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Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type ☆

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Abstract

Let $0 < \gamma < 1$, b be a BMO function and $I^m_{\gamma,b}$ the commutator of order m for the fractional integral. We prove two type of weighted L^p inequalities for $I^m_{\gamma,b}$ in the context of the spaces of homogeneous type. The first one establishes that, for A_∞ weights, the operator $I^m_{\gamma,b}$ is bounded in the weighted L^p norm by the maximal operator $M_\gamma(M^m)$, where M_γ is the fractional maximal operator and M^m is the Hardy–Littlewood maximal operator iterated m times. The second inequality is a consequence of the first one and shows that the operator $I^m_{\gamma,b}$ is bounded from $L^p[M_{\gamma p}(M^{[(m+1)p]}w)(x)\,d\mu(x)]$ to $L^p[w(x)\,d\mu(x)]$, where $[(m+1)p]$ is the integer part of $(m+1)p$ and no condition on the weight w is required. From the first inequality we also obtain weighted L^p–L^q estimates for $I^m_{\gamma,b}$ generalizing the classical results of Muckenhoupt and Wheeden for the fractional integral operator.
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Keywords: Commutators; Fractional integral; Spaces of homogeneous type; Weighted strong inequalities

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1. Introduction

Let us consider a space of homogeneous type (X, d, μ) which is a set X endowed with a quasi-distance d such that the balls $B(x,r) = \{y \in X\colon d(x,y) < r\}$ are open sets and a positive measure μ satisfying a doubling condition (we refer to Section 2 for a more complete definition). In this context we define the fractional integral operator as

$$I_\gamma f(x) = \int_X K_\gamma(x,y) f(y)\, d\mu(y), \quad 0 < \gamma < 1,$$

where

$$K_\gamma(x,y) = \begin{cases} [\mu(B(x,d(x,y)))]^{\gamma-1} & \text{if } x \neq y, \\ \mu(x)^{\gamma-1} & \text{if } x = y \text{ and } \mu(x) > 0. \end{cases} \tag{1.1}$$

Notice that when $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and μ is the Lebesgue measure, we recover the classical fractional integral operator. It is known that for the Muckenhoupt A_∞ weights the fractional integral is controlled in the p-norm, by the fractional maximal function

$$M_\gamma f(x) = \sup \mu(B)^{\gamma-1} \int_B \big|f(y)\big|\, d\mu(y), \quad 0 < \gamma < 1, \tag{1.2}$$

where the supremum is taken over all balls B such that $x \in B$ (when we take $\gamma = 0$ in (1.2) we get the Hardy–Littlewood maximal operator). More precisely, if $0 < p < \infty$ and $\omega \in A_\infty$ then there exists a constant $C > 0$ such that

$$\int_X \big|I_\gamma f(x)\big|^p \omega(x)\, d\mu(x) \leqslant C \int_X \big|M_\gamma f(x)\big|^p \omega(x)\, d\mu(x). \tag{1.3}$$

It was observed in [1] that the above inequality can be proved following the reasoning for $\mathbb{R}^n$ in [14].

On the other hand, Pérez and Wheeden [20] obtained other kind of weighted inequality for I_γ. They proved that if ω is a nonnegative measurable function and $1 < p < \infty$, then there exists a constant $C > 0$ such that

$$\int_X \big|I_\gamma f(x)\big|^p \omega(x)\, d\mu(x) \leqslant C \int_X \big|f(x)\big|^p M_{\gamma p}\big(M^{[p]} w\big)(x)\, d\mu(x), \tag{1.4}$$

where $M^{[p]}$ is the Hardy–Littlewood maximal operator M iterated $[p]$ times ($[p]$ meaning the integer part of p). The interesting point in this two-weighted inequality is that non-a-priori assumption on the weight ω is required. This estimate was previously proved in the Euclidean context in [17] to improve some results on weighted Sobolev inequalities.

Inequality (1.4) was proved in [20] for integral operators which include the fractional integral as a particular case. However, the spaces of homogeneous type considered in [20] satisfy that all the annuli are nonempty. This restriction implies that the spaces are of infinite measure and they have no atoms (i.e., points of positive measure).

The aim of this article is to study inequalities (1.3) and (1.4) for the commutators of the fractional integrals in the setting of the spaces of homogeneous type. These operators are not included in the integral operators considered in [20]. As far as we know, the results are new even in the case of the Euclidean space $\mathbb{R}^n$.

For a function $b \in \mathrm{BMO}(X)$ (see the definition in Section 2) we define the commutator of order $m \in \mathbb{N}$ for the fractional integral as

$$I^m_{\gamma,b} f(x) = \int_X \left[b(x) - b(y)\right]^m K_\gamma(x, y) f(y)\, d\mu(y), \quad 0 < \gamma < 1. \tag{1.5}$$

Actually, we shall obtain our results for the operators

$$\mathcal{I}^m_{\gamma,b} f(x) = \int_X \left|b(x) - b(y)\right|^m K_\gamma(x, y) f(y)\, d\mu(y), \quad 0 < \gamma < 1. \tag{1.6}$$

The operators $\mathcal{I}^m_{\gamma,b}$ are bigger than the operators $I^m_{\gamma,b}$ in the sense that for all $f \geqslant 0$ and all $x \in X$, $|I^m_{\gamma,b} f(x)| \leqslant \mathcal{I}^m_{\gamma,b} f(x)$. Concretely, we shall prove the following theorems.

Theorem 1.1. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$, $0 < p < \infty$, $0 < \gamma < 1$ and $m \in \mathbb{N} \cup \{0\}$. If $\omega \in A_\infty$ and $b \in \mathrm{BMO}(X)$ then there exists a constant C depending on m and on the A_∞ constant of ω, such that*

$$\int_X \left|\mathcal{I}^m_{\gamma,b} f(x)\right|^p \omega(x)\, d\mu(x) \leqslant C\|b\|^{mp}_{\mathrm{BMO}} \int_X \left[M_\gamma\left(M^m f\right)(x)\right]^p \omega(x)\, d\mu(x), \tag{1.7}$$

for all f such that the left-hand side of the previous inequality is finite.

Theorem 1.2. *Let (X, d, μ) be space of homogeneous type such that the continuous functions are dense in $L^1(X)$, ω any weight on X, $0 < \gamma < 1$, $1 < p < \infty$ and $m \in \mathbb{N} \cup \{0\}$. If $b \in \mathrm{BMO}(X)$ then there exists a constant C such that*

$$\int_X \left|\mathcal{I}^m_{\gamma,b} f(x)\right|^p \omega(x)\, d\mu(x) \leqslant C\|b\|^{mp}_{\mathrm{BMO}} \int_X \left|f(x)\right|^p M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\, d\mu(x). \tag{1.8}$$

We require the density of the continuous functions to be able to apply the Lebesgue differentiation theorem.

In order to prove Theorems 1.1 and 1.2, it is clear that we may replace the kernel K_γ by any kernel Q_γ equivalent to K_γ, in the sense that $1/cK_\gamma \leqslant Q_\gamma \leqslant cK_\gamma$ for some positive constant. We shall use an equivalent kernel having some smoothness properties that the fractional kernel K_γ does not have.

Inequalities (1.7) and (1.8) inclose important information about the behavior of the commutator $I^m_{\gamma,b}$. They show a higher order of singularity of the commutator when m increases since the maximal functions on the right-hand side of both inequalities need more iterations to control the left-hand side. This situation is similar to that of commutators for singular integral operators. For this last kind of commutators, results analogous to Theorems 1.1 and 1.2 were obtained in [18] on $\mathbb{R}^n$ and in [22] on spaces of homogeneous type.

As a consequence of Theorem 1.1 we can obtain the following weighted strong (p, q) inequality for $\mathcal{I}^m_{\gamma,b}$.

Corollary 1.3. *Let (X, d, μ) be space of homogeneous type such that the continuous functions are dense in $L^1(X)$. Given γ, $0 < \gamma < 1$, and p, $1 < p < 1/\gamma$, fix q so that $1/q = 1/p - \gamma$. Let ω a weight satisfying the following condition: there exists a positive constant C such that*

$$\left(\frac{1}{\mu(B)} \int_B \omega^q \, d\mu\right)^{1/q} \left(\frac{1}{\mu(B)} \int_B \omega^{-p'} \, d\mu\right)^{1/p'} \leqslant C,$$

for all balls B. Then, $\mathcal{I}^m_{\gamma,b}$ satisfies the strong (p, q) inequality

$$\left(\int_X \left|\mathcal{I}^m_{\gamma,b} f(x)\right|^q \omega^q(x) \, d\mu(x)\right)^{1/q} \leqslant C \|b\|^m_{\mathrm{BMO}} \left(\int_X \left|f(x)\right|^p \omega^p(x) \, d\mu(x)\right)^{1/p}.$$

The proof of the corollary follows easily from Theorem 1.1. Notice that the condition on the weight is equivalent to $\omega^q \in A_\beta$, $\beta = 1 + q/p'$ (therefore $\omega^q \in A_\infty$) and to the strong (p, q) inequality for M_γ (see, for instance, [9]). The condition on the weight implies also that $\omega^p \in A_p$. The corollary follows from these facts and (1.7).

Finally, we want to point out that our results improve the results in [20] for the fractional integral (case $m = 0$) since we consider more general spaces. Corollary 1.3 improves also the corresponding result in [3], where they obtain the result for $m = 1$, $\omega \equiv 1$ and spaces of homogeneous type satisfying certain property (P) (see [3, Theorem 2.11]).

