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VALIDITY IN A DIALETHEIST FRAMEWORK

Federico Pailos & Diego Tajer

Abstract

In this paper, we develop two theories of validity in a dialetheist framework, both based on Meadows (2014). The first one, *LPV**, has *LP*'s consequence relation but the validity predicate of Meadows' fixed point construction. The second theory, *DT* (the one we favour), is defined in terms of its validity predicate. Therefore, in *DT*, the validity predicate and the consequence relation coincide. Moreover, this theory, unlike Meadows' *VAL*, is reflexive.

Keywords: validity paradox, dialetheism, validity, paraconsistency

1. Introduction

One of the main aims of dialetheism is to achieve a semantically closed language. The acceptance of some contradictions, dialetheists argue, is the only way to solve the semantical paradoxes[1]. In recent years, Beall (2009) and Beall & Murzi (2013) tried to show that dialetheism, in the way we know it, is unable to express the concept of validity. The inexpressibility result goes as follows.

Let *Val* be a validity predicate characterized by the following rules:

VP If $A \vdash B$, then $\vdash Val(\ulcorner A \urcorner, \ulcorner B \urcorner)$

VD $A \wedge Val(\ulcorner A \urcorner, \ulcorner B \urcorner) \vdash B$

From *VP* and *VD*, we can get:

Internal detachment $\vdash Val(\ulcorner A \wedge Val(\ulcorner A \urcorner, \ulcorner B \urcorner)\urcorner, \ulcorner B \urcorner)$

If the theory allows strong diagonalization, we can get a proposition π definitionally equivalent to $Val(\ulcorner \pi \urcorner, \ulcorner \perp \urcorner)$, usually known as the 'Beall-Murzi sentence'[2].

[1] See for example Priest (2006) or Beall (2009).

[2] Depending on whether or not one works with arithmetic as a theory of syntax, the Beall-Murzi sentence may be a sentence π such that the diagonal lemma establishes the following: $\vdash \pi \leftrightarrow Val(\ulcorner \pi \urcorner, \ulcorner \perp \urcorner)$.

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As an instance of Internal detachment, we get:

$\vdash Val(\ulcorner \pi \wedge Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)\urcorner, \ulcorner \bot \urcorner)$

We will also use a principle of 'Equivalent Subformula Substitution' (ESS), which allows to replace any formula by an equivalent one also in an opaque context (i.e. under the scope of quotation marks[3]). And now we can argue as follows:

1. $Val(\ulcorner \pi \wedge Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)\urcorner, \ulcorner \bot \urcorner)$ (Internal detachment)
2. $Val(\ulcorner \pi \wedge \pi \urcorner, \ulcorner \bot \urcorner)$ (ESS, 1)
3. $Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)$ (ESS, 2)
4. π (ESS, 3)
5. $\bot$ (VD, 3,4)

In this way, we get triviality out of strong diagonalization and some very intuitive principles regarding validity. In this paper we show that, in spite of this result, dialetheism *can* express the concept of validity. In particular, we develop two paraconsistent theories of validity which validate *VP* and *VD*, and also avoid the Validity Paradox. In the next section, we describe the recent proposal of Meadows (2014), on which our proposals are based. In section 3, we develop the dialetheist theory *LPV**. Finally, in section 4 we propose and defend the theory *DT*. In this theory, the predicate and the concept of validity coincide.

It is important to point out that our theory is not necessarily a theory of purely logical validity. As Ketland (2012) argued, purely logical validity can be captured in first-order arithmetic. Following the observations of Shapiro & Murzi (2013), here we are using a broader notion of logical validity, which (for example) includes inferences that essentially involve the validity predicate.

2. A paracomplete validity predicate

In Meadows (2014), the author introduces a validity predicate which validates *VD* and *VP*, and also avoids the troubles generated by the Beall-Murzi sentence.

The general strategy he adopts is to build a Kripke fixed point construction for a validity predicate '*Val*'. He starts by taking any first-order extension of the language of arithmetic and adding the predicate '*Val*'. The next step

[3] This principle is admittedly controversial. In particular, Ketland (2012)'s solution to the Beall-Murzi paradox is based on rejecting the subtitution of equivalents under opaque contexts.

is to construct a partial model for the resulting language, where the arithmetical vocabulary is interpreted in the standard way, and the non-arithmetical vocabulary other than '*Val*' gets some classical interpretation. '*Val*' is assigned an extension and an anti-extension in the first step: the former contains just all instances of *VD* ($\langle \ulcorner \phi \wedge Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner)\urcorner, \ulcorner \psi \urcorner \rangle$), and the latter is the empty set.

