N}^n$, then $p = (p_1, p_2, \ldots, p_n)$, and p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ stands for $(p_1 + q_1, p_2 + q_2, \ldots, p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, \ldots, p_n \geqq q_n$. x^p entails $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$. We shall denote $|\, p \,| = \sum_{j=1}^{n} p_j$ and call D^p the differential operator $\partial^{p_1+p_2+\cdots+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} \ldots \partial {x_n}^{p_n}$.

For any natural k we define $x^k = x_1^k x_2^k \ldots x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k \ldots \partial x_n^k$.

The space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q \phi(x)|$ is bounded for any p and q, being defined [see Ref. ([29])] by means of the countably set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \le q \le p,\, x} e^{p|x|} \left| D^q \hat{\phi}(x) \right|, \quad p = 0, 1, 2, \ldots. \tag{A.1}$$

The space of continuous linear functionals defined on $\mathcal{H}$ is the space Λ_∞ of the distributions of exponential type given by (Ref. [29]).

$$T = \frac{\partial^k}{\partial x^k}\left[e^{k|x|} f(x)\right], \tag{A.2}$$

where k is an integer such that $k \geqq 0$ and $f(x)$ is a bounded continuous function. In addition we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (Ref. [30]).

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\, e^{iz\cdot x}\, dx. \tag{A.3}$$

According to Ref. [29], $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathfrak{H}$ the set of all such functions.

$$\mathfrak{H} = \mathcal{F}\{\mathcal{H}\}. \tag{A.4}$$

The topology in $\mathfrak{H}$ is defined by the countable set of semi-norms:

$$\|\phi\|_k = \sup_{z\in V_k} |z|^k |\phi(z)|, \tag{A.5}$$

where $V_k = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n :| Imz_j |\leqq k,\ 1 \leqq j \leqq n\}$.

The dual of $\mathfrak{H}$ is the space $\mathcal{U}$ of tempered ultradistributions [see Ref. ([29])]. In other words, a tempered ultradistribution is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis. Moreover, we have $\mathfrak{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{U}$.

$\mathcal{U}$ can also be characterized in the following way [see Ref. ([29])]: let $\mathcal{A}_\omega$ be the space of all functions $F(z)$ such that:

(**A**)- $F(z)$ is analytic for $\{z \in \mathbb{C}^n : |\mathrm{Im}(z_1)| > p, |\mathrm{Im}(z_2)| > p, \ldots, |\mathrm{Im}(z_n)| > p\}$.

(**B**)- $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |\mathrm{Im}(z_1)| \geqq p, |\mathrm{Im}(z_2)| \geqq p, \ldots, |\mathrm{Im}(z_n)| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $F(z)$.

Let Π be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\mathcal{U}$ is the quotient space

(**C**)- $\mathcal{U} = \mathcal{A}_\omega/\Pi$.

By a pseudo-polynomial we understand a function of z of the form $\sum_s z_j^s G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$ with $G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathcal{A}_\omega$.

Due to these properties it is possible to represent any ultradistribution as [see Ref. ([29])]

$$F(\phi) = \langle F(z), \phi(z)\rangle = \oint_\Gamma F(z)\phi(z)\, dz \tag{A.6}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \Gamma_n$, where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\mathrm{Im}(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\mathrm{Im}(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Eq. (A.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of the "Dirac formula" for ultradistributions [see Ref. ([28])]

$$F(z) = \frac{1}{(2\pi i)^n}\int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)\ldots(t_n - z_n)}\, dt \tag{A.7}$$

where the "density" $f(t)$ is such that

$$\oint_\Gamma F(z)\phi(z)\, dz = \int_{-\infty}^{\infty} f(t)\phi(t)\, dt. \tag{A.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (A.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z [29]

$$|F(z)| \leq C|z|^p, \tag{A.9}$$

where C and p depend on F.

The representation (A.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ does not alter the ultradistribution:

$$\oint_\Gamma \{F(z) + P(z)\}\phi(z)\, dz = \oint_\Gamma F(z)\phi(z)\, dz + \oint_\Gamma P(z)\phi(z)\, dz.$$

However,

$$\oint_\Gamma P(z)\phi(z)\, dz = 0.$$

As $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing), we obtain:

$$\oint_\Gamma \{F(z) + P(z)\}\phi(z)\, dz = \oint_\Gamma F(z)\phi(z)\, dz. \tag{A.10}$$

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