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# Damping of vortex waves in a superfluid

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**Abstract**

The damping of vortex cyclotron modes is investigated within a generalized quantum theory of vortex waves. Similarly to the case of Kelvin modes, the friction coefficient turns out to be essentially unchanged under such oscillations, but it is shown to be affected by appreciable memory corrections. On the other hand, the non-equilibrium dynamics of the vortex energy, which is investigated within the framework of linear response theory, shows that its memory corrections are negligible. The vortex response is found to be of the Debye type, with a relaxation frequency whose dependence on temperature and impurity concentration reflects the complexity of the heat bath and its interaction with the vortex.

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## 1. Introduction

The simplest vortex dynamics in a superfluid corresponds to the two-dimensional motion of a rectilinear vortex filament [1]. In fact, in an infinite superfluid a vortex 'charged' with one quantum of anticlockwise circulation will move like an electron in a uniform magnetic field, i.e. performing a circular cyclotron motion ruled by

$$m_v \ddot{\mathbf{r}} = \rho_s h \hat{z} \times \dot{\mathbf{r}}. \tag{1}$$

Here $m_v$ denotes the vortex effective mass per unit length, $\rho_s$ is the number density of the background superfluid at rest, $h$ is Planck's constant and $\mathbf{r} = (x, y)$ is the two-dimensional coordinate of the vortex core. We note that the Magnus force on the right-hand side of (1) is formally equivalent to the Lorentz force on a negative point charge in a uniform magnetic field parallel to the $z$ axis. Actually, this electromagnetic analogy is only a part of a whole mapping by which a 2D homogeneous superfluid can be mapped onto a (2+1)D electrodynamic system, with vortices and phonons playing the role of charges and photons, respectively [2]. For instance, any accelerated motion of a vortex would result in the radiation of sound waves

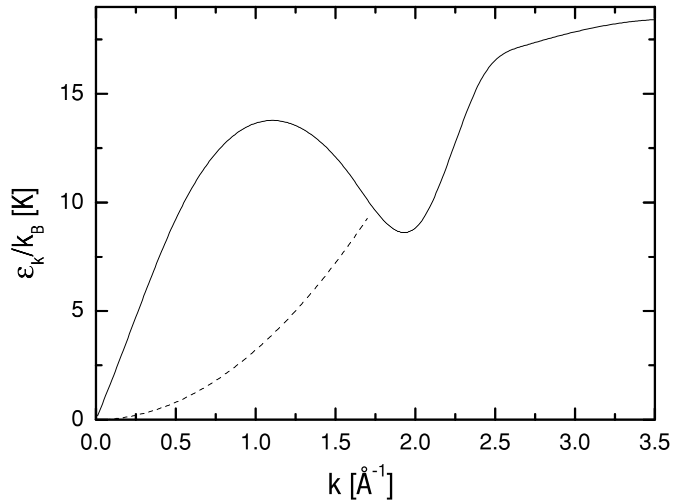


**Figure 1.** Dispersion curve for elementary excitations in superfluid $^4$He (full curve) and energy spectrum of solvated $^3$He atoms in ordinary helium (broken curve).

in the superfluid, i.e. the emission of phonons, a process which is entirely analogous to the photon radiation mechanism stemming from an accelerated charge in electrodynamics. In practice, however, this simple picture would only apply to a superfluid formed by $^4$He atoms, namely the boson isotope of helium, at low temperatures ($T < 0.4$ K). Ordinary helium, on the other hand, contains a small amount of impurity, fermion $^3$He atoms, which however produces a viscous drag force on a moving vortex at the lowest temperatures. In fact, below 0.4 K the scattering of thermal phonons by the vortex has negligible effects compared to the drag force due to $^3$He scattering [3]. In addition, at higher temperatures ($T > 0.5$ K), most of the elementary excitations of the superfluid $^4$He that collide with the vortex are *rotons*, i.e. quasi-particles having their momentum around the minimum of the dispersion curve. In fact (see figure 1), only the elementary excitations with momentum below 0.5 Å$^{-1}$ can be regarded as phonons arising from a linear dispersion relation $\omega = c_s k$, whereas the rest of the dispersion curve does not yield thermal elementary excitations, except for a small interval around the minimum ($\sim$1.9 Å$^{-1}$). Quasi-particles with momentum at the right (left) of this minimum have their group velocity parallel (antiparallel) to their momentum and are called $R^+$ ($R^-$) rotons. Above 0.5 K, the source of the drag force on a vortex in ordinary helium is roton scattering, the effect of $^3$He collisions being practically negligible. Whatever the source, however, such a drag force can be written as two additional terms on the right-hand side of equation (1), namely

$$m_v \ddot{\mathbf{r}} = (\rho_s h - D')\hat{z} \times \dot{\mathbf{r}} - D\dot{\mathbf{r}}, \tag{2}$$

where $D'$ and $D$ denote *transversal* and *longitudinal* friction coefficients, respectively [1, 4]. It is interesting to compare in figure 1 the dispersion curve for elementary excitations in superfluid $^4$He, with the energy spectrum of $^3$He atoms in ordinary helium. Such solvated atoms behave like heavier free particles ($\varepsilon_k = \hbar^2 k^2/2m^*$) with an effective mass $m^*$ which exceeds twice the mass of a bare $^3$He atom. As a final remark about figure 1 we note that the terminations of both curves are due to unstabilities caused by roton creation processes. That is, any elementary excitation exceeding twice the roton energy should be unstable against decay into two rotons, whereas $^3$He atoms exceeding the roton energy should decay into a low energy atom plus a roton.