We 'improve' the interpretation of '*Val*' by a jump operator, which forces us to include in the next step all pairs $\langle \ulcorner \phi \urcorner, \ulcorner \psi \urcorner \rangle$ in the extension if and only if ϕ is false or ψ is true in all models 'completed' by the previous interpretation of *Val*, and to put them in the anti-extension if and only if there is a model completed by the previous interpretation where ϕ is true and ψ is false. The process continues until we reach a fixed point (which we will, for cardinality considerations).

More formally, let L be any first-order language (without function symbols), $L_{Ar} = \langle 0, 1, M, P \rangle$, $L_{+} = L \cup L_{Ar}$ and, finally, let $L_V = L + Val$. Meadows defines his bases of models, *Mod*, as follows. Take an arbitrary language L. Let Mod_α be the set of all models M of $L \cup L_{Ar}$, where its domain is $\omega + \alpha$, the *total* domain. The non-logical vocabulary of L is interpreted taking objects of α, the *concrete* domain. The non-logical vocabulary of L_{Ar} is interpreted in the standard way, taking objects from ω, the *number* domain in the standard way. 'M' and 'P' are triadic predicates, where $M(k, m, n)$ iff $k+m = n$, and $P(k, m, n)$ iff $k \times m = n$[4].

Finally, we may define the set of all models:

$$Mod = \cup_{1 \leq \alpha \leq \omega + 1} Mod_\alpha$$

This gives us a way to interpret the vocabulary of L over every concrete domain of some countable cardinality.

Meadows then defines a valuation function which takes an extension/antiextension pair of sets of formulae Φ (of the validity predicate), a model M and the set of sentences $Sent_{L_V}$ of the language L, and returns a semantic value. We assume that L_V has names for all the objects in the domain of M, $|M|$. So let $\Phi = \langle \Phi^+, \Phi^- \rangle \in P(\omega^2)^2$ and $v_{\Phi,M}$: $P(Sent_{L_V})^2 \times Mod \times Sent_{L_V} \rightarrow \{0, 1, \frac{1}{2}\}$ be a partial function which gives to each sentence $\phi \in Sent_{L_V}(|M|)$ a value relative to a model and a pair of interpretation (an extension and an anti-extension) of the validity predicate. This is how the valuation function works:

$$v_{\Phi,M}(\phi) = 1 \text{ iff}$$

[4] Meadows introduces these two new predicates because of the technical problems that causes the fact that arithmetical functions no longer take arbitrary objects from the domain as arguments, and thus fail to be properly defined. So he replaced the functions + and × by these three-place function symbols, '*P*' and '*M*'.

$\phi = Ra_1, ..., a_n, \langle a_1^M, ..., a_n^M \rangle \in R^M$, for $R \in L$

$\phi = Ra_1, ..., a_n, \langle a_1^M, ..., a_n^M \rangle \in R^{\mathbb{N}}$, for $R \in L_{Ar}$

$\phi = Val(x, y)$, where $x = \ulcorner \psi \urcorner$ and $y = \ulcorner \chi \urcorner$ and $\langle \psi, \chi \rangle \in \Phi^+$

$\phi = \neg\psi$, $v_{\Phi,M}(\psi) = 0$

$\phi = \psi \wedge \chi$, $v_{\Phi,M}(\psi) = v_{\Phi,M}(\chi) = 1$

$\phi = \forall x\psi(x)$, $v_{\Phi,M}(\psi[a/x]) = 1$ for all a

$v_{\Phi,M}(\phi) = 0$ iff

$\phi = Ra_1, ..., a_n, \langle a_1^M, ..., a_n^M \rangle \notin R^M$, for $R \in L$

$\phi = Ra_1, ..., a_n, \langle a_1^M, ..., a_1^M \rangle \notin R^{\mathbb{N}}$, for $R \in L_{Ar}$

$\phi = Val(x, y)$, where $x = \ulcorner \psi \urcorner$ and $y = \ulcorner \chi \urcorner$ and $\langle \psi, \chi \rangle \in \Phi^-$

$\phi = \neg\psi$, $v_{\Phi,M}(\psi) = 1$

$\phi = \psi \wedge \chi$, $v_{\Phi,M}(\psi) = 0$ or $v_{\Phi,M}(\chi) = 0$

$\phi = \forall x\psi(x)$, $v_{\Phi,M}(\psi[a/x]) = 0$ for some a

$\frac{1}{2}$ otherwise.

