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On the Theory of Intermittency in 1D Maps

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The classical theory of intermittency developed for return maps assumes uniform density of points reinjected from the chaotic to laminar region. Recently, we reported how the reinjection probability density (RPD) can be generalized. Estimation of the universal RPD is based on fitting a linear function to experimental or numerical data. Here we present an analytical approach to estimate the RPD. After this, we can get an analytic evaluation of the characteristic exponent traditionally used to characterize the intermittency type. The proposed theoretical method is general and very simple to use. It is compared with numerical computation, showing a good agreement between both. Our analytical results are compared with some celebrated classical numerical results on intermittency theory.

Keywords: Chaos; intermittency; one-dimensional map.

1. Introduction

Intermittency is a particular route to the deterministic chaos characterized by spontaneous transitions between laminar and chaotic dynamics. For the first time this concept was introduced by Pomeau and Maneville in the context of the Lorenz system [Manneville & Pomeau, 1979; Pomeau & Manneville, 1980]. Later intermittency has been found in a variety of different systems including, for example, periodically forced nonlinear oscillators, Rayleigh–Bénard convection, derivative nonlinear Schrödinger (DNLS) equation, and the development of turbulence in hydrodynamics (see e.g. [Dubois *et al.*, 1983; del Rio *et al.*, 1994; Stavrinides *et al.*, 2008; Krause *et al.*, 2014; Sanchez-Arriaga *et al.*, 2007]).

Besides this, there are other types of intermittencies such as type V, X, on–off, eyelet and ring [Kaplan, 1992; Price & Mullin, 1991; Platt *et al.*, 1993; Pikovsky *et al.*, 1997; Hramov *et al.*, 2006]. A more general case of on–off intermittency is the so-called in–out intermittency. A complete review of on–off and in–out intermittencies can be found in [Stavrinides & Anagnostopoulos, 2013].

Proper qualitative and quantitative characterizations of intermittency based on experimental data are especially useful for studying problems with partial or complete lack of knowledge on exact governing equations, as it frequently happens for example, in Economics, Biology, and Medicine (see e.g. [Zebrowski & Baranowski, 2004; Chian, 2007]).

It is interesting to note that most of the above cited references are devoted to systems having more than one dimension. In spite of this, they can be described by one-dimensional map. This phenomenon is typical of systems that contract volume in phase space [Ott, 2008].

All cases of Pomeau and Maneville intermittency have been classified into three types called I, II, and III [Schuster & Just, 2005]. The local laminar dynamics of type-I intermittency evolves in a narrow channel, whereas the laminar behaviors of type-II and type-III intermittencies develop around a fixed point of its generalized Poincaré map.

Another characteristic attribute of intermittency is the *global reinjection mechanism* that maps trajectories of the system from the chaotic region back into the local laminar phase. This mechanism can be described by the corresponding reinjection probability density (RPD). The RPD is determined by the chaotic dynamics of the system and drives the statistical properties of the system. Note that in the linear region, the invariant density of the map is formed by the RPD and by the trajectories on reinjected point going out of the linear region. The RPD contributes to the invariant density of the map in the linear region only with the reinjected points.

Analytical expressions for the RPD are available for a few maps only, hence to describe main statistical properties of intermittency, different approximations of the RPD have been employed. The most common classical approach uses the uniform RPD, which, however, works fine in a few model cases only [Manneville, 1980; Kim *et al.*, 1994; Cho *et al.*, 2002]. Another approach for RPD deals with a δ-function limit as it considers reinjection into a given point in the presence of noise [Kim *et al.*, 1998; Kye & Kim, 2000; Koronovskii & Hramov, 2008]. In this paper we propose a simple method to get an analytical approximation of the RPD, in good agreement with the numerical result given in the literature.

The contents of the present paper is outlined as follows: After this introduction, in Sec. 2, we briefly describe the state of the art and how the RPD is generated by the map. Section 3 is devoted to present the main result of this work, and we propose an analytical method to estimated the RPD and the characteristic exponent β. A comparison between numerical and analytical results are presented in Sec. 4. Finally, in Sec. 5, we summarize the main results.

2. Theoretical Framework

First, let us briefly describe the theoretical framework that accounts for a wide class of dynamical systems exhibiting intermittency. We consider a general 1D map

$$x_{n+1} = F(x_n), \quad F : \mathbb{R} \to \mathbb{R} \tag{1}$$

which exhibits intermittency, hence it presents two different behaviors according to where the trajectory takes place: laminar and chaotic.

The local laminar dynamics of type-I intermittency is determined by the 1D map in the form:

$$x_{n+1} = \varepsilon + x_n + a x_n^p \quad \text{Type-I} \tag{2}$$

where $a > 0$ accounts for the weight of the nonlinear component and ε is a controlling parameter ($\varepsilon \ll 1$). The laminar behaviors of type-II and type-III intermittencies develop around the fixed point of maps:

$$x_{n+1} = (1+\varepsilon)x_n + a x_n^p \qquad \text{Type-II} \tag{3}$$

$$x_{n+1} = -(1+\varepsilon)x_n - a x_n^p \quad \text{Type-III.} \tag{4}$$

The laminar interval L is delimited by a small constant c as following: $L = (0, c)$ for type-I and type-II and by the symmetric interval $L = (-c, c)$ in the case of type-III.

In some pioneer papers devoted to type-I and type-II intermittencies, the nonlinear component is quadratic, (i.e. $p = 2$). In the same way, in the classical theory on type-III it was fixed as $p = 3$.