The article is organized in the following way: in Section 2 we give some definitions and preliminary results about the spaces of homogeneous type. Section 3 is devoted to establish definitions of the Orlicz spaces and some preliminary results about the maximal function associated to a Young function. In Sections 4 and 5 we state and prove two important previous lemmata. In the proofs of these lemmata appear the main differences with respect to previous results. Finally, by standard arguments, in Sections 6 and 7 we shall give the proofs of Theorems 1.1 and 1.2, respectively.

2. Spaces of homogeneous type: definitions and preliminary results

Given a set X, a function $d : X \times X \to \mathbb{R}^+_0$ is called a quasi-distance on X if the following conditions are satisfied:

(i) for every x and y in X, $d(x, y) \geqslant 0$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x, y) = d(y, x)$,
(iii) there exists a constant $K \geqslant 1$ such that

$$d(x, y) \leqslant K\big(d(x, z) + d(z, y)\big) \tag{2.1}$$

for every x, y and z in X.

We shall say that two quasi-distances d and d' on X are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 d'(x, y) \leqslant d(x, y) \leqslant c_2 d'(x, y)$ for all $x, y \in X$. In particular, equivalent quasi-distances induce the same topology on X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r) = \{y\colon d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu\big(B(x,2r)\big) \leqslant A\mu\big(B(x,r)\big) < \infty \tag{2.2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ), with d and μ as above, is called a *space of homogeneous type*. The constants K and A in (2.1) and (2.2) will be called the constants of the space.

The balls in a general space of homogeneous type are not necessarily open, but the next result of Macías and Segovia [10] shows that there is a continuous quasi-distance d' which is equivalent to d for which every ball is open.

Theorem 2.1. [10] *Let d be a quasi-distance on a set X. Then there exists a quasi-distance d' on X, a finite constant C and a number $0 < \theta < 1$, such that d' is equivalent to d and, for every x, y and z in X*

$$\big|d'(x,y) - d'(z,y)\big| \leqslant Cd'(x,z)^{\theta}\big(d'(x,y) + d'(z,y)\big)^{1-\theta}. \tag{2.3}$$

Moreover, the balls corresponding to d' are open sets in the topology induced by d'.

In this article we always assume that the quasi-distance d is continuous and the balls are open sets.

On the space of homogeneous type, if B is a ball and B' is another ball with center in B and radius smaller than that of B, the measure of $B \cap B'$ is not, in general, greater than a constant fraction of the measure of B', as it is the case in $\mathbb{R}^n$. In [11], Macías and Segovia, construct a quasi-distance equivalent to the original one, such that the balls defined by the new quasi-distance have the above property. We state the result in the following theorem.

Theorem 2.2. [11] *Let (X, d, μ) be a space of homogeneous type. There exists a quasi-distance δ on X which is equivalent to d such that, for some constant $C > 0$ depending only on the constants of the space, if $x \in X$, $0 < r \leqslant 2KR$ and $y \in B_\delta(x, R)$ then*

$$\mu\big(B_\delta(y,r) \cap B_\delta(x,R)\big) \geqslant C\mu\big(B_\delta(y,r)\big). \tag{2.4}$$

Moreover,

$$\delta(x,y) \leqslant d(x,y) \leqslant 3K^2\delta(x,y), \tag{2.5}$$

for every x and y in X.

The balls $B_\delta(x, R)$ endowed with the restrictions of the quasi-distance δ and the measure μ become bounded spaces of homogeneous type with constants K' and A', satisfying (2.1) *and* (2.2) *respectively, independent of $R > 0$ and $x \in X$.*

Remark 2.3. We notice that inequality (2.4) actually holds for $0 < r \leqslant 2K'R$, where $K' = 3K^3$ is the constant in the quasi-triangular inequality for δ. In fact, if $2KR < r \leqslant 2K'R$, let $n \in N$ such that $2^{n-1} < 3K^2 \leqslant 2^n$ and $r' = r/(3K^2)$. Then

$$\begin{aligned}\mu\big(B_\delta(y,r) \cap B_\delta(x,R)\big) &= \mu\big(B_\delta\big(y, 3K^2r'\big) \cap B_\delta(x,R)\big)\\ &\geqslant \mu\big(B_\delta(y,r') \cap B_\delta(x,R)\big) \geqslant C\mu\big(B_\delta(y,r')\big)\\ &= C\mu\big(B_\delta\big(y, r/\big(3K^2\big)\big)\big) \geqslant \frac{C}{(A')^n}\mu\big(B_\delta(y,r)\big).\end{aligned}$$

We observe that in the third line, since $0 < r' \leqslant 2KR$, we have used the result of Macías–Segovia [11] and in the last inequality we have used the doubling property with constant A'.

Our main results involve weighted strong type inequalities for the operators $\mathcal{I}^m_{\gamma,b}$ defined in (1.6). The argument that we shall use to prove them involves the estimate of the composition of the sharp function with the commutators. This estimate requires some smoothness property on the kernels of the operators. For that reason, we shall work with a suitable version of $\mathcal{I}^m_{\gamma,b}$ equivalent with the definition (1.6) for all $f \geqslant 0$.

In fact, notice that the function $\kappa(x, y)$ defined as $\mu(B(x, d(x, y)))$ if $x \neq y$ and $\kappa(x, x) = 0$, is not a quasi-distance because it might not be symmetric. However, it is easy to prove that the function

$$\rho(x, y) = \begin{cases} \frac{1}{2}[\mu(B(x, d(x, y))) + \mu(B(y, d(x, y)))] & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

is a quasi-distance equivalent to $\kappa(x, y)$. Now, let η be a continuous quasi-distance equivalent to ρ (the existence of η is guaranteed by Theorem 2.1). Associated to η we define the kernel

$$Q_\gamma(x, y) = \begin{cases} \eta(x, y)^{\gamma-1} & \text{if } x \neq y, \\ \mu(x)^{\gamma-1} & \text{if } x = y \text{ and } \mu(x) > 0, \end{cases}$$

and the operator

$$\mathcal{I}^m_{\gamma,b} f(x) = \int_X \left|b(x) - b(y)\right|^m Q_\gamma(x, y) f(y)\, d\mu(y). \tag{2.6}$$

It is clear that the above operator is equivalent to the one defined in (1.6). Consequently, we shall work in the proofs of Theorems 1.1 and 1.2 with the operator defined in (2.6).

Now, we recall some definitions and give some notations. Let (X, d, μ) be a space of homogeneous type, B a ball in X and $m_B(f) = \mu(B)^{-1} \int_B f\, d\mu$, we define the sharp function as

$$M^\sharp f(x) = \sup_{B:\, x\in B} \inf_{C_B\in\mathbb{R}} m_B\big(|f - C_B|\big) \approx \sup_{B:\, x\in B} m_B\big(\big|f - m_B(f)\big|\big).$$

A function f belongs to the space $\mathrm{BMO} = \mathrm{BMO}(X)$ of bounded mean oscillation functions if $M^\sharp f$ belongs to $L^\infty(X)$. A semi-norm in this space is defined by $\|f\|_{\mathrm{BMO}} = \|M^\sharp f\|_\infty$.

We will also use the notation $T_{(\delta)}(f)$, $0 < \delta < 1$, for the operator $[T(|f|^\delta)]^{1/\delta}$, where T will be a suitable operator.

Let us recall the definition of the Muckenhoupt class of weights A_p, $1 \leqslant p \leqslant \infty$. A weight ω is a nonnegative and locally integrable function on X. We say that $\omega \in A_1$ if there exists $C > 0$ such that

$$m_B(\omega) \leqslant C\omega(x),$$

for all balls B and $x \in B$, except for x that belongs to a set with zero μ-measure. We say that $\omega \in A_p$, $1 < p < \infty$, if there exists $C > 0$ such that

$$m_B(\omega)\big[m_B\big(\omega^{-1/(p-1)}\big)\big]^{p-1} \leqslant C,$$

for all balls B. Finally, we say that $\omega \in A_\infty(\mu)$ if there are positive constants C and ϵ such that

$$\frac{\omega(E)}{\omega(B)} \leqslant C\left(\frac{\mu(E)}{\mu(B)}\right)^\epsilon$$

for every ball B and all μ-measurable sets $E \subset B$, where $\omega(E)$ means $\int_E \omega\, d\mu$.

In the proofs of the theorems we shall need the next generalization of a result of Fefferman and Stein about the relationship in L^p-norm of the Hardy–Littlewood maximal function and the sharp function. The theorem was proved in [22] and is based on ideas of Aimar, who proved the result without weights.

Lemma 2.4. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$, $0 < \delta < 1$ and $w \in A_\infty$. Then, for every p, $1 < p < \infty$, there exists a constant C depending on the A_∞ constant of ω such that*

$$\|M_{(\delta)}(f)\|^p_{p,\omega} \leqslant \begin{cases} C\|M^\sharp_{(\delta)}(f)\|^p_{p,\omega} & \text{if } \mu(X)=\infty,\\ Cw(X)[m_X(|f|^\delta)]^{p/\delta} + C\|M^\sharp_{(\delta)}(f)\|^p_{p,\omega} & \text{if } \mu(X)<\infty \end{cases}$$

for all f such that $\|M_{(\delta)}f\|_{p,w} < +\infty$.