The jump operator (a central part of the inductive construction) is a function $j_v : P(Sent_{L_V})^2 \times P(Sent_{L_V})^2$ such that for $\Phi = \langle \Phi^+, \Phi^- \rangle$,

$$j_v(\phi) = \langle \{\langle \phi, \psi \rangle \in P(Sent_{L_V})^2 \mid \forall M \in Mod(v_{\Phi,M}(\phi) \in \{\tfrac{1}{2}, 1\} \rightarrow v_{\Phi,M}(\psi) = 1\},$$
$$\{\langle \phi, \psi \rangle \in P(Sent_{L_V})^2 \mid \exists M \in Mod(v_{\Phi,M}(\phi) = 1 \wedge v_{\Phi,M}(\psi) = 0\}\rangle$$

This allows Meadows to provide an inductive definition for his validity predicate. The goal is to ensure that it validates both *VD* and *VP*. He accomplishes the first aim by putting all instances of *VD* in the extension, in the first step of the construction. In the proof that this validity predicate so defined can have a consistent fixed point interpretation, he uses some additional notions:

1. $\sqcup_{\alpha<\beta} \Phi_\alpha = \langle \cup_{\alpha<\beta} \Phi_{\alpha+}, \cup_{\alpha<\beta} \Phi_{\alpha+} \rangle$
2. $\psi \sqcup \phi = \langle \psi^+ \cup \phi^+, \psi^- \cup \phi^- \rangle$
3. $\psi \sqsubseteq \phi$ iff $\psi^+ \subseteq \phi^+$ and $\psi^- \subseteq \phi^-$
4. ψ is *consistent* iff $\psi^+ \cap \phi^+ = \emptyset$

Now, let the hierarchy $\Gamma : On \rightarrow P(Sent_{L_V})^2$ be defined by transfinite recursion as follows:

$$\Gamma_0 = \langle \{\langle \phi \wedge Val(\ulcorner \psi \urcorner, \ulcorner \psi \urcorner), \psi \rangle \mid \phi, \psi \in Sent_{L_V}\}, \emptyset \rangle$$

$$\Gamma_\alpha = j_v\,(\sqcup_{\beta<\alpha} \Gamma_\beta) \sqcup_{\beta<\alpha} \Gamma_\beta$$

The contruction is evidently non-decreasing – it is designed for being so. But it still might be the case that for some $\phi, \psi \in Sent_{L_V}$, $\langle \phi, \psi \rangle \in \Gamma^+_\alpha$ and $\langle \phi, \psi \rangle \in \Gamma^-_\alpha$. Meadows has proven (by induction of the complexity of the formulae) that if $\Phi \sqsubseteq \Psi$, and both Φ and Ψ are consistent, then for each $\phi \in Sent_{L_V}$:

1. If $v_{\Phi,M}(\phi) = 1$, then $v_{\Psi,M}(\phi) = 1$, and
2. If $v_{\Phi,M}(\phi) = 0$, then $v_{\Psi,M}(\phi) = 0$.

Moreover, Meadows proves that Γ_α is consistent, for every ordinal α. And with these elements, he proves that there are no $\phi, \psi \in Sent_{L_V}$ such that $\langle \phi, \psi \rangle \in \Gamma^+_\alpha$ and $\langle \phi, \psi \rangle \in \Gamma^-_\alpha$. As a corollary, we get that the sequence $(\Gamma_\alpha)_{\alpha \in On}$ is well defined. Finally, by cardinality considerations, there is an ordinal α such that $\Gamma_\alpha = \Gamma_{\alpha+1}$. Let Γ_{Val} be the least of such sets – the minimal fixed point. This one will give us both the extension and the anti-extension of the validity predicate.

Maybe the most important thing to mention at this point is Meadows' definition of logical consequence:

(*VAL*) Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be finite, then $\Gamma \vDash_{VAL} \phi$ iff $\langle \gamma_1 \wedge ... \wedge \gamma_n, \phi \rangle \in \Gamma^+_{Val}$.
As a limit case of the previous definition, $\vDash_{VAL} \phi$ iff $\langle \top, \phi \rangle \in \Gamma^+_{Val}$.

We say that a system *S corresponds* to its validity predicate whenever: $A \vDash_S B$ iff $\vDash_S Val(\ulcorner A \urcorner, \ulcorner B \urcorner)$[5]. Meadows' system will obviously correspond to its validity predicate, because the former is defined in terms of the latter. This validity predicate does not intend to capture the consequence relation of K_3 or any other paracomplete system. It establishes a *sui generis* consequence relation.