In all of the cases $\varepsilon \gtrsim 0$, the point $x_0 = 0$ becomes unstable, and hence trajectories slowly escape from the origin preserving orientation for type-I and type-II and reversing orientation for type-III intermittency.

Regarding nonlinear dynamics, the so-called Reinjection Probability Density (RPD), describes the reinjection mechanism that maps back the trajectories from chaotic region of the map into the laminar region where the dynamics is determined by the local maps above referred. Notice that this mechanics is necessary to sustain the alternation between laminar and chaotic bursts. The RPD function is the most important one to determine the intermittent behavior of the system, but before embarking on a discussion concerning RPD, note the relationship between the RPD and the probability measure of an interval $S \subset [0, 1]$ given by

$$P(S) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I_S(x_n) \tag{5}$$

where $I_S(x)$ denotes characteristic function of the interval defined as

$$I_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S. \end{cases} \tag{6}$$

Thus the probability measure indicates the frequency of the signal in the corresponding interval of the attractor, and it is related with the invariant density $\rho(x)$ by

$$P(S) = \int_S \rho(x)dx. \tag{7}$$

It is clear that in our context $P([0,1]) = 1$. Let L be the laminar region and z an interval in the laminar region $z \subset L$. Before considering the probability $P(z)$ given by definition (5), let us split the whole data series into three subsets

$$\{x_n\} = \{x_{n'}\} \cup \{x_{n''}\} \cup \{x_{n'''}\} \tag{8}$$

having empty intersection between them

$$\begin{aligned} \{x_{n'}\} \cap \{x_{n''}\} &= \{x_{n''}\} \cap \{x_{n'''}\} \\ &= \{x_{n'''}\} \cap \{x_{n'}\} = \emptyset \end{aligned} \tag{9}$$

with the following definitions. Firstly, we choose $x_{n'} \in z$ and in the preceding period it already exists, that is, $I_z(x_{n'}) = 1$ and $I_z(x_{n'-1}) = 1$. For the next step, we have $x_{n''} \in z$ but in the preceding period it does not exist, that is $I_z(x_{n'}) = 1$ and $I_z(x_{n'-1}) = 0$. Finally, $x_{n'''} \notin z$.

Now it is clear that $P(z)$ is given by

$$\begin{aligned} P(z) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I_z(x_{n'}) + \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I_z(x_{n''}) \\ + \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I_z(x_{n'''}) \end{aligned} \tag{10}$$

where, in fact, the last term is not necessary because it is zero. The first term in Eq. (10) denotes the probability for the signal to be in z when in the preceding period it has already been there. The second term in Eq. (10) denotes the probability for the signal to be in z when in the preceding period it has not been there. The second term on the right-hand side of Eq. (10) defines the RPD, denoted here by $\phi(x)$, by the following relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} I_z(x_{n''}) = w \int_z \phi(x)dx \tag{11}$$

where the weight w is introduced because it is usual to normalize the function $\phi(x)$ over the whole laminar interval L as

$$\int_L \phi(x)dx = 1. \tag{12}$$

From Eq. (11) it is clear that the probability of finding a reinjected point between x and $x + dx$ is $\phi(x)dx$.

We note that $\phi(x)$ determines the fundamental characteristics of intermittency, as for instance, the probability density of the length of the laminar phase, denoted here by $\psi(l)$, and the characteristic exponent β, used to identify the intermittency type as it will be defined below.

From the mathematical RPD shape for each case it is possible to analytically estimate the fundamental characteristic of the intermittency, the probability density of the length of laminar phase $\psi(l)$, depending on l, that approximates the number of iterations in the laminar region, i.e. the length of the laminar phase. Note that the function $\psi(l)$ can be estimated from time series, as it is usual to characterize the intermittency type. The characteristic exponent β, depending on $\psi(l)$ through the relation $\bar{l} \propto \varepsilon^{-\beta}$, is also a good indicator of the intermittency type.

To illustrate how the RPD determines $\psi(l)$ and β, let us consider the type-II intermittency. For the other case, the methodology is similar. To estimate $\psi(l)$, we take, as usual, the approximation of the particular map for the local dynamics by the differential equation [Schuster & Just, 2005]. In the case of the map (3) we have

$$\frac{dx}{dl} = \varepsilon x + ax^p \tag{13}$$

where l approximates the number of iterations in the laminar region, i.e. the length of the laminar phase. Note that Eq. (13) provides a good approximation when $x_{n+1} - x_n$ is small enough. This fact determines the limit of the laminar interval defined by the constant c.

Solving (13) for l we get

$$l = \frac{1}{\varepsilon} \ln\left[\ln\left(\frac{x}{a}\right) - \frac{1}{p-1} \ln\left(\frac{ac^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon}\right)\right]. \tag{14}$$

Since x in (14) is a random variable described by $\phi(x)$, the statistics of l is also governed by the global properties of the RPD.

Let $\psi(l)$ be the probability density function of l, then it can be obtained by

$$\psi(l) = \phi(X(l)) \left| \frac{dX(l)}{dl} \right| \tag{15}$$

where $X(l)$ is the inverse function of (14), that is,

$$X(l) = \left[\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{p-1}}\right) e^{(p-1)\varepsilon l} - a} \right]^{\frac{1}{p-1}}. \tag{16}$$

Thus the probability density function of the length of the laminar phase is given by

$$\psi(l) = \phi(X(l))[\varepsilon X(l) + aX^p(l)]. \tag{17}$$

Using (17) we can determine the mean value of l which is an important property to characterize the intermittency

$$\bar{l} = \int_0^\infty \tau\psi(\tau)\mathrm{d}\tau. \tag{18}$$

Notice that $\bar{l}$ is directly related to the characteristic exponent β, commonly used to characterize any type of intermittency, by means of the characteristic relation $\bar{l} \propto \varepsilon^{-\beta}$, which describes, for small values of ε, how the length of the laminar phase increases as ε decreases.

