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# Numerical evaluation of integrals involving the product of two Bessel functions and a rational fraction arising in some elastodynamic problems

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ABSTRACT

This paper presents numerical techniques for evaluating integrals of the form

$$\int_0^\infty \frac{J_\beta(k\rho)\,J_\gamma(kR)}{k^\alpha(k-s)}dk.$$

These integrals arise during the application of the Hankel transform to pass the displacements of a layered soil profile from the wave number domain to the spatial domain in three-dimensional problems of elastodynamics. The objective here is to obtain solutions with an adequate accuracy from the engineering point of view to the integrals that arise in a first order formulation of a wave propagation model widely used for layered soils.



## 1. Introduction

The integrals considered here arise after expressing the flexibility matrix of a stratified soil profile layer as function of modal parameters of first order in the wave number domain [1]:

$$F_{ij}(k) = \begin{bmatrix} \sum_{\boldsymbol{r}=1}^{N_r} \frac{\varphi^i_{\rho,\boldsymbol{r}}\varphi^j_{\rho,\boldsymbol{r}}}{k-s_{\boldsymbol{r}}} & 0 & \sum_{\boldsymbol{r}=1}^{N_r} \frac{\varphi^i_{\rho,\boldsymbol{r}}\varphi^j_{z,\boldsymbol{r}}}{k-s_{\boldsymbol{r}}} \\ 0 & \sum_{\boldsymbol{l}=1}^{N_l} \frac{\varphi^i_{\theta,\boldsymbol{l}}\varphi^j_{\theta,\boldsymbol{l}}}{k-s_{\boldsymbol{l}}} & 0 \\ \sum_{\boldsymbol{r}=1}^{N_r} \frac{\varphi^i_{z,\boldsymbol{r}}\varphi^j_{\rho,\boldsymbol{r}}}{k-s_{\boldsymbol{r}}} & 0 & \sum_{\boldsymbol{r}=1}^{N_r} \frac{\varphi^i_{z,\boldsymbol{r}}\varphi^j_{z,\boldsymbol{r}}}{k-s_{\boldsymbol{r}}} \end{bmatrix} \tag{1}$$

where $k$ represents the wave number, $s$ and $\varphi$ are the eigenvalues and eigenvector components of different wave propagation modes, $i$ and $j$ represent two generic interfaces of the layered soil, $\rho, \theta$ and $z$ are the radial, azimuthal and vertical cylindrical coordinates, respectively, while $\boldsymbol{r}$ and $\boldsymbol{l}$ refer to the Rayleigh and Love waves, respectively. The flexibility matrix of a layer thus expressed represents a variant of first order as an alternative to the second order formulation presented for Kausel [2]

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and Kausel and Roesset [3], where this matrix is also described as a combination of different propagation modes of Rayleigh wave (coupled coordinates $\rho$ and $z$) and Love waves (coordinate $\theta$). The application of uniform loads or loads with linear variation on circular areas of radius $R$ at the interfaces of the layered profile generates the Bessel function of the 1st kind $J_\gamma(kR)$ in the expressions of displacements in the wave number domain, while the application of inverse Hankel transform to return to the spatial domain produces the Bessel function of the 1st kind $J_\beta(k\rho)$.

## 2. Integrals to solve

The integrals to solve have the following general form

$$V_{\alpha\beta\gamma}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}dk \tag{2}$$

using a similar nomenclature to that used by Hemsley [4]. The values adopted by the parameters $\alpha$, $\beta$ and $\gamma$ are the integer numbers 0, 1 and 2. The parameters $\rho$ and $R$ are nonnegative real numbers, while the eigenvalue $s$ is in general a complex number. The integral in (2) does not have closed analytic solution, with exceptions such as for $s = 0$ that corresponds to the static solution to the elastodynamic problem, and for $R = 0$ that it is related to point loads in this type of problems.

The numerical technique proposed in this paper is explicitly used for the evaluation of the integral (2) for $\Re(s) \le 0$ (2nd and 3rd quadrants of the complex plane), where the denominator of the integrand does not vanish for any value of $k$, so that the integrals become non-oscillating in function of the coordinate $\rho$. For $\Re(s) > 0$ (1st and 4th quadrants) the solution is expressed as the sum of an integral with analytic solution and other integral with numerical solution that is obtained by changing the sign of $s$ and using the technique for $\Re(s) \le 0$. To this end, the denominator of the integrand in (2) is replaced by

$$\frac{1}{k-s} = \frac{2s}{k^2-s^2} + \frac{1}{k+s} \tag{3}$$

for odd values of $(\alpha+\beta+\gamma)$, while this denominator is replaced by

$$\frac{1}{k-s} = \frac{2k}{k^2-s^2} - \frac{1}{k+s} \tag{4}$$

for even values of $(\alpha+\beta+\gamma)$. Thus, evaluation of the integral for $\Re(s) > 0$ is performed as

$$\begin{aligned} V_{\alpha\beta\gamma}(\rho, R, s) &= 2s\int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k^2-s^2)}dk + \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k+s)}dk \\ &= 2sW_{\alpha\beta\gamma}(\rho, R, s) + V_{\alpha\beta\gamma}(\rho, R, -s) \end{aligned} \tag{5}$$

for odd values of $(\alpha+\beta+\gamma)$, or as

$$\begin{aligned} V_{\alpha\beta\gamma}(\rho, R, s) &= 2\int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^{\alpha-1}(k^2-s^2)}dk - \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k+s)}dk \\ &= 2W_{(\alpha-1)\beta\gamma}(\rho, R, s) - V_{\alpha\beta\gamma}(\rho, R, -s) \end{aligned} \tag{6}$$

for even values of $(\alpha+\beta+\gamma)$. The application of this procedure is due to the fact that the integral

$$W_{\alpha\beta\gamma}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k^2-s^2)}dk \tag{7}$$

possesses an analytic solution only for odd values of $(\alpha+\beta+\gamma)$ as in cases analyzed by Kausel [2], who presents solutions for values of $s$ in the 4th quadrant of the complex plane. For the 1st quadrant it is required to replace the Hankel function of the 2nd kind for that of the 1st kind (as originally proposed by Watson [5]). The nature of the integrals (5) and (6) is oscillating due to the integral in (7), while the other integral of non-oscillating nature is solved by numerical techniques as proposed for $\Re(s) \le 0$.

The conditions to be satisfied by parameters $\alpha$, $\beta$ and $\gamma$ so that the integral becomes finite are given below. The integrand in (2) for $k \to 0$ results in

$$\left.\frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}\right|_{k\to 0} = \frac{\rho^\beta R^\gamma}{2^{\beta+\gamma}\beta!\,\gamma!}\frac{k^{\beta+\gamma-\alpha}}{(k-s)} \tag{8}$$

from which it follows that

$$\beta+\gamma-\alpha \ge 0. \tag{9}$$

The integrand in (2) for $k \to \infty$ yields

$$\left.\frac{J_\beta(k\rho)J_\gamma(kR)}{k^\alpha(k-s)}\right|_{k\to\infty} = \frac{2}{\pi\sqrt{\rho R}}\cos\left(k\rho - \frac{\pi}{4}(2\beta+1)\right)\cos\left(kR - \frac{\pi}{4}(2\gamma+1)\right)\frac{1}{k^{\alpha+2}} \tag{10}$$

from which it can be shown that

$$\begin{aligned} &\alpha > -2 \quad \text{for } \rho \neq R, \\ &\alpha > -1 \quad \text{for } \rho = R. \end{aligned} \tag{11}$$

In some cases, the transformation of the load function to the wave number domain generates the function $(1 - J_0(kR))$, so that the transformation to the spatial domain for obtaining the displacement takes the form

$$\bar{V}_{\alpha\beta}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)\ (1 - J_0(kR))}{k^\alpha (k - s)} dk. \tag{12}$$

Expanding the term in parentheses, this integral can be expressed as

$$\bar{V}_{\alpha\beta}(\rho, R, s) = V_{\alpha\beta 0}(\rho, 0, s) - V_{\alpha\beta 0}(\rho, R, s) \tag{13}$$

which not always satisfies the condition for $k \to 0$. In such cases, the integral in (12) must be solved jointly, for which it is required to know the solution of integrals of the form

$$\bar{W}_{\alpha\beta}(\rho, R, s) = \int_0^\infty \frac{J_\beta(k\rho)\ (1 - J_0(kR))}{k^\alpha (k^2 - s^2)} dk. \tag{14}$$

The integral taken as a starting point in this paper is $V_{011}(\rho, R, s)$. The evaluation of the remaining integrals is performed decomposing the integrands in a similar way to that used for (3) and (4) and/or applying repeated differentiation on the integrands [4] as

$$V_{111}(\rho, R, s) = \frac{1}{s}(H_{011}(\rho, R) - V_{011}(\rho, R, s)) \tag{15}$$

$$V_{211}(\rho, R, s) = \frac{1}{s}(H_{111}(\rho, R) - V_{111}(\rho, R, s)) \tag{16}$$

$$V_{001}(\rho, R, s) = \frac{\partial}{\partial r} V_{111}(\rho, R, s) + \frac{1}{r} V_{111}(\rho, R, s) \tag{17}$$

$$V_{010}(\rho, R, s) = \frac{\partial}{\partial R} V_{111}(\rho, R, s) + \frac{1}{R} V_{111}(\rho, R, s) \tag{18}$$

$$V_{101}(\rho, R, s) = \frac{\partial}{\partial r} V_{211}(\rho, R, s) + \frac{1}{r} V_{211}(\rho, R, s) \tag{19}$$

$$V_{110}(\rho, R, s) = \frac{\partial}{\partial R} V_{211}(\rho, R, s) + \frac{1}{R} V_{211}(\rho, R, s) \tag{20}$$

$$\begin{aligned} V_{000}(\rho, R, s) &= \frac{\partial}{\partial R} V_{101}(\rho, R, s) + \frac{1}{R} V_{101}(\rho, R, s) \\ &= \frac{\partial}{\partial r} V_{110}(\rho, R, s) + \frac{1}{r} V_{110}(\rho, R, s) \end{aligned} \tag{21}$$

where it has been resorted to the following auxiliary integrals that possess analytic solutions

$$H_{\alpha\beta\gamma}(\rho, R) = V_{\alpha\beta\gamma}(\rho, R, 0) = \int_0^\infty \frac{J_\beta(k\rho) J_\gamma(kR)}{k^{\alpha+1}} dk. \tag{22}$$

Although the integral $V_{000}(\rho, R, s)$ can be solved through the expressions given in (21), it is convenient to solve it independently following the same strategy as that used for $V_{011}(\rho, R, s)$ since simpler and more efficient numerical expressions are obtained.

Moreover, the following relationships

$$\begin{cases} J_2(k\rho) = \dfrac{2}{k\rho} J_1(k\rho) - J_0(k\rho) \\ J_2(kR) = \dfrac{2}{kR} J_1(kR) - J_0(kR) \end{cases} \tag{23}$$

allow to use the solved integrals to obtain the following ones

$$V_{002}(\rho, R, s) = \frac{2}{R} V_{101}(\rho, R, s) - V_{000}(\rho, R, s) \tag{24}$$

$$V_{020}(\rho, R, s) = \frac{2}{\rho} V_{110}(\rho, R, s) - V_{000}(\rho, R, s) \tag{25}$$

**Table 1**
Integrals evaluated in this paper.

| $W_{\alpha\beta\gamma}(\rho, R, s)$ | $V_{\alpha\beta\gamma}(\rho, R, s)$ | | $\bar{W}_{\alpha\beta}(\rho, R, s)$ | $\bar{V}_{\alpha\beta}(\rho, R, s)$ |
|---|---|---|---|---|
| $W_{-100}(\rho, R, s)$ | $V_{000}(\rho, R, s)$ | $V_{002}(\rho, R, s)$ | $\bar{W}_{-10}(\rho, R, s)$[a] | $\bar{V}_{00}(\rho, R, s)$[a] |
| $W_{-111}(\rho, R, s)$ | $V_{011}(\rho, R, s)$ | $V_{020}(\rho, R, s)$ | $\bar{W}_{01}(\rho, R, s)$[a] | $\bar{V}_{11}(\rho, R, s)$[a] |
| $W_{111}(\rho, R, s)$ | $V_{111}(\rho, R, s)$ | $V_{012}(\rho, R, s)$ | $\bar{W}_{10}(\rho, R, s)$ | $\bar{V}_{10}(\rho, R, s)$ |
| $W_{001}(\rho, R, s)$ | $V_{211}(\rho, R, s)$ | $V_{021}(\rho, R, s)$ | $\bar{W}_{21}(\rho, R, s)$ | $\bar{V}_{21}(\rho, R, s)$ |
| $W_{010}(\rho, R, s)$ | $V_{001}(\rho, R, s)$ | $V_{112}(\rho, R, s)$ | | |
| | $V_{010}(\rho, R, s)$ | $V_{121}(\rho, R, s)$ | | |
| | $V_{101}(\rho, R, s)$ | $\bar{V}_{001}(\rho, R, s)$ | | |
| | $V_{110}(\rho, R, s)$ | $\bar{V}_{010}(\rho, R, s)$ | | |

[a] Integrals can be solved by developing the parentheses in the numerator of (12) and (14), but they are evaluated explicitly because they are required as an intermediate step for the calculation of the remaining integrals of the corresponding column.

$$V_{012}(\rho, R, s) = \frac{2}{R} V_{111}(\rho, R, s) - V_{010}(\rho, R, s) \tag{26}$$

$$V_{021}(\rho, R, s) = \frac{2}{\rho} V_{111}(\rho, R, s) - V_{001}(\rho, R, s) \tag{27}$$

$$V_{112}(\rho, R, s) = \frac{2}{R} V_{211}(\rho, R, s) - V_{110}(\rho, R, s) \tag{28}$$

$$V_{121}(\rho, R, s) = \frac{2}{\rho} V_{211}(\rho, R, s) - V_{101}(\rho, R, s). \tag{29}$$

Other composed integrals arising in elastodynamic problems that may be convenient to solve jointly from the computational point of view are

$$\bar{V}_{001}(\rho, R, s) = V_{001}(\rho, R, s) - \frac{1}{\rho} V_{111}(\rho, R, s) \tag{30}$$

$$\bar{V}_{010}(\rho, R, s) = V_{010}(\rho, R, s) - \frac{1}{R} V_{111}(\rho, R, s). \tag{31}$$

The complete list of integrals evaluated in this paper is shown in Table 1.