3. Orlicz spaces and the maximal function associated: definitions and preliminary results

A function $\phi : [0,\infty) \to [0,\infty)$ is called a *Young function* if it is continuous, convex, increasing and satisfies $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. It follows that $\phi(t)/t$ is increasing and, in particular, that

$$\phi(st) \geqslant s\phi(t) \quad \text{if } s \geqslant 1 \text{ and } t \geqslant 0. \tag{3.1}$$

We shall say that ϕ is doubling if there exists $C > 0$ such that $\phi(2t) \leqslant C\phi(t)$ for all $t \geqslant 0$.

If ϕ is a Young function, we define the ϕ-average of a function f over a ball B by means of the Luxemburg norm:

$$\|f\|_{\phi,B} = \inf\left\{\lambda > 0\colon \frac{1}{\mu(B)}\int_B \phi\left(\frac{|f(y)|}{\lambda}\right)d\mu(y) \leqslant 1\right\}.$$

When $\phi(t) = t$, we recover $m_B(|f|)$. Each Young function ϕ has an associated complementary Young function $\tilde\phi$ satisfying

$$t \leqslant \phi^{-1}(t)\tilde\phi^{-1}(t) \leqslant 2t,$$

for all $t > 0$. There is a generalization of Hölder's inequality

$$\frac{1}{\mu(B)}\int_B |fg|\,d\mu \leqslant \|f\|_{\phi,B}\|g\|_{\tilde\phi,B}. \tag{3.2}$$

A further generalization of Hölder's inequality (see [15]) that will be useful later is the following. If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are Young functions and

$$\mathcal{A}^{-1}(x)\mathcal{B}^{-1}(x) \leqslant \mathcal{C}^{-1}(x)$$

then

$$\|fg\|_{\mathcal{C},B} \leqslant 2\|f\|_{\mathcal{A},B}\|g\|_{\mathcal{B},B}. \tag{3.3}$$

Associated to the ϕ-average of f is the following fractional maximal function defined for $0 \leqslant \gamma < 1$ as

$$M_{\gamma,\phi}f(x) = \sup_{B\colon x\in B} \mu(B)^\gamma\|f\|_{\phi,B},$$

where the supremum is taken over all balls containing x. When $\gamma = 0$ we denote it $M_\phi f$. Also, when $\phi(t) = t$, $M_{\gamma,\phi} = M_\gamma$.

For a Young function ϕ, the maximal operator M_ϕ satisfies the following weak type inequality

$$\mu\bigl(\bigl\{x \in X:\ M_\phi f(x) > \lambda\bigr\}\bigr) \leqslant C \int_X \phi\biggl(\frac{|f(x)|}{\lambda}\biggr)\, d\mu(x). \tag{3.4}$$

The proof of the above inequality is similar to that of the $(1,1)$-weak type inequality for the Hardy–Littlewood maximal operator (see [4]). By standard arguments, it follows from (3.4) that

$$\mu\bigl(\bigl\{x \in X:\ M_\phi f(x) > \lambda\bigr\}\bigr) \leqslant C \int_{\{x\in X:\ |f(x)|>\lambda/2\}} \phi\biggl(\frac{2|f(x)|}{\lambda}\biggr)\, d\mu(x), \tag{3.5}$$

for some constant C, all $\lambda > 0$ and all measurable function f.

Let $1 < p < \infty$. We say that a doubling Young function ϕ satisfies the B_p condition if there exists a positive constant c such that

$$\int_c^\infty \frac{\phi(t)}{t^p}\frac{dt}{t} \cong \int_c^\infty \biggl(\frac{t^{p'}}{\tilde{\phi}(t)}\biggr)^{p-1}\frac{dt}{t} < \infty.$$

Pradolini and Salinas [21] proved the following theorem for M_ϕ, generalizing a previous result in [20].

Theorem 3.1. *Let (X, d, μ) be a space of homogeneous type, $1 < p < \infty$ and ϕ a doubling Young function such that ϕ satisfies the condition B_p. Then, there exists a constant C such that*

$$\int_X \bigl[M_\phi f(x)\bigr]^p\, d\mu(x) \leqslant C \int_X \bigl|f(x)\bigr|^p\, d\mu(x),$$

for all measurable functions f.

The necessity of the condition B_p holds only if $\mu(X) = \infty$.

4. Equivalences between maximal functions

In this section we shall prove the next lemma.

Lemma 4.1. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$, $0 \leqslant \gamma < 1$, $k \in \mathbb{N}$ and $\phi_k(t) = t[\log(e + t)]^k$. Then, there exist constants $C_1, C_2 > 0$ such that*

$$M_\gamma\bigl(M^k f\bigr)(x) \leqslant C_1 M_{\gamma,\phi_k} f(x) \tag{4.1}$$

and

$$M_{\gamma,\phi_k} f(x) \leqslant C_2 M_\gamma\bigl(M^k f\bigr)(x), \tag{4.2}$$

for every $x \in X$.

This lemma was proved in [7] in the Euclidean context. In the framework of spaces of homogeneous type, inequality (4.1) was obtained in [5] in the case $\gamma = 0$ and $k = 1$. On the other hand, inequality (4.2) was proved in [20] under the assumption that the annuli on X are nonempty.

In order to prove inequality (4.1) we shall need the following lemma.

Lemma 4.2. *Let (X, d, μ) be a space of homogeneous type, ϕ a Young function, $B = B(x, R)$ a fixed ball and $\tilde{B} = B(x, 2KR)$. Then, there exists a constant $C > 0$, depending only on the constants of the space, such that*

$$\max\{M_{\gamma,\phi}(f\chi_{X\setminus\tilde{B}})(y), \mu(B)^{\gamma} M_{\phi}(f\chi_{X\setminus\tilde{B}})(y)\} \leqslant C \inf_{z\in B} M_{\gamma,\phi}(f\chi_{X\setminus\tilde{B}})(z),$$

for all $y \in B$.

Proof. Let z and y be two points of B and let $S = B(x_S, R_S)$ a ball such that $y \in S$ and $\|f\chi_{X\setminus\tilde{B}}\|_{\phi,S} \neq 0$. Then $S \cap (X \setminus \tilde{B}) \neq \emptyset$ and $B \subset \tilde{S}$ where $\tilde{S} = B(x_S, K(4K^2 + 1)R_S)$. Now, it is easy to prove that

$$\max\{\mu(S)^{\gamma}, \mu(B)^{\gamma}\}\|f\chi_{X\setminus\tilde{B}}\|_{\phi,S} \leqslant C\mu(\tilde{S})^{\gamma}\|f\chi_{X\setminus\tilde{B}}\|_{\phi,\tilde{S}} \leqslant CM_{\gamma,\phi}(f\chi_{X\setminus\tilde{B}})(z),$$

and, since z is arbitrary, the inequality follows. □

On the other hand, to prove (4.2) we shall need a result about the Hardy–Littlewood maximal function. Let (X, d, μ) be a space of homogeneous type. It is a well-known result that the Hardy–Littlewood maximal operator M is of weak type $(1, 1)$ (see, e.g., [4]). It follows that there exists $C > 0$ such that

$$\mu(\{x \in X\colon\ Mf(x) > \lambda\}) \leqslant \frac{C}{\lambda} \int\limits_{\{x\in X\colon\ |f(x)|>\lambda/2\}} |f(x)|\, d\mu(x).$$

To prove (4.2) we shall need a reverse inequality. The next lemma provides us a suitable version of this reverse.

Lemma 4.3. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ and let δ be the quasi-distance defined in Theorem 2.2. Let $B_\delta = B_\delta(x, R)$ be a fixed ball on X. Then, there exist positive constants C and D, depending only on the constants of the space, such that*

$$\frac{1}{\lambda} \int\limits_{\{y\in B_\delta\colon\ f(y)>\lambda\}} f(y)\, d\mu(y) \leqslant C\mu(\{y \in B_\delta\colon\ Mf(y) > D\lambda\}),$$

for any $\lambda > m_{B_\delta}(f)$ and all nonnegative integrable functions f on B_δ.

Proof. Given a nonnegative integrable f on B_δ and $\lambda > m_{B_\delta}(f)$, we apply a Calderón–Zygmund decomposition to f at the level λ on the space of homogeneous type (B_δ, δ, μ), of the type found in [11]. That is, there exists a sequence $\{x_i\} \subset B_\delta$ and disjoint δ-balls $S_i = B_\delta(x_i, r_i) \cap B_\delta$ in this space such that if $\tilde{S}_i = B_\delta(x_i, 5(K')^2 r_i) \cap B_\delta$ then

(a) $m_{\tilde{S}_i}(f) \leqslant \lambda < m_{S_i}(f)$, and
(b) $f(x) \leqslant \lambda$ for almost every $x \in B_\delta \setminus \bigcup_i \tilde{S}_i$.

