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Squeezed states from a quantum deformed oscillator Hamiltonian

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ABSTRACT

The spectrum and the time evolution of a system, which is modeled by a non-hermitian quantum deformed oscillator Hamiltonian, is analyzed. The proposed Hamiltonian is constructed from a non-standard realization of the algebra of Heisenberg. We show that, for certain values of the coupling constants and for a range of values of the deformation parameter, the deformed Hamiltonian is a pseudo-hermitic Hamiltonian. We explore the conditions under which the Hamiltonian is similar to a Swanson Hamiltonian. Also, we show that the lowest eigenstate of the system is a squeezed state. We study the time evolution of the system, for different initial states, by computing the corresponding Wigner functions.

1. Introduction

The fundamental mathematical aspects of the so-called quantum deformed algebras have been studied intensively [1,2]. From the early appearance of quantum groups, great effort has been devoted to he search for physical inspired hamiltonians. Though concrete applications of the formalism have been explored [3–12], the field is still open.

Recently, the analysis of the properties of q-deformed potentials has renewed interest in connection with the description of different molecular systems [13–16]. The essential properties of these systems, are modeled through attractive finite range potentials [17–20,14]. As an example, the authors of [17,18] have reported the construction of squeezed coherent states for the hydrogen chloride molecule $^{1}H^{35}Cl$. In [19], a q-deformed hamiltonian has been constructed to model a Pöschl–Teller potential, the behavior of the spectrum, particularly the uncertainty relations of the eigenstates have been analyzed. In the same line of work, the authors of [20], by using the f-deformed oscillator formalism, have introduced a class of squeezed coherent states for a Morse potential system. A common feature of these works is the looking for states that optimizes the Heisenberg Uncertainty Relations.

In a series of papers, Wess et al. [21–25], and Zang [26–28] have studied different non-standard q-deformation schemes of the Heisenberg–Weyl algebra, and they have obtained the corresponding an-harmonic hamiltonians. In [29], we have applied these ideas to study of possible correspondence between q-deformations and boundary conditions. We have compared the spectrum of the q-deformed harmonic oscillator with that of different finite range potentials, i.e. the Woods–Saxon potential and the a Pöschl–Teller potential. We have performed a similar analysis in [30], by constructing hamiltonians from the non-standard algebra reported in [31–39].

In this work, we continue the analysis of the q-deformed Hamiltonian of [30]. We are interested in the description of the squeezing properties of the corresponding eigenstates. Also, we aim to discuss the time evolution of different initial states.

The work is organized as follows. In Section 1 we present the model Hamiltonian as well as the adopted formalism. In Subsection 2.1, we briefly reviewed the concept of pseudo-hermicity. In Subsection 2.2 we discussed the time evolution of a pseudo-hermitian hamiltonian for different initial states. Analytical and numerical results are presented in Section 3. Conclusions are drawn in Section 4.

2. Formalism

In recent works [40–42], the interest in the study of the harmonic oscillator has been renewed in connection with the construction of minimum-uncertainty squeezed states. The squeezed harmonic oscillator Hamiltonian can be written as

$$H_{sq} = \eta\{a^{\dagger}, a\} + \zeta(a^{\dagger 2} + a^{2}), \tag{1}$$

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where the $\{N,\ a^{\dagger},\ a\}$ are the generators of the Heisenberg oscillator algebra. That is

$$[N,a]=-a, \quad \left[N,a^{\dagger}\right]=a^{\dagger}, \quad \left[a,a^{\dagger}\right]=\mathbf{I}, \quad [\mathbf{I},\cdot]=0. \tag{2}$$

As has been reported in [40–42], the eigenvalues and eigenfunctions of Eq. (1), for $|\zeta|<|\eta|$, can be obtained analytically, and the lowest eigenstate is a squeezed state.

We shall generalized the Hamiltonian of Eq. (1) by replacing the generators of the Heisenberg algebra of Eq. (2) by the generators of the non-standard $U_{\lambda}^{n}(h4)$ oscillator algebra [36].

The generators of the Hopf $U_{\lambda}^{n}(h4)$ algebra, $\{A_+,\ A_-,\ N,\ M\}$, obey the following commutation relations [36]

$$\begin{aligned}&[N,A_+]=\frac{e^{\lambda A_+}-1}{\lambda},\ [N,A_-]=-A_-,\\ &[A_-,A_+]=Me^{\lambda A_+}, \quad [M,\cdot]=0,\end{aligned} \tag{3}$$

and the corresponding Casimir operator, C_{λ}, is given by

$$C_{\lambda}=NM-\frac{1}{2}\left\{\frac{1-e^{-\lambda A_+}}{\lambda}A_-+A_-\frac{1-e^{-\lambda A_+}}{\lambda}\right\}. \tag{4}$$

The general Hamiltonian, which we propose to study, reads [30]

$$H_{\lambda}=\eta\{A_+,A_-\}+\zeta(A_+^2+A_-^2). \tag{5}$$

Among the possible boson realizations of the non-standard algebra, we shall adopt the exponential form [30,43]

$$\begin{aligned}&A_+=a^{\dagger}, \quad A_-=\delta e^{\lambda a^{\dagger}}a+\delta\beta z e^{\lambda a^{\dagger}},\\ &N=\frac{e^{\lambda a^{\dagger}}-1}{\lambda}a+\beta\frac{e^{\lambda a^{\dagger}}+1}{2},\\ &M=\delta\mathbf{I}.\end{aligned} \tag{6}$$

In the previous equations, $a^{\dagger}$ and a are boson creation and annihilation operators, respectively. They obey the usual commutation relation of Eq. (2). Without loss of generality, we shall take $\delta=1$ and $\beta=0$ [30].