Unfortunately, as Meadows observed, his system does not validate unrestricted versions of reflexivity, transitivity and monotonicity. The latter fails because we didn't add the instantes of *VD* with extra premises at the beginning. If we do so, we recover the property (this is stated without proof in Meadows' paper). But the failure of the other two structural rules is intrinsic to the proposal. If we have a sentence such as π (the Beall-Murzi sentence), which receives value $\frac{1}{2}$ in every valuation and every model, then we can't have reflexivity, because $\langle \pi, \pi \rangle \notin \Gamma^+_{Val}$, and so $\pi \nvDash_{VAL} \pi$. Something similar happens with transitivity. As we validate all instances of *VD* by 'brute force' – they get inside the extension at the first step of the construction –, we will have $\pi \wedge Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner) \vDash_{VAL} \bot$ and also $\bot \vDash_{VAL} \phi$, but at the

[5] This seems to be, at least, a necessary condition for a predicate to capture the consequence relation of the system. If there is correspondence, the predicate shows in the system how its consequence relation works. However, we are not trying to argue that this is also a sufficient condition.

same time $\pi \wedge Val(\ulcorner\pi\urcorner, \ulcorner\bot\urcorner) \nvDash_{VAL} \phi$. Moreover, Modus Ponens is invalid in *VAL*. For example, $\pi, \pi \rightarrow \pi \nvDash_{VAL} \pi$.

So, how does Meadows treat the Beall-Murzi sentence in order to avoid the contradiction? $\langle\pi, \bot\rangle$ can't be in the extension of the validity predicate, because it would imply $\bot$. If π or $\langle\pi, \bot\rangle$ were in the extension, then it would also force $\langle\pi, \bot\rangle$ into the anti-extension, causing the construction to fail (if it were at stage n of the construction, then in $n + 1$ there will be a valuation such that π gets 1, but $\bot$ obviously doesn't). For similar reasons, it can't be in the anti-extension, because that would force the pair into the extension (if it were at stage n of the construction, then in $n + 1$ there would be no valuation such that π gets 1 or $\frac{1}{2}$ and $\bot$ gets value 1). Therefore, this problematic pair is neither in the extension nor in the anti-extension of the validity predicate.

How does this help to establish the possibility of a dialetheist theory of validity? We will see how in the next section.

3. A paraconsistent validity predicate: first version

In this section, we will develop a paraconsistent theory for the validity predicate. The strategy is, in a nutshell, to adopt Meadows' predicate while changing the underlying logic into a paraconsistent one.

Let *LP** be *LP* formulated in L_{+}, and let $LPV^* = LP^* + Val$ (with a fixed point interpretation). *LPV** (like *LP*) has a tolerant consequence relation, i.e. the inference $\Gamma|\phi$ is valid in *LPV** iff for all valuations v and models M, if for every $\gamma \in \Gamma$, $v(\gamma) = 1$ or $\frac{1}{2}$, then $v(\phi) = 1$ or $\frac{1}{2}$ The valuation matrixes of *LPV** are Strong Kleene's ones, like the matrixes of *VAL*.

Now we need to tell when a pair of formulae belongs to the extension and when it belongs to the anti-extension of the validity predicate. The short answer is: in the same cases as in Meadows' system. Because what belongs to them is fixed by (i) the initial step of the construction, (ii) how a valuation works, and (iii) which jump operator one decides to use. But (i) the initial step of the contruction will be the same – nothing but all instances of *VD* will be in the extension of the validity predicate, (ii) for all valuations v, v is a Meadows' valuation iff v is a *LPV** valuation, and (iii) the jump operator j_v will be the same.

Anyway, the analogies stop there. The system *VAL* is defined in terms of its validity predicate, so they are co-extensional by definition. Whereas *LPV** has a tolerant consequence relation, defined independently of how the validity predicate is constructed. As we will soon see, $\Gamma_{LPV^*} = \langle\Gamma^{+}_{LPV^*}, \Gamma^{-}_{LPV^*}\rangle$, the pair of the extension and anti-extension of *LPV**'s validity predicate, will not correspond in every case to what *LPV** establishes.

Let's take a look at other features of *LPV**.