We claim that, there exists $D > 0$ such that for all i

$$S_i \subset \{y \in B_\delta\colon\ Mf(y) > D\lambda\}. \tag{4.3}$$

Notice that the definition of $S_i = B_\delta(x_i, r_i) \cap B_\delta$ and the fact that $\lambda > m_{B_\delta}(f)$ imply that $0 < r_i \leqslant 2K'R$. To prove (4.3) let us consider $y \in S_i$. By item (a), inequality (2.4), Remark 2.3, inequality (2.5) and the doubling property of μ we get that

$$\lambda < \frac{1}{\mu(S_i)} \int_{S_i} f\,d\mu \leqslant \frac{1}{C\mu(B_\delta(x_i, r_i))} \int_{B_\delta(x_i,r_i)} f\,d\mu$$
$$\leqslant \frac{C'}{C\mu(B_d(x_i, 3K^2r_i))} \int_{B_d(x_i,3K^2r_i)} f\,d\mu \leqslant \frac{C'}{C} Mf(y),$$

and thus (4.3) holds with $D = C'/C$. Finally, by (4.3) and items (a) and (b) we get that

$$\mu(\{y \in B_\delta\colon\ Mf(y) > D\lambda\}) \geqslant \sum_i \mu(S_i) \geqslant C \sum_i \mu(\tilde{S}_i) \geqslant \frac{C}{\lambda} \sum_i \int_{\tilde{S}_i} f\,d\mu \geqslant \frac{C}{\lambda} \int_{\bigcup_i \tilde{S}_i} f\,d\mu$$
$$\geqslant \frac{C}{\lambda} \int_{\{y \in B_\delta\colon\ f(y) > \lambda\}} f(y)\,d\mu(y).$$

Thus, the proof is done. □

Now, we are in condition to prove Lemma 4.1.

Proof of (4.1). Without loss of generality, we may assume that $f \geqslant 0$. We begin proving the case $\gamma = 0$. We shall proceed by induction. As we mention above, the case $k = 1$ and $\gamma = 0$ was proved in [5]. Let $k > 1$ and let us assume that (4.1) holds with $\gamma = 0$ and $k - 1$ instead of k. We claim that

$$\frac{1}{\mu(B)} \int_B M^k f(y)\,d\mu(y) \leqslant C\|f\|_{\phi_k, B}, \tag{4.4}$$

for all f such that $\operatorname{supp}(f) \subset B$. In fact, by an homogeneity argument we may assume that $\|f\|_{\phi_k,B} = 1$ and, thus,

$$\frac{1}{\mu(B)} \int_B f(y)\big[\log\big(e + f(y)\big)\big]^k\,d\mu(y) \leqslant 1. \tag{4.5}$$

On the other hand, by induction hypothesis, (3.5), integration and (4.5) it follows that

$$\int_B M^k f(y)\,d\mu(y) \leqslant C \int_B M_{\phi_{k-1}} f(y)\,d\mu(y) = C \int_0^\infty \mu(\{x \in B\colon\ M_{\phi_{k-1}} f(x) > \lambda\})\,d\lambda$$
$$\leqslant C\mu(B) + C \int_1^\infty \int_{\{x \in X\colon\ f(x) > \lambda/2\}} \phi_{k-1}\left(\frac{2f(x)}{\lambda}\right) d\mu(x)\,d\lambda$$
$$\leqslant C\mu(B) + C \int_B f(x)\big[\log\big(e + f(y)\big)\big]^k\,d\mu(x) \leqslant C\mu(B).$$

Then, (4.4) follows for f such that $\operatorname{supp}(f) \subset B$. For an arbitrary $f \geqslant 0$, let $x \in X$, $B = B(z, R)$ a ball such that $x \in B$ and $\tilde{B} = B(x, 2KR)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$. Then

$$\begin{aligned}\frac{1}{\mu(B)}\int_B M^k f(y)\,d\mu(y) &\leqslant \frac{1}{\mu(B)}\int_B M^k f_1(y)\,d\mu(y) + \frac{1}{\mu(B)}\int_B M^k f_2(y)\,d\mu(y)\\ &= I + II.\end{aligned}\tag{4.6}$$

By (4.4) and the doubling property,

$$I \leqslant C\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}} M^k f_1(y)\,d\mu(y) \leqslant C\|f\|_{\phi_k,\tilde{B}} \leqslant CM_{\phi_k} f(x).$$

On the other hand, by induction hypothesis and Lemma 4.2 we get that

$$\begin{aligned}II &\leqslant \frac{C}{\mu(B)}\int_B M_{\phi_{k-1}} f_2(y)\,d\mu(y)\\ &\leqslant C\inf_{z\in B} M_{\phi_{k-1}} f_2(z) \leqslant CM_{\phi_{k-1}} f_2(x) \leqslant CM_{\phi_k} f(x).\end{aligned}$$

Thus, we obtain (4.1) in the case $\gamma = 0$. Now, the case $0 < \gamma < 1$ can be proved easily. In fact, let I and II be as in (4.6). Then, by (4.4),

$$\mu(\tilde{B})^\gamma I \leqslant C\frac{\mu(\tilde{B})^\gamma}{\mu(\tilde{B})}\int_{\tilde{B}} M^k f_1(y)\,d\mu(y) \leqslant C\mu(\tilde{B})^\gamma\|f\|_{\phi_k,\tilde{B}} \leqslant CM_{\gamma,\phi_k} f(x).$$

On the other hand, by (4.1) with $\gamma = 0$ and Lemma 4.2,

$$\begin{aligned}\mu(\tilde{B})^\gamma II &\leqslant \frac{C}{\mu(B)}\int_B \mu(B)^\gamma M_{\phi_{k-1}} f_2(y)\,d\mu(y)\\ &\leqslant C\inf_{z\in B} M_{\gamma,\phi_{k-1}} f_2(z) \leqslant CM_{\gamma,\phi_{k-1}} f_2(x) \leqslant CM_{\gamma,\phi_k} f(x).\end{aligned}$$

Then, taking into account (4.6) we get that

$$\frac{\mu(B)^\gamma}{\mu(B)}\int_B M^k f(y)\,d\mu(y) \leqslant CM_{\gamma,\phi_k} f(x),$$

and (4.1) follows taking supremum on the balls B containing x. □

Proof of (4.2). The proof of (4.2) follows the lines of the one given in [20] (see [20, Lemma 8.5]), but we shall not use the hypothesis of nonempty annuli. In order to avoid this hypothesis we shall use Lemma 4.3 instead of [20, Lemma 8.1].

We may assume again that $f \geqslant 0$. Let $B = B(z, R)$ be any ball on X such that $x \in B$ and $\tilde{B} = B(x, 3K^2R)$. Notice that it is enough to show that there is a constant $\tilde{C}_k$ such that

$$\|f\|_{\phi_k,B} \leqslant \frac{\tilde{C}_k}{\mu(\tilde{B})}\int_{\tilde{B}} M^k f(y)\,d\mu(y).\tag{4.7}$$

Let δ be the quasi-distance equivalent to d defined in Theorem 2.2. If $B_\delta = B_\delta(z, R)$, let

$$\lambda_k = \lambda_k(f) = \frac{1}{\mu(B_\delta)} \int_{B_\delta} M^k f \, d\mu.$$

To prove (4.7) it will be enough to show that there is a constant $C_k > 1$ such that

$$\frac{1}{\mu(B_\delta)} \int_{B_\delta} \phi_k\left(\frac{f}{C_k \lambda_k}\right) d\mu \leqslant 1. \tag{4.8}$$

In fact, from (2.5) we get that $B \subset B_\delta \subset \tilde{B}$. On the other hand, $\mu(\tilde{B}) \leqslant \tilde{C}\mu(B)$ for some universal constant $\tilde{C} \geqslant 1$. It follows from (3.1) that if (4.8) holds then

$$\begin{aligned} \frac{1}{\mu(B)} \int_B \phi_k\left(\frac{f}{\tilde{C} C_k \lambda_k}\right) &\leqslant \frac{\mu(\tilde{B})}{\mu(B)} \frac{1}{\mu(B_\delta)} \int_{B_\delta} \phi_k\left(\frac{f}{\tilde{C} C_k \lambda_k}\right) \leqslant \frac{\tilde{C}}{\mu(B_\delta)} \int_{B_\delta} \phi_k\left(\frac{f}{\tilde{C} C_k \lambda_k}\right) \\ &\leqslant \frac{1}{\mu(B_\delta)} \int_{B_\delta} \phi_k\left(\frac{f}{C_k \lambda_k}\right) \leqslant 1. \end{aligned}$$

Thus,

$$\|f\|_{\phi_k, B} \leqslant \tilde{C} C_k \lambda_k = \tilde{C} \frac{C_k}{\mu(B_\delta)} \int_{B_\delta} M^k f \, d\mu \leqslant \frac{\tilde{C}^2 C_k}{\mu(\tilde{B})} \int_{\tilde{B}} M^k f \, d\mu,$$

and we get inequality (4.7) with $\tilde{C}_k = \tilde{C}^2 C_k$.