An approximation to the integral (2) was proposed by Hemsley [4] for 3 values of $s$ as an intermediate step to solve the following integral

$$G_{\alpha\beta\gamma}(\rho, R) = \int_0^\infty \frac{k^\alpha J_\beta(k\rho) J_\gamma(kR)}{k^3 + 1} dk. \tag{32}$$

This integral arises during the evaluation of bending stresses in an infinite plate on a half-space subjected to an axial-symmetrical load. The denominator of the integrand in (32) may be decomposed into partial fractions as follows

$$\frac{1}{k^3 + 1} = \frac{1}{3} \left( \frac{1}{k+1} + \frac{c}{k+c} + \frac{c^*}{k+c^*} \right) \quad \text{where } c = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \tag{33}$$

$c^*$ is the complex conjugate of $c$. Thus, the 3 values of $s$ for which the integral in (2) was solved are $-1, -c$ y $-c^*$. Note that the term $k^\alpha$ is in the denominator of (2) instead of keeping it in the numerator of (32) as proposed by Hemsley. This is to the effect that the values of $\alpha$ do not take negative values for most cases of interest here.

The technique proposed by Hemsley produces accurate results only for relatively low values of $\rho$, $R$ and $s$. Therefore, an efficient alternative technique even for values of $\rho$ and $s$ that tend to infinity is presented in what follows. On the other hand, integrals of type (12) appearing in some elastodynamic problems and which have not been found in the literature are also solved. Some integrals similar to those shown in this paper have also been solved by Van Deun and Cools [6].

## 3. Integrals of type $W_{\alpha\beta\gamma}(\rho, R, s)$

These integrals can be solved analytically. In fact, the work of Kausel [2] includes the solutions of the following integrals which are then used in solving the integrals of type $V_{\alpha\beta\gamma}(\rho, R, s)$ for $\Re(s) > 0$:

$$W_{-100}(\rho, R, s) = \pm \frac{i\pi}{2} J_0(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) \tag{34}$$

$$W_{-111}(\rho, R, s) = \pm \frac{i\pi}{2} J_1(\bar{s}\bar{\rho}) H_1^{(1,2)}(\bar{s}) \tag{35}$$

$$W_{111}(\rho, R, s) = \pm\frac{i\pi}{2s^2}J_1(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) - \frac{\bar{\rho}}{2s^2} \tag{36}$$

$$W_{001}(\rho, R, s) = \begin{cases} \pm\frac{i\pi}{2s}J_0(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) - \frac{1}{s\bar{s}} & \rho \leq R \\ \pm\frac{i\pi}{2s}J_1(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s}) & \rho > R \end{cases} \tag{37}$$

$$W_{010}(\rho, R, s) = \begin{cases} \pm\frac{i\pi}{2s}J_1(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s}) & \rho \leq R \\ \pm\frac{i\pi}{2s}J_0(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) - \frac{1}{s\bar{s}} & \rho > R \end{cases} \tag{38}$$

where the following dimensionless variables are used

$$\left.\begin{array}{l}\bar{s} = sR \\ \bar{\rho} = \rho/R\end{array}\right\} \quad \text{for } \rho \leq R \quad \text{and} \quad \left.\begin{array}{l}\bar{s} = s\rho \\ \bar{\rho} = R/\rho\end{array}\right\} \quad \text{for } \rho > R. \tag{39}$$

In these expressions it should be used positive sign and Hankel function of the 1st kind $H_\nu^{(1)}(\bar{s})$ for $\Im(s) > 0$, and negative sign and Hankel function of the 2nd kind $H_\nu^{(2)}(\bar{s})$ for $\Im(s) \leq 0$.

Note that these integrals produce complex results even for real values of $s$, since a discontinuity is presented respect to the positive axis of $\Re(s)$, and in such case one should adopt the value corresponding to $\Im(s) \to 0^-$.

## 4. Integrals of type $V_{\alpha\beta\gamma}(\rho, R, s)$

### 4.1. Integral $V_{000}(\rho, R, s)$

The form that the integral takes in this case is

$$V_{000}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho)J_0(kR)}{k - s}dk. \tag{40}$$

A suitable replacement for the product of the cylindrical functions suggested by Hemsley [4] is based on the following expression given by Watson [5]

$$J_0(k\rho)J_0(kR) = \frac{1}{\pi}\int_0^\pi J_0(kr)d\theta \tag{41}$$

where

$$r = \sqrt{\rho^2 + R^2 - 2\rho R\cos\theta}. \tag{42}$$

By introducing (41) and (42) in (40), and changing the integration order, the following expression is obtained

$$V_{000}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \int_0^\infty \frac{J_0(kr)}{k - s}dkd\theta. \tag{43}$$

The integration of variable $k$ for $\Re(s) \leq 0$ produces [5]

$$V_{000}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \left(\frac{\pi}{2}\left[\boldsymbol{H}_0(-sr) - Y_0(-sr)\right]\right)d\theta \tag{44}$$

where $Y_0(-sr)$ is the Bessel function of the 2nd kind and $\boldsymbol{H}_0(-sr)$ is the Struve function. Note that for $\rho = 0$ results $r = R$ becoming the integrand independent of $\theta$

$$V_{000}(0, R, s) = \frac{\pi}{2}\left[\boldsymbol{H}_0(-sR) - Y_0(-sR)\right]. \tag{45}$$

A convenient form to express the integral in (44) involves replacing the argument of the cylindrical functions as follows

$$V_{000}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \left(\frac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right]\right)d\theta \tag{46}$$

where

$$z = -sr = -\bar{s}\bar{r} = \bar{z}\bar{r} \qquad \bar{z} = -\bar{s} \qquad \bar{r} = \sqrt{\bar{\rho}^2 + 1 - 2\bar{\rho}\cos\theta} \tag{47}$$

with the dimensionless variables defined in (39).

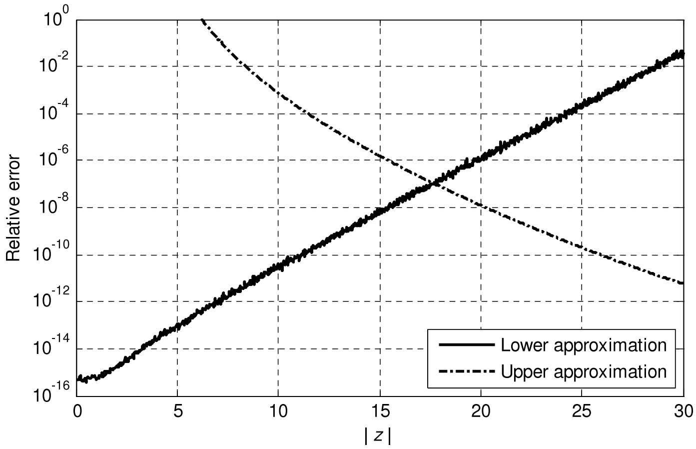


**Fig. 1.** Relative error for both approximations for $(\pi/2\ [\boldsymbol{H}_0(z) - Y_0(z)])$.

The extreme values of parameter $z$ are

$$z_{\mathrm{L}} = \bar{z}\,(1-\bar{\rho}) \quad \text{for } \theta = 0, \qquad z_{\mathrm{U}} = \bar{z}\,(1+\bar{\rho}) \quad \text{for } \theta = \pi. \tag{48}$$

Then, it follows that

$$z_{\mathrm{L}} = \bar{z} \quad \text{and} \quad z_{\mathrm{U}} = \bar{z} \quad \text{for } \bar{\rho} = 0\ (\rho = 0), \qquad z_{\mathrm{L}} = 0 \quad \text{and} \quad z_{\mathrm{U}} = 2\bar{z} \quad \text{for } \bar{\rho} = 1\ (\rho = R). \tag{49}$$

Cylindrical functions in (46) are expressed through the series expansion presented by Abramowitz & Stegun [7]. The approximation used for low values of the arguments is

$$\left(\frac{\pi}{2}\,[\boldsymbol{H}_0(z) - Y_0(z)]\right)_L = \sum_{n=0}^{N_L} (-1)^n \left(\frac{z^{2n+1}}{(2n+1)!!^2} - (\ln(z/2) + \gamma - \psi(n))\,\frac{z^{2n}}{2^{2n} n!^2}\right) \tag{50}$$

where the summation extends to a finite number of terms $N_L$. In this expansion,

$$\psi(n) = \sum_{l=1}^{n} \frac{1}{l} \quad \text{where } \psi(0) = 0 \tag{51}$$

$\gamma$ is Euler's constant and $(\ldots)!!$ represents double factorial.

On the other hand, the approximation used for high values of the arguments is

$$\left(\frac{\pi}{2}\,[\boldsymbol{H}_0(z) - Y_0(z)]\right)_U = \sum_{n=0}^{N_U} (-1)^n (2n-1)!!^2 z^{-2n-1} \tag{52}$$

where the summation extends to a finite number of terms $N_U$.

Fig. 1 shows the relative error for both approximations as a function of the modulus of the argument of the cylindrical functions, considering double precision (1e−16) and taking the maximum value for phase angles of $z$ ranging from −90° to 90° with steps of 0.5°. The reference values of the cylindrical functions for estimating the relative error were calculated using Maple. The maximum number of terms for the upper approximation is set at $N_U = 8$ since from this value the approximation begins to diverge in an oscillating manner. It is noted that the maximum relative error considering in both approximations is about 1e−7, which is considered acceptable from the engineering point of view. It also suggests that a suitable value for separating the lower and upper approximations is $|z| = 18$.

Fig. 2 shows the number of terms required to achieve an accuracy of 1e−8 adopting a maximum of $N_L = 30$ for the lower approximation and maintaining $N_U = 8$ for the upper approximation. In this way, errors slightly higher than 1e−8 are recognized and accepted in a range between approximately $|z| = 15$ and $|z| = 20$.

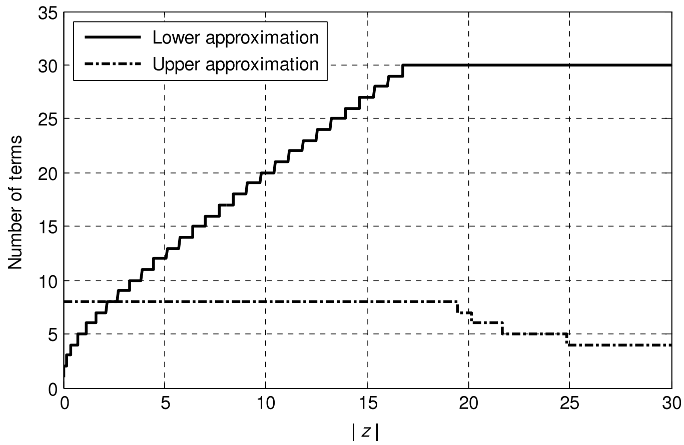


**Fig. 2.** Number of terms to achieve an accuracy of 1e−8 for $(\pi/2\ [\boldsymbol{H}_0(z) - Y_0(z)])$.

By introducing the expressions (50) and (52) in (46) one obtains

$$V_{000}(\rho,R,s) = \frac{1}{\pi}\left(\begin{matrix}\sum_{n=0}^{N_L}(-1)^n\left(\begin{matrix}\frac{\bar{z}^{2n+1}}{(2n+1)!!^2}\int_0^{\bar{\theta}}\bar{r}^{2n+1}d\theta - \cdots \\ \frac{\bar{z}^{2n}}{2^{2n}n!^2}\int_0^{\bar{\theta}}\ln(\bar{r})\,\bar{r}^{2n}d\theta - \cdots \\ (\ln(\bar{z}/2)+\gamma-\psi(n))\,\frac{\bar{z}^{2n}}{2^{2n}n!^2}\int_0^{\bar{\theta}}\bar{r}^{2n}d\theta\end{matrix}\right) + \cdots \\ \sum_{n=0}^{N_U}(-1)^n(2n-1)!!^2\bar{z}^{-2n-1}\int_{\bar{\theta}}^{\pi}\bar{r}^{-2n-1}d\theta\end{matrix}\right) \tag{53}$$

where integration limits between both approximations depend on the extreme values of $z$

$$\begin{array}{lll} |z_{\mathrm{U}}| \le 18 & \Rightarrow & \bar{\theta}=\pi \quad \text{(complete lower solution)} \\ |z_{\mathrm{L}}| \ge 18 & \Rightarrow & \bar{\theta}=0 \quad \text{(complete upper solution)} \\ |z_{\mathrm{L}}| < 18 < |z_{\mathrm{U}}| & \Rightarrow & \bar{\theta}=\mathrm{acos}\left(\left(\bar{\rho}^2+1-(18/|\bar{z}|)^2\right)/(2\bar{\rho})\right) \quad \text{(mixed solution).}\end{array} \tag{54}$$

The integrals in (53) can be solved using numerical integration techniques of Gauss–Kronrod type since they do not have singularities, except the following for $n = 0$ which is solved as indicated

$$\int_0^{\bar{\theta}}\ln(\bar{r})\,d\theta = \int_0^{\pi}\ln(\bar{r})\,d\theta - \int_{\bar{\theta}}^{\pi}\ln(\bar{r})\,d\theta \tag{55}$$

where the 1st term of the 2nd member is null (Gradshteyn & Ryzhik, [8]).