The boson image of H_{λ} can be written as [30]

$$\begin{aligned}H_{\lambda}=&(\eta-\zeta)(p^2-i\ \{p,\Theta_{\lambda}\})+\zeta\Theta_{\lambda}^2\\ &+(\eta+\zeta)(x^2+\{x,\Theta_{\lambda}\}),\end{aligned} \tag{7}$$

with

$$p=\frac{i}{\sqrt{2}}(a^{\dagger}-a), \quad x=\frac{1}{\sqrt{2}}(a^{\dagger}+a), \tag{8}$$

and

$$\Theta_{\lambda}=\frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}a^{\dagger k}a. \tag{9}$$

In what follows, we shall study the properties of the Hamiltonian of Eq. (7) up to order $O(\lambda^3)$. After reordering terms via Wick Theorem, we obtain the Hamiltonian

$$H=H_0+H_{res}, \tag{10}$$

with

$$\begin{aligned}H_0=&2(\eta+\zeta\lambda^2)(a^{\dagger}a+1/2)-\lambda^2\zeta\\ &+\zeta a^2+(\zeta+\eta\lambda^2/2)a^{\dagger 2}+\lambda(\eta a^{\dagger}+\zeta a),\end{aligned} \tag{11}$$

and

$$H_{res}=2\lambda(\eta a^{\dagger 2}a+\zeta a^{\dagger}a^2)+\lambda^2(2\zeta a^{\dagger 2}a^2+\eta a^{\dagger 3}a). \tag{12}$$

Clearly, H of Eq. (10) is a non-hermitian hamiltonian. We shall show that the Hamiltonian H is a pseudo-hermitian hamiltonian [44–46]. In the next subsection, we briefly review the essentials of the formalism of pseudo-hermitian operators.

2.1. Pseudo-hermitian hamiltonians

A non-hermitian hamiltonian H is pseudo-hermitian, if it is similar to an hermitian hamiltonian h [44–46]. That is

$$h=\Upsilon H\Upsilon^{-1}, \qquad h=h^{\dagger}. \tag{13}$$

As it can be demonstrated from the previous definition, h and H are iso-spectral hamiltonians

$$\begin{aligned}&h|\phi\rangle=\epsilon|\phi\rangle,\\ &H|\widetilde{\phi}\rangle=\epsilon|\widetilde{\phi}\rangle,\end{aligned} \tag{14}$$

with $|\phi\rangle=\Upsilon|\widetilde{\phi}\rangle$.

As h is an hermitian hamiltonian, $h=h^{\dagger}$, there exists a pseudo-metric operator U, such that $H^{\dagger}U=UH$. The explicit form of U can be derived from Eq. (13). The pseudo-metric operator U can be written as $U=\Upsilon^{\dagger}\Upsilon$. Consequently

$$\begin{aligned}&H|\widetilde{\psi}\rangle=E|\widetilde{\psi}\rangle,\\ &H^{\dagger}|\overline{\psi}\rangle=E|\overline{\psi}\rangle,\end{aligned} \tag{15}$$

and the eigenvector of H ($|\widetilde{\psi}\rangle$) and of $H^{\dagger}(|\widetilde{\psi}\rangle)$, corresponding to the same eigenvalue, are connected by the metric operator U, $|\overline{\psi}\rangle=U|\widetilde{\psi}\rangle$.

The Hamiltonian H is not an hermitian operator in the Hilbert space $\mathcal{H}=(\mathcal{H},\langle .|.\rangle)$, where $\langle .|.\rangle$ is the usual inner product. However, as H is a pseudo-hermitian operator we can define a new inner product over $\mathcal{H}$ [46]

$$\langle\ .|.\ \rangle_U:\mathcal{H}\times\mathcal{H}\rightarrow C, \quad \langle\psi|\phi\rangle_U:=\langle\psi U|\phi\rangle, \tag{16}$$

where $\langle\ .|.\ \rangle$ is the usual inner product in $\mathcal{H}$. The Hilbert space $\mathcal{H}$ equipped with the inner product $\langle .|.\rangle_U$ is the new physical Hilbert space $\mathcal{H}_U:=(\mathcal{H},\langle .|.\rangle_U)$.

The set $\{|\overline{\psi}_{\alpha}\rangle,|\widetilde{\psi}_{\beta}\rangle\}$ forms a bi-orthonormal basis for $\mathcal{H}$ [47], with

$$\langle\widetilde{\psi}_{\alpha}|\widetilde{\psi}_{\beta}\rangle_U=\langle\overline{\psi}_{\alpha}|\widetilde{\psi}_{\beta}\rangle=\delta_{\alpha\beta}, \tag{17}$$

and the Identity operator $\mathbb{1}$ is – in $\mathcal{H}$ – given by

$$\mathbb{1}=\sum_{\alpha}|\widetilde{\psi}_{\alpha}\rangle\langle\overline{\psi}_{\alpha}|. \tag{18}$$

In the Hilbert space $\mathcal{H}_U$, the mean values of a general pseudo-hermitian operator $\hat{O}=\Upsilon^{-1}\hat{o}\Upsilon$, with $\hat{o}=\hat{o}^{\dagger}$, are obtained as [46]

$$\langle\widetilde{\psi}|\hat{O}|\widetilde{\phi}\rangle_U=\langle\widetilde{\psi}|U\hat{O}|\widetilde{\phi}\rangle=\langle\widetilde{\psi}|\Upsilon^{\dagger}\hat{o}\Upsilon|\widetilde{\phi}\rangle. \tag{19}$$

2.2. Time evolution

We shall construct the time evolution of a general initial state, $|\widetilde{I}\rangle$ in the physical space $\mathcal{H}_U$.

In terms of the eigenvectors of H the initial state can be written as

$$|\widetilde{I}\rangle=\sum_k c_k\ |\widetilde{\Phi}_k\rangle. \tag{20}$$

We shall assume that the initial state is normalized, that is $\langle\widetilde{I}|\widetilde{I}\rangle_U=1$.