At this point, we mention that $\psi(l)$ and β depend on the RPD, hence the main task here is to determine the RPD. To describe the reinjection mechanisms present in a wide class of dynamical systems exhibiting intermittency, the RPD given by the following power law

$$\phi(x) = b(x - \hat{x})^\alpha \tag{19}$$

has been recently introduced.

The RPD (19) has been observed in a number of maps having intermittencies of types I–III [del Rio & Elaskar, 2010; Elaskar *et al.*, 2011; del Rio *et al.*, 2014]. Note that the RPD (19) includes the classical approach as the particular case $\alpha = 0$. The free parameters $\hat{x}$ and α are determined by the dynamics in the chaotic region. The parameter $\hat{x}$ corresponds with the lower bound of reinjection (LBR), the exponent α is generated by the trajectories within the chaotic regime in the vicinity of a point in the map with infinite or zero tangent, and b is a normalization constant. Later it has been shown that the value of α is robust against the external noise making possible to calculate α in the noisy RPD [del Rio *et al.*, 2012; Krause *et al.*, 2013; Elaskar *et al.*, 2011].

The free parameters $\hat{x}$ and α can be numerically computed even for a relatively small data set or data with high level of noise [del Rio & Elaskar, 2010; Elaskar *et al.*, 2011; del Rio *et al.*, 2013].

On the other hand, $\hat{x}$ can be obtained from the definition of the map, however, there is not a general method to get the parameter α from the expression of the map.

Regarding the characteristic exponent, for $\hat{x} \approx 0$ in Eq. (19), β has been established as a function of α, as follows:

In type-I case we have [del Rio *et al.*, 2014]

$$\beta = \frac{p - \alpha - 2}{p}. \tag{20}$$

For type-II, the expression for β is given by [del Rio & Elaskar, 2010]

$$\beta = \frac{p - \alpha - 2}{p - 1}. \tag{21}$$

Finally, for type-III, the characteristic exponent is given by Eq. (21) [Elaskar *et al.*, 2011].

2.1. *Assessment of RPD function*

Let us introduce some analytical features on the RPD, that can help us in the following section. The RPD function depends on the particular shape of $F(x)$. To fix ideas, let us introduce an illustrating model

$$x_{n+1} = F(x_n) \equiv \begin{cases} f_1(x_n), & x_n < x_m \\ f_2(x_n), & x_n \geq x_m \end{cases} \tag{22}$$

where $f_1(x) = (1 + \varepsilon)x_n + (1 - \varepsilon)x_n^p$, and x_m is the root of the equation $f_1(x_m) = 1$ (see Fig. 1). Note that f_1 drives the laminar dynamics whereas f_2 drives the reinjection mechanism from the chaotic region into the laminar region as represented in Fig. 1 by a green arrow.

In his seminal paper, Manneville [1980] reported uniform reinjection for the map (22) with $f_2(x_n) = f_1(x_n) - 1$ and $p = 2$, whereas del Rio and Elaskar [2010] proposed the generalization to the map $f_2(x_n) = (f_1(x_n) - 1)^\gamma$ giving the RPD of Eq. (19).

For $\gamma = 1$ the original map was recovered, however, for $\gamma \neq 1$ it is interesting to estimate some characteristic of $\phi(x)$ in a neighborhood of $\hat{x}$ (LBR). Without loss of generality, we set $\hat{x} = 0$ in the map (22). The value of $\hat{x}$ is determined by $x_m = F^{-1}(\hat{x})$ where x_m is an extreme point, hence

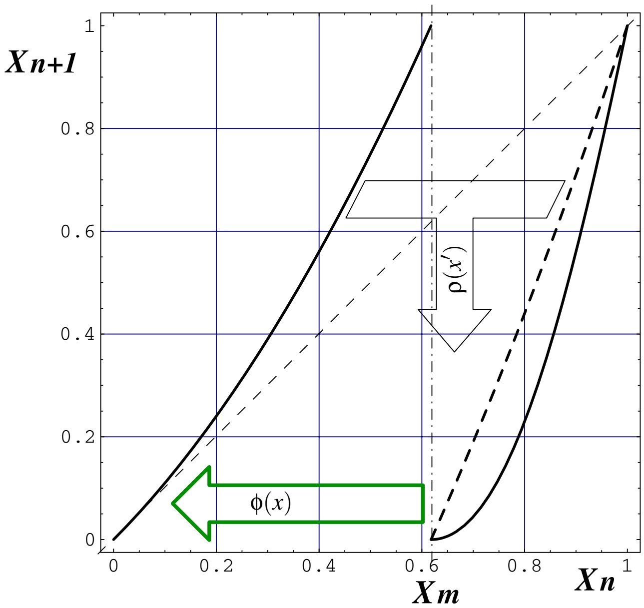


Fig. 1. Map of Eq. (22) for $p = 2$ with the bisecting line. The vertical dashed line indicates x_m. Cases of $f_2(x_n) = f_1(x_n) - 1$ and $f_2(x) = 1 - \cos(\pi\frac{f_1(x)-1}{2})$ correspond to dashed and solid lines respectively. The green arrow indicates the corresponding reinjection mechanism into the laminar zone. The reinjection around x_m is indicated by the black arrow.