Here it is also important to take into account another consequence of the above scattering processes, apart from the friction itself, that is the thermal excitation of vortex waves [1, 5]. These are helical waves in which each vortex line element executes a circular motion about the undisturbed line ($z$ axis). The radius of such a circle is assumed to be much smaller than the wavelength, so the above elements will remain almost parallel to the $z$ axis, fulfilling an equation of motion like (1). In fact, such an equation has to be generalized to the situation where there exists an external superfluid flow of velocity $\mathbf{v}_s$:

$$m_v \ddot{\mathbf{r}} = \rho_s h \hat{z} \times (\dot{\mathbf{r}} - \mathbf{v}_s). \tag{3}$$

Such an 'external' superflow corresponds in our case to the local self-induced velocity generated by the vortex line curvature [1, 6], $\mathbf{v}_s = -v_i \hat{\theta}$, which points in a direction opposite to that of the superfluid velocity field generated by the undisturbed vortex line. This self-induced velocity, being proportional to the line displacement $|\mathbf{r}|$ from the $z$ axis, can be written as $v_i = \omega_- |\mathbf{r}|$, where

$$\omega_-(k) \simeq -\frac{\hbar k^2}{2m_4} [\ln(|k|a) + 0.116] \tag{4}$$

corresponds to the well-known dispersion relation for Kelvin waves of long wavelength $\lambda$, $|k|a \ll 1$, with $|k| = 2\pi/\lambda$, $a \sim 1$ Å = vortex core parameter and $m_4$ = mass of a $^4$He atom. Note that the wave vector $k$, which points along the $z$ axis, can take positive or negative values depending on the two ways of generating the vortex helix. Then, equation (3) can be rewritten as

$$\ddot{\mathbf{r}} = \Omega \hat{z} \times \dot{\mathbf{r}} - \Omega \omega_- \mathbf{r}, \tag{5}$$

where

$$\Omega = \rho_s h / m_v \tag{6}$$

corresponds to the cyclotron frequency arising from the equation of motion (1). Actually, it is easy to check that equation (5) allows both possible directions for circular motion, since the replacement $|\mathbf{r}|$ = const, $\dot{\mathbf{r}} = \omega |\mathbf{r}| \hat{\theta}$ in (5) leads to a quadratic equation in the angular frequency $\omega$ with solutions:

$$\omega^{(\pm)} = \frac{\Omega}{2} \left[ 1 \pm \sqrt{1 + \frac{4\omega_-}{\Omega}} \right]. \tag{7}$$

That is, in the limit $\omega_-/\Omega \ll 1$ we have either the anticlockwise cyclotron motion of frequency $\omega^{(+)} \simeq \Omega$, or the usual clockwise polarization of Kelvin waves $\omega^{(-)} \simeq -\omega_-$ (cf (4)), for vortices of anticlockwise circulation and negligible mass ($\Omega \to \infty$). Actually, there are no experimental data about vortex trajectories, so the value of the vortex mass $m_v$ and hence of $\Omega$ can only be extracted from theoretical considerations. If $m_v$ is calculated from a classical hydrodynamical model, $\Omega$ should be about 3 ps$^{-1}$ [1], whereas more recent theories, for which $m_v$ should be logarithmically divergent with the system size, lead to $\Omega$ of order 0.1–0.01 ps$^{-1}$, for typical experimental conditions [7, 8]. All these figures are consistent with the approximation $\Omega \gg \omega_-$ for long wavelengths $|k|a \ll 1$, where the dependence on $k$ of the frequency $\omega^{(+)}$ can be neglected [1].

Finally, the vortex equation of motion is obtained by adding both effects, friction (2) and oscillations (5), together:

$$\ddot{\mathbf{r}} = \Omega \left[ \left( 1 - \frac{D'}{\rho_s h} \right) \hat{z} \times \dot{\mathbf{r}} - \frac{D}{\rho_s h} \dot{\mathbf{r}} - \omega_- \mathbf{r} \right]. \tag{8}$$

Note that in addition to the time dependence of $\mathbf{r}$, one should take into account a parametric dependence $\mathbf{r}(z)$ following the helix curvature.

Up to this point we have regarded the vortex coordinates as classical time-dependent variables, but it is important to observe in this respect that the above theoretical estimates of the cyclotron frequency yield values of $\hbar\Omega/k_B T$ greater than $\sim$0.1 for $T < 1$ K. This seems to indicate that a classical treatment cannot be wholly satisfactory. Moreover, even in the case of purely low-frequency Kelvin waves, the need for a quantum mechanical analysis was pointed out early by Fetter [9]. Such a theory was in fact used to study phonon scattering by a vortex [10], and it is our purpose to present in this paper a more general treatment in which the vortex mass, and hence the cyclotron frequency, are included in the theory and assigned finite values. Our starting point will be a vortex Hamiltonian from which the equation of motion (5) derives. Then, after quantization of the vortex variables, we will show that such a Hamiltonian consists of independent harmonic oscillator modes of frequencies $\Omega$ and $\omega_-$, which interact with the heat bath represented by the ordinary helium at a finite temperature. Such an interaction is modelled through a generic momentum-conserving scattering Hamiltonian, which is used to study the dissipative dynamics of the cyclotron modes. Thus, we shall show that the friction turns out to be essentially unaffected by such oscillations, allowing us to extend our previous conclusions on the memory effects on straight vortex lines [11]. Similar behaviour was reported long ago by Fetter [10] and Sonin [12] for the low-frequency Kelvin modes, showing that the dissipation of such modes remains essentially equal to that of strictly rectilinear vortices.

Our main objective in the present paper will be to analyse, within the framework of the linear response theory [13], the non-equilibrium dynamics and the equilibrium quantum fluctuations of the energy of the cyclotron modes. There exists an extensive literature on quantal Brownian motion of harmonic oscillators [14], but it is important to realize that our problem presents a number of distinctive features that are not found in previous treatments, namely

(i) Most of such previous studies have focused on the coordinates or the momentum of the oscillator, rather than on the energy, e.g. treatments of the time correlation function of the energy are rather uncommon.

(ii) It is evident that we are dealing with a very special heat bath, since in addition to being formed by Fermi particles and Bose quasi-particles, such bosons are characterized by a complex dispersion relationship which gives rise to different species (phonons, rotons $R^+$ and $R^-$).

(iii) The drag force on vortices that has been experimentally detected arises from scattering, thus we leave aside from our study the phonon radiation damping [2, 11]. We note that such a scattering interaction Hamiltonian is also unusual since it must be nonlinear in the heat bath operators. In fact, most of the Brownian motion models assume that the heat bath couples *linearly* to the harmonic oscillator, but in our case it is easy to realize that the scattering events must involve products of creation and annihilation operators of the particles that collide with the vortex.