As H is non-hermitian, the mean value of an operator O in $\mathcal{H}_U$ is evaluated as

$$\begin{aligned}\langle\widetilde{I}(t)|O|\widetilde{I}(t)\rangle_U&=\langle\widetilde{I}(0)|e^{iH^{\dagger}t}U\,O\,e^{-iHt}|\widetilde{I}(0)\rangle=\\ &=\sum_{n,m}c_nc_m^{*}e^{i(E_m-E_n)t}\langle\widetilde{\Phi}_m|\Upsilon^{\dagger}o\Upsilon|\widetilde{\Phi}_n\rangle.\end{aligned} \tag{21}$$

2.2.1. Initial state

In this work, we shall discuss the time evolution of the Gazeau–Klauder (GK) [48,49] coherent state associated to H of Eq. (10). The GK coherent can be written as

$$|GK\rangle = \mathcal{N} \sum_n \frac{z_\gamma^n}{\sqrt{\rho_n}} e^{i\gamma E_n} |\widetilde{\Phi}_n\rangle, \tag{22}$$

where $|\widetilde{\Phi}_n\rangle$ are the eigenstates of H, E_n the corresponding eigenvalues, and $\rho_n = \prod_n E_n$. Notice that, as has been demonstrated in [48], for the usual harmonic oscillator the $|GK\rangle$ state reduces to

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{23}$$

with $z = z_\gamma e^{-i\gamma}$, and being $|n\rangle$ the eigenstates of the harmonic oscillator Hamiltonian.

As a second example, we shall discuss the time evolution of the state constructed as a superposition of the ground state and first excited state of H. That is

$$|\widetilde{I}_1\rangle = \frac{1}{\sqrt{2}}(|\widetilde{\Phi}_0\rangle + |\widetilde{\Phi}_1\rangle). \tag{24}$$

2.2.2. Wigner distribution function

We shall study the time evolution of the system in phase space we shall compute the corresponding Wigner function distribution [50]. In $\mathcal{H}_U$, the Wigner distribution for the initial state $|I\rangle$ is given by

$$W(x,p,t) = \frac{1}{2\pi} \int e^{ipy} \langle \widetilde{I}(t)|\Upsilon^\dagger |x - \frac{y}{2}\rangle\langle x + \frac{y}{2}|\Upsilon|\widetilde{I}(t)\rangle dy. \tag{25}$$

The Wigner function of Eq. (25) is normalized in phase space. That means

$$\int W(x,p,t)\, dx\, dp = 1. \tag{26}$$

3. Results an discussion

We shall, in the first place, study the quadratic sector of the Hamiltonian of Eq. (10). In a second step, we shall analyze the contribution of H_{res}, which involves normal order products of three and four boson operators.

3.1. Quadratic Hamiltonian

Let us consider the properties of a system that evolves in time through the Hamiltonian H_0 of Eq. (11). In order to facilitate the algebra, we shall rewrite H_0 as

$$H_0 = \omega(\widetilde{a}^\dagger \widetilde{a} + \frac{1}{2}) + \alpha \widetilde{a}^2 + \beta(\widetilde{a}^\dagger)^2 + H_{00}, \tag{27}$$

with $\widetilde{a}^\dagger = a^\dagger - a_1$, $\widetilde{a} = a^\dagger - a_0$, and

$$\begin{aligned} \omega &= 2\eta + 2\zeta\lambda^2, \quad (28)\\ \alpha &= \zeta,\\ \beta &= \frac{\lambda^2\eta}{2} + \zeta,\\ a_0 &= -\frac{2\eta^2\lambda - 2\zeta^2\lambda}{2(2\eta^2 - 2\zeta^2 + 3\eta\zeta\lambda^2)},\\ a_1 &= -\frac{1}{2(2\eta^2 - 2\zeta^2 + 3\eta\zeta\lambda^2)},\\ H_{00} &= -\frac{5}{4}\zeta\lambda^2. \quad (29) \end{aligned}$$

It can be proved that the Hamiltonian of Eq. (11), is related to the Swanson Hamiltonian [47] by a similarity transformation V,

$$V = \exp\left(-\frac{a_1 - a_0}{2} x\right) \exp\left(-i\frac{a_1 + a_0}{2} p\right), \tag{30}$$

such that $H_0 = V H_{Sw} V^{-1}$. H_{SW} reads

$$H_{Sw} = \omega\left(a^\dagger a + \frac{1}{2}\right) + \alpha a^2 + \beta a^{\dagger 2} - H_{00}, \tag{31}$$

with $\alpha \neq \beta$. As it is well known from the literature [47], there exists an hermitian Hamiltonian h similar to H_{Sw}. The similarity transformation is not unique. We shall choose

$$W = \exp\left(-\frac{\alpha - \beta}{4} p^2\right). \tag{32}$$

$W H_{Sw} W^{-1} = h$.

Thus, the hermitian Hamiltonian h is similar to H_0 of Eq. (11), $h = \Upsilon H_0 \Upsilon^{-1}$, being

$$\Upsilon = W V^{-1}. \tag{33}$$

The explicit form of h can be calculated straightforward

$$h = \gamma\{a^\dagger, a\} + \varrho(a^{\dagger 2} + a^2) - H_{00}, \tag{34}$$

γ and ϱ can be expressed as

$$\begin{aligned} \gamma &= \frac{1}{4}\left(\omega + \alpha + \beta + \frac{\sqrt{\omega^2 - 4\alpha\beta}}{\omega + \alpha + \beta}\right),\\ \varrho &= \frac{1}{4}\left(\omega + \alpha + \beta - \frac{\sqrt{\omega^2 - 4\alpha\beta}}{\omega + \alpha + \beta}\right). \end{aligned} \tag{35}$$

Finally, the pseudo-metric operator U can be written in terms of Υ of Eq. (33) as $U = \Upsilon^\dagger \Upsilon$.