$F'(x_m)$ is zero. Note that all points reinjected in the laminar region which is defined by the interval $(0, c)$, come from the points close to x_m as the green arrow shows in Fig. 1. That is, for $x' \gtrsim x_m$, all points in the interval $(x', x' + dx')$ are directly mapped into the interval $(f_2(x'), f_2(x' + dx')) \approx (x, x + f_2'(x')dx')$, where $x = f_2(x')$, hence the probability to find a point in $(x', x' + dx')$ is the same as that to be reinjected into the interval $(x, x + f_2'(x')dx')$, where f_2' indicates the derivative of the function f_2, hence we have $K\rho(x')dx' = \phi(x)f_2'(x')dx'$. The constant K account of $\rho(x)$ is normalized on the whole interval $[0, 1]$ whereas $\phi(x)$ is normalized only on the laminar interval, that is, $\int_0^c \phi(\tau)d\tau = 1$. Finally, we can approximate $\phi(x)$ as follows

$$\phi(x) = \rho(x')\frac{K}{\left.\frac{\mathrm{d}f_2(\tau)}{\mathrm{d}\tau}\right|_{\tau=x'}}. \tag{23}$$

In [del Rio & Elaskar, 2010] the case $f_2(x) = (f_1(x) - 1)^\gamma$ was investigated. The expression (23) in this case gives

$$\phi(x) = \frac{K\rho(x')}{\gamma f_1'(x')}x^{\frac{1}{\gamma}-1}, \tag{24}$$

where f_1' indicates the derivative of the function f_1. In the linear approximation of f_1 in the interval $(x_m, f_2^{-1}(c))$, we can consider f_1' as a constant. Now, if the density $\rho(x')$ is uniform, we get that for the reinjection probability density, the aforementioned power law $\phi(x) = bx^\alpha$ where

$$\alpha = \frac{1}{\gamma} - 1 \tag{25}$$

was verified in [del Rio & Elaskar, 2010]. It is interesting to note that the power law $\phi(x) = bx^\alpha$ has already been verified in a wide class of 1D maps even in some classical "pathological" cases that deviate significantly from the classical predictions [del Rio *et al.*, 2013]. Regarding the classical hypothesis of uniform RPD, it holds for the map (22), only in the case of $\gamma = 1$, where x_m is not an extreme point, however it is false for $\gamma \neq 1$ where the RPD is given by Eq. (19). This means that whereas the hypothesis of uniform reinjection does not work in general, it usually works for $\rho(x')$ when it is generated in no extreme points as indicated by the black arrow in Fig. 1.

In this scenario, where $\phi(x)$ is generated around the point x_m as Eq. (24) describes, the parameter γ determines $\phi(x)$ following Eq. (25). Note that whereas in this case the parameter γ appears explicitly in the definition of the map, this is not the general case. In this way, a further generalization of the reinjection mechanism can be proposed as follows: $f_2(x_n) = f(f_1(x_n) - 1)$ with $f(0) = 0$ and $f(1) = 1$. In the next section, we develop a method to associate a value of γ to a general reinjection mechanism, hence we extend the application of Eq. (25) to practically any map.

Concerning type-III intermittency, the scenario can be more complicated. To illustrate this point, let us consider a map having type-III intermittency

$$\begin{aligned} x_{n+1} &= F_{\mathrm{III}}(x_n) \\ &= -(1 + \varepsilon)x_n - ax_n^3 + dx_n^6 \sin(x_n) \\ &\qquad \text{with } a > 0. \end{aligned} \tag{26}$$

Figure 2 shows the map of Eq. (26) with the reinjection mechanism depending on the value of $F_{\mathrm{III}}(x_m)$ at the extreme points x_m satisfying $\mathrm{d}F_{\mathrm{III}}(x)/\mathrm{d}x = 0$. In spite of our odd map having two extreme points, for simplicity, only the maximum is indicated in Fig. 2. As the number of iterations increases, any point x_n close to the origin goes away in a process driven by the parameters ε and a in the

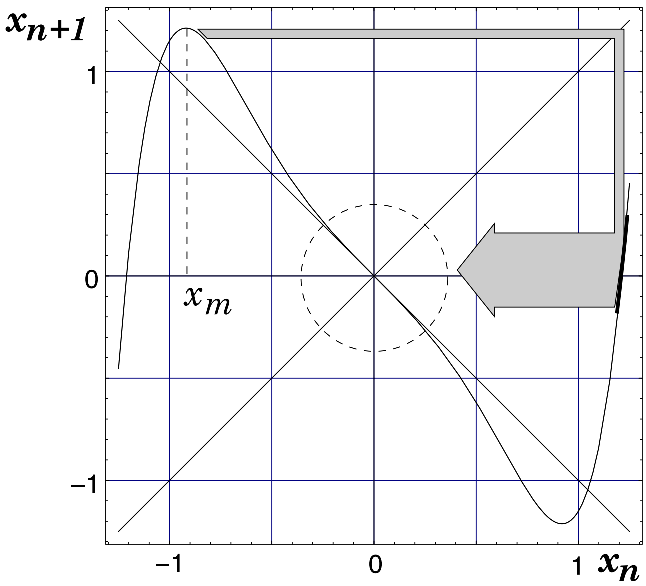


Fig. 2. Map (26) exhibiting type-III intermittency. Thick arrow illustrates mapping of points from the chaotic region (around the maximum of $F_{\rm III}(x)$) into the laminar region. The parameters used are $a = 1$, $\varepsilon = 0.01$ and $d = 1.1$.