(iv) Our recent study [11] has shown that the drag force could be affected by appreciable memory effects, so it will be important to extend our treatment beyond the usual Markovian approximation to explore such possible effects.

A suitable formalism to handle the above items can be found in [15], where a non-Markovian calculation of the energetic susceptibility of a harmonic oscillator, weakly coupled to boson and fermion environments, was carried out. So, we shall base our treatment on the above formalism.

This paper is organized as follows: in the following section we describe our quantum model for the dissipative vortex dynamics which leads to an analysis of vortex oscillations and memory effects. In section 3, we summarize the main results of the linear response theory applied to the vortex energy. In section 4, based on the previous calculation of the harmonic oscillator susceptibility, we analyse memory corrections to the Markov approximation. In section 5 we calculate the response and time correlation functions, and study the dependence of the relaxation frequency on temperature and impurity concentration. Finally, in section 6 we gather the summary and main conclusions of our study.

## 2. Quantum model for vortex dynamics

We start from a vortex Hamiltonian given by

$$H_v(z) = \frac{m_v}{2}(\mathbf{v}^2 + \Omega\omega_-\mathbf{r}^2), \tag{9}$$

where

$$\mathbf{v} = \mathbf{p}/m_v + \frac{\Omega}{2}\hat{z}\times\mathbf{r} \tag{10}$$

corresponds to the vortex velocity $\dot{\mathbf{r}}$ and $\mathbf{p}$ denotes the vortex canonical momentum. The second term in (10) corresponds to that of the vector potential (central gauge) in the electromagnetic analogy, and the $z$ dependence in (9) which arises from $\mathbf{r}(z)$ and $\mathbf{p}(z)$ corresponds to the rotation in the $x$–$y$ plane parametrized by $z$, which results from following the helix path. Note that both the canonical momentum $\mathbf{p}$ and the Hamiltonian (9) are given per unit length of the $z$ axis. Then, it is easy to verify that the Hamilton equations lead from (9) to the equation of motion (5).

The two-dimensional coordinate $\mathbf{r}$ of the vortex core can be written as the sum of the centre coordinate $\mathbf{r}_0$ of the cyclotron circle plus the relative coordinate $\mathbf{r}'$ from such a centre. Then, the quantization of such variables straightforwardly arises from the electromagnetic analogy:

$$\mathbf{r}_0 = \frac{1}{\sqrt{4\pi\rho_s L}}[(\beta^\dagger + \beta)\hat{x} + \mathrm{i}(\beta^\dagger - \beta)\hat{y}] \tag{11}$$

$$\mathbf{r}' = \frac{1}{\sqrt{4\pi\rho_s L}}[(a^\dagger + a)\hat{x} + \mathrm{i}(a - a^\dagger)\hat{y}], \tag{12}$$

where $a^\dagger$ ($\beta^\dagger$) denotes a creation operator of right (left) circular quanta [16] and $L$ denotes the vortex line length. The $z$ dependence of $\mathbf{r}$ arises from the replacements $\beta^\dagger \to \exp(-\mathrm{i}kz)\beta_k^\dagger, a^\dagger \to \exp(\mathrm{i}kz)a_k^\dagger$ in (11) and (12), and correspondingly for the annihilation operators. Actually, Fetter's theory identifies $\mathbf{r}_0(z)$ as the whole displacement from the $z$ axis (see [10], equation (14)). Analogous quantization for the canonical momentum $\mathbf{p}$ leads through (10) and (9) to a vortex Hamiltonian

$$\int_0^L \mathrm{d}z H_v(z) = \hbar(\Omega + \omega_-)\left(a_k^\dagger a_k + \frac{1}{2}\right) + \hbar\omega_-\left(\beta_k^\dagger\beta_k + \frac{1}{2}\right) + \hbar\omega_-\left(a_k^\dagger\beta_k^\dagger + a_k\beta_k\right), \tag{13}$$

where it is worthwhile to note that $\mathbf{v}^2$ and $\mathbf{r}^2$ in (9) turn out to be independent of $z$, as expected. The above Hamiltonian can be written to first order in $\omega_-/\Omega$ as

$$\hbar\Omega\left(a_k^\dagger a_k + \tfrac{1}{2}\right) + \hbar\omega_-\left(\beta_k^\dagger\beta_k + \tfrac{1}{2}\right), \tag{14}$$

where both polarizations (cyclotron and Kelvin modes) become decoupled. To prove this approximation, we first note that the set of eigenfunctions of (14) are represented by wavefunctions corresponding to well-defined values of both numbers of circular quanta, right and left [16]. On the other hand, the Schrödinger equation for the Hamiltonian (13) can be easily solved by noting that the term proportional to $\mathbf{r}^2$ in expression (9) can be added to the corresponding term arising from $\mathbf{v}^2$, yielding a Schrödinger equation formally equivalent to that with $\omega_- = 0$, whose solution is well known. Thus, we find that to the first order in $\omega_-/\Omega$ we obtain the same spectrum of eigenvalues and eigenfunctions as from (14), except for a slight correction in the radial coordinate of the wavefunctions, which has to be multiplied by the factor $1 + \omega_-/\Omega$.

The Hamiltonian (14) corresponds to helical oscillations of fixed wavelength $\lambda = 2\pi/|k|$. In the final step of our quantization procedure we shall assume that the system obeys periodic boundary conditions over a length $L$ along the $z$ axis, so $k$ will be restricted to values $2\pi s/L$, where $s$ is an integer. Thus, the complete vortex Hamiltonian is obtained by summing up expression (14) over all these values of $k$:

$$H_v = \sum_k \hbar\Omega(k)\left(a_k^\dagger a_k + \frac{1}{2}\right) + \hbar\omega_-(k)\left(\beta_k^\dagger \beta_k + \frac{1}{2}\right). \tag{15}$$

The above Hamiltonian differs from that of Fetter's theory by the presence of the cyclotron modes of frequency $\Omega(k) \simeq \Omega$ (cf [10], equation (11)).