3.1.1. Eigenstates and eigenvalues

The eigenvalues and eigenvectors of H_0 can be obtained by evaluating the corresponding eigenvalues and eigenvectors of h, and using similarity transformation Υ of Eq. (33). An alternative approach is the one proposed by S.M. Swanson [47]. Following the work of [47] we shall perform a general Bogoliubov transformation

$$\tilde{d} = g_4\, \tilde{a} - g_2\, \tilde{a}^\dagger, \tag{36}$$

$$\tilde{c} = -g_3\, \tilde{a} + g_1\, \tilde{a}^\dagger \tag{37}$$

with the condition

$$\left[\tilde{d}, \tilde{c}\right] = 1. \tag{38}$$

Clearly $\tilde{c} \neq \tilde{d}^\dagger$. After some algebra, H_0 can be written as

$$H_0 = H_{00} + \Omega\left(\tilde{c}\tilde{d} + \frac{1}{2}\right), \tag{39}$$

with

$$\Omega = \sqrt{D(\eta,\zeta,\lambda)}, \quad D(\eta,\zeta,\lambda) = \omega^2 - 4\alpha\beta. \tag{40}$$

In order to obtained real eigenvalues, we ask $D(\eta,\zeta,\lambda) > 0$. Thus, the eigenvalues and eigenvectors of H_0 can easily be calculated

$$H_0|\widetilde{\Phi}\rangle = E_n|\widetilde{\Phi}\rangle, \tag{41}$$

with

$$E_n = \left(n + \frac{1}{2}\right)\Omega, \quad |\widetilde{\Phi}\rangle = \frac{1}{\sqrt{n!}}\tilde{c}^n|0_{\tilde{d}}\rangle. \tag{42}$$

The explicit form of $|0_{\tilde{d}}\rangle$ can be derived from the condition $\tilde{d}|0_{\tilde{d}}\rangle = 0$. It reads

$$|0_{\tilde{d}}\rangle = \mathcal{N}_d \, \exp(\tau \tilde{a}^{\dagger 2})|0\rangle, \quad \tilde{a}|0\rangle = 0, \tag{43}$$

where $\tau = (\Omega - \omega)/(4\alpha)$.

In the physical space $\mathcal{H}_U$, the fluctuation of the operator $\hat{O}$ on the eigenstate $|\widetilde{\Phi}_n\rangle$ of H_0 is computed as

$$\Delta_U^2(\hat{O}) = \langle \widetilde{\Phi}_n|\Upsilon^\dagger \, \hat{o}^2 \, \Upsilon|\widetilde{\Phi}_n\rangle - (\langle \widetilde{\Phi}_n|\Upsilon^\dagger \, \hat{o} \, \Upsilon|\widetilde{\Phi}_n\rangle)^2. \tag{44}$$

Let us calculate the fluctuations of the operators X ($X = \Upsilon^{-1}x\Upsilon$) and P ($P = \Upsilon^{-1}p\Upsilon$) on $|0_{\tilde{d}}\rangle$, they can be expressed as

$$\Delta_U^2 \hat{\mathrm{X}} = \frac{e^{2\sigma}}{2}, \quad \Delta_U^2 \hat{\mathrm{P}} = \frac{e^{-2\sigma}}{2}, \tag{45}$$

with $\sigma = (\tau/|\tau|)\tanh(|\tau|)$. We shall define the corresponding squeezing parameters as

$$Q(x,p) = 2\Delta_U^2 \hat{\mathrm{X}}, \quad Q(p,x) = 2\Delta_U^2 \hat{\mathrm{P}}. \tag{46}$$

Clearly, from this result, we can say that the state $|0_{\tilde{d}}\rangle$ is an squeezed state [42,51], which minimizes the uncertainty relations for the operators X and P in the physical space $\mathcal{H}_U$.

3.1.2. Numerical results

We shall present some numerical results concerning the behavior of the system model by the Hamiltonian of Eq. (11).

The eigenvalues of H_0 of Eq. (27) are real if $D(\eta,\zeta,\lambda) > 0$, see Eq. (40). In terms of the deformation parameter, λ, $D(\eta,\zeta,\lambda)$ of Eq. (40) can be expressed as

$$D(\eta,\zeta,\lambda) = 4\eta^2\left(1 - \left(\frac{\zeta}{\eta}\right)^2 + \frac{3}{2}\left(\frac{\zeta}{\eta}\right)\lambda^2 + \left(\frac{\zeta}{\eta}\right)^2\lambda^4\right), \tag{47}$$

if $\lambda = 0$, the eigenvalues of H_0, of Eq. (42), are given by $E_n = 2\eta\sqrt{1 - \left(\frac{\zeta}{\eta}\right)^2}\left(n + \frac{1}{2}\right)$. This result is similar to the expression presented in [40,41]. It indicates that the system displayed real energy spectra if $|\zeta/\eta| < 1$.

In Fig. 1, we show the values of $D(\eta,\zeta,\lambda)$ of Eq. (40), as a function of the ratio of the coupling constants, ζ/η, and of the deformation parameter λ. As the Hamiltonian H_0 has been obtained from H_λ of Eq. (5) to order $O(\lambda^3)$, we have considered values of λ, in the range $-1 < \lambda < 1$. The dark-gray zone corresponds to $D(\eta,\zeta,\lambda) < 0$. The light-gray region corresponds to values of $D(\eta,\zeta,\lambda) > 0$. For the range of parameters compatible with $D(\eta,\zeta,\lambda) > 0$, the Hamiltonian H_0 has real eigenvalues, as seen from Eq. (40). As can be observed in Fig. 1, there exist a lower and a upper limits in ζ/η as a function of λ. From Eq. (47), it can be calculated straightforwardly, and it reads

$$\frac{3\lambda^2 - 4\sqrt{1 - \frac{7\lambda^4}{16}}}{4\left(1-\lambda^4\right)} < \frac{\zeta}{\eta} < \frac{3\lambda^2 + 4\sqrt{1 - \frac{7\lambda^4}{16}}}{4\left(1-\lambda^4\right)}. \tag{48}$$

Thus, the inclusion of the parameter λ modifies the boundary limits that can reach the ratio of coupling constants ζ/η in order to ensure pseudo-hermicity. Also, notice that if $\lambda = 0$, the energy spectrum of H_0, which coincides with that of H_{sq} of Eq. (1), does not depend on the sign of the coupling constant ζ. For $0 < |\lambda| < 1$ the level spacing, $E_{n+1} - E_n = \sqrt{D(\eta,\zeta,\lambda)}$, is smaller than the level spacing for $\lambda = 0$ if $\zeta < 0$. While for $\zeta > 0$ and $0 < |\lambda| < 1$, the level spacing becomes greater than the one of the case $\lambda = 0$. In this sense, we shall say that the case $\zeta < 1$ correspond to an attractive interaction, while $\zeta > 1$ models a repulsive interaction. The inclusion in H_0 of terms depending on the parameter λ can be used to adjust the spectrum spacing without modifying the parameters of the original harmonic oscillator. As an example, we shall choose $\eta = 1$ and $\zeta = -0.45$, in arbitrary units of energy. This correspond to an energy spacing varying from $E_{n+1} - E_n = 1.78606$ for $\lambda = 0$, to $E_{n+1} - E_n = 1.14018$ for $\lambda = 1$, in arbitrary units of energy.