Assessing the integral in (40) for $\Re(s) > 0$ is performed using (6)

$$V_{000}(\rho,R,s) = 2\,W_{-100}(\rho,R,s) - V_{000}(\rho,R,-s) \tag{56}$$

where $W_{-100}(\rho,R,s)$ is presented in Eq. (34). An identical expression is obtained through the following analytic continuation relations [5,9]

$$\begin{cases}\boldsymbol{H}_0(-sr) = -\boldsymbol{H}_0(sr) \\ Y_0(-sr) = Y_0(sr) \mp 2iJ_0(sr) = -Y_0(sr) \mp 2i\,H_0^{(1,2)}(sr).\end{cases} \tag{57}$$

Introducing these expressions in (44) one obtains

$$\begin{aligned} V_{000}(\rho,R,s) &= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\left[-\boldsymbol{H}_0(sr) + Y_0(sr) \pm 2i\,H_0^{(1,2)}(sr)\right]\right)d\theta \\ &= \pm i\pi J_0(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s}) - V_{000}(\rho,R,-s) \end{aligned} \tag{58}$$

where [8]

$$\int_0^{\pi} H_0^{(1,2)}(sr)\,d\theta = \pi J_0(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s}). \tag{59}$$

### 4.2. Integral $V_{011}(\rho, R, s)$

The form that the integral takes in this case is

$$V_{011}(\rho, R, s) = \int_0^\infty \frac{J_1(k\rho)J_1(kR)}{k-s} dk. \tag{60}$$

A suitable replacement for the product of the cylindrical functions suggested by Hemsley [4] is based on the following expression given by Watson [5]

$$J_1(k\rho)J_1(kR) = \frac{k\rho R}{\pi} \int_0^\pi J_1(kr) \frac{\sin^2(\theta)}{r} d\theta \tag{61}$$

where $r$ is defined in (42). By introducing (61) in (60) and changing the integration order one obtains

$$V_{011}(\rho, R, s) = \frac{\rho R}{\pi} \int_0^\pi \int_0^\infty \frac{kJ_1(kr)}{k-s} dk \frac{\sin^2(\theta)}{r} d\theta. \tag{62}$$

The integration of variable $k$ for $\Re(s) \le 0$ produces [5]

$$\begin{aligned} V_{011}(\rho, R, s) &= \frac{s\rho R}{\pi} \int_0^\pi \left( \frac{\pi}{2} \left[ \boldsymbol{H}_{-1}(-sr) - Y_{-1}(-sr) \right] \right) \frac{\sin^2(\theta)}{r} d\theta \\ &= -\frac{s\rho R}{\pi} \int_0^\pi \left( \frac{\pi}{2} \left[ \boldsymbol{H}_1(-sr) - Y_1(-sr) \right] - 1 \right) \frac{\sin^2(\theta)}{r} d\theta \end{aligned} \tag{63}$$

which vanishes for $\rho = 0$. A convenient form to express the integral in (63) involves replacing the argument of the cylindrical functions as follows

$$V_{011}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{\pi} \int_0^\pi \left( \frac{\pi}{2} \left[ \boldsymbol{H}_1(z) - Y_1(z) \right] - 1 \right) \frac{\sin^2(\theta)}{\bar{r}} d\theta \tag{64}$$

where the dimensionless variables involved are defined in (39) and (47).

Cylindrical functions in (64) are expressed through the series expansion presented by Abramowitz & Stegun [7]. The approximation used for low arguments is

$$\left( \frac{\pi}{2} \left[ \boldsymbol{H}_1(z) - Y_1(z) \right] - 1 \right)_L = \left( \frac{1}{z} - 1 \right) + \sum_{n=1}^{N_L} (-1)^n \left( (\ln(z/2) + \gamma - \bar{\psi}(n)) \frac{(z/2)^{2n-1}}{(n-1)!n!} - \frac{z^{2n}}{(2n-1)!!(2n+1)!!} \right) \tag{65}$$

where the summation extends to a finite number of terms $N_L$. In this expansion,

$$\bar{\psi}(n) = \sum_{l=1}^{n} \frac{1}{l} - \frac{1}{2n} \tag{66}$$

while $(\ldots)!!$ represents double factorial.

On the other hand, the approximation used for high arguments is

$$\left( \frac{\pi}{2} \left[ \boldsymbol{H}_1(z) - Y_1(z) \right] - 1 \right)_U = \sum_{n=0}^{N_U} (-1)^n (2n-1)!!(2n+1)!! z^{-2n-2} \tag{67}$$

where the summation extends to a finite number of terms $N_U$.

Fig. 3 shows the relative error for both approximations as a function of the modulus of the argument of the cylindrical functions, considering double precision (1e−16) and taking the maximum value for phase angles of $z$ ranging from −90° to 90° with steps of 0.5°. The reference values of the cylindrical functions for estimating the relative error were calculated using Maple. The maximum number of terms for the upper approximation is set at $N_U = 8$ since from this value the approximation begins to diverge in an oscillating manner. It is noted that the maximum relative error considering in both approximations is about 1e−8. It also suggests that a suitable value for separating the lower and upper approximations is $|z| = 18$.

Fig. 4 shows the number of terms required to achieve an accuracy of 1e−8 adopting a maximum of $N_L = 30$ for the lower approximation and maintaining $N_U = 8$ for the upper approximation. Anyway, the limit of $|z| = 18$ between both approximations leads to a maximum number of terms for the upper approximation that does not exceed $N_U = 5$.

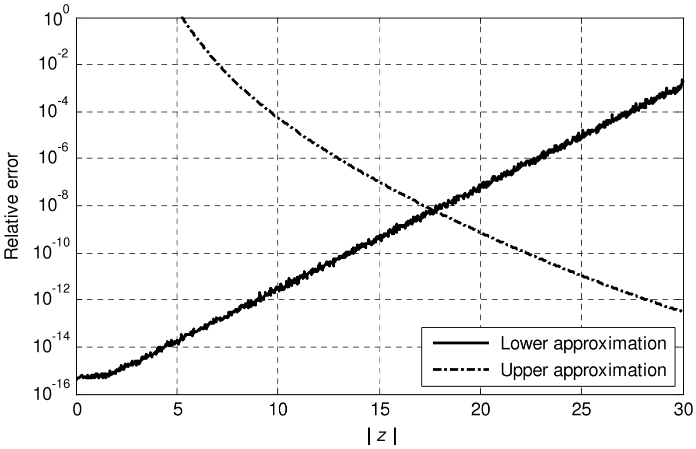


**Fig. 3.** Relative error for both approximations for $(\pi/2\ [\boldsymbol{H}_1(z) - Y_1(z)] - 1)$.

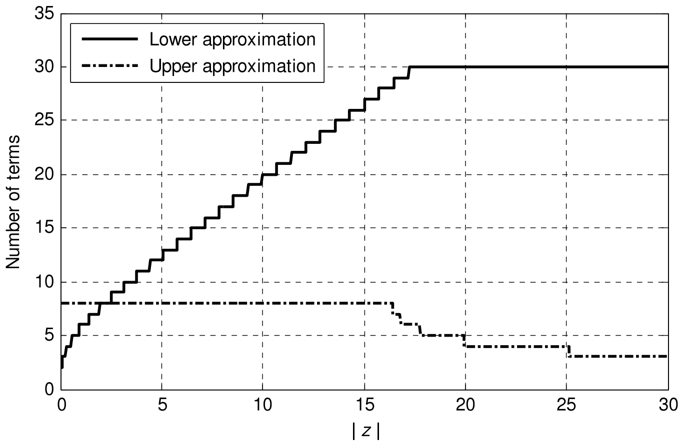


**Fig. 4.** Number of terms to achieve an accuracy of 1e−8 for $(\pi/2\ [\boldsymbol{H}_1(z) - Y_1(z)] - 1)$.

Introducing the expressions (65) and (67) in (64) one obtains:

$$V_{011}(\rho,R,s)=\frac{\bar{z}\bar{\rho}}{\pi}\begin{pmatrix}\dfrac{1}{\bar{z}}\displaystyle\int_0^{\bar{\theta}}\bar{r}^{-2}\sin^2\theta\,d\theta-\int_0^{\bar{\theta}}\bar{r}^{-1}\sin^2\theta\,d\theta+\cdots\\ \displaystyle\sum_{n=1}^{N_L}(-1)^n\begin{pmatrix}\dfrac{(\ln(\bar{z}/2)+\gamma-\bar{\psi}(n))\,\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}}\bar{r}^{2n-2}\sin^2\theta\,d\theta+\cdots\\ \dfrac{\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}}\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\,d\theta-\cdots\\ \dfrac{\bar{z}^{2n}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar{\theta}}\bar{r}^{2n-1}\sin^2\theta\,d\theta\end{pmatrix}+\cdots\\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-2}\int_{\bar{\theta}}^{\pi}\bar{r}^{-2n-3}\sin^2\theta\,d\theta\end{pmatrix} \tag{68}$$

where integration limits between both approximations depend on the extreme values of $z$ given in (54). The integrals in (68) do not have singularities and can be solved using numerical integration techniques of Gauss–Kronrod type.

The evaluation of the integral in (60) for $\Re(s) > 0$ is performed using (6)

$$V_{011}(\rho,R,s)=2\,W_{-111}(\rho,R,s)-V_{011}(\rho,R,-s) \tag{69}$$

where $W_{-111}(\rho,R,s)$ is presented in Eq. (35). An identical expression is obtained through the following analytic continuation relations [5,9]

$$\begin{cases}\boldsymbol{H}_1(-sr)=\boldsymbol{H}_1(sr)\\ Y_1(-sr)=-Y_1(sr)\pm 2iJ_1(sr)=Y_1(sr)\pm 2iH_1^{(1,2)}(sr).\end{cases} \tag{70}$$

Introducing these expressions in (63) one obtains

$$\begin{aligned} V_{011}(\rho, R, s) &= -\frac{s\rho R}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(sr) - Y_1(sr) \mp 2iH_1^{(1,2)}(sr) - 1\right]\right)\frac{\sin^2(\theta)}{r}d\theta \\ &= \pm i\pi J_1(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) - V_{011}(\rho, R, -s) \end{aligned} \tag{71}$$

where [8]

$$\int_0^{\pi} H_1^{(1,2)}(sr)\frac{\sin(\theta)^2}{r}d\theta = \frac{\pi}{s\rho R}J_1(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}). \tag{72}$$

### 4.3. Integral $V_{111}(\rho, R, s)$

The form that the integral takes in this case is

$$V_{111}(\rho, R, s) = \int_0^{\infty}\frac{J_1(k\rho)J_1(kR)}{k\,(k-s)}dk. \tag{73}$$

The argument of this integral can be expressed as

$$\frac{J_1(k\rho)J_1(kR)}{k\,(k-s)} = \frac{1}{s}\left(\frac{J_1(k\rho)J_1(kR)}{(k-s)} - \frac{J_1(k\rho)J_1(kR)}{k}\right). \tag{74}$$

Thereby

$$V_{111}(\rho, R, s) = \frac{1}{s}(V_{011}(\rho, R, s) - H_{011}(\rho, R)) \tag{75}$$

where the auxiliary integral possessing the form defined in (22) is

$$H_{011}(\rho, R) = \int_0^{\infty}\frac{J_1(k\rho)J_1(kR)}{k}dk = \frac{\bar{\rho}}{2}. \tag{76}$$

A more efficient alternative to solve the integral in (73) is by introducing (61) into (76) and changing the integration order

$$H_{011}(\rho, R) = \frac{\rho R}{\pi}\int_0^{\pi}\int_0^{\infty}J_1(kr)dk\frac{\sin^2(\theta)}{r}d\theta = \frac{\rho R}{\pi}\int_0^{\pi}\frac{\sin^2(\theta)}{r^2}d\theta. \tag{77}$$

Introducing (63) and (77) into (75) one arrives for $\Re(s) \leq 0$ at

$$V_{111}(\rho, R, s) = -\frac{\rho R}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(-sr) - Y_1(-sr)\right] - 1 + \frac{1}{sr}\right)\frac{\sin^2(\theta)}{r}d\theta. \tag{78}$$

Using the dimensionless variables defined in (39) and (47) one obtains

$$V_{111}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{s\pi}\int_0^{\pi}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - 1 - \frac{1}{z}\right)\frac{\sin^2(\theta)}{\bar{r}}d\theta. \tag{79}$$

Introducing (65) and (67) into (79) one arrives at

$$V_{111}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{s\pi}\left(\begin{array}{l} -\int_0^{\bar{\theta}}\bar{r}^{-1}\sin^2\theta\,d\theta - \frac{1}{\bar{z}}\int_{\bar{\theta}}^{\pi}\bar{r}^{-2}\sin^2\theta\,d\theta + \cdots \\ \sum_{n=1}^{N_L}(-1)^n\left(\begin{array}{l} \frac{(\ln(\bar{z}/2) + \gamma - \bar{\psi}(n))\,\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\int_0^{\bar{\theta}}\bar{r}^{2n-2}\sin^2\theta\,d\theta + \cdots \\ \frac{\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\int_0^{\bar{\theta}}\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\,d\theta - \cdots \\ \frac{\bar{z}^{2n}}{(2n-1)!!(2n+1)!!}\int_0^{\bar{\theta}}\bar{r}^{2n-1}\sin^2\theta\,d\theta \end{array}\right) + \cdots \\ \sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-2}\int_{\bar{\theta}}^{\pi}\bar{r}^{-2n-3}\sin^2\theta\,d\theta \end{array}\right) \tag{80}$$

where integration limits between both approximations depend on the extreme values of $z$ given in (54). Once again, the integrals in (80) do not have singularities and can be solved using numerical integration techniques of Gauss–Kronrod type.

The evaluation of the integral for $\Re(s) > 0$ is performed using (5)

$$V_{111}(\rho, R, s) = 2s\, W_{111}(\rho, R, s) + V_{111}(\rho, R, -s) \tag{81}$$

where $W_{111}(\rho, R, s)$ is presented in Eq. (36). In this way

$$V_{111}(\rho, R, s) = \pm\frac{i\pi}{s} J_1(\bar{s}\bar{\rho}) H_1^{(1,2)}(\bar{s}) - \frac{\bar{\rho}}{s} + V_{111}(\rho, R, -s). \tag{82}$$

An identical expression may be obtained through analytic continuation relations.