The heat bath Hamiltonian is given by

$$H_B = \sum_{\mathbf{k}} \hbar\omega_k b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{q},\sigma} \epsilon_q c_{\mathbf{q},\sigma}^\dagger c_{\mathbf{q},\sigma}, \tag{16}$$

where $b_{\mathbf{k}}^\dagger$ denotes a creation operator of $^4$He quasi-particle excitations of momentum $\hbar\mathbf{k}$ and frequency $\omega_k$, and $c_{\mathbf{q},\sigma}^\dagger$ denotes a creation operator of solvated $^3$He atoms of momentum $\hbar\mathbf{q}$, energy $\epsilon_q$ and spin 1/2 projection $\sigma$. Note that we disregard any interaction between the heat bath particles themselves, since we shall work at low enough temperature and impurity concentration, so that such particles remain dilute allowing their treatment as a non-interacting gas.

To model the scattering interaction Hamiltonian, we will consider a generic momentum-conserving form:

$$\int_0^L \mathrm{d}z \sum_{\mathbf{k},\mathbf{q},\sigma} \left[\Lambda_{\mathbf{kq}}^{(k)} b_{\mathbf{k}}^\dagger b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k)} c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}\right] \mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}, \tag{17}$$

where $\Lambda_{\mathbf{kq}}^{(k)}$ and $\Gamma_{\mathbf{kq}}^{(k)}$ denote scattering amplitudes depending on the momentum of the heat bath scatterers and the wave vector $k\hat{z}$ of the vortex wave. Recalling that the vortex coordinate can be written as $\mathbf{r} = \mathbf{r}_0(z) + \mathbf{r}'(z) + z\hat{z}$ and taking into account that $\mathbf{r}_0(z)$ and $\mathbf{r}'(z)$ commute, the exponential factor in (17) can be factorized as $\mathrm{e}^{-\mathrm{i}(k_z-q_z)z}\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}'(z)}\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}_0(z)}$. Since the amplitude of the vortex wave was assumed to be very small, it is tempting to expand the last two exponentials retaining only first-order terms in $\mathbf{r}'(z)$ and $\mathbf{r}_0(z)$. This procedure was analysed by Fetter [10] for $\mathbf{r}_0(z)$, finding that it leads to divergences at long wavelengths. The physical reason for this result can be understood by recalling that $\mathbf{r}_0(z)$ is linear in the creation and destruction operators, $\beta_k^\dagger$ and $\beta_k$. Consequently, an expansion in powers of $\mathbf{r}_0(z)$ is bound to fail whenever the energy per quantum $\hbar\omega_-$ becomes very small at long wavelengths, as the transitions should involve many of these 'soft' quanta [10]. Note that this argument does not apply to $\mathbf{r}'(z)$ since the operators $a_k^\dagger$ and $a_k$ correspond to the high-frequency cyclotron quanta. This means that the treatment of Kelvin modes represented by $\mathbf{r}_0(z)$ turns out to be considerably more complicated than that of the cyclotron modes represented by $\mathbf{r}'(z)$. In fact, only the phonon drag force arising from $\mathbf{r}_0(z)$ could be analysed by Fetter [10], but we shall see that all sources of friction acting on $\mathbf{r}'(z)$ can be studied. To this aim, let us set $\mathbf{r}_0(z) = 0$ in (17) while retaining only the first-order term in $\mathbf{r}'(z)$. Then, using the second-quantized expression for $\mathbf{r}'(z)$ (cf (12)), performing the integral in $z$ and recalling the above-mentioned periodic boundary conditions, the interaction (17) reads

$$L \sum_{\mathbf{k},\mathbf{q},\sigma} \left[\Lambda_{\mathbf{kq}}^{(k)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k)} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{q},\sigma}\right] \left\{ \delta_{k_z,q_z} + \frac{1}{\sqrt{4\pi\rho_s L}} \left[ \mathrm{i}(q_x - k_x)\left(\delta_{k,k_z-q_z} a_k^{\dagger} + \delta_{k,q_z-k_z} a_k\right) + (q_y - k_y)\left(\delta_{k,k_z-q_z} a_k^{\dagger} - \delta_{k,q_z-k_z} a_k\right)\right]\right\}, \tag{18}$$

where the Kronecker-delta factors represent $z$-momentum conservation and the scattering amplitudes were assumed to be independent of the sign of $k$, i.e. the interaction should be the same for both possible directions of a helical deformation. Note that the first term between braces in (18) does not contribute to the interaction, so it should be added to the heat bath Hamiltonian (16). Finally, summing up expression (18) over $k$ we obtain the interaction Hamiltonian:

$$H_{\text{int}} = \sqrt{\frac{L}{4\pi\rho_s}} \sum_{\mathbf{k},\mathbf{q},\sigma} \left[\Lambda_{\mathbf{kq}}^{(k_z-q_z)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k_z-q_z)} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{q},\sigma}\right] \times \left[\mathrm{i}(q_x - k_x)\left(a_{k_z-q_z}^{\dagger} + a_{q_z-k_z}\right) + (q_y - k_y)\left(a_{k_z-q_z}^{\dagger} - a_{q_z-k_z}\right)\right]. \tag{19}$$

From (15) and (19), we may realize that each cyclotron mode of the unperturbed Hamiltonian

$$H_k = \hbar\Omega(k)\left(a_k^{\dagger} a_k + \tfrac{1}{2}\right), \tag{20}$$

will evolve independently, interacting with the heat bath through the following terms of (19):

$$H_{\text{int}}^{(k)} = \sqrt{\frac{L}{4\pi\rho_s}} \left\{ \sum_{\mathbf{k},\mathbf{q},\sigma}^{(+)} \left[\Lambda_{\mathbf{kq}}^{(k)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k)} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{q},\sigma}\right]\left[(q_y - k_y) + \mathrm{i}(q_x - k_x)\right] a_k^{\dagger} + \sum_{\mathbf{k},\mathbf{q},\sigma}^{(-)} \left[\Lambda_{\mathbf{kq}}^{(k)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k)} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{q},\sigma}\right]\left[(k_y - q_y) + \mathrm{i}(q_x - k_x)\right] a_k \right\}, \tag{21}$$