Let us then prove (4.8) by induction on k. Let $k = 1$, $g = \frac{f}{C_1 \lambda_1}$ and $\phi(t) = t \log(e + t)$. Then

$$\begin{aligned} \frac{1}{\mu(B_\delta)} \int_{B_\delta} \phi_1\left(\frac{f}{C_1 \lambda_1}\right) d\mu &= \frac{1}{\mu(B_\delta)} \int_{B_\delta} g\big[\log(e + g)\big] \, d\mu \\ &= \frac{1}{\mu(B_\delta)} \int_{1-e}^{\infty} \frac{1}{e + \lambda} \int_{\{x \in B_\delta:\ g(x) > \lambda\}} g \, d\mu \, d\lambda \\ &= \frac{1}{\mu(B_\delta)} \left(\int_{1-e}^{1} + \int_{1}^{\infty} \right) \frac{1}{e + \lambda} \int_{\{x \in B_\delta:\ g(x) > \lambda\}} g \, d\mu \, d\lambda = I + II. \end{aligned}$$

By the Lebesgue differentiation theorem,

$$\begin{aligned} I &\leqslant \frac{\log(1 + e) g(B_\delta)}{\mu(B_\delta)} = \frac{\log(1 + e)}{\mu(B_\delta) C_1 \lambda_1} \int_{B_\delta} f(y) \, d\mu(y) \\ &= \frac{\log(1 + e) \int_{B_\delta} f(y) \, d\mu(y)}{C_1 \int_{B_\delta} M f(y) \, d\mu(y)} \leqslant \frac{\log(1 + e)}{C_1} < 1, \end{aligned}$$

if we choose $C_1 > \log(1 + e)$. On the other hand, by Lemma 4.3, since $\lambda > 1 > m_{B_\delta}(g)$ there exists constants C and D independent of f such that

$$\begin{aligned}
II &= \frac{1}{\mu(B_\delta)} \int_1^\infty \frac{1}{e+\lambda}\biggl(\int_{\{x\in B_\delta:\ g>\lambda\}} g\,d\mu\biggr)\,d\lambda \\
&\leqslant \frac{C}{\mu(B_\delta)} \int_1^\infty \frac{\lambda}{e+\lambda}\mu\bigl(\{x\in B_\delta:\ Mg > D\lambda\}\bigr)\,d\lambda \\
&\leqslant \frac{C}{D\mu(B_\delta)} \int_0^\infty \mu\bigl(\{x\in B_\delta:\ Mg > \lambda\}\bigr)\,d\lambda \leqslant \frac{C}{D\mu(B_\delta)} \int_{B_\delta} Mg\,d\mu \\
&\leqslant \frac{C}{D\lambda_1 C_1 \mu(B_\delta)} \int_{B_\delta} Mf\,d\mu = \frac{C}{DC_1} < 1,
\end{aligned}$$

by choosing $C_1 > C/D$. Thus we have proved the case $k = 1$. Suppose that $k > 1$ and the result holds for $k-1$. If $g = f/(C_k\lambda_k)$ then

$$\begin{aligned}
\frac{1}{\mu(B_\delta)} \int_{B_\delta} \phi_k\biggl(\frac{f}{C_k\lambda_k}\biggr)\,d\mu &= \frac{1}{\mu(B_\delta)} \int_{B_\delta} g\bigl[\log(e+g)\bigr]^k\,d\mu \\
&= \frac{k}{\mu(B_\delta)} \int_{1-e}^\infty \frac{[\log(e+\lambda)]^{k-1}}{e+\lambda} \int_{\{x\in B_\delta:\ g(x)>\lambda\}} g\,d\mu\,d\lambda \\
&= \frac{k}{\mu(B_\delta)}\biggl(\int_{1-e}^{e} + \int_e^\infty\biggr) = I + II.
\end{aligned}$$

By the Lebesgue differentiation theorem,

$$I \leqslant \frac{[\log(2e)]^k}{\mu(B_\delta)} g(B_\delta) = \frac{[\log(2e)]^k}{C_k \int_{B_\delta} M^k f} \int_{B_\delta} f \leqslant \frac{[\log(2e)]^k}{C_k} < 1,$$

if we choose $C_k > [\log(2e)]^k$. Notice that by this election of C_k, $m_{B_\delta}(g) = g(B_\delta)/\mu(B_\delta) \leqslant 1$. Then, applying Lemma 4.3 we get

$$\begin{aligned}
II &= \frac{k}{\mu(B_\delta)} \int_e^\infty \frac{[\log(e+\lambda)]^{k-1}}{e+\lambda} \int_{\{x\in B_\delta:\ g(x)>\lambda\}} g\,d\mu\,d\lambda \\
&\leqslant \frac{kC}{\mu(B_\delta)} \int_e^\infty \frac{\lambda[\log(e+\lambda)]^{k-1}}{e+\lambda} \mu\bigl(\{x\in B_\delta:\ Mg > D\lambda\}\bigr)\,d\lambda \\
&\leqslant \frac{kC}{\mu(B_\delta)} \int_e^\infty \bigl[\log(e+\lambda)\bigr]^{k-1} \mu\bigl(\{x\in B_\delta:\ Mg > D\lambda\}\bigr)\,d\lambda \\
&\leqslant \frac{kC}{\mu(B_\delta)} \int_0^\infty \phi'_{k-1}(\lambda)\mu\bigl(\bigl\{x\in B_\delta:\ D^{-1}Mg > \lambda\bigr\}\bigr)\,d\lambda
\end{aligned}$$

$$\leqslant \frac{kC}{\mu(B_\delta)} \int\limits_{B_\delta} \phi_{k-1}\big(D^{-1}Mg\big)\,d\mu \leqslant \frac{kC}{\mu(B_\delta)} \int\limits_{B_\delta} \phi_{k-1}\left(\frac{Mf}{DC_k\lambda_k(f)}\right) d\mu$$

$$\leqslant \frac{kC'}{\mu(B_\delta)} \int\limits_{B_\delta} \phi_{k-1}\left(\frac{Mf}{C_k\lambda_k(f)}\right) d\mu.$$

Since $\lambda_k(f) = \lambda_{k-1}(Mf)$, by the induction hypothesis and the fact that $\phi_{k-1}(t)/t$ is increasing we obtain that

$$II \leqslant C'k\frac{C_{k-1}}{C_k}\frac{1}{\mu(B_\delta)} \int\limits_{B_\delta} \phi_{k-1}\left(\frac{Mf}{C_{k-1}\lambda_{k-1}(Mf)}\right)$$

$$\leqslant C'k\frac{C_{k-1}}{C_k} < 1$$

if we choose $C_k > C'kC_{k-1}$. In this way, inequality (4.8) is proved and also inequality (4.2). □

5. A pointwise estimate

As in the case of commutators of singular integrals, a key-point in the proof of Theorem 1.1 is the following pointwise estimate.

Lemma 5.1. *Let (X, d, μ) be a space of homogeneous type, $0 < \gamma < 1$, $b \in \mathrm{BMO}$, $m \in \mathbb{N}$, $\phi_m(t) = t[\log(e + t)]^m$ and $0 < \delta < \epsilon < 1$. Then there exists a constant C such that*

$$M^\sharp_{(\delta)}\big(\mathcal{I}^m_{\gamma,b} f\big)(x) \leqslant C\left(\sum_{k=0}^{m-1} \|b\|^{m-k}_{\mathrm{BMO}} M_{(\epsilon)}\big(\mathcal{I}^k_{\gamma,b} f\big)(x) + \|b\|^m_{\mathrm{BMO}} M_{\gamma,\phi_m} f(x)\right),$$

for all $x \in X$ and all $f \geqslant 0$.

The above lemma was proved in [7] and in [6] for $m = 1$ in the Euclidean context. The corresponding estimate for commutators of singular integrals was obtained in [19] (Euclidean case) and in [5] (space of homogeneous type and $m = 1$).

In order to prove Lemma 5.1 we shall need two previous results. The first one is the following result due to Macías and Torrea (see [13, Lemma 2.5]) and the other one is a technical lemma.

Lemma 5.2. [13] *Let (X, d, μ) be a space of homogeneous type. For $r > 0$ we denote*

$$E(x, r) = \big\{y \in X\colon\ \mu\big(\bar{B}\big(x, d(x, y)\big)\big) \leqslant r\big\},$$

where $\bar{B}(x, R) = \{y\colon\ d(x, y) \leqslant R\}$ and let $R^x_r = \sup\{d(x, y)\colon\ y \in E(x, r)\}$. Then

(i) $B(x, R^x_r) \subset E(x, r) \subset \bar{B}(x, R^x_r)$,
(ii) $\mu(E(x, r)) \leqslant r$, *and*
(iii) $\mu(\bar{B}(x, R^x_r)) \leqslant Cr$, *where C is a constant depending only on the constants of the space.*

We observe that if we denote by $B_\kappa(x, r)$ the set $\{y \in X\colon\ \kappa(x, y) \leqslant r\}$ (recall that $\kappa(x, y) = \mu(B(x, d(x, y)))$, if $x \neq y$ and $\kappa(x, x) = 0$) we can easily prove that

$$B_\kappa(x, r/A) \subset E(x, r), \tag{5.1}$$

for all $r \geqslant \mu(\{x\})$, where A is the constant in (2.2). The following properties of the sets $B_\kappa(x,r)$ were proved in [12]:

(i) $A^{-2}r \leqslant \mu(B_\kappa(x,r))$, if $r < \mu(X)$, and
(ii) $B_\kappa(x,r) = X$, if $\mu(X) \leqslant r$.

These properties and (5.1) imply that

$$A^{-1}r \leqslant \mu\big(E(x,r)\big), \quad \text{if } \mu\big(\{x\}\big) \leqslant r < A\mu(X), \quad \text{and} \tag{5.2}$$

$$E(x,r) = X, \quad \text{if } r \geqslant A\mu(X). \tag{5.3}$$

Now, we state and prove the technical lemma.

Lemma 5.3. *Let (X,d,μ) be a space of homogeneous type, $0 < \gamma < 1$, $b \in \mathrm{BMO}$, $m \in \mathbb{N}$ and $c, \lambda \in \mathbb{R}$. Then there exists a constant $C = C(m)$ such that*

$$\big|\mathcal{I}^m_{\gamma,b} f(y) - c\big| \leqslant C \sum_{k=0}^{m-1} \big|b(y) - \lambda\big|^{m-k} \mathcal{I}^k_{\gamma,b} f(y) + \big|I_\gamma\big(|b-\lambda|^m f\big)(y) - c\big|, \tag{5.4}$$

for all $f \geqslant 0$ and $y \in X$.