Modus Ponens. As we previously mentioned, Modus Ponens will not be in general part of the extension of the validity predicate. So it is not the case that for all ϕ and ψ, $\langle \phi \wedge (\phi \rightarrow \psi), \psi \rangle \in \Gamma^{+}_{LPV*}$. Just take $\phi = \pi$ (i.e. the Beall-Murzi sentence), $\psi = \perp$. Then for all v, $v(\phi) = v(\pi) = \frac{1}{2}$ and $v(\psi) = 0$. Thus, this pair cannot belong to the extension of *Val*. However, no instance of Modus Ponens stays in the anti-extension of our validity predicate, since no valuation can give to the premises value 1, and value 0 to the conclusion. Therefore, it can just have the intermediate value. This means that $\vDash_{LPV*} Val(\ulcorner \phi \wedge (\phi \rightarrow \psi)\urcorner, \ulcorner \psi \urcorner)$.

The Beall-Murzi sentence. We have seen how Meadows treats the Beall-Murzi sentence. A similar reasoning is available for *LPV**. The pair $\langle \pi, \perp \rangle$ can be neither in the extension nor in the anti-extension of the validity predicate. So the only stable assignment of truth value to π is the one that gives $\frac{1}{2}$ to it. Then all valuations must give $\frac{1}{2}$ to π. Therefore, $\vDash_{LPV*} \pi$, but obviously $\nvDash_{LPV*} \perp$.

The relation between the validity predicate and the notion of consequence. Meadows' system has a validity predicate that captures its consequence relation correctly. The reason is that he defined the latter in terms of the former. As a result of this, Meadows' consequence relation is not reflexive, nor transitive, nor monotonic.

*LPV** has a validity predicate that only partially captures its consequence relation. It's still true that if $\phi \vDash_{LPV*} \psi$, then $\vDash_{LPV*} Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner)$. Nevertheless, it is possible that for some pair of formulae ϕ and ψ, and for all valuations v, $v(Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner)) = \frac{1}{2}$, and so $\vDash_{LPV*} Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner)$, but $\phi \nvDash_{LPV*} \psi$. Take the following example: $\vDash_{LPV*} Val(\ulcorner \pi \urcorner, \ulcorner \perp \urcorner)$, but $\pi \nvDash_{LPV*} \perp$.

The consequence relation of *LPV**, which is just the one of *LP*, is defined independently of the way the validity predicate works. Since it is based on a structural consequence relation, it is transitive, reflexive and monotonic.

*Detachment: a major problem for LPV**. *VD* has this form: $\phi \wedge Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner) \vdash \psi$. This isn't valid in *LPV**. For example, it fails when ϕ is the Beall-Murzi sentence and ψ is $\perp$. This seems to be a big problem for *LPV**'s validity predicate, because it's not just that its extension doesn't include all the pairs corresponding to a valid inference, but that it includes some pairs corresponding to invalid inferences – its validity predicate is unsound, so to say. We can still fix this problem by taking all instances of *VD* (or at least all invalid ones) out the initial step of the construction of the validity predicate. But then we lose our initial goal: to have a dialetheist theory of validity that validates both *VP* and *VD* unrestrictedly.

4. A paraconsistent validity predicate: second version

In this section, we will explore an alternative way of getting a paraconsistent validity predicate. Once again, the strategy is to adopt Meadows' construction, while changing the definition of the validity predicate. His system *VAL* corresponds by definition with the extension of the Validity predicate:

(*VAL*) Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be finite, then $\Gamma \vDash_{VAL} \phi$ iff $\langle \gamma_1 \wedge ... \wedge \gamma_n, \phi \rangle \in \Gamma^{+}_{Val}$. As a limit case of the previous definition, $\vDash_{VAL} \phi$ iff $(T, \phi) \in \Gamma^{+}_{Val}$.

We can modify this notion of validity and make it dialetheist, in the following way:

(*DT*) Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be finite, then $\Gamma \vDash_{DT} \phi$ iff $\langle \gamma_1 \wedge ... \wedge \gamma_n, \phi \rangle \notin \Gamma^{-}_{Val}$. As a limit case of the previous definition, $\vDash_{DT} \phi$ iff $\langle T, \phi \rangle \notin \Gamma^{-}_{Val}$.

We say that *DT* is dialetheist because it validates some sentences and their negations: a clear example is $\vDash_{DT} \pi$ and $\vDash_{DT} \neg\pi$. A great advantage of this proposal is that *DT*, unlike *VAL*, validates Reflexivity, for clearly no sentence can have value 1 and 0 at the same time in the fixed point; therefore, no pair $\langle \phi, \phi \rangle$ can belong to Γ^{-}_{Val}. But let's see how *DT* behaves with respect to other paradigmatic cases.