cubic term of the map. For large enough values of x_n, the influence of the third term on the right-hand side of Eq. (26) increases and x_n approaches the maximum x_m rendering the reinjection into the laminar zone indicated in Fig. 2 by a dashed circle. Note that in this case, points in the neighborhood of x_m need two map iterations to be reinjected into the laminar region, however, this class of maps still has a RPD given by Eq. (19), hence it should be possible to obtain from Eq. (26) a value of γ providing the exponent α by means of Eq. (25). In the next section we propose an analytical method to approximate the parameter γ in all of these cases. We also apply the result to maps having type-I intermittency. The reference [Elaskar & del Rio, 2016] provides more detailed information on this topic.

3. Estimation of the RPD

In this section, we propose a method to estimate analytically the RPD by means of parameter γ of Eq. (25).

Let us start with a direct reinjection from the extreme point that the map (22) shows. Based on the argument of the preceding section, we approximate the map (22) around the extreme point x_m by $F(x) \approx F(x_m) + d(x - x_m)^\gamma$, where d is a suitable constant. Hence the value of γ is given by the next limit

$$\gamma = \lim_{x \to x_m} \frac{\ln(F(x) - F(x_m)) - \ln d}{\ln(x - x_m)}. \qquad (27)$$

To convert the limit (27) from indeterminate forms to an expression that can be evaluated, we can apply the L'Hopital rule, but usually we get an indeterminate limit again. In the general case, for $x \to x_m$ we have $F^{(i)}(x) \to 0$ for $i \le q$ where $F^{(i)}(x)$ denotes the i-derivative of the function $F(x)$, so by using the L'Hopital theorem $q + 1$ times we have

$$\gamma = q + \lim_{x \to x_m} F^{(q+1)}(x) \frac{x - x_m}{F^{(q)}(x)}. \qquad (28)$$

Now, if the derivative $F^{(q+1)}$ exists, the limit (28) gives $\gamma = q + 1$ and according to Eq. (25) we have

$$\alpha = -\frac{q}{q+1}. \qquad (29)$$

It is interesting to note that the values of q in Eq. (29) are confined to odd natural numbers, a condition imposed because x_m is an extreme point. In this context, if $F'(x)$ is a polynomial having x_m as a root of multiplicity q, then the exponent α is given by Eq. (29).

Note that the approximation (29) gives two natural limits. For $q = 0$ we recover the uniform reinjection and on the other hand, for $q \to \infty$, $\alpha \to -1$ and the RPD collapses into a δ-function.

Let us consider a map $F(x)$ defined by a composition of single maps $\{F_i\}$. See for instance [Hirsch *et al.*, 1982a; Kim *et al.*, 1994]. Even in the case of a not-composed map as in Eq. (26), to reinject into laminar region a point lying in the vicinity of the maximum or minimum, there are necessarily two iterations of the map, hence the point $x = F_{\rm III}(F_{\rm III}(x_m))$ lies in the laminar region. In this case, whereas the intermittency is referred to the map $x_{n+1} = F_{\rm III}(x_n)$, the limit (27) must be referred to the functional composition of maps $x_{n+1} = F(x_n) \equiv F_{\rm III} \circ F_{\rm III}(x_n)$. For a general view, we consider the map

$$x_{n+1} = F(x_n) \equiv F_r \circ F_{r-1} \circ \cdots \circ F_1(x_n), \qquad (30)$$

where the function $F_1(x)$ has an extreme point at x_m, that will be mapped into the laminar region by successive application of the single functions F_i forming the composed map (30).

Let us demonstrate that even in this case, the RPD will be approximated by applying Eq. (28) just to function $F_1(x_n)$, instead of applying it to the complete function $F(x)$.

Theorem 1. *Let $F(x)$ be the function defined by Eq. (30). If for a positive integer q, $F_1^{(i)}(x_m) = 0$ with $i \leq q$ and $F_1^{(q+1)}(x_m) \neq 0$, where $F_1^{(i)}$ indicates the i-derivative of the function F_1, the following equality holds*:

$$\lim_{x\to x_m} F^{(q+1)}(x)\frac{x-x_m}{F^{(q)}(x)} = \lim_{x\to x_m} F_1^{(q+1)}(x)\frac{x-x_m}{F_1^{(q)}(x)}. \tag{31}$$

Proof. Let us prove the equality (31) for $q = 1$, that is

$$\lim_{x\to x_m} F''(x)\frac{x-x_m}{F'(x)} = \lim_{x\to x_m} F_1''(x)\frac{x-x_m}{F_1'(x)}. \tag{32}$$

To prove Eq. (32) let us evaluate $F'(x)$ from its definition in Eq. (30),

$$F'(x) = \prod_{i=1}^{r} F_i' \circ G_{i-1} \tag{33}$$

where we define $G_0(x) \equiv x$ and $G_i(x) \equiv F_i \circ \cdots \circ F_1(x)$. Note that $F'(x_m) = 0$ because $F_1'(x_m) = 0$, hence x_m is an extreme point also for $F(x)$. The second derivative is given by

$$F''(x) = \sum_{j=1}^{r}\left(F_j'' \circ G_{j-1}\prod_{i\neq j} F_i' \circ G_{i-1}\right). \tag{34}$$