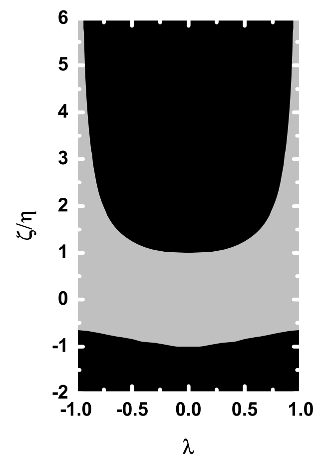

Fig. 1. Values of $D(\eta,\zeta,\lambda)$ of Eq. (40), as a function of the ratio of the coupling constants, ζ/η, and of the deformation parameter λ. The dark-gray zone corresponds to $D(\eta,\zeta,\lambda) < 0$, while the light-gray region corresponds to values of $D(\eta,\zeta,\lambda) > 0$.

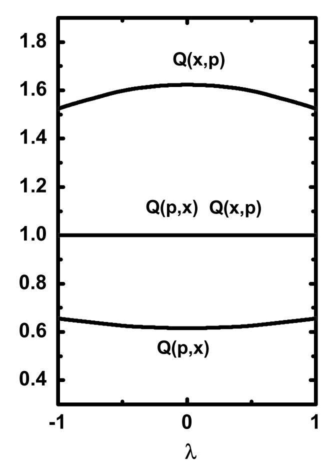

Fig. 2. Fluctuations of the operators X and P as a function of the deformation parameter λ, see Eq. (44), calculated for the lowest eigenstate of H_0 of Eq. (43). The coupling constants have been fixed to the values $\eta = 1.0$ and $\zeta = -0.45$.

In Fig. 2, we display the fluctuations of the operators X and P as a function of the deformation parameter λ, see Eq. (44), calculated for the first eigenstate of H_0 of Eq. (27). Clearly, from the figure the first state of H_0, Eq. (43) is a squeezed state and it minimizes the uncertainty relations.

To complete the analysis of the spectra of the Hamiltonian H_0 of Eq. (11), in Table 1 we present the values of the first energy levels with respect to the ground state, $E_n - E_0$, and the fluctuations of X and P in the corresponding eigenstate.

In order to study the time evolution of the system, we shall fixed the deformation parameter at the value $\lambda = 0.1$. In Figs. 3–5 we displayed the results corresponding to the time evolution of the GK-coherent state, $|GK\rangle$. The mean value of P, $\langle P(t)\rangle$, as a function of the mean value of X, $\langle X(t)\rangle$, is displayed in Fig. 3. The phase space trajectory is periodic. The period is given by $T = 2\pi/\Omega$, in units of $h = 1$.

In Fig. 4, we show the behavior of the fluctuations of the operators X and P as a function of time, (t/T), for the GK-coherent state of Eq. (22). As seen from the figure, the GK-coherent state is squeezed in P, and the uncertainty relation takes the minimum

Table 1
Spectrum of H_0 of Eq. (11), and their corresponding fluctuations. In the first column, the eigenvalues of H_0 respect the lowest eigenvalue, $E_0 = 0.89488$ in arbitrary units of energy, (for $\eta = 1$, $\zeta = -0.45$ and $\lambda = 0.1$) are displayed. In columns 2 to 4, the fluctuation of each state, computing from the corresponding eigenvector, is presented.

$E_n - E_0$	$2\Delta^2 X$	$2\Delta^2 P$	$4\Delta^2 X \Delta^2 P$
0.00000	1.62272	0.61625	1.00000
1.77851	4.86815	1.84875	9.00000
3.55701	8.11359	3.08125	25.00000
5.33552	11.35902	4.31375	49.00000
7.11402	14.60446	5.54625	81.00000
8.89253	17.84989	6.77875	121.00000
10.67103	21.09533	8.01125	169.00000

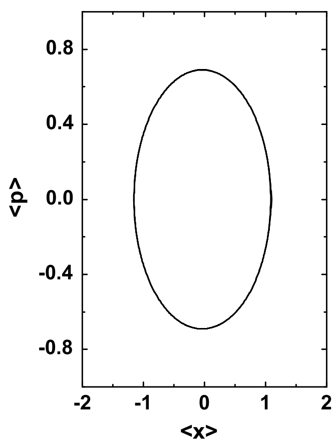

Fig. 3. Mean value of P, $\langle P(t)\rangle$, as a function of the mean value of X, $\langle X(t)\rangle$, for the GK-coherent state of Eq. (22). The GK-coherent state is constructed from the eigenvalues of H_0 of Eq. (27). The coupling constants are the same as those of Fig. 2. We have fixed the deformation parameter to the value $\lambda = 0.1$.

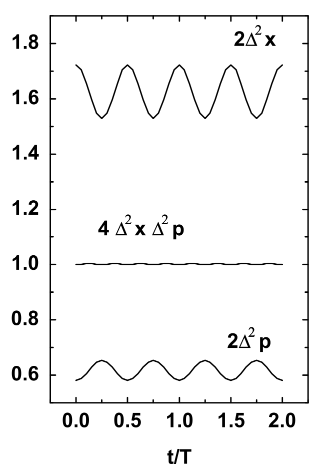

Fig. 4. Fluctuations of the operators X and P as a function of time (t/T), for the GK-coherent of H_0 of Eq. (27). The parameters are the same as those of Fig. 3.

value. The Wigner function of the system is presented in Fig. 5, for $t/T = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$, respectively. The results presented in Fig. 5, shows that GK coherent states for general oscillator Hamiltonians are a natural extension of the usual harmonic oscillator coherent state.