where the $(\pm)$ sign above each summation symbol indicates that only the terms with $k_z - q_z = \pm k$ must be considered. We recall, however, that for long wavelengths we have $k \ll a^{-1} \sim 1$ Å$^{-1}$, so it will be valid to neglect $k$ in such $z$-momentum conservation relationships, except at extremely low temperatures ($\hbar c_s k_z \sim k_B T$). Thus, $H_{\text{int}}^{(k)}$ becomes

$$H_{\text{int}}^{(k)} = \sqrt{\frac{L}{4\pi\rho_s}} \sum_{\mathbf{k},\mathbf{q},\sigma} \delta_{k_z,q_z} \left[\Lambda_{\mathbf{kq}}^{(k)} b_{\mathbf{k}}^{\dagger} b_{\mathbf{q}} + \Gamma_{\mathbf{kq}}^{(k)} c_{\mathbf{k},\sigma}^{\dagger} c_{\mathbf{q},\sigma}\right] \left\{ \left[(q_y - k_y) + \mathrm{i}(q_x - k_x)\right] a_k^{\dagger} + \left[(k_y - q_y) + \mathrm{i}(q_x - k_x)\right] a_k \right\} \tag{22}$$

and the time evolution of $a_k^{\dagger}$ will be ruled by the Hamiltonian given by the sum of (20), (16) and (22). To study the time evolution of the vortex coordinate $\mathbf{r}' = (x', y')$, it will be convenient to use a complex form $R' = x' + \mathrm{i}y'$ since

$$R'(z) = \frac{1}{\sqrt{\pi\rho_s L}} \sum_k \mathrm{e}^{\mathrm{i}kz} a_k^{\dagger} \tag{23}$$

is simply written as a linear combination of $a_k^{\dagger}(t)$. In [11] we studied the cyclotron dynamics of a rigid rectilinear vortex; this being equivalent to considering a single term with $k \to 0$

in (23). We derived, within a weak-coupling approximation, a non-Markovian equation of motion for the mean value of the vortex position operator, finding that cyclotron frequency values within the range 0.01−0.03 ps$^{-1}$ lead to a very good agreement with the experimental determinations of the longitudinal friction coefficient $D$ (equation (2)), versus temperature and $^3$He concentration. We showed that memory effects could represent up to ∼10% of the $D$ value as the number of heat bath scatterers is increased, that is, such effects are found to be increasing with temperature and impurity concentration. The scattering amplitudes leading to such results read [11, 17]

$$\Lambda^{(0)}_{\mathbf{kq}} = \frac{2\pi\hbar^2}{m_4 V c_s}\sqrt{\frac{19}{140}|\omega'_k||\omega'_q|} \tag{24}$$

$$\Gamma^{(0)}_{\mathbf{kq}} = \frac{3\hbar^2}{m^* V}\sqrt{\frac{\pi}{32}\sigma_0 (kq)^{\frac{1}{4}}}, \tag{25}$$

where $V$ denotes the volume of the system, $\omega'_k$ denotes the quasi-particle group velocity and $\sigma_0 = 18.54$ Å corresponds to an effective cross section for vortex–$^3$He scattering. Note that these amplitudes are in fact negligible with respect to the heat bath single-particle levels, $L\Lambda^{(0)}_{\mathbf{kk}} \ll \hbar\omega_k$ and $L\Gamma^{(0)}_{\mathbf{kk}} \ll \epsilon_k$, for experimental sizes [18] and not extremely low temperatures.

The thermal excitation of vortex waves can be made consistent with the above experimental data, if we assume that the scattering amplitudes $\Lambda^{(k)}_{\mathbf{kq}}$ and $\Gamma^{(k)}_{\mathbf{kq}}$ in (22) are well approximated by the $k = 0$ values, (24) and (25), respectively. Note that this approximation is similar to the previous one, $k_z - q_z = \pm k \to 0$ (below equation (21)) and also to $\Omega(k) \simeq \Omega$ in (20). Thus, each $a^\dagger_k(t)$ in (23) will present the same dissipative evolution as $a^\dagger_0(t)$, i.e., the same friction coefficient $D$ should be ascribed to all long-wavelength cyclotron modes. This generalizes the previous result [10, 12] that the phonon friction coefficient associated with low-frequency Kelvin modes turns out to be essentially the same as that of strictly rectilinear vortices.

## 3. The vortex energy in the linear response theory

According to the standard framework of the linear response theory [13], we will assume that, having the vortex reached thermal equilibrium with the heat bath before $t = 0$, a weak perturbing time-dependent scalar field $\lambda(t)$ is coupled to the vortex Hamiltonian from $t = 0$ onwards. Then, the Hamiltonian of the whole system can be written as

$$H(t) = H_v + H_B + H_{\text{int}} - \lambda(t) H_v, \tag{26}$$

where $H_B$ is given by (16), the vortex Hamiltonian is given by

$$H_v = \hbar\Omega \sum_k \left( a^\dagger_k a_k + \frac{1}{2} \right) \tag{27}$$

and the interaction Hamiltonian is given by

$$H_{\text{int}} = \sqrt{\frac{L}{4\pi\rho_s}} \sum_{k,\mathbf{k},\mathbf{q},\sigma} \delta_{k_z,q_z} \left[ \Lambda^{(0)}_{\mathbf{kq}} b^\dagger_{\mathbf{k}} b_{\mathbf{q}} + \Gamma^{(0)}_{\mathbf{kq}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{q},\sigma} \right]$$
$$\times \left\{ [(q_y - k_y) + \mathrm{i}(q_x - k_x)] a^\dagger_k + [(k_y - q_y) + \mathrm{i}(q_x - k_x)] a_k \right\}. \tag{28}$$