### 4.4. Integral $V_{211}(\rho, R, s)$

The form that the integral takes in this case is

$$V_{211}(\rho, R, s) = \int_0^\infty \frac{J_1(k\rho)J_1(kR)}{k^2(k-s)} dk. \tag{83}$$

The argument of this integral can be expressed as

$$\frac{J_1(k\rho)J_1(kR)}{k^2(k-s)} = \frac{1}{s}\left(\frac{J_1(k\rho)J_1(kR)}{k\,(k-s)} - \frac{J_1(k\rho)J_1(kR)}{k^2}\right). \tag{84}$$

Thereby

$$V_{211}(\rho, R, s) = \frac{1}{s}\left(V_{111}(\rho, R, s) - H_{111}(\rho, R)\right) \tag{85}$$

where the auxiliary integral defined in (22) is

$$H_{111}(\rho, R) = \int_0^\infty \frac{J_1(k\rho)J_1(kR)}{k^2} dk = \frac{2}{3\pi}\frac{\bar{s}}{s\bar{\rho}}\left((\bar{\rho}^2+1)E(\bar{\rho}) + (\bar{\rho}^2-1)K(\bar{\rho})\right) \tag{86}$$

where $E(\ldots)$ and $K(\ldots)$ represent complete elliptic integrals of 1st and 2nd kind, respectively (these functions usually appear in static solutions for $s = 0$).

A more efficient alternative to solve the integral in (83) is by introducing (61) into (86) and changing the integration order

$$H_{111}(\rho, R) = \frac{\rho R}{\pi}\int_0^\pi \int_0^\infty \frac{J_1(kr)}{k} dk \frac{\sin^2(\theta)}{r} d\theta = \frac{\rho R}{\pi}\int_0^\pi \frac{\sin^2(\theta)}{r} d\theta. \tag{87}$$

Introducing (78) and (87) into (85) one arrives for $\Re(s) \le 0$ at

$$V_{211}(\rho, R, s) = -\frac{\rho R}{s\pi}\int_0^\pi \left(\frac{\pi}{2}\left[\boldsymbol{H}_1(-sr) - Y_1(-sr)\right] + \frac{1}{sr}\right)\frac{\sin^2(\theta)}{r} d\theta. \tag{88}$$

Using the dimensionless variables defined in (47) one obtains

$$V_{211}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{s^2\pi}\int_0^\pi \left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - \frac{1}{z}\right)\frac{\sin^2(\theta)}{\bar{r}} d\theta. \tag{89}$$

Introducing (65) and (67) into (89) one arrives at

$$V_{211}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{s^2\pi}\left(\begin{array}{l}
\displaystyle\int_{\bar{\theta}}^\pi \bar{r}^{-1}\sin^2\theta\, d\theta - \frac{1}{\bar{z}}\int_{\bar{\theta}}^\pi \bar{r}^{-2}\sin^2\theta\, d\theta + \cdots \\
\displaystyle\sum_{n=1}^{N_L}(-1)^n\left(\begin{array}{l}
\displaystyle\frac{(\ln(\bar{z}/2)+\gamma-\bar{\psi}(n))\,\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\int_0^{\bar{\theta}} \bar{r}^{2n-2}\sin^2\theta\, d\theta + \cdots \\
\displaystyle\frac{\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\int_0^{\bar{\theta}} \ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\, d\theta - \cdots \\
\displaystyle\frac{\bar{z}^{2n}}{(2n-1)!!(2n+1)!!}\int_0^{\bar{\theta}} \bar{r}^{2n-1}\sin^2\theta\, d\theta
\end{array}\right) + \cdots \\
\displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-2}\int_{\bar{\theta}}^\pi \bar{r}^{-2n-3}\sin^2\theta\, d\theta
\end{array}\right) \tag{90}$$

where integration limits between both approximations depend on the extreme values of $z$ given in (54). The integrals in (90) do not have singularities and can be solved using numerical integration techniques of Gauss–Kronrod type.

The evaluation of the integral for $\Re(s) > 0$ is performed using (6)

$$V_{211}(\rho, R, s) = 2\,W_{111}(\rho, R, s) - V_{211}(\rho, R, -s) \tag{91}$$

where $W_{111}(\rho, R, s)$ is presented in Eq. (36). In this way

$$V_{211}(\rho, R, s) = \pm\frac{i\pi}{s^2} J_1(\bar{s}\bar{\rho}) H_1^{(1,2)}(\bar{s}) - \frac{\bar{\rho}}{s^2} - V_{211}(\rho, R, -s). \tag{92}$$

An identical expression is obtained through analytic continuation relations.

### 4.5. Integrals $V_{001}(\rho, R, s)$ and $V_{010}(\rho, R, s)$

The form taken by the first of these integrals is

$$V_{001}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho) J_1(kR)}{k - s} dk. \tag{93}$$

This integral can be expressed as

$$V_{001}(\rho, R, s) = \frac{\partial}{\partial \rho} V_{111}(\rho, R, s) + \frac{1}{\rho} V_{111}(\rho, R, s). \tag{94}$$

The derivative of $V_{111}(\rho, R, s)$ is obtained by differentiation of the integrand in (78) as follows

$$\frac{\partial}{\partial \rho} V_{111}(\rho, R, s) = \cdots \frac{1}{\pi} \int_0^\pi \begin{pmatrix} \dfrac{sR(\rho^2 - R^2 + r^2)}{2r^2} \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_0(-sr) - Y_0(-sr) \right] + \dfrac{1}{sr} \right) + \cdots \\ \dfrac{R(\rho^2 - R^2)}{r^3} \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_1(-sr) - Y_1(-sr) \right] - 1 + \dfrac{1}{sr} \right) \end{pmatrix} \sin^2(\theta)\, d\theta. \tag{95}$$

Hence, for $\Re(s) \leq 0$ Eq. (94) yields

$$V_{001}(\rho, R, s) = \cdots \frac{1}{\pi} \int_0^\pi \begin{pmatrix} \dfrac{sR(\rho^2 - R^2 + r^2)}{2r^2} \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_0(-sr) - Y_0(-sr) \right] + \dfrac{1}{sr} \right) + \cdots \\ \dfrac{R(\rho^2 - R^2 - r^2)}{r^3} \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_1(-sr) - Y_1(-sr) \right] - 1 + \dfrac{1}{sr} \right) \end{pmatrix} \sin^2(\theta)\, d\theta. \tag{96}$$

Using dimensionless variables one obtains for $\rho \leq R$

$$V_{001}(\rho, R, s) = \frac{1}{\pi} \int_0^\pi \begin{pmatrix} (\bar{\rho}\cos\theta - \bar{\rho}^2) \left( z\dfrac{\pi}{2} \left[ \boldsymbol{H}_0(z) - Y_0(z) \right] - 1 \right) + \cdots \\ 2\,(\bar{\rho}\cos\theta - 1) \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_1(z) - Y_1(z) \right] - 1 - \dfrac{1}{z} \right) \end{pmatrix} \frac{\sin^2\theta}{\bar{r}^3}\, d\theta \tag{97}$$

while for $\rho > R$

$$V_{001}(\rho, R, s) = \frac{\bar{\rho}}{\pi} \int_0^\pi \begin{pmatrix} (\bar{\rho}\cos\theta - 1) \left( z\dfrac{\pi}{2} \left[ \boldsymbol{H}_0(z) - Y_0(z) \right] - 1 \right) + \cdots \\ 2\,(\bar{\rho}\cos\theta - \bar{\rho}^2) \left( \dfrac{\pi}{2} \left[ \boldsymbol{H}_1(z) - Y_1(z) \right] - 1 - \dfrac{1}{z} \right) \end{pmatrix} \frac{\sin^2\theta}{\bar{r}^3}\, d\theta. \tag{98}$$

The function $(z\pi/2\,[\boldsymbol{H}_0(z) - Y_0(z)] - 1)$ should be assessed eliminating the first term in (52) for the purpose of avoiding numerical problems.

The solution is divided into two parts

$$V_{001}(\rho, R, s) = V_{001}^0(\rho, R, s) + V_{001}^1(\rho, R, s). \tag{99}$$

Using the adequate approximations for low and high arguments the following expressions are obtained

$$V_{001}^{0}(\rho,R,s)=\frac{1}{\pi}\begin{pmatrix}-\displaystyle\int_0^{\bar\theta} f_0(\theta)\,\bar r^{-3}\sin^2\theta\,d\theta+\cdots\\ \displaystyle\sum_{n=0}^{N_L}(-1)^n\begin{pmatrix}\dfrac{\bar z^{2n+2}}{(2n+1)!!^2}\displaystyle\int_0^{\bar\theta} f_0(\theta)\,\bar r^{2n-1}\sin^2\theta\,d\theta-\cdots\\ \dfrac{\bar z^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar\theta} f_0(\theta)\,\ln(\bar r)\,\bar r^{2n-2}\sin^2\theta\,d\theta-\cdots\\ \dfrac{f_a(\bar z)\,\bar z^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar\theta} f_0(\theta)\,\bar r^{2n-2}\sin^2\theta\,d\theta\end{pmatrix}+\cdots\\ \displaystyle\sum_{n=1}^{N_U}(-1)^n(2n-1)!!^2\bar z^{-2n}\int_{\bar\theta}^{\pi} f_0(\theta)\,\bar r^{-2n-3}\sin^2\theta\,d\theta\end{pmatrix} \tag{100}$$

$$V_{001}^{1}(\rho,R,s)=\frac{1}{\pi}\begin{pmatrix}-\displaystyle\int_0^{\bar\theta} f_1(\theta)\,\bar r^{-3}\sin^2\theta\,d\theta-\frac{1}{\bar z}\int_{\bar\theta}^{\pi} f_1(\theta)\,\bar r^{-4}\sin^2\theta\,d\theta+\cdots\\ \displaystyle\sum_{n=1}^{N_L}(-1)^n\begin{pmatrix}\dfrac{f_b(\bar z)\,\bar z^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar\theta} f_1(\theta)\,\bar r^{2n-4}\sin^2\theta\,d\theta+\cdots\\ \dfrac{\bar z^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar\theta} f_1(\theta)\,\ln(\bar r)\,\bar r^{2n-4}\sin^2\theta\,d\theta-\cdots\\ \dfrac{\bar z^{2n}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar\theta} f_1(\theta)\,\bar r^{2n-3}\sin^2\theta\,d\theta\end{pmatrix}+\cdots\\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar z^{-2n-2}\int_{\bar\theta}^{\pi} f_1(\theta)\,\bar r^{-2n-5}\sin^2\theta\,d\theta\end{pmatrix} \tag{101}$$

where

$$\left.\begin{aligned}f_a(\bar z)&=\ln(\bar z/2)+\gamma-\psi(n)\\ f_b(\bar z)&=\ln(\bar z/2)+\gamma-\bar\psi(n)\end{aligned}\right\} \tag{102}$$

$$\begin{aligned}&\left.\begin{aligned}f_0(\theta)&=\bar\rho\cos\theta-\bar\rho^2\\ f_1(\theta)&=2\,(\bar\rho\cos\theta-1)\end{aligned}\right\}\quad \rho\le R\\ &\left.\begin{aligned}f_0(\theta)&=\bar\rho\,(\bar\rho\cos\theta-1)\\ f_1(\theta)&=2\bar\rho\left(\bar\rho\cos\theta-\bar\rho^2\right)\end{aligned}\right\}\quad \rho> R.\end{aligned} \tag{103}$$

The evaluation of this integral for $\Re(s)>0$ is performed using (5)

$$V_{001}(\rho,R,s)=2s\,W_{001}(\rho,R,s)+V_{001}(\rho,R,-s) \tag{104}$$

where $W_{001}(\rho,R,s)$ is obtained from (37). In this way

$$V_{001}(\rho,R,s)=\begin{cases}\pm i\pi J_0(\bar s\bar\rho)H_1^{(1,2)}(\bar s)-\dfrac{2}{\bar s}+V_{001}(\rho,R,-s) & \rho\le R\\ \pm i\pi J_1(\bar s\bar\rho)H_0^{(1,2)}(\bar s)+V_{001}(\rho,R,-s) & \rho> R.\end{cases} \tag{105}$$

The remaining integral is solved considering that

$$V_{010}(\rho,R,s)=V_{001}(R,\rho,s). \tag{106}$$

### *4.6. Integrals $V_{101}(\rho,R,s)$ and $V_{110}(\rho,R,s)$*

The form taken by the first of these integrals is

$$V_{101}(\rho,R,s)=\int_0^{\infty}\frac{J_0(k\rho)J_1(kR)}{k(k-s)}dk. \tag{107}$$

This integral can be expressed as

$$V_{101}(\rho,R,s)=\frac{\partial}{\partial\rho}V_{211}(\rho,R,s)+\frac{1}{\rho}V_{211}(\rho,R,s). \tag{108}$$

The derivative of $V_{211}(\rho, R, s)$ is obtained by differentiation of the integrand in (88) as follows

$$\frac{\partial}{\partial\rho}V_{211}(\rho,R,s)=\frac{1}{\pi}\int_0^{\pi}\begin{pmatrix}\dfrac{R(\rho^2-R^2+r^2)}{2r^2}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(-sr)-Y_0(-sr)\right]\right)+\cdots\\ \dfrac{R(\rho^2-R^2)}{sr^3}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(-sr)-Y_1(-sr)\right]+\dfrac{1}{sr}\right)\end{pmatrix}\sin^2\theta\,d\theta. \tag{109}$$

Hence, for $\Re(s) \le 0$ Eq. (107) yields

$$V_{101}(\rho,R,s)=\frac{1}{\pi}\int_0^{\pi}\begin{pmatrix}\dfrac{R(\rho^2-R^2+r^2)}{2r^2}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(-sr)-Y_0(-sr)\right]\right)+\cdots\\ \dfrac{R(\rho^2-R^2-r^2)}{sr^3}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(-sr)-Y_1(-sr)\right]+\dfrac{1}{sr}\right)\end{pmatrix}\sin^2\theta\,d\theta. \tag{110}$$