In Figs. 6–8 we display the results corresponding to the time evolution of the state of Eq. (24), $|\widetilde{I}_1\rangle$. In Fig. 6, the mean value of X, $\langle X(t)\rangle$, as a function of the mean value of P, $\langle P(t)\rangle$, is plotted. Also in this case the phase space trajectory is periodic. The period is given by $T = 2\pi/\Omega$, in units of $h = 1$. In Fig. 7, we show the behavior of the fluctuations of the operators X and P as a function of time (t/T). As seen from the figure, the initial state of Eq. (24)

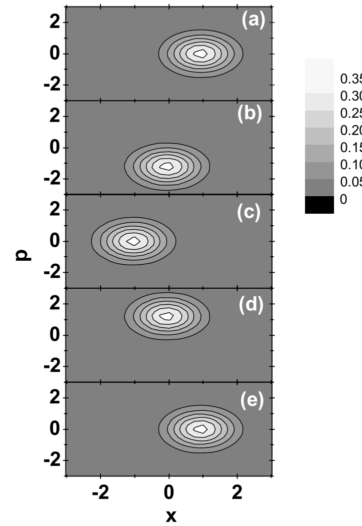

Fig. 5. Wigner Function for the GK-coherent (constructed from the eigenvectors of H_0 of Eq. (27)) state of Eq. (22), at different times. Insets (a), (b), (c), (d) and (e) correspond to $t/T = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$, respectively.

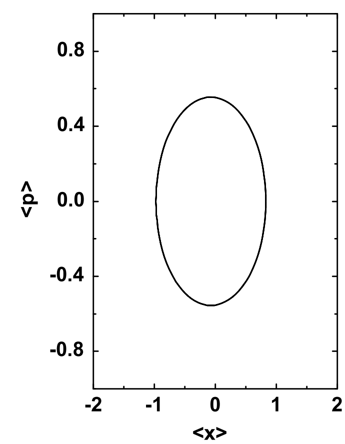

Fig. 6. Idem Fig. 3, for the initial state $|\widetilde{I}_1\rangle$, of Eq. (24).

shows a pattern of revivals of squeezing in P, but the uncertainty relation does not take the minimum value. The Wigner function of the system is presented in Fig. 8, for $t/T = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$, respectively. It resembles the Wigner Function of a hybrid entanglement system as the one reported in [53]. In this sense, we can conjecture that the Hamiltonian of Eq. (5) is an effective hamiltonian [52], and that the deformation parameter is related to the classical degrees of freedom of the system reported in [53].

3.2. General Hamiltonian

Finally, we shall analyzed the behavior of the system when the term H_{res} of Eq. (12) is taken into account. The Hamiltonian H of Eq. (10) can be written in terms of the operators p and x of Eq. (8) as

$$
\begin{aligned}
H = {} & \frac{\zeta\lambda^2}{2} + (\eta - \zeta\lambda^2)(x^2 + p^2) + \zeta(x^2 - p^2) \\
& + \frac{\eta\lambda^2}{4}(x^2 - p^2)(x^2 + p^2) - \mathrm{i}\frac{\eta\lambda^2}{4}\{x, p\}(x^2 + p^2) \\
& + \frac{\lambda\eta}{\sqrt{2}}(x - \mathrm{i}\,p)(x^2 + p^2) + \frac{\lambda\zeta}{\sqrt{2}}(x^2 + p^2)(x + \mathrm{i}\,p) \\
& + \zeta\frac{\lambda^2}{2}(x^2 + p^2)(x^2 + p^2). \qquad (49)
\end{aligned}
$$

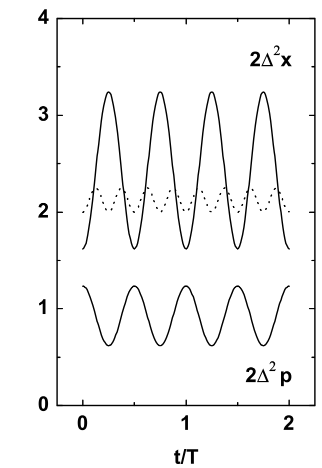

Fig. 7. Idem Fig. 4, for the initial state $|\widetilde{I}_1\rangle$, of Eq. (24). Dotted lines correspond to the value $4\Delta^2 x\Delta^2 p$.

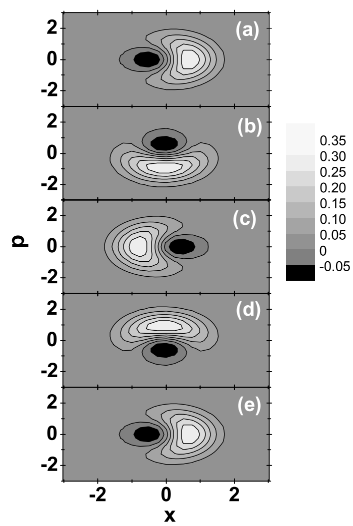

Fig. 8. Idem Fig. 5, for the initial state $|\widetilde{I}_1\rangle$, of Eq. (24).

We look for a similarity transformation, so that the Hamiltonian h ($h = \Upsilon H \Upsilon^{-1}$) being an hermitian hamiltonian. We have chosen

$$\Upsilon = \mathrm{e}^{-F(p)}\mathrm{e}^{-G(x)}, \tag{50}$$

with

$$\begin{aligned} G(x) &= g_1(\theta,\lambda)x^2 + g_2(\theta,\lambda)x^3 + g_3(\theta,\lambda)x^4, \\ F(p) &= f_1(\theta,\lambda)p^2 - \mathbf{i} f_2(\theta,\lambda)p^3 + f_3(\theta,\lambda)p^4. \end{aligned} \tag{51}$$