Then, the mean value of the vortex energy can be written to first order in $\lambda(t)$ as

$$\langle H_v(t) \rangle = \langle H_v \rangle_{\text{eq}} + \int_0^t \mathrm{d}\tau\, \lambda(t-\tau) \alpha(\tau), \tag{29}$$

where $\langle H_v \rangle_{\rm eq} = N\hbar\Omega\left\{[\exp(\hbar\Omega/k_B T) - 1]^{-1} + \frac{1}{2}\right\}$ corresponds to the canonical equilibrium value ($N$ = total number of long-wavelength cyclotron modes), and the function $\alpha(\tau)$ embodies the vortex response to the applied field. In particular, for a Dirac delta impulse $\lambda(t) = \tau_0\delta(t - t_0)$, the above equation yields

$$\frac{\langle H_v(t)\rangle - \langle H_v\rangle_{\rm eq}}{\tau_0} = \alpha(t - t_0) \tag{30}$$

that is, $\alpha(\tau)$ represents the energy displacement from the equilibrium value, per unit strength of a pulse acting at $\tau = 0$. Now, if we assume a constant field $\lambda(t) = \lambda_0$, the so-called *response function* [13] is given by

$$\Psi(t) \equiv \lim_{\lambda_0\to 0}[\langle H_v(t)\rangle - \langle H_v\rangle_{\rm eq}]/\lambda_0 = \int_0^t {\rm d}\tau\, \alpha(\tau). \tag{31}$$

Finally, if the field is oscillatory $\lambda(t) = \lambda_0 \cos(\omega t)$ ($t \geqslant 0$), the *non-transient* regime [13] can be described by setting $t = \infty$ in the upper limit of the integral in (29),

$$\begin{aligned}\langle H_v(t)\rangle_{\rm NT} - \langle H_v\rangle_{\rm eq} &= \lambda_0 \int_0^\infty {\rm d}\tau \cos[\omega(t-\tau)]\alpha(\tau)\\ &= \lambda_0\, {\rm Re}[\tilde\alpha(\omega)\exp(-{\rm i}\omega t)],\end{aligned} \tag{32}$$

where $\tilde\alpha(\omega)$ may be defined as a complex *generalized susceptibility*, which is given by the Fourier–Laplace transform of the pulse response $\alpha(\tau)$,

$$\tilde\alpha(\omega) = \int_0^\infty \exp({\rm i}\omega\tau)\alpha(\tau)\,{\rm d}\tau. \tag{33}$$

Then, according to (33) and (31), the static susceptibility $\tilde\alpha(0)$ is given by

$$\tilde\alpha(\omega\to 0) = \Psi(t\to\infty) = \frac{N(\hbar\Omega)^2}{4k_B T}\left[\sinh\left(\frac{\hbar\Omega}{2k_B T}\right)\right]^{-2}, \tag{34}$$

where the right-hand side arises from taking into account that $\langle H_v(t \to \infty)\rangle$ in (31) corresponds to the canonical distribution of a vortex with a Hamiltonian$(1 - \lambda_0)H_v$ (or equivalently, a vortex with the Hamiltonian $H_v$ at the effective temperature $T/(1 - \lambda_0)$).

As a final remark we note that from the susceptibility $\tilde\alpha(\omega)$, one can readily get the equilibrium time correlation function

$$C(t) = \tfrac{1}{2}\langle H_v(t)H_v(0) + H_v(0)H_v(t)\rangle_{\rm eq} - \langle H_v\rangle_{\rm eq}^2 \tag{35}$$

via the fluctuation–dissipation theorem [19]:

$$\bar C(\omega) = \hbar\coth\left(\frac{\hbar\Omega}{2k_B T}\right){\rm Im}[\tilde\alpha(\omega)], \tag{36}$$

where $\bar C(\omega) = \int_{-\infty}^{\infty}{\rm d}t\,\exp({\rm i}\omega t)C(t)$ denotes the Fourier transform of $C(t)$.

## 4. Analytic continuation of the generalized susceptibility: study of memory effects

Being a Laplace transform, the generalized susceptibility (33) can be regarded as a function of a complex variable $z$, $\tilde\alpha(z)$, which must be analytic in the upper half-plane, ${\rm Im}\, z > 0$. The important information, however, lies in the lower half-plane, where the spectrum of singularities of its analytic continuation yields the set of characteristic frequencies in the time evolution of the pulse response $\alpha(\tau)$ [15]. Our calculation of $\tilde\alpha(z)$ is completely analogous to that leading to the harmonic oscillator susceptibility in [15]. Thus, we refer the reader to that paper for the technical details, and only quote here the final result (cf [15], equation (2.25)):

$$\tilde\alpha(z) = \frac{N\hbar^2\Omega^2[q(z) - q(0)]/z}{[1 - \exp(-\hbar\Omega/k_B T)][z + {\rm i}\nu(z)]}, \tag{37}$$

where $q(z)$ and $v(z)$ are Cauchy integrals,

$$q(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega - z} F(\omega) \tag{38}$$

$$v(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega - z} \nu(\omega), \tag{39}$$

with kernels (cf [15], equations (2.31) and (2.32)),

$$F(\omega) = \frac{R(\omega)n(\Omega+\omega)}{n(\omega)} - \frac{R(-\omega)n(\Omega-\omega)}{n(-\omega)} \tag{40}$$

$$\nu(\omega) = \frac{\hbar}{\mathrm{i}}[R(\omega) + R(-\omega)], \tag{41}$$

being

$$n(\omega) = [\exp(\hbar\omega/k_B T) - 1]^{-1} \tag{42}$$

and

$$R(\omega) = \frac{2\mathrm{i}}{m_v \hbar \Omega}(\Omega+\omega)D(\Omega+\omega). \tag{43}$$