Using dimensionless variables one obtains for $\rho \le R$

$$V_{101}(\rho,R,s)=\frac{\bar{z}}{s\pi}\int_0^{\pi}\begin{pmatrix}(\bar{\rho}\cos\theta-\bar{\rho}^2)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z)-Y_0(z)\right]\right)+\cdots\\ \dfrac{2}{z}(\bar{\rho}\cos\theta-1)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z)-Y_1(z)\right]-\dfrac{1}{z}\right)\end{pmatrix}\frac{\sin^2\theta}{\bar{r}^2}\,d\theta \tag{111}$$

while for $\rho > R$

$$V_{101}(\rho,R,s)=\frac{\bar{z}\bar{\rho}}{s\pi}\int_0^{\pi}\begin{pmatrix}(\bar{\rho}\cos\theta-1)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z)-Y_0(z)\right]\right)+\cdots\\ \dfrac{2}{z}(\bar{\rho}\cos\theta-\bar{\rho}^2)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z)-Y_1(z)\right]-\dfrac{1}{z}\right)\end{pmatrix}\frac{\sin^2\theta}{\bar{r}^2}\,d\theta. \tag{112}$$

The solution is divided into two parts

$$V_{101}(\rho,R,s)=V_{101}^0(\rho,R,s)+V_{101}^1(\rho,R,s). \tag{113}$$

Using the adequate approximations for low and high arguments the following expressions are obtained

$$V_{101}^0(\rho,R,s)=\frac{\bar{z}}{s\pi}\begin{pmatrix}\displaystyle\sum_{n=0}^{N_L}(-1)^n\begin{pmatrix}\dfrac{\bar{z}^{2n+1}}{(2n+1)!!^2}\displaystyle\int_0^{\bar{\theta}}f_0(\theta)\,\bar{r}^{2n-1}\sin^2\theta\,d\theta-\cdots\\ \dfrac{\bar{z}^{2n}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}}f_0(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\,d\theta-\cdots\\ \dfrac{f_a(\bar{z})\,\bar{z}^{2n}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}}f_0(\theta)\,\bar{r}^{2n-2}\sin^2\theta\,d\theta\end{pmatrix}+\cdots\\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!^2\bar{z}^{-2n-1}\int_{\bar{\theta}}^{\pi}f_0(\theta)\,\bar{r}^{-2n-3}\sin^2\theta\,d\theta\end{pmatrix} \tag{114}$$

$$V_{101}^1(\rho,R,s)=\frac{\bar{z}}{s\pi}\begin{pmatrix}\dfrac{1}{\bar{z}}\displaystyle\int_{\bar{\theta}}^{\pi}f_1(\theta)\,\bar{r}^{-3}\sin^2\theta\,d\theta-\dfrac{1}{\bar{z}^2}\int_{\bar{\theta}}^{\pi}f_1(\theta)\,\bar{r}^{-4}\sin^2\theta\,d\theta+\cdots\\ \displaystyle\sum_{n=1}^{N_L}(-1)^n\begin{pmatrix}\dfrac{f_b(\bar{z})\,\bar{z}^{2n-2}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}}f_1(\theta)\,\bar{r}^{2n-4}\sin^2\theta\,d\theta+\cdots\\ \dfrac{\bar{z}^{2n-2}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}}f_1(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-4}\sin^2\theta\,d\theta-\cdots\\ \dfrac{\bar{z}^{2n-1}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar{\theta}}f_1(\theta)\,\bar{r}^{2n-3}\sin^2\theta\,d\theta\end{pmatrix}+\cdots\\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-3}\int_{\bar{\theta}}^{\pi}f_1(\theta)\,\bar{r}^{-2n-5}\sin^2\theta\,d\theta\end{pmatrix} \tag{115}$$

where the expressions in (102) and (103) remain applicable.

The evaluation of this integral for $\Re(s) > 0$ is performed using (6)

$$V_{101}(\rho,R,s)=2\,W_{001}(\rho,R,s)-V_{101}(\rho,R,-s) \tag{116}$$

where $W_{001}(\rho, R, s)$ is obtained from (37). In this way

$$V_{101}(\rho, R, s) = \begin{cases} \pm\frac{i\pi}{s} J_0(\bar{s}\bar{\rho}) H_1^{(1,2)}(\bar{s}) - \frac{2}{s\bar{s}} - V_{101}(\rho, R, -s) & \rho \leq R \\ \pm\frac{i\pi}{s} J_1(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) - V_{101}(\rho, R, -s) & \rho > R. \end{cases} \quad (117)$$

The remaining integral is solved considering that

$$V_{110}(\rho, R, s) = V_{101}(R, \rho, s). \quad (118)$$

### 4.7. Integrals $V_{002}(\rho, R, s)$ and $V_{020}(\rho, R, s)$

The form taken by the first of these integrals in this case is

$$V_{002}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho) J_2(kR)}{(k-s)} dk. \quad (119)$$

This integral can be expressed as

$$V_{002}(\rho, R, s) = \frac{2}{R} V_{101}(\rho, R, s) - V_{000}(\rho, R, s). \quad (120)$$

Hence, for $\Re(s) \leq 0$ Eq. (119) yields

$$V_{002}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \begin{pmatrix} \left(\frac{(\rho^2 - R^2 + r^2)\sin^2\theta}{r^2} - 1\right)\left(\frac{\pi}{2}\left[\boldsymbol{H}_0(-sr) - Y_0(-sr)\right]\right) + \cdots \\ \frac{2(\rho^2 - R^2 - r^2)\sin^2\theta}{sr^3}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(-sr) - Y_1(-sr)\right] + \frac{1}{sr}\right) \end{pmatrix} d\theta. \quad (121)$$

Using dimensionless variables one obtains for $\rho \leq R$

$$V_{002}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \begin{pmatrix} \left(2\left(\bar{\rho}^2 - \bar{\rho}\cos\theta\right)\frac{\sin^2\theta}{\bar{r}^2} - 1\right)\left(\frac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right]\right) + \cdots \\ \frac{4}{z}(1 - \bar{\rho}\cos\theta)\frac{\sin^2\theta}{\bar{r}^2}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - \frac{1}{z}\right) \end{pmatrix} d\theta \quad (122)$$

while for $\rho > R$

$$V_{002}(\rho, R, s) = \frac{1}{\pi}\int_0^\pi \begin{pmatrix} \left(2\left(1 - \bar{\rho}\cos\theta\right)\frac{\sin^2\theta}{\bar{r}^2} - 1\right)\left(\frac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right]\right) + \cdots \\ \frac{4}{z}(\bar{\rho}^2 - \bar{\rho}\cos\theta)\frac{\sin^2\theta}{\bar{r}^2}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - \frac{1}{z}\right) \end{pmatrix} d\theta. \quad (123)$$

The solution is divided into two parts

$$V_{002}(\rho, R, s) = V_{002}^0(\rho, R, s) + V_{002}^1(\rho, R, s). \quad (124)$$

Using the adequate approximations for low and high arguments one obtains

$$V_{002}^0(\rho, R, s) = \frac{1}{\pi}\begin{pmatrix} \sum_{n=0}^{N_L}(-1)^n \begin{pmatrix} \frac{\bar{z}^{2n+1}}{(2n+1)!!^2}\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n+1} d\theta - \cdots \\ \frac{\bar{z}^{2n}}{2^{2n} n!^2}\int_0^{\bar{\theta}} f_0(\theta)\,\ln(\bar{r})\,\bar{r}^{2n} d\theta - \cdots \\ \frac{f_a(\bar{z})\,\bar{z}^{2n}}{2^{2n} n!^2}\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n} d\theta \end{pmatrix} + \cdots \\ \sum_{n=0}^{N_U}(-1)^n (2n-1)!!^2 \bar{z}^{-2n-1}\int_{\bar{\theta}}^{\pi} f_0(\theta)\,\bar{r}^{-2n-1} d\theta \end{pmatrix} \quad (125)$$

$$V^1_{002}(\rho,R,s)=\frac{1}{\pi}\left(\begin{array}{l}\frac{1}{\bar z}\int_{\bar\theta}^{\pi}f_1(\theta)\,\bar r^{-1}d\theta-\frac{1}{\bar z^2}\int_{\bar\theta}^{\pi}f_1(\theta)\,\bar r^{-2}d\theta+\cdots\\ \sum_{n=1}^{N_L}(-1)^n\left(\begin{array}{l}\frac{f_b(\bar z)\,\bar z^{2n-2}}{2^{2n-1}(n-1)!n!}\int_0^{\bar\theta}f_1(\theta)\,\bar r^{2n-2}d\theta+\cdots\\ \frac{\bar z^{2n-2}}{2^{2n-1}(n-1)!n!}\int_0^{\bar\theta}f_1(\theta)\,\ln(\bar r)\,\bar r^{2n-2}d\theta-\cdots\\ \frac{\bar z^{2n-1}}{(2n-1)!!(2n+1)!!}\int_0^{\bar\theta}f_1(\theta)\,\bar r^{2n-1}d\theta\end{array}\right)+\cdots\\ \sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar z^{-2n-3}\int_{\bar\theta}^{\pi}f_1(\theta)\,\bar r^{-2n-3}d\theta\end{array}\right)\qquad(126)$$

where the expressions in (102) remain applicable, while

$$\left.\begin{array}{l}f_0(\theta)=2\,(\bar\rho^2-\bar\rho\cos\theta)\sin^2\theta/\bar r^2-1\\ f_1(\theta)=4\,(1-\bar\rho\cos\theta)\sin^2\theta/\bar r^2\end{array}\right\}\quad\rho\le R$$
$$\left.\begin{array}{l}f_0(\theta)=2\,(1-\bar\rho\cos\theta)\sin^2\theta/\bar r^2-1\\ f_1(\theta)=4\,(\bar\rho^2-\bar\rho\cos\theta)\sin^2\theta/\bar r^2\end{array}\right\}\quad\rho> R.\qquad(127)$$

The evaluation of this integral for $\Re(s)>0$ is performed as

$$V_{002}(\rho,R,s)=\begin{cases}\pm i\pi J_0(\bar s\bar\rho)H_2^{(1,2)}(\bar s)-\frac{4}{\bar s^2}-V_{002}(\rho,R,-s) & \rho\le R\\ \pm i\pi J_2(\bar s\bar\rho)H_0^{(1,2)}(\bar s)-V_{002}(\rho,R,-s) & \rho>R.\end{cases}\qquad(128)$$

The remaining integral is solved considering that

$$V_{020}(\rho,R,s)=V_{002}(R,\rho,s).\qquad(129)$$

### 4.8. Integrals $V_{012}(\rho,R,s)$ and $V_{021}(\rho,R,s)$

The form taken by the first of these integrals in this case is

$$V_{012}(\rho,R,s)=\int_0^{\infty}\frac{J_1(k\rho)J_2(kR)}{(k-s)}dk.\qquad(130)$$

This integral can be expressed as

$$V_{012}(\rho,R,s)=\frac{2}{R}V_{111}(\rho,R,s)-V_{010}(\rho,R,s).\qquad(131)$$

Hence, for $\Re(s)\le 0$ Eq. (130) yields

$$V_{012}(\rho,R,s)=\frac{1}{\pi}\int_0^{\pi}\left(\begin{array}{l}\frac{-s\rho(R^2-\rho^2+r^2)}{2r^2}\left(\frac{\pi}{2}\left[\boldsymbol{H}_0(-sr)-Y_0(-sr)\right]+\frac{1}{sr}\right)-\cdots\\ \frac{\rho(R^2-\rho^2+r^2)}{r^3}\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(-sr)-Y_1(-sr)\right]-1+\frac{1}{sr}\right)\end{array}\right)\sin^2\theta\,d\theta.\qquad(132)$$

Using dimensionless variables one obtains for $\rho\le R$

$$V_{012}(\rho,R,s)=\frac{\bar\rho}{\pi}\int_0^{\pi}\left(\begin{array}{l}(1-\bar\rho\cos\theta)\left(z\frac{\pi}{2}\left[\boldsymbol{H}_0(z)-Y_0(z)\right]-1\right)-\cdots\\ 2(1-\bar\rho\cos\theta)\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z)-Y_1(z)\right]-1-\frac{1}{z}\right)\end{array}\right)\frac{\sin^2\theta}{\bar r^3}\,d\theta\qquad(133)$$

while for $\rho>R$

$$V_{012}(\rho,R,s)=\frac{1}{\pi}\int_0^{\pi}\left(\begin{array}{l}(\bar\rho^2-\bar\rho\cos\theta)\left(z\frac{\pi}{2}\left[\boldsymbol{H}_0(z)-Y_0(z)\right]-1\right)-\cdots\\ 2\,(\bar\rho^2-\bar\rho\cos\theta)\left(\frac{\pi}{2}\left[\boldsymbol{H}_1(z)-Y_1(z)\right]-1-\frac{1}{z}\right)\end{array}\right)\frac{\sin^2\theta}{\bar r^3}\,d\theta.\qquad(134)$$

The solution is divided into two parts

$$V_{012}(\rho, R, s) = V_{012}^{0}(\rho, R, s) - V_{012}^{1}(\rho, R, s). \tag{135}$$