Proof. If λ is an arbitrary constant, we write (see [19])

$$I^m_{\gamma,b} f(x) = \sum_{k=0}^{m-1} C_{k,m}\big(b(x)-\lambda\big)^{m-k} I^k_{\gamma,b} f(x) + I_\gamma\big((\lambda - b)^m f\big)(x), \tag{5.5}$$

where $C_{k,m}$ are constants proceeding from the Newton's formula.

Clearly, $I^m_{\gamma,b} f(x) = \mathcal{I}^m_{\gamma,b} f(x)$ for even m. Then (5.4) follows from (5.5). Now, let us assume that m is odd. Writing $b(x) - b(y) = (b(x) - \lambda) + (\lambda - b(y))$ it is easy to show that

$$\mathcal{I}^m_{\gamma,b} f(x) \leqslant \mathcal{I}^{m-1}_{\gamma,b}\big(|b-\lambda| f\big)(x) + \big|b(x)-\lambda\big| \mathcal{I}^{m-1}_{\gamma,b} f(x) \tag{5.6}$$

and

$$\mathcal{I}^m_{\gamma,b} f(x) \geqslant \mathcal{I}^{m-1}_{\gamma,b}\big(|b-\lambda| f\big)(x) - \big|b(x)-\lambda\big| \mathcal{I}^{m-1}_{\gamma,b} f(x). \tag{5.7}$$

Since $m-1$ is even, using (5.5) we get that

$$\mathcal{I}^{m-1}_{\gamma,b}\big(|b-\lambda| f\big)(x) \leqslant \sum_{k=0}^{m-2} |C_{k,m}| \big|b(x)-\lambda\big|^{m-1-k} \mathcal{I}^k_{\gamma,b}\big(|b-\lambda| f\big)(x) + I_\gamma\big(|b-\lambda|^m f\big)(x).$$

On the other hand, we get that

$$\mathcal{I}^k_{\gamma,b}\big(|b-\lambda| f\big)(x) \leqslant \big|b(x)-\lambda\big| \mathcal{I}^k_{\gamma,b} f(x) + \mathcal{I}^{k+1}_{\gamma,b} f(x). \tag{5.8}$$

Then, from the above inequalities we get that

$$\mathcal{I}^m_{\gamma,b} f(x) \leqslant \sum_{k=0}^{m-1} \tilde{C}_{k,m} \big|b(x)-\lambda\big|^{m-k} \mathcal{I}^k_{\gamma,b} f(x) + I_\gamma\big(|b-\lambda|^m f\big)(x), \tag{5.9}$$

where $\tilde{C}_{k,m}$ are positive constants that also come from the Newton's formula. Now, notice that from (5.5) we also have that

$$\begin{aligned}\mathcal{I}^{m-1}_{\gamma,b}\big(|b-\lambda|f\big)(x) \geqslant{}& -\sum_{k=0}^{m-2}|C_{k,m}|\big|b(x)-\lambda\big|^{m-1-k}\mathcal{I}^{k}_{\gamma,b}\big(|b-\lambda|f\big)(x)\\ &+ I_\gamma\big(|b-\lambda|^m f\big)(x).\end{aligned}$$

Then, using (5.8) we get that from (5.7) and the above inequality that

$$\mathcal{I}^{m}_{\gamma,b}f(x) \geqslant -\sum_{k=0}^{m-1}\tilde{C}_{k,m}\big|b(x)-\lambda\big|^{m-k}\mathcal{I}^{k}_{\gamma,b}f(x) + I_\gamma\big(|b-\lambda|^m f\big)(x). \tag{5.10}$$

Now, (5.4) follows for the case m odd from (5.9) if $\mathcal{I}^{m}_{\gamma,b}f(x) - c > 0$ and from (5.10) if $\mathcal{I}^{m}_{\gamma,b}f(x) - c < 0$. □

Proof of Lemma 5.1. Let $B = B(z,r)$ be an arbitrary ball containing x. Since $0 < \delta < 1$ implies $||a|^\delta - |c|^\delta| \leqslant |a-c|^\delta$ for $a, c \in \mathbb{R}$, it is enough to show that for some constant C_B there exists $C > 0$ such that

$$\begin{aligned}&\left(\frac{1}{\mu(B)}\int_B \big|\mathcal{I}^{m}_{\gamma,b}f(y) - C_B\big|^\delta d\mu(y)\right)^{1/\delta}\\ &\quad\leqslant C\left(\sum_{k=0}^{m-1}\|b\|^{m-k}_{\mathrm{BMO}}M_{(\epsilon)}\big(\mathcal{I}^{k}_{\gamma,b}f\big)(x) + \|b\|^{m}_{\mathrm{BMO}}M_{\gamma,\phi_m}f(x)\right).\end{aligned}$$

Let $B^* = B(z, 2Kr)$. First, we shall assume that $\mu(B^*\setminus B) > 0$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{B^*}$, $\lambda = m_{B^*}(b)$ and $C_B = m_B(I_\gamma(|b-m_{B^*}(b)|^m f_2))$. Then applying Lemma 5.3 we get that

$$\left(\frac{1}{\mu(B)}\int_B \big|\mathcal{I}^{m}_{\gamma,b}f(y) - C_B\big|^\delta d\mu(y)\right)^{1/\delta} \leqslant I_1 + I_2 + I_3, \tag{5.11}$$

where

$$I_1 = C\sum_{k=0}^{m-1}\left(\frac{1}{\mu(B)}\int_B \big|b(y)-m_{B^*}(b)\big|^{(m-k)\delta}\big|\mathcal{I}^{k}_{\gamma,b}f(y)\big|^\delta\, d\mu(y)\right)^{1/\delta},$$

$$I_2 = C\left(\frac{1}{\mu(B)}\int_B \big|I_\gamma\big(\big|b-m_{B^*}(b)\big|^m f_1\big)(y)\big|^\delta\, d\mu(y)\right)^{1/\delta}, \quad \text{and}$$

$$I_3 = \left(\frac{1}{\mu(B)}\int_B \big|I_\gamma\big(\big|b-m_{B^*}(b)\big|^m f_2\big)(y) - m_B\big(I_\gamma\big(\big|b-m_{B^*}(b)\big|^m f_2\big)\big)\big|^\delta\, d\mu(y)\right)^{1/\delta}.$$

Using Hölder's inequality with exponents r and r' so that $1 < r < \epsilon/\delta$ and the John–Nirenberg theorem (see [4]) it follows that

$$\begin{aligned}I_1 &\leqslant C\sum_{k=0}^{m-1}\left(\frac{1}{\mu(B)}\int_B \big|b-m_{B^*}(b)\big|^{(m-k)\delta r'}\right)^{1/\delta r'}\left(\frac{1}{\mu(B)}\int_B \big|\mathcal{I}^{k}_{\gamma,b}f(y)\big|^{\delta r}\right)^{1/\delta r}\\ &\leqslant C\sum_{k=0}^{m-1}\|b\|^{m-k}_{\mathrm{BMO}}M_{(\delta r)}\big(\mathcal{I}^{k}_{\gamma,b}f\big)(x) \leqslant C\sum_{k=0}^{m-1}\|b\|^{m-k}_{\mathrm{BMO}}M_{(\epsilon)}\big(\mathcal{I}^{k}_{\gamma,b}f\big)(x).\end{aligned}$$

Now, we estimate II. Since I_γ is of weak type $(1,(1-\gamma)^{-1})$ (see [2, Lemma 2.1]), Kolmogorov's inequality and (3.2) yield

$$I_2 \leqslant \frac{C}{\mu(B)^{1-\gamma}} \int\limits_{B^*} \left|b - m_{B^*}(b)\right|^m f(y)\,d\mu(y)$$
$$\leqslant C\mu(B)^\gamma \left\| \left|b - m_{B^*}(b)\right|^m \right\|_{\tilde{\phi}_m, B^*} \|f\|_{\phi_m, B^*},$$

where $\tilde{\phi}_m(t) = \tilde{\phi}(t^{1/m})$ with $\tilde{\phi} = \exp(t)$. On the other hand, by John–Nirenberg Theorem it is easy to see that there is $C > 0$ such that $\|b - m_B(b)\|_{\tilde{\phi}, B} \leqslant C\|b\|_{\mathrm{BMO}}$, for all balls B in X. Then

$$I_2 \leqslant C\left\|b - m_{B^*}(b)\right\|_{\tilde{\phi}, B^*}^m M_{\gamma,\phi_m} f(x) \leqslant C\|b\|_{\mathrm{BMO}}^m M_{\gamma,\phi_m} f(x).$$

To estimate I_3 we shall use Lemma 5.2 following the ideas in [2]. Notice that, by using Jensen's inequality I_3 is dominated by

$$\frac{1}{\mu(B)^2} \int\limits_B \int\limits_B \int\limits_{X\setminus B^*} \left|Q_\gamma(y,w) - Q_\gamma(v,w)\right| \left|b(w) - m_{B^*}(b)\right|^m f(w)\,d\mu(w)\,d\mu(v)\,d\mu(y).$$