The function $D$ in the above equation has been studied in previous works [11, 17], since $D(\Omega)$ corresponds to the longitudinal friction coefficient in the Markovian approximation. It reads [11]

$$D(\Omega) = \frac{L\pi}{2\hbar\Omega} \sum_{\mathbf{k},\mathbf{q}} \delta_{k_z q_z} (\mathbf{k}-\mathbf{q})^2 \Big[ \big|\Lambda^{(0)}_{\mathbf{kq}}\big|^2 (n_q - n_k)\delta(\omega_k - \omega_q - \Omega) + 2\big|\Gamma^{(0)}_{\mathbf{kq}}\big|^2 (f_q - f_k)\delta(\epsilon_k/\hbar - \epsilon_q/\hbar - \Omega)\Big], \tag{44}$$

where $n_k = [\exp(\hbar\omega_k/k_B T) - 1]^{-1}$ and $f_k = \{\exp[(\epsilon_k - \mu)/k_B T] + 1\}^{-1}$ respectively denote the thermal equilibrium Bose and Fermi occupation numbers for the corresponding scatterers. The expression (37) for the susceptibility, on the other hand, is fully non-Markovian and each pole $z_j$ of it yields a term proportional to $\exp(-\mathrm{i}z_j\tau)$ in $\alpha(\tau)$. Then, the Markov approximation consists in neglecting the set of such poles which are located far enough from the origin, so that they yield rapidly vanishing terms, i.e. terms which decay faster than any observational timescale. This is the case for the set of poles arising from $q(z)$ in (37), actually poles of $n(\Omega \pm \omega)$ in (40) [15], which are of the form $\pm\Omega - \mathrm{i}n2\pi k_B T/\hbar$ $(n = 1, 2, \ldots)$. Such poles give rise to exponentially decaying terms in the expression of $\alpha(\tau)$, which have lifetimes shorter than $\hbar/k_B T$. This thermal timescale turns out to be much smaller than that arising from the friction coefficient, i.e. $\hbar/k_B T \ll m_v/D(\Omega) = [\rho_s h/D(\Omega)]\Omega^{-1}$ for $T < 1.5$ K and $\Omega$ of order 0.01 $\mathrm{ps}^{-1}$. So, the above set of thermal poles can be safely ignored. It is then clear that we should look for poles of order $\Omega[D(\Omega)/\rho_s h]$ and, in the following, we shall see that they arise from the equation $z + \mathrm{i}v(z) = 0$. In fact, looking for a solution close to the origin, one may begin with the ansatz $z_0 = -\mathrm{i}v(0)$ and next proceed iteratively, i.e. $z_1 = -\mathrm{i}v(z_0)$ and so on. This leads to a very rapid convergence to a solution $z_s$ that is better worked out in terms of the Taylor expansion of $v(z)$ around the origin:

$$z_s = -\mathrm{i}v(0)[1 - \mathrm{i}v'(0) - v''(0)v(0)/2], \tag{45}$$

where the second and third terms inside the square brackets represent first- and second-order corrections to the zeroth-order solution, respectively. The Cauchy integral (39) and its derivatives in (45) can be written as [15]

$$v(0) = \nu(0)/2 = 2\Omega D(\Omega)/\rho_s h \tag{46}$$

$$\begin{aligned} v'(0) &= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega^2} [\nu(\omega) - \nu(0)] \\ &= \frac{2}{\mathrm{i}\pi\rho_s h} \int_0^{\infty} \frac{\mathrm{d}\omega}{\omega^2} [(\Omega+\omega)D(\Omega+\omega) + (\Omega-\omega)D(\Omega-\omega) - 2\Omega D(\Omega)] \end{aligned} \tag{47}$$

$$v''(0) = \nu''(0)/2 = \frac{2}{\rho_s h}[2D'(\Omega) + \Omega D''(\Omega)]. \tag{48}$$

We see from (46) that the zeroth-order solution $z_0 = -\mathrm{i}v(0)$ has in fact the expected dependence and, moreover, the factor 2 in the expression of $v(0)$ is easily interpreted if we recall that a damping in the velocity like $\exp(-\Omega t D(\Omega)/\rho_s h)$ should give rise to twice as fast energy damping. The first- and second-order corrections in (45) are easily evaluated from (47) and (48), and they are always negligible, e.g. for ordinary helium at $T = 0.67$ K ($\Omega = 0.01$ ps$^{-1}$) we have $-\mathrm{i}v'(0) = 1.52 \times 10^{-5}$ and $-v''(0)v(0)/2 = 1.32 \times 10^{-9}$. In conclusion, we have found that the Markovian pole $-\mathrm{i}2\Omega D(\Omega)/\rho_s h$ is unaffected by memory corrections. This is to be contrasted with the appreciable memory corrections to the friction coefficient seen in section 2. Therefore, the Markovian approximation for the generalized susceptibility (37) reads

$$\tilde{\alpha}_M(z) = \frac{N\hbar^2\Omega^2 q'(0)}{[1 - \exp(-\hbar\Omega/k_B T)][z + \mathrm{i}v(0)]} \tag{49}$$

which should be valid for $z$ inside a circle with a radius $r_0$ fulfilling $v(0) < r_0 \ll k_B T/\hbar$. Note that we have replaced the expression $[q(z) - q(0)]/z$ in (37) by the derivative $q'(0)$. This approximation may be readily tested if one considers the first-order term $q''(0)z/2$ at $z = z_0 = -\mathrm{i}v(0)$. In fact, we have [15]

$$q'(0) = F'(0)/2 = -n(\Omega)z_0/k_B T \tag{50}$$

and

$$q''(0) = \frac{2}{\pi \mathrm{i}} \int_0^{\infty} \frac{\mathrm{d}\omega}{\omega^3}[F(\omega) - F'(0)\omega], \tag{51}$$

so, we may find again that the first-order correction is totally negligible, e.g. for ordinary helium at $T = 0.67$ K ($\Omega = 0.01$ ps$^{-1}$) it represents $\sim 10^{-6}$ of the zeroth order $q'(0)$. Then, replacing (50) in (49) we have the final expression,

$$\tilde{\alpha}_M(z) = \frac{N\hbar^2\Omega^2 \mathrm{i}2\Omega D(\Omega)/\rho_s h}{4k_B T \sinh^2(\hbar\Omega/2k_B T)[z + \mathrm{i}2\Omega D(\Omega)/\rho_s h]} \tag{52}$$

which immediately reproduces the result (34) for the static susceptibility $\tilde{\alpha}(0)$.

## 5. Response and time correlation functions

From (52) one easily extracts $\alpha(\tau)$ and the response function (31),

$$\Psi(t) = \frac{N(\hbar\Omega)^2}{4k_B T}\left[\sinh\left(\frac{\hbar\Omega}{2k_B T}\right)\right]^{-2}[1 - \exp(-2\Omega t D(\Omega)/\rho_s h)]. \tag{53}$$

This kind of response, characterized by a single relaxation time, is well known in theories of dielectric and magnetic relaxation and goes under the name of *Debye response* [13].