Modus Ponens. *DT* does validate Modus Ponens, because for every ϕ and every ψ, $\langle \phi \wedge (\phi \rightarrow \psi), \psi \rangle \notin \Gamma^{-}_{Val}$. In the three-valued schema we are using, it is not possible for a premise of a Modus Ponens to have value 1 while the conclusion is having value 0.

However, *DT* does not validate Modus Ponens as a meta-rule. For example, $\vDash_{DT} \pi$, and $\vDash_{DT} \pi \rightarrow \perp$, but $\nvDash_{DT} \perp$. But this, which seems to be an undesirable result, is what allows us to block the derivation of the Validity Paradox. Step 5 – in which we arrive to $\perp$ – will not be derivable from 3 and 4 by this external Modus Ponens rule. Therefore, if one thinks that it's desirable that the conditional of a theory validates Modus Ponens, then *DT* does it better, in this respect, than *VAL* and *LPV**.

The Beall-Murzi sentence. Since the Beall-Murzi sentence π does not belong to the anti-extension of the validity predicate, it is valid. This does not mean that the pair of formulae with T as the first member, and the Beall-Murzi sentence as the second one, will belong to the extension of the validity predicate – it will not, because that would lead to contradiction.

The relation between the validity predicate and the notion of consequence. As *VAL*, the system *DT* corresponds by definition to its own validity predicate, i.e. $A \vDash_{DT} B$ if and only if $\vDash_{DT} Val(\ulcorner A \urcorner, \ulcorner B \urcorner)$. However, Meadows' consequence relation is not reflexive, nor transitive, nor monotonic. While we have seen that *DT* 's consequence relation is, at least, reflexive. Moreover, it is also monotonic. If $\Gamma \vDash_{DT} \phi$, then there is no valuation v such that $v(\gamma_1 \wedge ... \gamma_n \wedge \alpha) = 1$ for every $\gamma_i \in \Gamma$ and some eventual extra premise α, but $v(\phi) = 0$. For suppose there is. For the conjunction of the premises to have value 1, every sentence γ_i must have value 1 too. Therefore, this valuation v would give 1 to $\gamma_1 \wedge ... \wedge \gamma_n$, and 0 to ϕ, contradicting the assumption that $\Gamma \vDash_{DT} \phi$. Therefore, $\Gamma \cup \{\alpha\} \vDash_{DT} \phi$.

The case of the Beall-Murzi sentence shows the failure of *transitivity*. It is clear that $\langle \pi \wedge (Val(\pi, \bot), \bot\rangle \notin \Gamma^{-}_{DT}$. Therefore, $Val(\ulcorner \pi \wedge Val(\pi, \bot)\urcorner, \ulcorner \bot \urcorner) \vDash_{DT} \bot$. So $Val(\ulcorner \pi \wedge \pi \urcorner, \ulcorner \bot \urcorner) \vDash_{DT} \bot$. Therefore, $Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner) \vDash_{DT} \bot$. Then it follows that $\pi \vDash_{DT} \bot$, and $\vDash_{DT} \pi$. But of course, $\nvDash_{DT} \bot$. The failure of transitivity is clearly the most important cost that *DT* must pay in order to represent validity adequately.

Detachment. Every instance of *VD* is valid in *DT*, because every instance of *VD* belongs to the extension of the predicate in the first stage of the hierarchy, and remains there in the minimal fixed point. However, *DT* does not validate *VD* as a meta-rule. For some cases, it happens that $\vDash_{DT} \phi$ and $\vDash_{DT} Val(\ulcorner \phi \urcorner, \ulcorner \psi \urcorner)$, but $\nvDash_{DT} \psi$. Take the following counterexample: $\vDash_{DT} \pi$ and $\vDash_{DT} Val(\ulcorner \pi \urcorner, \ulcorner \bot \urcorner)$, but $\nvDash_{DT} \bot$. The same problem affects *LPV**, so it cannot be seen as a specific flaw of *DT* but as a general flaw of paraconsistent approaches to validity.