To evaluate the limit (31) at $x = x_m$, we point out that all brackets in (34) with $i \neq 1$ are zero because $F_1'(x_m) = 0$, so we have

$$\lim_{x\to x_m} F''(x)\frac{x-x_m}{F'(x)} = \lim_{x\to x_m} F_1''\prod_{i\neq 1}(F_i' \circ G_{i-1})\frac{x-x_m}{\prod_{i=1}^{r} F_i' \circ G_{i-1}} \tag{35}$$

so after eliminating the common factor in the fraction we have Eq. (32). In the case of $q > 1$, Eq. (31) is provided by means of L'Hopital rule applied to Eq. (32). ■

A direct consequence of Theorem 1 is that the estimation of γ in the case of a map defined like Eq. (30) is given by

$$\gamma = q + \lim_{x\to x_m} F_1^{(q+1)}(x)\frac{x-x_m}{F_1^{(q)}(x)}. \tag{36}$$

Note that the function F_1 in Eq. (36) is just F_{III} in the map (26), but in the general case the function F_1 can be quite different from the function used to define the map. Examples of this point will be shown in the next section.

Once γ is estimated, by means of Eq. (25) we have the exponent α for the power law (19) which describes the RPD. In the next section we apply this result to 1D map having intermittency.

Regarding $\bar{l}$, by using Eq. (29), we get an analytical expression for β depending on q as follow:

In type-I case, Eq. (20) becomes

$$\beta = 1 - \frac{q+2}{p(q+1)} \tag{37}$$

and for type-II, the expression (21) transforms into

$$\beta = 1 - \frac{1}{(p-1)(q+1)}. \tag{38}$$

Finally, for type-III, the characteristic exponent is given by Eq. (38) with $p = 3$ [Elaskar *et al.*, 2011].

4. Numerical Results

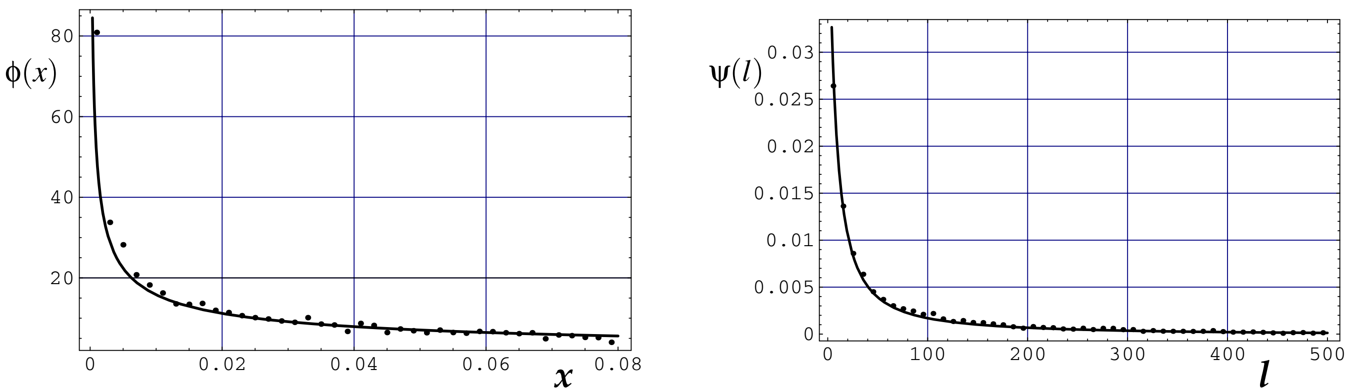
We apply the proposed method to 1D maps having types I–III intermittency to approach their RPDs and the characteristic exponents β.

Firstly we study the map (22) having type-II intermittency. In the case of $f_2(x_n) = (f_1(x_n) - 1)^\gamma$ it is clear that the value of the limit (28) is equal to the parameter γ and the expression (25) was already checked in [del Rio & Elaskar, 2010]. Here, we focus on a more general function as, for instance,

$$f_2(x) = 1 - \cos\left(\pi\frac{f_1(x)-1}{2}\right). \tag{39}$$

The reinjection mechanism proposed by Eq. (39) is represented by a solid line in Fig. 1. Following the last section, from Eq. (39) we have $f_2'(x_m) = 0$ and the second derivative exists with $f_2''(x_m) \neq 0$, hence we can use Eq. (29) with $q = 1$ giving $\alpha = -\frac{1}{2}$ and according to Eq. (38) the predicted value of β is 1/2.

It is interesting to note that the argument is quite general and the fundamental values of α and β do not depend on the particular parameter values of the map. To evaluate this analytical prediction, we set $p = 2$ and $\varepsilon = 0.01$ in the map (22). Note that for this map $\hat{x} = 0$, hence α is the only free parameter to determine $\phi(x)$ and $\psi(l)$ of Eqs. (19)



(a) (b)

Fig. 3. Dots refer to the numerical estimation obtained from 10^6 iterations of the map (22)–(39) having 9074 reinjections into the laminar region $(0, 0.1)$. Curves superimpose the analytical prediction using $\alpha = -1/2$ for: (a) $\phi(x)$ and (b) $\psi(l)$. The other parameters are $p = 2$ and $\varepsilon = 0.01$.

and (15) respectively. In Figs. 3(a) and 3(b) we compare these functions using $\alpha = -\frac{1}{2}$ with the numerical estimation.