In what follows, we shall introduce the parameter $\theta = \zeta/\eta$. The expressions of $f_j(\theta,\lambda)$ and $g_j(\theta,\lambda)$, up to order $O(\lambda^3)$, for $\theta \neq -3$, 0, 1, can be written as

$$\begin{aligned} g_1(\theta,\lambda) &= -\frac{1+\theta}{2(\theta-1)} + \frac{\theta^2(1+\theta)\lambda}{(\theta-1)^2} - \\ &\quad \frac{\theta(-1+\theta+4\theta^2+8\theta^3+4\theta^4)\lambda^2}{2(\theta-1)^3}, \\ g_2(\theta,\lambda) &= \frac{\theta(3+2\theta+7\theta^2)\lambda}{3\sqrt{2}(-1+\theta)^3(3+\theta)} + \\ &\quad \frac{2\sqrt{2}\theta^2(-3-13\theta-23\theta^2-15\theta^3+6\theta^4)\lambda^2}{3(\theta-1)^4(3+\theta)^2}, \\ g_3(\theta,\lambda) &= \frac{\left(\theta^4+14\theta^3-2\theta^2+14\theta-3\right)\lambda^2}{16(\theta-1)^4(\theta+3)}, \\ f_1(\theta,\lambda) &= \frac{\theta-1}{4\theta} + \frac{\lambda}{2} - \frac{\left(4\theta^4+4\theta^3+\theta-1\right)\lambda^2}{8(\theta-1)\theta} \\ f_2(\theta,\lambda) &= \frac{(3+\theta+13\theta^2-9\theta^3)\lambda}{12\sqrt{2}\theta(1+\theta)^2(3+\theta)} + \\ &\quad \frac{(9+8\theta^2+70\theta^3-9\theta^4-46\theta^5)\lambda^2}{6\sqrt{2}(\theta-1)(1+\theta)^2(3+\theta)^2} \\ f_3(\theta,\lambda) &= \frac{1}{32\theta(\theta-1)^2(\theta+1)^3(\theta+3)^2}\Big(2\theta^7 + \\ &\quad 129\theta^6 - 128\theta^5 + 23\theta^4 - \\ &\quad 246\theta^3 + 31\theta^2 - 12\theta + 9\Big)\lambda^{3/2}. \end{aligned} \tag{52}$$

After some algebra, the explicit form of $h = \Upsilon H \Upsilon^{-1}$ can be obtained

$$\begin{aligned} h_0 &= h_0(\theta,\lambda) + h_1(\theta,\lambda)p^2 + h_2(\theta,\lambda)x^2 + h_3(\theta,\lambda)p^4 + \\ &\quad h_4(\theta,\lambda)x^4 + h_5(\theta,\lambda)\{x^2,p^2\} + h_6(\theta,\lambda)\{x,p^2\} + \\ &\quad h_7(\theta,\lambda)x + h_8(\theta,\lambda)x^3, \end{aligned} \tag{53}$$

with

$$\begin{aligned} h_0(\theta,\lambda) &= \frac{1}{2}\eta\lambda^2\big(-48 f_1(\theta,\lambda)g_3(\theta,\lambda)(\theta-1) + \\ &\quad 3\sqrt{2} f_2(\theta,\lambda) - 3\sqrt{2} g_2(\theta,\lambda) + \theta\big), \\ h_1(\theta,\lambda) &= \eta(1-\theta) - \\ &\quad 4 f_1(\theta,\lambda)\eta\lambda\left(f_1(\theta,\lambda)(\theta+1) + 2g_1(\theta,\lambda)(\theta-1)\right) - \\ &\quad \frac{1}{2}\eta\lambda^2\Big(32 f_1(\theta,\lambda)^2 g_1(\theta,\lambda)^2(\theta-1) + \\ &\quad 3\sqrt{2} f_2(\theta,\lambda) + 2\theta - 1\Big), \\ h_2(\theta,\lambda) &= \eta(\theta+1) + 4\eta g_1(\theta,\lambda)^2(\theta-1)\lambda + \\ &\quad \frac{1}{2}\eta\lambda^2\Big(3\sqrt{2} g_2(\theta,\lambda) - 2\theta - 1\Big), \\ h_3(\theta,\lambda) &= \frac{1}{4}\eta\lambda^2\big(64 f_3(\theta,\lambda)\big(f_1(\theta,\lambda) \\ &\quad (-(\theta+1)) - g_1(\theta,\lambda)(\theta-1)\big) + \\ &\quad 36 f_2(\theta,\lambda)^2(\theta+1) + 6\sqrt{2} f_2(\theta,\lambda) + 2\theta - 1\big), \\ h_4(\theta,\lambda) &= \frac{1}{4}\eta\lambda^2\Big(36 g_2(\theta,\lambda)^2(\theta-1) + \\ &\quad 64 g_1(\theta,\lambda)g_3(\theta,\lambda)(\theta-1) - \\ &\quad 6\sqrt{2} g_2(\theta,\lambda) + 2\theta + 1\Big), \\ h_5(\theta,\lambda) &= \frac{1}{4}\eta\lambda^2\big(-96 f_1(\theta,\lambda)g_3(\theta,\lambda)(\theta-1) + \\ &\quad 9\sqrt{2} f_2(\theta,\lambda) - 9\sqrt{2} g_2(\theta,\lambda) + 2\theta\big), \\ h_6(\theta,\lambda) &= \frac{1}{4}\eta\lambda^{3/2}\big(-48 f_1(\theta,\lambda)g_2(\theta,\lambda)\theta + \\ &\quad 48 f_1(\theta,\lambda)g_2(\theta,\lambda) + \\ &\quad 4\sqrt{2} f_1(\theta,\lambda) - 6\sqrt{2} g_1(\theta,\lambda)\big) + \\ &\quad \frac{1}{4}\eta\lambda^2\Big(48 f_2(\theta,\lambda)g_1(\theta,\lambda)^2\theta + \\ &\quad 16\sqrt{2} f_1(\theta,\lambda)g_1(\theta,\lambda) + \sqrt{2}\theta - \\ &\quad 48 f_2(\theta,\lambda)g_1(\theta,\lambda)^2 - 12\sqrt{2} f_1(\theta,\lambda)^2\Big) + \\ &\quad \frac{1}{4}\eta\lambda\Big(12 f_2(\theta,\lambda)\theta + 12 f_2(\theta,\lambda) + \sqrt{2}\Big), \end{aligned}$$

Table 2
Idem Table 1, for H of Eq. (10), with $E_0 = 0.89303$ in arbitrary units of energy.