Using the adequate approximations for low and high arguments one obtains

$$V_{012}^{0}(\rho, R, s) = \frac{\bar{\rho}}{\pi} \begin{pmatrix} -\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{-3}\sin^2\theta\,d\theta + \cdots \\ \displaystyle\sum_{n=0}^{N_L}(-1)^n \begin{pmatrix} \dfrac{\bar{z}^{2n+2}}{(2n+1)!!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-1}\sin^2\theta\,d\theta - \cdots \\ \dfrac{\bar{z}^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\,d\theta - \cdots \\ \dfrac{f_a(\bar{z})\,\bar{z}^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-2}\sin^2\theta\,d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=1}^{N_U}(-1)^n(2n-1)!!^2\bar{z}^{-2n}\int_{\bar{\theta}}^{\pi} f_0(\theta)\,\bar{r}^{-2n-3}\sin^2\theta\,d\theta \end{pmatrix} \tag{136}$$

$$V_{012}^{1}(\rho, R, s) = \frac{\bar{\rho}}{\pi} \begin{pmatrix} -\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{-3}\sin^2\theta\,d\theta - \frac{1}{\bar{z}}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-4}\sin^2\theta\,d\theta + \cdots \\ \displaystyle\sum_{n=1}^{N_L}(-1)^n \begin{pmatrix} \dfrac{f_b(\bar{z})\,\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-4}\sin^2\theta\,d\theta + \cdots \\ \dfrac{\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-4}\sin^2\theta\,d\theta - \cdots \\ \dfrac{\bar{z}^{2n}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-3}\sin^2\theta\,d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-2}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-2n-5}\sin^2\theta\,d\theta \end{pmatrix} \tag{137}$$

where the expressions in (102) remain applicable, while

$$\left.\begin{aligned} f_0(\theta) &= \bar{\rho}\,(1 - \bar{\rho}\cos\theta) \\ f_1(\theta) &= 2f_0(\theta) \end{aligned}\right\} \quad \rho \le R$$
$$\left.\begin{aligned} f_0(\theta) &= \bar{\rho}^2 - \bar{\rho}\cos\theta \\ f_1(\theta) &= 2f_0(\theta) \end{aligned}\right\} \quad \rho > R. \tag{138}$$

The evaluation of this integral for $\Re(s) > 0$ is performed as

$$V_{012}(\rho, R, s) = \begin{cases} \pm i\pi J_1(\bar{s}\bar{\rho})H_2^{(1,2)}(\bar{s}) - \dfrac{2\bar{\rho}}{\bar{s}} + V_{012}(\rho, R, -s) & \rho \le R \\ \pm i\pi J_2(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) + V_{012}(\rho, R, -s) & \rho > R. \end{cases} \tag{139}$$

The remaining integral is solved considering that

$$V_{021}(\rho, R, s) = V_{012}(R, \rho, s). \tag{140}$$

### 4.9. Integrals $V_{112}(\rho, R, s)$ and $V_{121}(\rho, R, s)$

The form taken by the first of these integrals is

$$V_{112}(\rho, R, s) = \int_0^{\infty} \frac{J_1(k\rho)J_2(kR)}{k(k-s)}dk. \tag{141}$$

This integral can be expressed as

$$V_{112}(\rho, R, s) = \frac{2}{R}V_{211}(\rho, R, s) - V_{110}(\rho, R, s). \tag{142}$$

Hence, for $\Re(s) \leq 0$ Eq. (141) yields

$$V_{112}(\rho, R, s) = -\frac{1}{\pi}\int_0^{\pi} \begin{pmatrix} \dfrac{\rho(R^2-\rho^2+r^2)}{2r^2}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(-sr) - Y_0(-sr)\right]\right) + \cdots \\ \dfrac{\rho(R^2-\rho^2+r^2)}{sr^3}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(-sr) - Y_1(-sr)\right] + \dfrac{1}{sr}\right) \end{pmatrix} \sin^2\theta \, d\theta. \tag{143}$$

Using dimensionless variables one obtains for $\rho \leq R$

$$V_{112}(\rho, R, s) = \frac{\bar{z}\bar{\rho}}{s\pi}\int_0^{\pi} \begin{pmatrix} (1-\bar{\rho}\cos\theta)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right]\right) - \cdots \\ \dfrac{2}{z}(1-\bar{\rho}\cos\theta)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - \dfrac{1}{z}\right) \end{pmatrix} \frac{\sin^2\theta}{\bar{r}^2} \, d\theta \tag{144}$$

while for $\rho > R$

$$V_{112}(\rho, R, s) = \frac{\bar{z}}{s\pi}\int_0^{\pi} \begin{pmatrix} (\bar{\rho}^2-\bar{\rho}\cos\theta)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right]\right) - \cdots \\ \dfrac{2}{z}(\bar{\rho}^2-\bar{\rho}\cos\theta)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - \dfrac{1}{z}\right) \end{pmatrix} \frac{\sin^2\theta}{\bar{r}^2} \, d\theta. \tag{145}$$

The solution is divided into two parts

$$V_{112}(\rho, R, s) = V_{112}^0(\rho, R, s) - V_{112}^1(\rho, R, s). \tag{146}$$

Using the adequate approximations for low and high arguments one arrives at:

$$V_{112}^0(\rho, R, s) = \frac{\bar{z}}{s\pi}\begin{pmatrix} \displaystyle\sum_{n=0}^{N_L}(-1)^n \begin{pmatrix} \dfrac{\bar{z}^{2n+1}}{(2n+1)!!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-1}\sin^2\theta\,d\theta - \cdots \\ \dfrac{\bar{z}^{2n}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\,d\theta - \cdots \\ \dfrac{f_a(\bar{z})\,\bar{z}^{2n}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-2}\sin^2\theta\,d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!^2\bar{z}^{-2n-1}\int_{\bar{\theta}}^{\pi} f_0(\theta)\,\bar{r}^{-2n-3}\sin^2\theta\,d\theta \end{pmatrix} \tag{147}$$

$$V_{112}^1(\rho, R, s) = \frac{\bar{z}}{s\pi}\begin{pmatrix} \dfrac{1}{\bar{z}}\displaystyle\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-3}\sin^2\theta\,d\theta - \dfrac{1}{\bar{z}^2}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-4}\sin^2\theta\,d\theta + \cdots \\ \displaystyle\sum_{n=1}^{N_L}(-1)^n \begin{pmatrix} \dfrac{f_b(\bar{z})\,\bar{z}^{2n-2}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-4}\sin^2\theta\,d\theta + \cdots \\ \dfrac{\bar{z}^{2n-2}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-4}\sin^2\theta\,d\theta - \cdots \\ \dfrac{\bar{z}^{2n-1}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-3}\sin^2\theta\,d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-3}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-2n-5}\sin^2\theta\,d\theta \end{pmatrix} \tag{148}$$

where the expressions in (102) and (138) remain applicable.

The evaluation of this integral for $\Re(s) > 0$ is performed as

$$V_{112}(\rho, R, s) = \begin{cases} \pm\dfrac{i\pi}{s}J_1(\bar{s}\bar{\rho})H_2^{(1,2)}(\bar{s}) - \dfrac{2\bar{\rho}}{s\bar{s}} - V_{112}(\rho, R, -s) & \rho \leq R \\ \pm\dfrac{i\pi}{s}J_2(\bar{s}\bar{\rho})H_1^{(1,2)}(\bar{s}) - V_{112}(\rho, R, -s) & \rho > R. \end{cases} \tag{149}$$

The remaining integral is solved considering that

$$V_{121}(\rho, R, s) = V_{112}(R, \rho, s). \tag{150}$$

### 4.10. Integrals $\bar{V}_{001}(\rho, R, s)$ and $\bar{V}_{010}(\rho, R, s)$

The form taken by the first of these integrals is defined in (30), resulting for $\Re(s) \le 0$ in

$$\bar{V}_{001}(\rho, R, s) = \frac{1}{\pi}\int_0^{\pi} \begin{pmatrix} \dfrac{sR(\rho^2 - R^2 + r^2)}{2r^2}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_0(-sr) - Y_0(-sr)\right] + \dfrac{1}{sr}\right) + \cdots \\ \dfrac{R(\rho^2 - R^2)}{r^3}\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(-sr) - Y_1(-sr)\right] - 1 + \dfrac{1}{sr}\right) \end{pmatrix} \sin^2(\theta)\, d\theta. \tag{151}$$

Using dimensionless variables one obtains for $\rho \le R$

$$\bar{V}_{001}(\rho, R, s) = \frac{1}{\pi}\int_0^{\pi} \begin{pmatrix} (\bar{\rho}\cos\theta - \bar{\rho}^2)\left(z\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right] - 1\right) + \cdots \\ (\bar{\rho}^2 - 1)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - 1 - \dfrac{1}{z}\right) \end{pmatrix} \frac{\sin^2(\theta)}{\bar{r}^3}\, d\theta \tag{152}$$

while for $\rho > R$

$$\bar{V}_{001}(\rho, R, s) = \frac{\bar{\rho}}{\pi}\int_0^{\pi} \begin{pmatrix} (\bar{\rho}\cos\theta - 1)\left(z\dfrac{\pi}{2}\left[\boldsymbol{H}_0(z) - Y_0(z)\right] - 1\right) + \cdots \\ (1 - \bar{\rho}^2)\left(\dfrac{\pi}{2}\left[\boldsymbol{H}_1(z) - Y_1(z)\right] - 1 - \dfrac{1}{z}\right) \end{pmatrix} \frac{\sin^2(\theta)}{\bar{r}^3}\, d\theta. \tag{153}$$

The solution is divided into two parts

$$\bar{V}_{001}(\rho, R, s) = \bar{V}^0_{001}(\rho, R, s) + \bar{V}^1_{001}(\rho, R, s). \tag{154}$$

Using the adequate approximations for low and high arguments one obtains

$$\bar{V}^0_{001}(\rho, R, s) = \frac{1}{\pi}\begin{pmatrix} -\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{-3}\sin^2\theta\, d\theta + \cdots \\ \displaystyle\sum_{n=0}^{N_L}(-1)^n \begin{pmatrix} \dfrac{\bar{z}^{2n+2}}{(2n+1)!!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-1}\sin^2\theta\, d\theta - \cdots \\ \dfrac{\bar{z}^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-2}\sin^2\theta\, d\theta - \cdots \\ \dfrac{f_a(\bar{z})\,\bar{z}^{2n+1}}{2^{2n}n!^2}\displaystyle\int_0^{\bar{\theta}} f_0(\theta)\,\bar{r}^{2n-2}\sin^2\theta\, d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=1}^{N_U}(-1)^n(2n-1)!!^2\bar{z}^{-2n}\int_{\bar{\theta}}^{\pi} f_0(\theta)\,\bar{r}^{-2n-3}\sin^2\theta\, d\theta \end{pmatrix} \tag{155}$$

$$\bar{V}^1_{001}(\rho, R, s) = \frac{1}{\pi}\begin{pmatrix} -\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{-3}\sin^2\theta\, d\theta - \frac{1}{\bar{z}}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-4}\sin^2\theta\, d\theta + \cdots \\ \displaystyle\sum_{n=1}^{N_L}(-1)^n \begin{pmatrix} \dfrac{f_b(\bar{z})\,\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-4}\sin^2\theta\, d\theta + \cdots \\ \dfrac{\bar{z}^{2n-1}}{2^{2n-1}(n-1)!n!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\ln(\bar{r})\,\bar{r}^{2n-4}\sin^2\theta\, d\theta - \cdots \\ \dfrac{\bar{z}^{2n}}{(2n-1)!!(2n+1)!!}\displaystyle\int_0^{\bar{\theta}} f_1(\theta)\,\bar{r}^{2n-3}\sin^2\theta\, d\theta \end{pmatrix} + \cdots \\ \displaystyle\sum_{n=0}^{N_U}(-1)^n(2n-1)!!(2n+1)!!\bar{z}^{-2n-2}\int_{\bar{\theta}}^{\pi} f_1(\theta)\,\bar{r}^{-2n-5}\sin^2\theta\, d\theta \end{pmatrix} \tag{156}$$

where the expressions in (102) remain applicable, while

$$\left.\begin{aligned} f_0(\theta) &= \bar{\rho}\cos\theta - \bar{\rho}^2 \\ f_1(\theta) &= \bar{\rho}^2 - 1 \end{aligned}\right\}\quad \rho \le R$$
$$\left.\begin{aligned} f_0(\theta) &= \bar{\rho}\,(\bar{\rho}\cos\theta - 1) \\ f_1(\theta) &= \bar{\rho}\left(1 - \bar{\rho}^2\right) \end{aligned}\right\}\quad \rho > R. \tag{157}$$

The evaluation of this integral for $\Re(s) > 0$ is performed as

$$\bar{V}_{001}(\rho, R, s) = \begin{cases} \pm i\pi \left( J_0(\bar{s}\bar{\rho}) - \frac{1}{\bar{s}\bar{\rho}} J_1(\bar{s}\bar{\rho}) \right) H_1^{(1,2)}(\bar{s}) - \frac{1}{\bar{s}} + \bar{V}_{001}(\rho, R, -s) & \rho \le R \\ \pm i\pi J_1(\bar{s}\bar{\rho}) \left( H_0^{(1,2)}(\bar{s}) - \frac{1}{\bar{s}} H_1^{(1,2)}(\bar{s}) \right) + \frac{\bar{\rho}}{\bar{s}} + \bar{V}_{001}(\rho, R, -s) & \rho > R. \end{cases} \tag{158}$$

The remaining integral defined in (31) is solved considering that

$$\bar{V}_{010}(\rho, R, s) = \bar{V}_{001}(R, \rho, s). \tag{159}$$

## 5. Integrals of type $\bar{W}_{\alpha\beta}(\rho, R, s)$

### 5.1. Integral $\bar{W}_{-10}(\rho, R, s)$

This integral can be solved without problems developing the parentheses in the numerator of (14) and using expression (34)

$$\begin{aligned} \bar{W}_{-10}(\rho, R, s) &= \int_0^\infty \frac{kJ_0(k\rho)\left(1 - J_0(kR)\right)}{(k^2 - s^2)} dk \\ &= W_{-100}(\rho, 0, s) - W_{-100}(\rho, R, s) \end{aligned} \tag{160}$$

that results in

$$\bar{W}_{-10}(\rho, R, s) = \begin{cases} \pm \frac{i\pi}{2} \left( H_0^{(1,2)}(\bar{s}\bar{\rho}) - J_0(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) \right) & \rho \le R \\ \pm \frac{i\pi}{2} H_0^{(1,2)}(\bar{s}) \left(1 - J_0(\bar{s}\bar{\rho})\right) & \rho > R. \end{cases} \tag{161}$$