Note that, on the contrary to the previously published results on this kind of intermittencies, dots and curves are plotted in Fig. 3 without any fitting analysis. We just superimpose the numerical estimations with our analytical predictions.

Concerning the characteristic relation, in Fig. 4 are represented the numerical estimations for $\bar{l}$ depending on ε. The slope of the line of the log–log plot should approach $-\beta$, hence Fig. 4 displays a good agreement with our analytical prediction for β.

Regarding type-III intermittency, we consider the map (26). Note that in this case the power law (19) is generated in the neighborhood of the maximum x_m (also around the minimum) but this

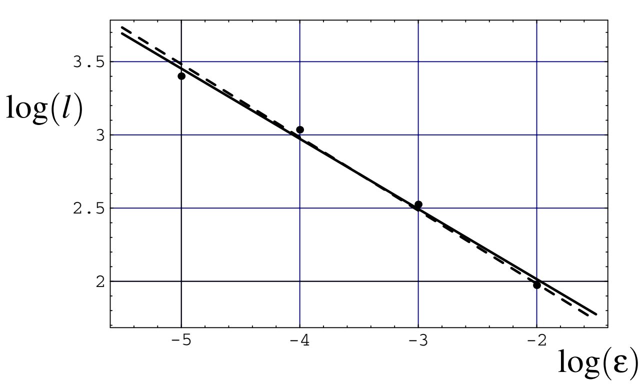


Fig. 4. Characteristic relations. Dots show numerical data for the map (22)–(39) with $p = 2$, while the solid line represents the least squares straight fitting with slope $-\beta = -0.48$, in good agreement with the analytical prediction giving $\beta = 0.5$ indicated by a dashed line.

region is not mapped into the laminar region by the map $x_{n+1} = F_{\mathrm{III}}(x_n)$, however, it is done by the composed map $F(x) \equiv F_{\mathrm{III}} \circ F_{\mathrm{III}}(x)$. According to Theorem 1, we can still apply Eq. (28) to the function F_{III}. With these considerations, we get a similar case as the previous one, hence $q = 1$ so we have $\alpha = -1/2$ and according to (38) evaluated for $p = 3$ we get $\beta = 3/4$, close to the reported value $\beta \approx 0.8$ in [Elaskar *et al.*, 2011]. The error around 6% between both data is due to the large value $c = 0.6$ used to delimit the laminar interval. It affects our approximation around the extreme point. Note that to deal with Eq. (38), following [Schuster & Just, 2005], we approximate in the laminar region a finite difference equation by a continuous differential equation, and this approximation also introduces error in the case of a large length of the laminar region.

Concerning type-I intermittency, we consider the classical cases reported in [Kwon *et al.*, 1996], where the authors study the three maps shown in Fig. 5. The mentioned paper is one of the pioneer works reporting *nonstandard* characteristic exponents β, where a uniform RPD fails to explain these values.

The authors evaluated the RPD by solving numerically the Shaw relation [Lichtenberg & Lieberman, 1983]. In all of the cases the reinjection from the extreme point into the laminar region is done by the composed maps $x_{n+1} = F \circ F(x_n)$ as sketched in Fig. 5. For the three cases, by Taylor expansion the map in the laminar region can be approximated by f_1 of Eq. (22) with $p = 2$ (see [Kwon *et al.*, 1996]).