$E_n - E_0$	$2\Delta^2 x$	$2\Delta^2 p$	$4\Delta^2 x\Delta^2 p$
0.00000	1.62250	0.61697	1.00103
1.78123	4.84558	1.84478	8.93902
3.55681	8.03784	3.06776	24.65863
5.32575	11.19851	4.28234	47.96595
7.08700	14.22259	5.53824	78.76815
8.83946	16.61043	7.03846	116.91183
10.58196	16.66588	8.99761	149.95307

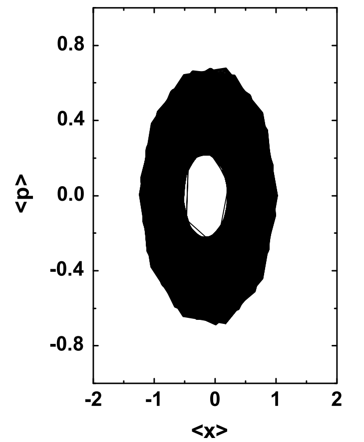

Fig. 9. Mean value of P, $\langle P(t)\rangle$, as a function of the mean value of X, $\langle X(t)\rangle$, for the GK-coherent state of Eq. (22). The GK-coherent state is constructed from the eigenvectors of H of Eq. (10). The parameters are the same as those of Fig. 3.

$$h_7(\theta,\lambda) = \sqrt{2}\eta g_1(\theta,\lambda)\lambda^{3/2} - \frac{\eta\theta\lambda^2}{\sqrt{2}} - \frac{\eta\lambda}{\sqrt{2}},$$
$$h_8(\theta,\lambda) = \eta g_1(\theta,\lambda)\lambda^{3/2}\left(-12 g_2(1-\theta) - \sqrt{2}\right) + \frac{\eta\lambda^2\left(\theta - 4g_1(\theta,\lambda)^2\right)}{\sqrt{2}} + \frac{\eta\lambda}{\sqrt{2}}. \quad (54)$$

Notice that, the terms in $\{x^2, p^2\}$, p^4 and x^4, contribute to a Kerr-like term $(x^2+p^2)(x^2+p^2)$ [54].

Let us discuss some numerical results. We have adopted the parameter of the previous sections, $\eta = 1$, $\zeta = -0.45$, $\lambda = 0.1$. In Table 2, we display the behavior of the spectra of H of Eq. (10). The table shows the values of the first energy levels with respect to the ground state, $E_n - E_0$, and the fluctuations of X and P in the corresponding eigenstate. Clearly the lowest eigenstate is a squeezed state, and its uncertainty is close to the minimum value. Also, while for H_0 the level spacing is constant ($E_{n+1} - E_n = \sqrt{D(\eta,\zeta,\lambda)}$), for the general case, the level spacing becomes compressed. In this sense, the inclusion of the additional terms in the Hamiltonian can be used to fit the spectra of different finite range attractive potentials, i.e the Morse potential [17,18].

We computed the time evolution of GK-coherent state in presence of an interaction model by H of Eq. (10). The behavior of $\langle X(t)\rangle$ as a function of $\langle P(t)\rangle$ is drawn in Fig. 9. As it can be inferred from the figure, the trajectory in phase space is not closed, though it is bounded. The fluctuations of the *GK* state, as a function of time, are presented in Fig. 10. They show a modulate pattern of revivals. It can seen from the figure that the GK-coherent state is squeezed in P, though the uncertainty relation does not reach its minimum value.

The previous results, indicate that the lowest eigenstate of the generalization of the harmonic squeezed oscillator presented in Eq. (5), in the regime of parameters adopted, is a squeezed states. Moreover, the GK-state constructed as the coherent superposition of the eigenstates of H, see Eq. (22), evolves in time as a squeezed state.

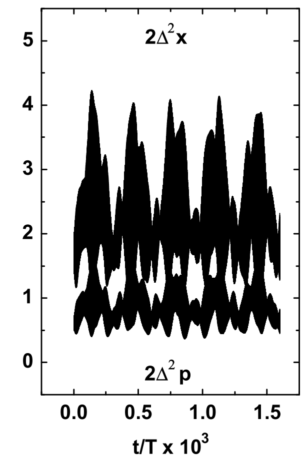

Fig. 10. Fluctuations of the operators X and P as a function of time (t/T), for the GK-coherent of H of Eq. (10). The parameters are the same as those of Fig. 3.

4. Conclusions

In this work, we have studied the spectrum and the time evolution of a system modeled by the non-hermitian Hamiltonian H_λ of Eq. (5). As reported previously [30], this Hamiltonian is obtained as a generalization of the squeezed harmonic oscillator, bay adopting the non-standard $U_\lambda^n(h4)$ Hopf algebra. We have considered the system modeled by the Hamiltonian H_λ to order $O(\lambda^3)$. We have proved that the Hamiltonian H_0 of Eq. (11) is similar to a Swanson Hamiltonian. We have verified that, for the regime of real spectrum, the lowest eigenstate is a squeezed state that minimizes the Heisenberg Uncertainty Relations. Also, we have study the time evolution of the Gazeau–Klauder coherent state constructed from the eigenstates of H_0, and we have shown that it evolves in time as a squeezed state, and it minimizes the uncertainty relations. Finally, we have found a similarly transformation for the Hamiltonian H of Eq. (10), so that $h = \Upsilon H \Upsilon^{-1}$, being h an hermitian Hamiltonian. We have shown that the first eigenstate of the general Hamiltonian H, behaves as a squeezed state. The Gazeau–Klauder coherent state constructed from the eigenstates of H displays a pattern of revivals as it evolves in time, and shows the behavior of a squeezed state, though the uncertainty relations does not reach its minimum value. From the properties of the spectra of H, it is concluded that it can be used as an effective hamiltonian to model the behavior of different finite range attractive potentials [17,18], and to construct states which optimizes the associated uncertainty relations.

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