In general, we take *DT* to be an improvement of *VAL* and *LPV**. Like *VAL*, *DT* corresponds to its validity predicate. But is has many virtues that *VAL* lacks: *DT* is reflexive and monotonic. On the other hand, even though *LPV** is an interesting theory (for it is structural), Modus Ponens and Detachment are invalid in the system. Those inferences are valid in *DT*. The only important problem of *DT* is the failure of transitivity; but it seems to us that it is reasonable price to pay given its overall virtues. The following table illustrates the main differences between the forementioned systems:

	VAL	*LPV**	*DT*
Modus Ponens	No	No	Yes
Detachment	Yes	No	Yes
Validity Correspondence	Yes	No	Yes
Reflexivity	No	Yes	Yes
Transitivity	No	Yes	No

Relation to other theories

Some recent papers (Shapiro (2011; 2015), Priest (2015)) developed responses to the V-Curry paradox in which $A, B \vDash C$ does not imply $A \wedge B \vDash C$. In particular, one cannot infer *VD* from VD_*:

$(VD_*)\ A, Val(\ulcorner A \urcorner, \ulcorner B \urcorner) \vDash B$

$(VD)\ A \wedge Val(\ulcorner A \urcorner, \ulcorner B \urcorner) \vDash B$

Shapiro (2015) develops a system with an additive conjunction, but without structural contraction. Priest (2015) claims that the conjunction in *VD* should not be the usual conjunction but *fusion*, a multiplicative connective which does not allow idempotence, thus avoiding the V-Curry paradox. Zardini

(2013) accepts the step from VD_* to VD, but he rejects the equivalence between $\phi \wedge \phi$ and ϕ.

The non-contractive approaches constitute a whole different family of responses to V-Curry, so for reasons of space is it not possible to criticize them here. Our theory accepts the move from VD_* to VD (unlike Shapiro or Priest), and the equivalence between ϕ and $\phi \wedge \phi$ (unlike Zardini). According to our view, the standard meaning of conjunction is fairly intuitive, so it can be abandoned only as a last resource.

The theory *DT* has some similarities with Weber's (2013) fixed point theory of validity. He develops a proof-theoretical hierarchy of validity rules for operators $\Rightarrow_i$:

- If $A \vdash_n B$, then $\vdash_{n+1} A \Rightarrow_n B$.
- If $\vdash_{n+1} A \Rightarrow_n B$, then $A \vdash_n B$.

Weber appeals to a fixed point theorem to obtain a general operator $\Rightarrow$, where $A \Rightarrow B$ whenever $A \Rightarrow_i B$ for some step i in the hierarchy. The operator $\Rightarrow$ does not validate this rule of Contraction: $A \Rightarrow (A \Rightarrow B) / A \Rightarrow B$.

It may be observed that the predicate version of Weber's rule of contraction, i.e. $Val(\ulcorner\ulcorner A\urcorner, \ulcorner Val(\ulcorner A\urcorner, \ulcorner B\urcorner)\urcorner) / Val(\ulcorner A\urcorner, \ulcorner B\urcorner)$ is *valid* in *DT*: for $Val(\ulcorner A\urcorner, \ulcorner B\urcorner)$ to have value 0, A must have value 1 and B value 0, thus the premise should also have value 0.

Moreover, our proposal has two further differences with Weber's theory. On the one hand, the meta-theoretical fixed point construction we offer is developed in classical logic, not in paraconsistent logic. On the other hand, our approach is essentially semantical, and does not emerge from a hierarchy of rules but from a hierarchy of models.

More importantly, as one anonymous referee observed, our theory *DT* strongly resembles the theory *ST* of Ripley (2012), which is also non-transitive and does not validate the meta-rule of Modus Ponens[6]. Indeed, Ripley proves that his sequent calculus is complete with respect to a three-valued semantics and a strict-tolerant consequence relation (similar to our fixed point construction). However, Ripley's approach in Ripley (2012) is mainly focused on truth, and does not address the problem of including a Validity predicate.

In a different paper, Ripley (2013) briefly mentions a proof-theoretical way of introducing the validity predicate in his non-transitive system, through the rules *VP* and *VD*: let's call that system *STV*. Our theory *DT* could be seen as a semantic counterpart of *STV*: the two *Val*-rules of *STV* are valid according to *DT*. Nevertheless, as a theory of validity, *DT* is

[6] For a less technical defense of *ST*, see *Cobreros et al* (2013).

stronger than *STV*[7]. For example, *DT* validates some internalized metarules such as:

$$(\wedge Val)\ Val(\ulcorner\phi\urcorner, \ulcorner\psi\urcorner), Val(\ulcorner\phi\urcorner, \ulcorner\chi\urcorner) \vDash Val(\ulcorner\phi\urcorner, \ulcorner\psi \wedge \chi\urcorner).$$

According to the semantics of *DT*, $\wedge Val$ is valid: if the conclusion has value 0, then there is a valuation where ϕ has value 1 and $\psi \wedge \chi$ has value 0. That is a valuation where ϕ has value 1 and ψ has value 0, or where ϕ has value 1 and χ has value 0. In any case, if this valuation exists, one of the premises must have value 0.