### 5.2. Integral $\bar{W}_{01}(\rho, R, s)$

This integral can also be solved developing the parentheses in the numerator of (14) and using expression (38)

$$\begin{aligned} \bar{W}_{01}(\rho, R, s) &= \int_0^\infty \frac{J_1(k\rho)\left(1 - J_0(kR)\right)}{(k^2 - s^2)} dk \\ &= W_{010}(\rho, 0, s) - W_{010}(\rho, R, s) \end{aligned} \tag{162}$$

resulting in

$$\bar{W}_{01}(\rho, R, s) = \begin{cases} \pm \frac{i\pi}{2s} \left( H_1^{(1,2)}(\bar{s}\bar{\rho}) - J_1(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) \right) - \frac{1}{s\bar{s}\bar{\rho}} & \rho \le R \\ \pm \frac{i\pi}{2s} H_1^{(1,2)}(\bar{s}) \left(1 - J_0(\bar{s}\bar{\rho})\right) & \rho > R. \end{cases} \tag{163}$$

### 5.3. Integral $\bar{W}_{10}(\rho, R, s)$

This integral cannot be solved by developing the parentheses in the numerator of (14) since the condition (9) is not satisfied

$$\bar{W}_{10}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho)\left(1 - J_0(kR)\right)}{k(k^2 - s^2)} dk. \tag{164}$$

Anyway, the parenthesis in the numerator of the integrand can be expressed as

$$\left(1 - J_0(kR)\right) = k \int_0^R J_1(ka) da. \tag{165}$$

By introducing (165) in (164), and changing the integration order, the following expression is obtained

$$\bar{W}_{10}(\rho, R, s) = \int_0^R \int_0^\infty \frac{J_0(k\rho) J_1(ka)}{(k^2 - s^2)} dk\, da \tag{166}$$

in which the following integral, whose solution is given in (37), can be identified

$$W_{001}(\rho, a, s) = \int_0^\infty \frac{J_0(k\rho)J_1(ka)}{(k^2 - s^2)} dk. \tag{167}$$

Thus, the integral in (164) can be computed as

$$\bar{W}_{10}(\rho, R, s) = \begin{cases} \displaystyle\int_0^\rho W_{001}(\rho, a, s)_{a \le \rho} da + \int_\rho^R W_{001}(\rho, a, s)_{a > \rho} da & \rho \le R \\ \displaystyle\int_0^R W_{001}(\rho, a, s)_{a \le \rho} da & \rho > R \end{cases} \tag{168}$$

resulting in

$$\bar{W}_{10}(\rho, R, s) = \begin{cases} \pm \dfrac{i\pi}{2s^2} \left( H_0^{(1,2)}(\bar{s}\bar{\rho}) - J_0(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) \right) + \dfrac{\ln(\bar{\rho})}{s^2} & \rho \le R \\ \pm \dfrac{i\pi}{2s^2} H_0^{(1,2)}(\bar{s}) \left(1 - J_0(\bar{s}\bar{\rho})\right) & \rho > R. \end{cases} \tag{169}$$

### *5.4. Integral $\bar{W}_{21}(\rho, R, s)$*

This integral cannot be solved by developing the parentheses in the numerator of (14)

$$\bar{W}_{21}(\rho, R, s) = \int_0^\infty \frac{J_1(k\rho)\left(1 - J_0(kR)\right)}{k^2(k^2 - s^2)} dk. \tag{170}$$

However, introducing (165) in (170) and changing the integration order one obtains

$$\bar{W}_{21}(\rho, R, s) = \int_0^R \int_0^\infty \frac{J_1(k\rho)J_1(ka)}{k(k^2 - s^2)} dk da \tag{171}$$

in which the following integral, whose solution is given in (36), can be identified

$$W_{111}(\rho, a, s) = \int_0^\infty \frac{J_1(k\rho)J_1(ka)}{k(k^2 - s^2)} dk. \tag{172}$$

Thus, the integral in (170) can be computed as

$$\bar{W}_{21}(\rho, R, s) = \begin{cases} \displaystyle\int_0^\rho W_{111}(\rho, a, s)_{a \le \rho} da + \int_\rho^R W_{111}(\rho, a, s)_{a > \rho} da & \rho \le R \\ \displaystyle\int_0^R W_{111}(\rho, a, s)_{a \le \rho} da & \rho > R \end{cases} \tag{173}$$

resulting in

$$\bar{W}_{21}(\rho, R, s) = \begin{cases} \left( \begin{array}{l} \pm \dfrac{i\pi}{2s^3} \left( H_1^{(1,2)}(\bar{s}\bar{\rho}) - J_1(\bar{s}\bar{\rho}) H_0^{(1,2)}(\bar{s}) \right) - \cdots \\ \dfrac{1}{\bar{s}\bar{\rho}\, s^3} - \dfrac{\bar{s}\bar{\rho}}{4s^3} \left(1 - 2\ln(\bar{\rho})\right) \end{array} \right) & \rho \le R \\ \pm \dfrac{i\pi}{2s^3} H_1^{(1,2)}(\bar{s}) \left(1 - J_0(\bar{s}\bar{\rho})\right) - \dfrac{\bar{s}\bar{\rho}^2}{4s^3} & \rho > R. \end{cases} \tag{174}$$

## 6. Integrals of type $\bar{V}_{\alpha\beta}(\rho, R, s)$

### *6.1. Integral $\bar{V}_{00}(\rho, R, s)$*

This integral can be solved by developing the parentheses in the numerator of (12)

$$\bar{V}_{00}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho)\left(1 - J_0(kR)\right)}{(k - s)} dk = V_{000}(\rho, 0, s) - V_{000}(\rho, R, s). \tag{175}$$

For $\Re(s) \leq 0$ is used (53) together with

$$V_{000}(\rho, 0, s) = \int_0^\infty \frac{J_0(k\rho)}{(k-s)} dk = \frac{\pi}{2}\left[\boldsymbol{H}_0(-s\rho) - Y_0(-s\rho)\right] = \begin{cases} \frac{\pi}{2}\left[\boldsymbol{H}_0(\bar{z}\bar{\rho}) - Y_0(\bar{z}\bar{\rho})\right] & \rho \leq R \\ \frac{\pi}{2}\left[\boldsymbol{H}_0(\bar{z}) - Y_0(\bar{z})\right] & \rho > R \end{cases} \quad (176)$$

while for $\Re(s) > 0$ according to (4) and (14) results in

$$\bar{V}_{00}(\rho, R, s) = 2\bar{W}_{-10}(\rho, R, s) - \bar{V}_{00}(\rho, R, -s) = \begin{cases} \pm i\pi\left(H_0^{(1,2)}(\bar{s}\bar{\rho}) - J_0(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s})\right) - \bar{V}_{00}(\rho, R, -s) & \rho \leq R \\ \pm i\pi H_0^{(1,2)}(\bar{s})\left(1 - J_0(\bar{s}\bar{\rho})\right) - \bar{V}_{00}(\rho, R, -s) & \rho > R. \end{cases} \quad (177)$$

### 6.2. Integral $\bar{V}_{11}(\rho, R, s)$

This integral can also be solved by developing the parentheses in the numerator of (12)

$$\bar{V}_{11}(\rho, R, s) = \int_0^\infty \frac{J_1(k\rho)\left(1 - J_0(kR)\right)}{k(k-s)} dk = V_{110}(\rho, 0, s) - V_{110}(\rho, R, s). \quad (178)$$

For $\Re(s) \leq 0$ are used (113)–(115) considering (118), together with

$$V_{110}(\rho, 0, s) = \int_0^\infty \frac{J_1(k\rho)}{k(k-s)} dk = -\frac{1}{s}\left(\frac{\pi}{2}\left(\boldsymbol{H}_1(-s\rho) - Y_1(-s\rho)\right) + \frac{1}{s\rho}\right) = \begin{cases} -\frac{1}{s}\left(\frac{\pi}{2}\left(\boldsymbol{H}_1(\bar{z}\bar{\rho}) - Y_1(\bar{z}\bar{\rho})\right) - \frac{1}{\bar{z}\bar{\rho}}\right) & \rho \leq R \\ -\frac{1}{s}\left(\frac{\pi}{2}\left(\boldsymbol{H}_1(\bar{z}) - Y_1(\bar{z})\right) - \frac{1}{\bar{z}}\right) & \rho \geq R \end{cases} \quad (179)$$

while for $\Re(s) > 0$ according to (4) and (14) results in

$$\bar{V}_{11}(\rho, R, s) = 2\bar{W}_{01}(\rho, R, s) - \bar{V}_{11}(\rho, R, -s) = \begin{cases} \pm\frac{i\pi}{s}\left(H_1^{(1,2)}(\bar{s}\bar{\rho}) - J_1(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s})\right) - \frac{2}{s\bar{s}\bar{\rho}} - \bar{V}_{11}(\rho, R, -s) & \rho \leq R \\ \pm\frac{i\pi}{s}H_1^{(1,2)}(\bar{s})\left(1 - J_0(\bar{s}\bar{\rho})\right) - \bar{V}_{11}(\rho, R, -s) & \rho > R. \end{cases} \quad (180)$$

### 6.3. Integral $\bar{V}_{10}(\rho, R, s)$

This integral cannot be solved by developing the parentheses in the numerator of (12) since the condition (9) is not satisfied

$$\bar{V}_{10}(\rho, R, s) = \int_0^\infty \frac{J_0(k\rho)\left(1 - J_0(kR)\right)}{k(k-s)} dk. \quad (181)$$

However, this integral can be expressed as

$$\bar{V}_{10}(\rho, R, s) = \frac{1}{s}\left(\bar{V}_{00}(\rho, R, s) - \bar{H}_{00}(\rho, R)\right). \quad (182)$$

For $\Re(s) \leq 0$ is used (175) together with (see expression (22))

$$\bar{H}_{00}(\rho, R) = \int_0^\infty \frac{J_0(k\rho)\left(1 - J_0(kR)\right)}{k} dk = \begin{cases} -\ln(\bar{\rho}) & \rho \leq R \\ 0 & \rho > R \end{cases} \quad (183)$$

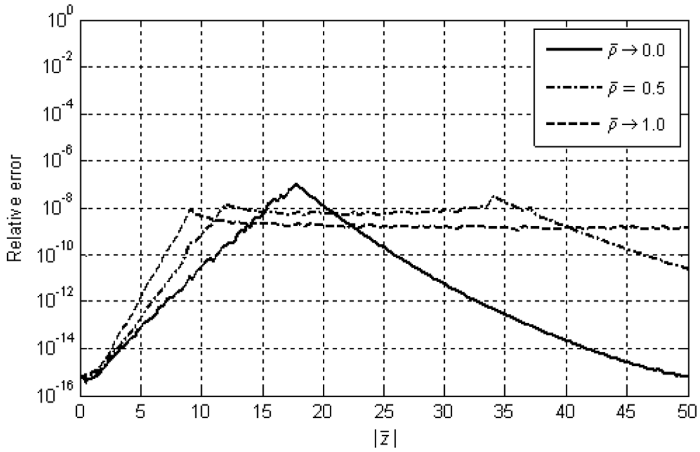


**Fig. 5.** Relative error for integral $V_{000}(\rho, R, s)$.

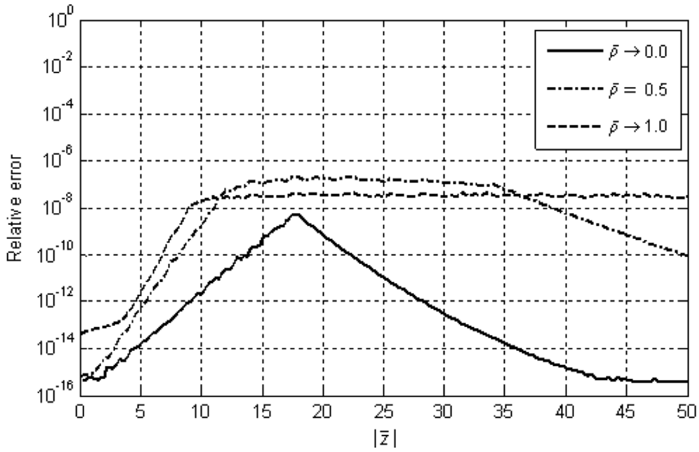


**Fig. 6.** Relative error for integral $V_{011}(\rho, R, s)$.

while for $\Re(s) > 0$ according to (3) and (14) results in

$$\bar{V}_{10}(\rho, R, s) = 2s\,\bar{W}_{10}(\rho, R, s) + \bar{V}_{10}(\rho, R, -s)$$
$$= \begin{cases} \left( \pm\dfrac{i\pi}{s}\left(H_0^{(1,2)}(\bar{s}\bar{\rho}) - J_0(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s})\right) + \cdots \atop \dfrac{2\ln(\bar{\rho})}{s} + \bar{V}_{10}(\rho, R, -s) \right) & \rho \le R \\ \pm\dfrac{i\pi}{s}H_0^{(1,2)}(\bar{s})\left(1 - J_0(\bar{s}\bar{\rho})\right) + \bar{V}_{10}(\rho, R, -s) & \rho > R. \end{cases} \tag{184}$$

### 6.4. *Integral* $\bar{V}_{21}(\rho, R, s)$

This integral cannot be solved by developing the parentheses in the numerator of (12)

$$\bar{V}_{21}(\rho, R, s) = \int_0^\infty \frac{J_1(k\rho)\left(1 - J_0(kR)\right)}{k^2(k - s)}dk. \tag{185}$$

However, this integral can be expressed as

$$\bar{V}_{21}(\rho, R, s) = \frac{1}{s}\left(\bar{V}_{11}(\rho, R, s) - \bar{H}_{11}(\rho, R)\right). \tag{186}$$