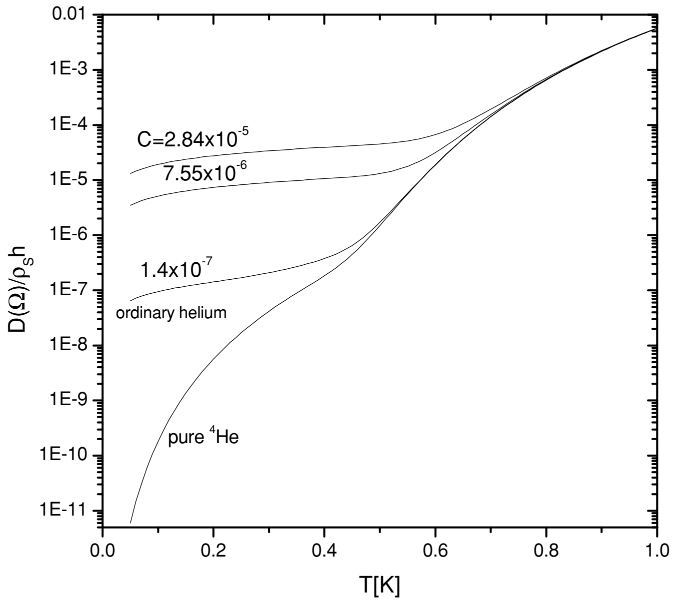


**Figure 2.** Relative value of the friction coefficient $D(\Omega)$ with respect to the Magnus force coefficient $\rho_s h$, versus temperature for several $^3$He concentrations $C$.

The Fourier transform of the time correlation function arises from (36) and (52):

$$\bar{C}(\omega) = \frac{N(\hbar\Omega)^2}{4k_BT}\left[\sinh\left(\frac{\hbar\Omega}{2k_BT}\right)\right]^{-2}\frac{\hbar\coth(w)w\epsilon}{\epsilon^2+w^2}, \tag{54}$$

where $\epsilon = [\Omega D(\Omega)/\rho_s h]/[k_BT/\hbar]$ and $w = \hbar\omega/2k_BT$. With $\epsilon \ll 1$, the function of $w$, $\epsilon/(\epsilon^2+w^2)$ in (54), turns out to be sharply peaked around $w = 0$, so we may approximate $\coth(w)w \simeq 1$ and thus get the antitransform:

$$C(t) = \frac{N(\hbar\Omega)^2\exp[-2\Omega|t|D(\Omega)/\rho_s h]}{4\sinh^2(\hbar\Omega/2k_BT)}. \tag{55}$$

Therefore, the time correlation function is ruled by the same relaxation time as the response function, as expected. Note that $C(0)$ in (55) actually corresponds to $\langle H_v^2\rangle_{\text{eq}} - \langle H_v\rangle_{\text{eq}}^2$, as can be easily verified by an elementary calculation of $\langle H_v^2\rangle_{\text{eq}}$ in the canonical ensemble.

To conclude, it is interesting to analyse how the relaxation frequency $2\Omega D(\Omega)/\rho_s h$ depends on temperature and impurity concentration. In figure 2, $D(\Omega)/\rho_s h$ is plotted against temperature for several $^3$He concentrations [3]. Such curves actually correspond to $\Omega = 0.01$ ps$^{-1}$, but it is important to remark that the dependence on $\Omega$ turns out to be negligible for $\Omega$ within $10^{-2}$ ps$^{-1}$ or less [11, 17]. It is also worth noting that consistently with our weak-coupling approximation [11, 15, 17], the Markovian friction coefficient $D(\Omega)$ always remains small compared to the coefficient $\rho_s h$ of the Magnus force in (1) (actually $\rho_s h$ has virtually no dependence on temperature for $T < 1$ K). The lowest curve in figure 2 corresponds to pure $^4$He and it displays two well separated regimes [17],

$$D(\Omega) \sim \begin{cases} T^5, & \text{for} \quad T < 0.4\ \text{K (phonon domain)} \\ \exp(-\Delta/k_BT), & \text{for} \quad T > 0.5\ \text{K (roton domain),} \end{cases}$$

where $\Delta/k_B = 8.62$ K corresponds to the height of the roton minimum in the dispersion curve of figure 1. Note that the phonon–roton transition clearly manifests itself as an intermediate region of positive second derivative. The remaining curves in figure 2 correspond to finite $^3$He concentrations that have been experimentally studied [3]. Such concentrations are low enough to allow a Maxwell–Boltzmann approximation for the $^3$He statistics in (44). Then, the low-temperature regime of $D(\Omega)$, which is now dominated by impurity scattering, turns out to be proportional to $\sqrt{T}$ and $^3$He concentration [3, 11]. We may see from figure 2 that phonon effects are completely hidden in ordinary helium, since the $^3$He domain extends as far as $T \simeq 0.4$ K. For higher concentrations such a domain reaches higher temperatures also hiding the first portion of the roton curve.

Finally, it is worth observing that relaxation frequencies with any temperature dependence rarely appear in models of quantal Brownian motion of harmonic oscillators, since most of them assume, in contrast to our scattering model, linear coupling in the heat bath operators [14, 15].

## 6. Summary and conclusions

A generalization of the quantum theory of vortex waves [9, 10] has been proposed to study the damping of cyclotron modes. We have shown that the friction values should be practically unaffected by such oscillations, a result which was already known in the case of phonon scattering of low-frequency Kelvin modes [10, 12]. All sources of dissipation arising in ordinary helium, namely phonons, rotons and $^3$He atoms, were considered, showing that appreciable memory effects must be taken into account in the evaluation of the friction coefficient. We have also analysed memory corrections to the Markov approximation in the case of the non-equilibrium energetics of cyclotron modes, finding this time that they are negligible. We have shown that the vortex response is of the Debye type, i.e. it is ruled by a single relaxation frequency which governs the time correlation function as well. Such a relaxation frequency is shown to embody all the complexities of the heat bath, in that very well separated regimes belonging to the different species comprising the superfluid helium are recognized from its dependence on temperature and impurity concentration.

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