It follows from the definition of Q_γ, the Mean Value Theorem and condition (2.3) that

$$\begin{aligned}\left|Q_\gamma(y,w) - Q_\gamma(v,w)\right| &= \left|\eta(y,w)^{(\gamma-1)} - \eta(v,w)^{(\gamma-1)}\right| \\ &\leqslant C\frac{|\eta(v,w)^{(1-\gamma)} - \eta(y,w)^{(1-\gamma)}|}{\eta(z,w)^{2(1-\gamma)}} \\ &\leqslant C\frac{\eta(y,v)^\theta}{\eta(z,w)^{1-\gamma+\theta}} \leqslant C\frac{\mu(B)^\theta}{\mu(B(z,d(z,w)))^{1-\gamma+\theta}}\end{aligned}$$

for every $y, v \in B$ and every $\omega \in X \setminus B^*$, where z is the center of B and C is independent of B. Then

$$I_3 \leqslant C\mu(B)^\theta \int\limits_{X\setminus B^*} \frac{|b(w) - m_{B^*}(b)|^m f(w)}{\mu(B(z,d(z,w)))^{1-\gamma+\theta}}\,d\mu(w).$$

Let us write $R_0 = \mu(B) = \mu(B(z,r))$. With the notation of Lemma 5.2 we define $E_i(z) = E(z, 2^i R_0)$ and $B_i = B(z, R^z_{2^{i+1}R_0})$. Then, since $\mu(B^* \setminus B) > 0$, by the definitions of the sets $E_i(z)$ we get that

$$\begin{aligned} I_3 &\leqslant C\mu(B)^\theta \int\limits_{X\setminus E_0(z)} \frac{|b(w) - m_{B^*}(b)|^m f(w)}{\mu(B(z,d(z,w)))^{1-\gamma+\theta}}\,d\mu(w) \\ &\leqslant C\mu(B)^\theta \sum_{i=0}^\infty \int\limits_{E_{i+1}(z)\setminus E_i(z)} \frac{|b(w) - m_{B^*}(b)|^m f(w)}{\mu(B(z,d(z,w)))^{1-\gamma+\theta}}\,d\mu(w).\end{aligned}$$

From Lemma 5.2(iii) we get that $\mu(B_i) \leqslant C2^{i+1}R_0$ and if $w \notin E_i(z)$ we have that

$$\mu\big(B\big(z,d(z,w)\big)\big) \geqslant \frac{\mu(\bar{B}(z,d(z,w)))}{A} \geqslant \frac{2^i R_0}{A} \geqslant \frac{\mu(B_i)}{2CA} \geqslant \frac{\mu(2B_i)}{2CA^2},$$

where $2B_i = B(z, 2R^z_{2^{i+1}R_0})$. Thus, by Lemma 5.2(i)

$$\begin{aligned}
I_3 &\leqslant C\mu(B)^\theta \sum_{i=0}^{\infty} \frac{(2^i R_0)^{-\theta}}{\mu(2B_i)^{1-\gamma}} \int\limits_{2B_i} \left|b(w) - m_{B^*}(b)\right|^m f(w)\, d\mu(w) \\
&= C\sum_{i=0}^{\infty} 2^{-i\theta} \frac{\mu(2B_i)^\gamma}{\mu(2B_i)} \int\limits_{2B_i} \left|b(w) - m_{B^*}(b)\right|^m f(w)\, d\mu(w) \\
&\leqslant C\sum_{i=0}^{\infty} 2^{-i\theta} \frac{\mu(2B_i)^\gamma}{\mu(2B_i)} \int\limits_{2B_i} \left|b(w) - m_{2B_i}(b)\right|^m f(w)\, d\mu(w) \\
&\quad + C\sum_{i=0}^{\infty} 2^{-i\theta} \frac{\mu(2B_i)^\gamma}{\mu(2B_i)} \left|m_{2B_i}(b) - m_{B^*}(b)\right|^m \int\limits_{2B_i} f(w)\, d\mu(w) \\
&\leqslant C\sum_{i=0}^{\infty} 2^{-i\theta} \mu(2B_i)^\gamma \left\|b - m_{2B_i}(b)\right\|_{\tilde{\phi},2B_i}^m \|f\|_{\phi_m,2B_i} \\
&\quad + C\sum_{i=0}^{\infty} 2^{-i\theta} \left|m_{2B_i}(b) - m_{B^*}(b)\right|^m M_\gamma f(x). \qquad (5.12)
\end{aligned}$$

Let us observe that for all $i \geqslant 1$

$$\begin{aligned}
\left|m_{2B_i}(b) - m_{2B_{i-1}}(b)\right| &\leqslant \frac{1}{\mu(2B_{i-1})} \int\limits_{2B_{i-1}} \left|b(y) - m_{2B_i}(b)\right| d\mu(y) \\
&\leqslant \frac{\mu(2B_i)}{\mu(2B_{i-1})} \left(\frac{1}{\mu(2B_i)} \int\limits_{2B_i} \left|b(y) - m_{2B_i}(b)\right| d\mu(y)\right).
\end{aligned}$$

Now, from Lemma 5.2, (5.2) and (5.3) (notice that $\mu(\{z\}) \leqslant R_0$) we get that

$$\left|m_{2B_i}(b) - m_{2B_{i-1}}(b)\right| \leqslant C\frac{\mu(E_{i+1}(z))}{\mu(E_{i-1}(z))} \|b\|_{\mathrm{BMO}} \leqslant C\|b\|_{\mathrm{BMO}}. \qquad (5.13)$$

On the other hand, we have that $B \subset 2B_0$. Then

$$\begin{aligned}
\left|m_{2B_0}(b) - m_{B^*}(b)\right| &\leqslant \left|m_{2B_0}(b) - m_B(b)\right| + \left|m_B(b) - m_{B^*}(b)\right| \\
&\leqslant \frac{1}{\mu(B)} \int\limits_B \left|b(y) - m_{2B_0}(b)\right| + \frac{1}{\mu(B)} \int\limits_B \left|b(y) - m_{B^*}(b)\right| \\
&\leqslant C\|b\|_{\mathrm{BMO}}. \qquad (5.14)
\end{aligned}$$

Thus, using (5.13) and (5.14) in (5.12) and the inequality $\|b - m_{2B_i}(b)\|_{\tilde{\phi},2B_i}^m \leqslant C\|b\|_{\mathrm{BMO}}^m$ we get that

$$I_3 \leqslant C\sum_{i=0}^{\infty} 2^{-i\theta} \|b\|_{\mathrm{BMO}}^m \big(M_{\gamma,\phi_m} f(x) + i M_\gamma f(x)\big) \leqslant C\|b\|_{\mathrm{BMO}}^m M_{\gamma,\phi_m} f(x),$$

which finishes the proof in the case $\mu(B^* \setminus B) > 0$. Now, let us assume that $\mu(B^* \setminus B) = 0$ and $\mu(X \setminus B^*) \neq 0$ (if $\mu(X \setminus B^*) = 0$ then the term I_3 in (5.11) is equal to zero and the terms I_1 and I_2 can be estimated as before). Then, we can chose a ball B' such that $\mu(B') = \mu(B)$ and $\mu((B')^* \setminus B') > 0$. Now, the proof of the lemma follows as in the previous case taking the constant $C_{B'}$ in (5.11) instead of C_B. □

6. Proof of Theorem 1.1

Without loss of generality we may assume that $f \geqslant 0$. We shall prove the theorem by using an induction argument. First, notice that (1.7) for the case $m = 0$ is the inequality

$$\|\mathcal{I}_\gamma f\|_{p,\omega} \leqslant C\|M_\gamma f\|_{p,\omega},$$

which already holds as it was observed in Section 1. Suppose now that (1.7) is true for $0, 1, \ldots,$ $m-1$. Then applying the Lebesgue Differentiation Theorem and Lemma 2.4 we get for $\delta > 0$ that

$$\begin{aligned}\left\|\mathcal{I}^m_{\gamma,b} f\right\|^p_{p,\omega} &\leqslant \left\|M_{(\delta)}\left(\mathcal{I}^m_{\gamma,b} f\right)\right\|^p_{p,\omega}\\ &\leqslant \begin{cases} C\|M^\sharp_{(\delta)}(\mathcal{I}^m_{\gamma,b} f)\|^p_{p,\omega} & \text{if } \mu(X) = \infty,\\ Cw(X)(m_X(|\mathcal{I}^m_{\gamma,b} f|^\delta))^{p/\delta} + C\|M^\sharp_{(\delta)}(\mathcal{I}^m_{\gamma,b} f)\|^p_{p,\omega} & \text{if } \mu(X) < \infty.\end{cases}\end{aligned}$$

Applying Lemma 5.1 we get for $\delta < \epsilon$ that

$$\left\|M^\sharp_{(\delta)}\left(\mathcal{I}^m_{\gamma,b} f\right)\right\|^p_{p,\omega} \leqslant C\sum_{k=0}^{m-1}\|b\|^{(m-k)p}_{\mathrm{BMO}}\left\|M_{(\epsilon)}\left(\mathcal{I}^k_{\gamma,b} f\right)\right\|^p_{p,\omega} + C\|b\|^{mp}_{\mathrm{BMO}}\|M_{\gamma,\phi_m} f\|^p_{p,\omega}.$$