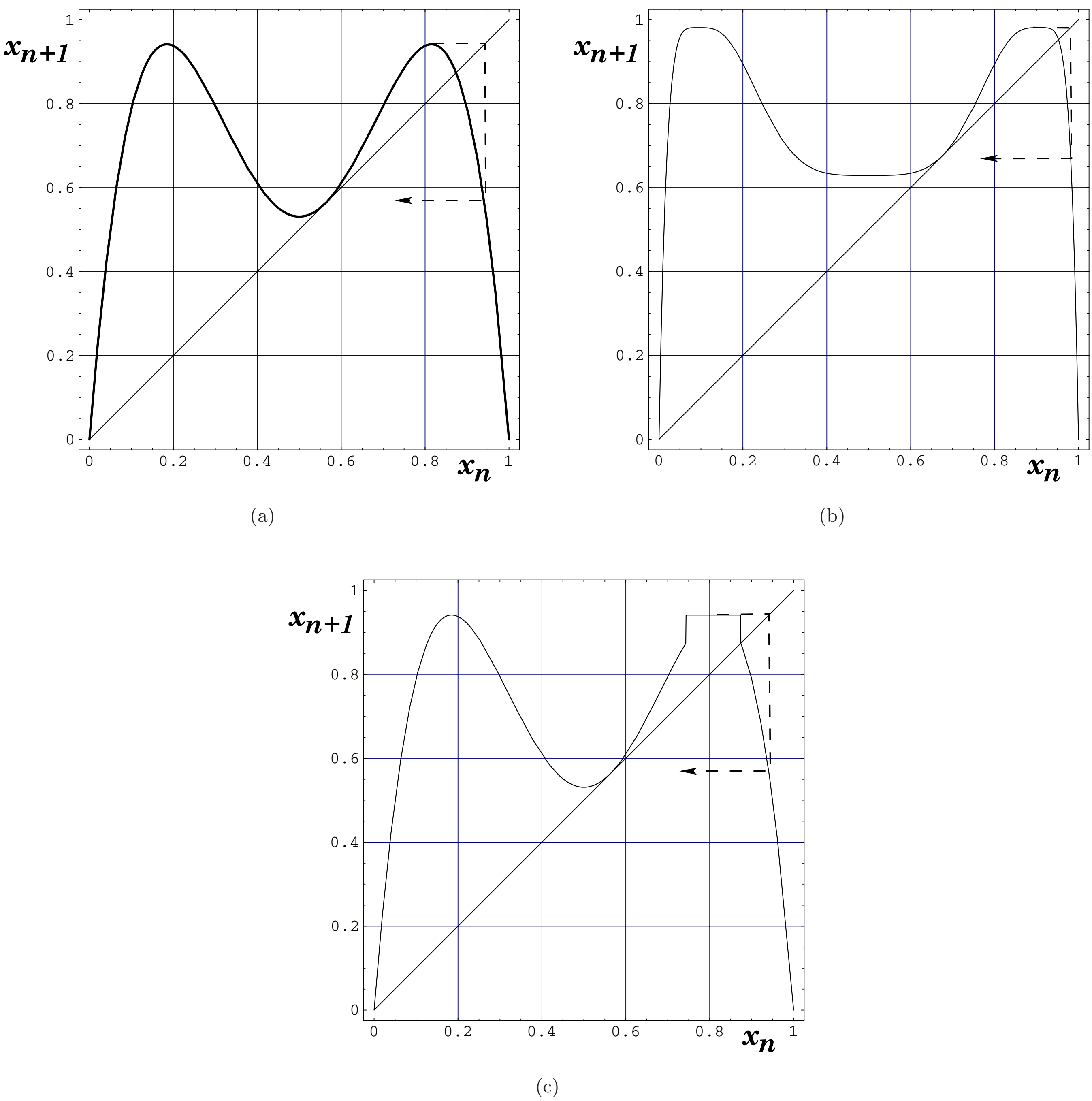


Fig. 5. Maps of [Kwon *et al.*, 1996] exhibiting type-I intermittency. Arrow illustrates mapping of points from the chaotic region into the laminar region. The parameter values are the same as that used in [Kwon *et al.*, 1996] as follows: (a) $A = 0.9416195$, (b) $A = 0.98115325$ and (c) $A = 0.9416$, $B = 0.83023023$, $a = 0.743$, $b = 0.874$ and $x^* = 0.9414793$.

Let us apply the method proposed in the previous section to the maps investigated in [Kwon *et al.*, 1996]. The first map is the next composition of logistic maps $F(x) = f^{(A)}[f^{(B)}(x)]$ (we follow the same notation as in the cited reference) where $f^{(A)}(x) = 4Ax(1-x)$ and $f^{(B)}(x) = 4Bx(1-x)$. By Theorem 1, we can study only the map $f^{(B)}(x)$ at its extreme point $x_m = 0.5$. As $f^{(B)}(x)$ is a second-order polynomial, the second derivative of $f^{(B)}$ must be different from zero hence $q = 1$, the first odd natural number, and according to Eq. (28) we have $\alpha = -1/2$ that from Eq. (37) we get $\beta = 1/4$. In this simple way we get the same result as reported in [Kwon *et al.*, 1996].

The second map is defined again by a composed function in a similar form as the first one except that $f^{(A)}(x) = A(1 - 16(x - 1/2)^4)$ and $f^{(B)}(x) = B(1 - 16(x - 1/2)^4)$. By applying the same argument as in the first case to the fourth-order polynomial of $f^{(B)}$ we conclude that now $q = 3$, the second odd natural number, hence $\alpha = -3/4$ and $\beta = 3/8$, getting the same values as reported by the mentioned reference.

Finally, the third map has a similar shape as the first one except in the range $a < x < b$ where the map is a flat constant at value $x*$ (see Fig. 5). In this case, the derivative $F^{(q)}$ is zero for all values of q, hence in the limit $q \to \infty$, we get $\alpha \to -1$ and $\beta = 1/2$, that coincides again with the value reported in [Kwon *et al.*, 1996].

In other cases reported in the literature (see for instance [Hirsch *et al.*, 1982b]) the map is constructed by a number of composed identical logistic maps, but the reinjection into the laminar region comes from several regions of the map without local extreme points hence the RPD is like a constant reinjection.

5. Conclusions

Based on the well confirmed result for many 1D maps, the reinjection probability density RPD can be approached by the power law given by Eq. (19). In this work we investigate how that RPD is generated in return maps. We show that the RPD is generated around the extreme points of the map. Based on this fact, we propose an analytical method providing the RPD, hence by simple calculus it is possible to estimate the RPD given by Eq. (19). In particular, we propose a formula that provides the parameter α as a function of the number of null derivatives of the map at the extreme point. The extreme condition of the point imposes that this number must be odd, hence there is a restriction on the possible values of α.

Once the value of α is determined, the value of the characteristic exponent is also determined, depending on the intermittency type. This analytical prediction has been compared with numerical estimation showing good agreement between both. For the maps investigated in the celebrated paper [Kwon *et al.*, 1996], our method provides the exact value of the characteristic exponent β in agreement with the numerically reported values in the mentioned reference.

We emphasize that the values of α and β found by our method usually do not depend on the parameter values of the map, in particular on ε.

Notice that the intermittency type (I, II, III) is determined by laminar map and, on the other hand, the RPD is determined by the chaotic region of the map. As a consequence, the proposed method to determine the RPD can be used in systems having other intermittencies such as type V, X, on–off or ring ones.

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