However, *STV* cannot prove $\wedge Val$. A natural proof would be as follows (we omit the quotes for readability):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{Val(\phi, \psi), \phi \Rightarrow \psi}\,VD}{Val(\phi, \psi), Val(\phi, \chi), \phi \Rightarrow \psi}\,LW \qquad \dfrac{\dfrac{}{Val(\phi, \chi), \phi \Rightarrow \chi}\,VD}{Val(\phi, \psi), Val(\phi, \chi), \phi \Rightarrow \chi}\,LW}{Val(\phi, \psi), Val(\phi, \chi), \phi \Rightarrow \psi \wedge \chi}\,R\wedge}{Val(\phi, \psi), Val(\phi, \chi) \Rightarrow Val(\phi, \psi \wedge \chi)}\,VP??$$

The rule *VP* cannot be applied in the last step, since the original rule needs an empty context in the left-hand side. There are many ways of strenghtening *STV* (for example, admitting *VP* with a non-empty context at the left-hand side), but none of them is straightforward[8]. Our theory *DT* might correspond to one extension of *STV*, but the issue is far from trivial and too complex to be covered in this paper, so we leave it as an open question.

5. Conclusion

Authors such as Beall (2009) and Beall & Murzi (2013) argued that dialetheism was unable to express the concept of validity. In this paper, we showed that it is possible to construct a validity predicate in a dialetheist framework, which validates both *VD* and *VP*. We first proposed the dialetheic theory *LPV**, whose consequence notion is defined independently of its validity predicate. Even though *LPV** is structural (reflexive, transitive, monotonic), its validity predicate doesn't capture its notion of consequence adequately. So we proposed a second theory, *DT*, in which the validity predicate and the notion of consequence coincide. The theory has two crucial advantanges: it is reflexive (unlike *VAL*) and monotonic.

[7] It is hard to determine whether *DT* is strictly speaking stronger than *STV*, since Ripley and Meadows develop the self-reference apparatus in different ways. Here we just claim that *DT* validates more principles about validity than *STV*.

[8] A detailed analysis of the theories of validity in cut-free contexts can be found at *Barrio et al* (2016).

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References

[1] Barrio, E., Rosenblatt, L. and Tajer, D. (2016), ‘Capturing Naive Validity in a Cut-Free approach’, manuscript.

[2] Beall, J.C. (2009), *Spandrels of truth*, Oxford University Press, Oxford.

[3] Beall, J.C. and Murzi, J. (2013), ‘Two flavors of Curry’s paradox’, *Journal of Philosophy* **110(3)**, 143-165.

[4] Cobreros, P., Egre, P., Ripley, D. & van Rooij, R. (2013), ‘Reaching transparent truth’, *Mind* **122(488)**, 841-866.

[5] Ketland, J. (2012), ‘Validity as a primitive’, *Analysis* **72(3)**, 421-430.

[6] Kripke, S. (1975), ‘Outline of a theory of truth’, *Journal of Philosophy* **72(19)**, 690-716.

[7] Meadows, T. (2014) ‘Fixed Points for Consequence Relations’, *Logique et Analyse* **57(227)**, 333-357.

[8] Ripley, D. (2012) ‘Conservatively extending classical logic with transparent truth’, *Review of Symbolic Logic* **5(2)**, 354-378.

[9] Ripley, D. (2013) ‘Paradoxes and failures of cut’, *Australasian Journal of Philosophy* **91(1)**, 139-164.

[10] Murzi, J. & Shapiro, L. (2013) ‘Validity and truth-preservation’, in: Achourioti, T. *et al.* (2013) *Unifying the philosophy of truth*, Springer, 431-460.

[11] Priest, G. (2006), *In contradiction*, Oxford University Press, Oxford.

[12] Priest, G. (2015), ‘Fusion and confusion’, *Topoi* **34**, 55-61.

[13] Shapiro, L. (2011), ‘Deflating logical consequence’, *The Philosophical Quarterly* **61(243)**, 320-342.

[14] Shapiro, L. (2015), ‘Naive structure, Contraction and Paradox’, *Topoi* **34(1)**, 75-87.

[15] Weber, Z. (2013) ‘Naive Validity’, *Philosophical Quarterly* **64(254)**, 99-114.

[16] Zardini, E. (2013) ‘Naive *Modus Ponens*’, *Journal of Philosophical Logic* **42**, 575-593.

Federico Pailos & Diego Tajer
University of Buenos Aires & CONICET
Argentina