Since $\omega \in A_\infty$ then there exists $r > 1$ so that $\omega \in A_r$. Choosing ϵ with $0 < \epsilon < p/r$, it follows that $\omega \in A_{p/\epsilon}$ and $M_{(\epsilon)}$ is bounded in $L^p(\omega)$. From this fact, applying Lemma 4.1, the induction and the inequality $M_\gamma(M^k f) \leqslant M_\gamma(M^m f)$ for $k \leqslant m$, we get that

$$\begin{aligned}\left\|M^\sharp_{(\delta)}\left(\mathcal{I}^m_{\gamma,b} f\right)\right\|_{p,\omega} &\leqslant C\sum_{k=0}^{m-1}\|b\|^{(m-k)p}_{\mathrm{BMO}}\left\|\mathcal{I}^k_{\gamma,b} f\right\|^p_{p,\omega} + C\|b\|^{mp}_{\mathrm{BMO}}\left\|M_\gamma\left(M^m f\right)\right\|^p_{p,\omega}\\ &\leqslant C\|b\|^{mp}_{\mathrm{BMO}}\left\|M_\gamma\left(M^m f\right)\right\|^p_{p,\omega}.\end{aligned}$$

Now, we estimate the term $w(X)(m_X(|\mathcal{I}^m_{\gamma,b} f|^\delta))^{p/\delta}$ when $\mu(X) < \infty$. It is known that $\mu(X) < \infty$ implies that X is bounded, i.e., there exists a ball B such that $X = B$. Applying (5.5) with $\lambda = m_X(b)$ and proceeding as in the proof of Lemma 5.1 (see boundedness of I_1 and I_2 in (5.11) with X instead of B^*) we get that for $0 < \delta < \epsilon < 1$

$$\begin{aligned}m_X\left(\left|\mathcal{I}^m_{\gamma,b} f\right|^\delta\right) &\leqslant C\sum_{k=0}^{m-1}\left(\frac{1}{\mu(X)}\int_X\left|b(y) - m_X(b)\right|^{(m-k)\delta}\left|\mathcal{I}^k_{\gamma,b} f(y)\right|^\delta\right)\\ &\quad + C\left(\frac{1}{\mu(X)}\int_X\left|I_\gamma\left(\left|b - m_X(b)\right|^m f\right)(y)\right|^\delta d\mu(y)\right)\\ &\leqslant C\sum_{k=0}^{m-1}\|b\|^{(m-k)\delta}_{\mathrm{BMO}}\frac{1}{\mu(X)}\int_X\left[M_{(\epsilon)}\left(\mathcal{I}^k_{\gamma,b} f\right)(x)\right]^\delta d\mu(x)\\ &\quad + C\mu(X)^{\gamma\delta}\|b\|^{m\delta}_{\mathrm{BMO}}\|f\|^\delta_{\phi_m,X}.\end{aligned}\tag{6.1}$$

Choosing ϵ as in the boundedness of the term $\|M^\sharp_{(\delta)}(\mathcal{I}^m_{\gamma,b}) f\|_{p,\omega}$, by Hölder's inequality and the fact that $\omega \in A_{p/\epsilon}$ implies $\omega \in A_{p/\delta}$ for $\delta < \epsilon$, we have that

$$\frac{\omega(X)^{\delta/p}}{\mu(X)}\int\limits_X \left|M_{(\epsilon)}\left(\mathcal{I}^k_{\gamma,b}f\right)(x)\right|^\delta d\mu(x)$$
$$\leqslant \frac{\omega(X)^{\delta/p}}{\mu(X)}\left(\int\limits_X w^{-p/(p-\delta)}\right)^{1-\delta/p}\left(\int\limits_X \left|M_{(\epsilon)}\left(\mathcal{I}^k_{\gamma,b}f\right)(x)\right|^p \omega(x)\,d\mu(x)\right)^{\delta/p}$$
$$\leqslant C\left(\int\limits_X \left|\mathcal{I}^k_{\gamma,b}f(x)\right|^p \omega(x)\,d\mu(x)\right)^{\delta/p}. \tag{6.2}$$

On the other hand, by definition we have that

$$\omega(X)^{\delta/p}\mu(X)^{\gamma\delta}\|f\|^\delta_{\phi_m,X} \leqslant \left(\int\limits_X |M_{\gamma,\phi_m}f|^p w\right)^{\delta/p}. \tag{6.3}$$

Replacing (6.2) and (6.3) in (6.1), using the induction hypothesis and Lemma 4.1 we have

$$w(X)\left[m_X\left(\left|\mathcal{I}^m_{\gamma,b}f\right|^\delta\right)\right]^{p/\delta} \leqslant C\sum_{k=0}^{m-1}\|b\|^{(m-k)p}_{\mathrm{BMO}}\left\|\mathcal{I}^k_{\gamma,b}f\right\|^p_{p,\omega} + C\|b\|^{mp}_{\mathrm{BMO}}\|M_{\gamma,\phi_m}f\|^p_{p,\omega}$$
$$\leqslant C\|b\|^{pm}_{\mathrm{BMO}}\left\|M_\gamma\left(M^m f\right)\right\|^p_{p,\omega}.$$

Thus the proof of the theorem is finished. □

7. Proof of Theorem 1.2

First, let us observe that if $\gamma p \geqslant 1$ then the inequality (1.8) holds trivially. In fact, if $\gamma p > 1$ then for all $g \neq 0$ we have that $M_{\gamma p}g(x) = \infty$ up to a set of μ-measure zero. The same holds if $\gamma p = 1$ and $g \notin L^1(X)$ which is the case when $g = M^{[(m+1)p]}\omega$. Then, we shall only consider the case $\gamma p < 1$. By a duality argument, it is enough to show that

$$\int \left|\mathcal{I}^m_{\gamma,b}f(x)\right|^{p'}\left[M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\right]^{1-p'} d\mu(x) \leqslant C\int \left|f(x)\right|^{p'}\omega(x)^{1-p'}\,d\mu(x).$$

Notice first that for $0 < \gamma < 1$, $0 \leqslant \delta < 1$ the function $(M_\gamma g)^\delta$ belongs to the Muckenhoupt class of weights A_1. The proof of this fact follows as in the case $\gamma = 0$ (see, for example, [8]) by using that M_γ is of weak type $(1, (1-\gamma)^{-1})$ and Lemma 4.2. Thus, choosing $r > p'$ and $\delta = (p'-1)/(r-1)$,

$$\left[M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\right]^{1-p'} = \left\{\left[M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\right]^{\frac{p'-1}{r-1}}\right\}^{1-r} \in A_r \subset A_\infty.$$

Applying Theorem 1.1 we get

$$\int \left|\mathcal{I}^m_{\gamma,b}f(x)\right|^{p'}\left[M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\right]^{1-p'} d\mu(x)$$
$$\leqslant C\int \left[M_\gamma\left(M^m f\right)(x)\right]^{p'}\left[M_{\gamma p}\left(M^{[(m+1)p]}\omega\right)(x)\right]^{1-p'} d\mu(x), \tag{7.1}$$

and then, by Lemma 4.1, it is enough to show that

$$\int \left[M_{\gamma,\phi_m}f(x)\right]^{p'}\left[M_{\gamma p,\phi_{[(m+1)p]}}\omega(x)\right]^{1-p'} d\mu(x) \leqslant C\int \left|f(x)\right|^{p'}\omega(x)^{1-p'}\,d\mu(x).$$

By defining $g = f\omega^{-1/p}$, the above inequality may be stated as

$$\int \left[M_{\gamma,\phi_m}\left(g\omega^{1/p}\right)(x)\right]^{p'}\left[M_{\gamma p,\phi_{[(m+1)p]}}\omega(x)\right]^{1-p'} d\mu(x) \leqslant C\int \left|g(x)\right|^{p'} d\mu(x).$$

Now, as in the case of commutators of singular integral operators (see [18]), we get for $\phi_m(t) = t[\log(e+t)]^m$ and for large t that

$$\phi_m^{-1}(t) \approx \frac{t}{[\log(e+t)]^m} = \frac{t^{1/p}}{[\log(e+t)]^{m+(p-1+\epsilon)/p}} t^{1/p'}\left[\log(e+t)\right]^{(p-1+\epsilon)/p} = \psi^{-1}(t)\varphi^{-1}(t),$$

where $\psi(t) \approx t^p[\log(e+t)]^{(m+1)p-1+\epsilon}$ and $\varphi(t) \approx t^{p'}[\log(e+t)]^{-(1+(p'-1)\epsilon)}$ (see [16]). Thus, by (3.3),

$$\mu(B)^\gamma \left\|g\omega^{1/p}\right\|_{\phi_m,B} \leqslant \mu(B)^\gamma \|g\|_{\varphi,B}\left\|\omega^{1/p}\right\|_{\psi,B}.$$

Choosing $\epsilon > 0$ so that $(m+1)p - 1 + \epsilon = [(m+1)p]$ we have that

$$\mu(B)^\gamma \left\|g\omega^{1/p}\right\|_{\phi_m,B} \leqslant \|g\|_{\varphi,B}\left[\mu(B)^{\gamma p}\|\omega\|_{\phi_{[(m+1)p]},B}\right]^{1/p},$$

so that

$$M_{\gamma,\phi_m}\left(g\omega^{1/p}\right)(x) \leqslant M_\varphi g(x)\left[M_{\gamma p,\phi_{[(m+1)p]}}\omega(x)\right]^{1/p}.$$

Moreover, since φ satisfies condition $B_{p'}$ we apply Theorem 3.1 to get

$$\int \left[M_{\gamma,\phi_m}\left(g\omega^{1/p}\right)(x)\right]^{p'}\left[M_{\gamma p,\phi[(m+1)p]}\omega(x)\right]^{1-p'} d\mu(x) \leqslant C\int \left|M_\varphi g(x)\right|^{p'} d\mu(x) \leqslant C\int \left|g(x)\right|^{p'} d\mu(x),$$

and we are done.

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