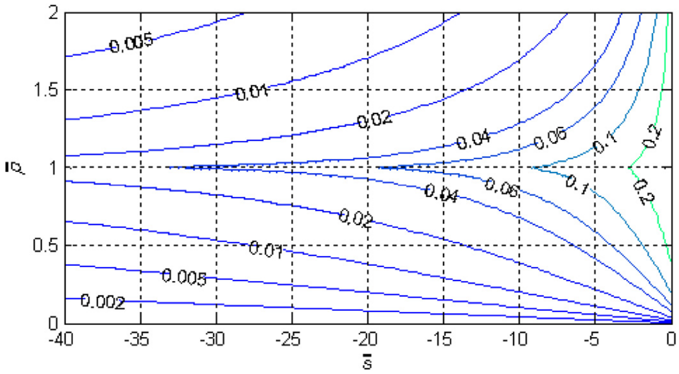


**Fig. 7.** $V_{011}$ for real and negative values of $\bar{s}$.

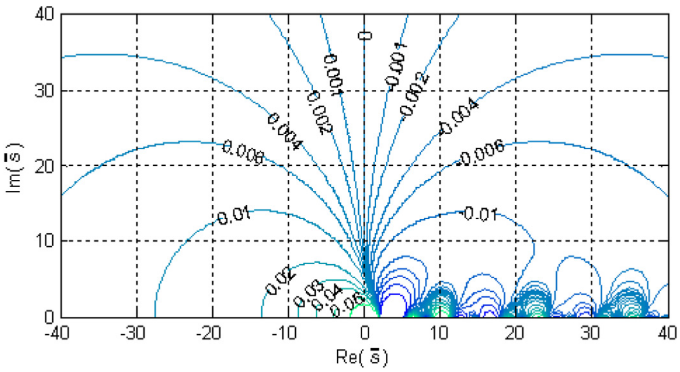


**Fig. 8.** Real part of $V_{011}$ for $\bar{\rho} = 0.5$.

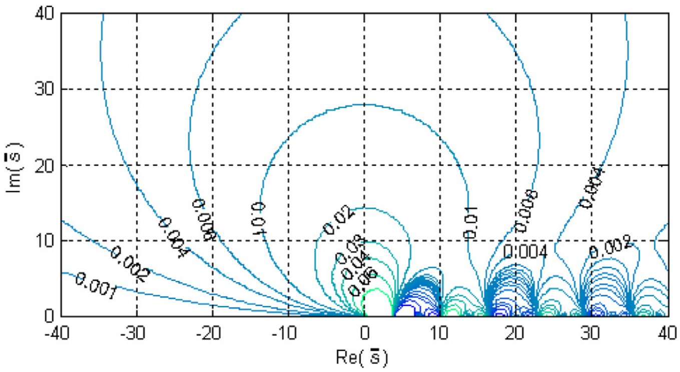


**Fig. 9.** Imaginary part of $V_{011}$ for $\bar{\rho} = 0.5$.

For $\Re(s) \leq 0$ is used (178) together with

$$\bar{H}_{11}(\rho, R) = \int_0^\infty \frac{J_1(k\rho)\,(1 - J_0(kR))}{k^2} dk = \begin{cases} \dfrac{\bar{s}\bar{\rho}}{4s}\,(1 - 2\ln(\bar{\rho})) & \rho \leq R \\ \dfrac{\bar{s}\bar{\rho}^2}{4s} & \rho > R \end{cases} \tag{187}$$

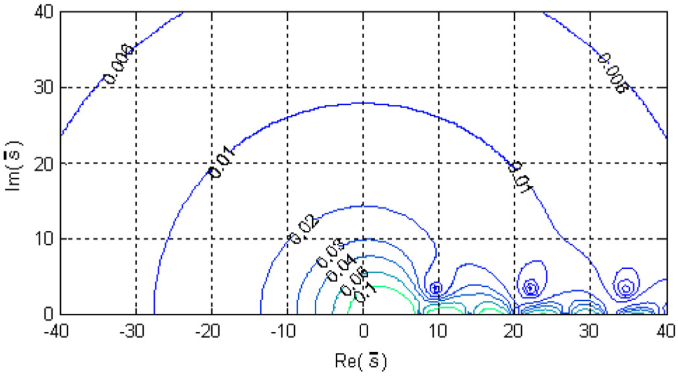


**Fig. 10.** Modulus of $V_{011}$ for $\bar{\rho} = 0.5$.

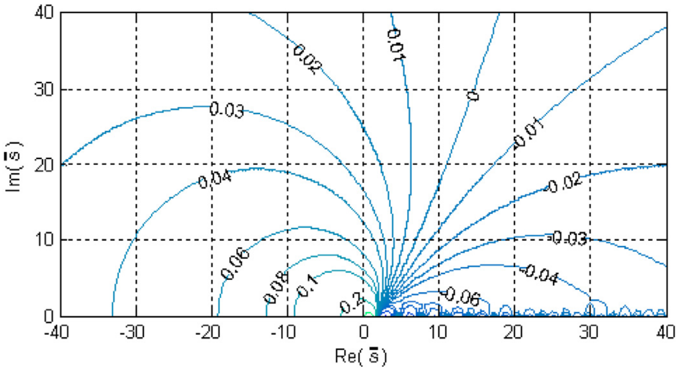


**Fig. 11.** Real part of $V_{011}$ for $\bar{\rho} = 1$.

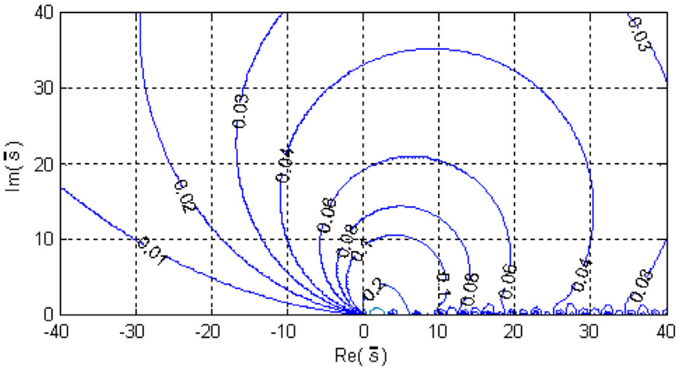


**Fig. 12.** Imaginary part of $V_{011}$ for $\bar{\rho} = 1$.

while for $\Re(s) > 0$ according to (3) and (14) results in

$$\begin{aligned}\bar{V}_{21}(\rho, R, s) &= 2s\,\bar{W}_{21}(\rho, R, s) + \bar{V}_{21}(\rho, R, -s)\\ &= \begin{cases} \left( \pm\dfrac{i\pi}{s^2}\left(H_1^{(1,2)}(\bar{s}\bar{\rho}) - J_1(\bar{s}\bar{\rho})H_0^{(1,2)}(\bar{s})\right) - \cdots \right. \\ \left. \dfrac{2}{\bar{s}\bar{\rho}\,s^2} - \dfrac{\bar{s}\bar{\rho}}{2s^2}\left(1 - 2\ln(\bar{\rho})\right) + \bar{V}_{21}(\rho, R, -s) \right) & \rho \le R \\ \pm\dfrac{i\pi}{s^2} H_1^{(1,2)}(\bar{s})\left(1 - J_0(\bar{s}\bar{\rho})\right) - \dfrac{\bar{s}\bar{\rho}^2}{2s^2} + \bar{V}_{21}(\rho, R, -s) & \rho > R. \end{cases}\end{aligned} \tag{188}$$

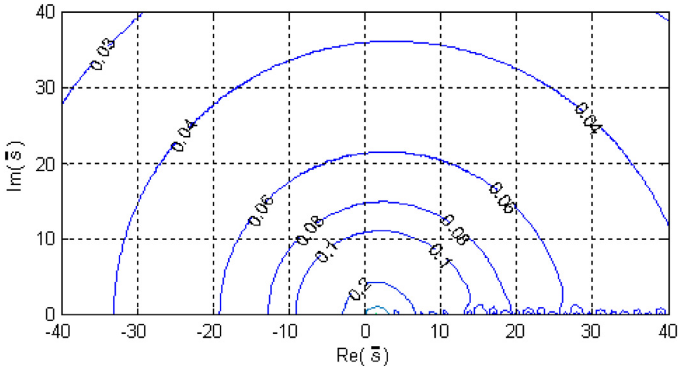


**Fig. 13.** Modulus of $V_{011}$ for $\bar{\rho} = 1$.

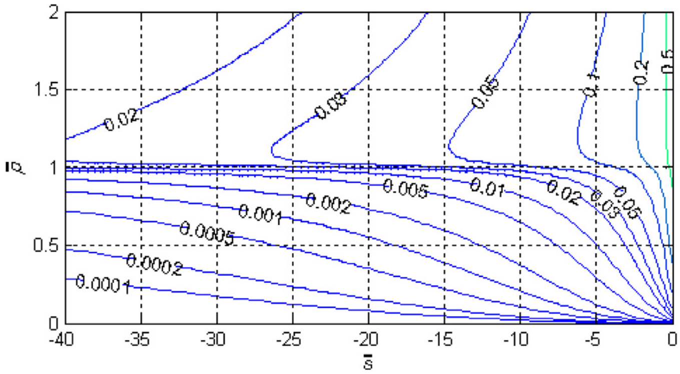


**Fig. 14.** $V_{010}$ for real and negative values of $\bar{s}$.

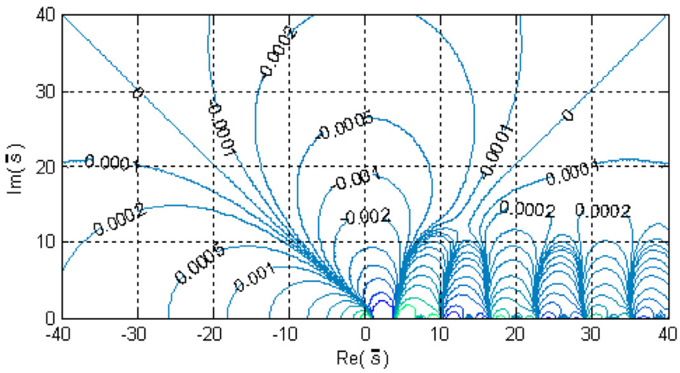


**Fig. 15.** Real part of $V_{010}$ for $\bar{\rho} = 0.5$.

## 7. Accuracy assessment for some integrals

For the purpose to estimate the accuracy of some of the described integral ADVANPIX [10] has been used. This toolbox equips MATLAB [11] with a multiple precision floating-point numeric type and extensive set of mathematical functions that are capable of computing with arbitrary precision.

Figs. 5 and 6 show relative errors for $V_{000}$ and $V_{011}$ integrals that arise from the comparison of results obtained with standard functions using double precision and with the ADVANPIX toolbox using 90 digits of precision. The maximum values for phase angles of $\bar{z}$ ranging from $-180°$ and $180°$ each $2°$ were taken. The calculations for $\bar{\rho} \to 0$ were made with $\bar{\rho} = 0.001$ while those for $\bar{\rho} \to 1$ were made with $\bar{\rho} = 0.999$. In both figures it can be seen that relative errors for $\bar{\rho} \to 0$ matches those of Figs. 1 and 3 since the integrands of the involved integrals become independent of $\theta$. In fact, the values of $V_{000}$ for $\bar{\rho} = 0$ are obtained through (45). Relative errors for $\bar{\rho} = 0.5$ and $\bar{\rho} \to 1$ can be sensitive to the parameters defined for numerical integration through techniques of Gauss–Kronrod type. For the figures of this section *quadgk* MATLAB

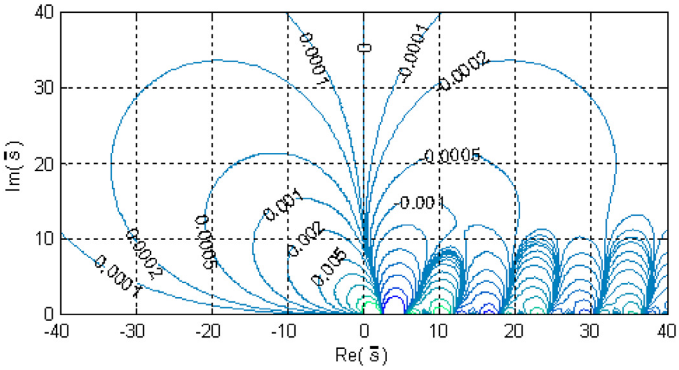


**Fig. 16.** Imaginary part of $V_{010}$ for $\bar{\rho} = 0.5$.

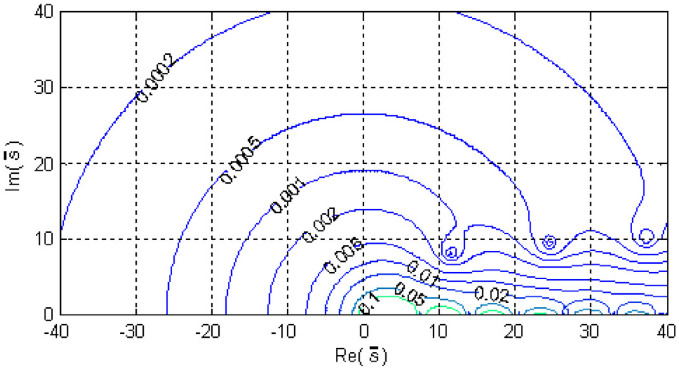


**Fig. 17.** Modulus of $V_{010}$ for $\bar{\rho} = 0.5$.

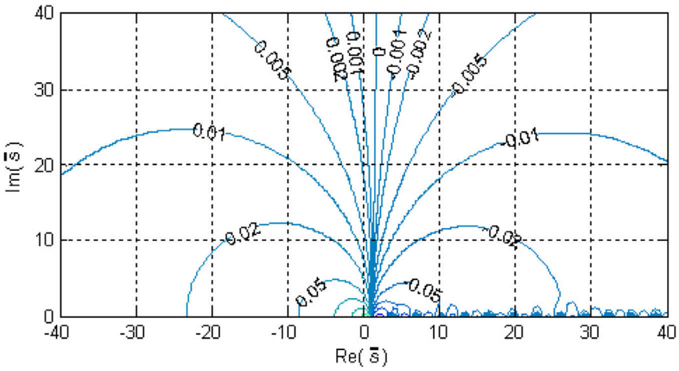


**Fig. 18.** Real part of $V_{010}$ for $\bar{\rho} = 1$.

toolbox has been used with the following relative tolerances: $RelTol = 1\text{e} - 13$ for double precision and $RelTol = 1\text{e} - 20$ for 90 digits of precision.

## 8. Graphical representation of some integrals

By way of example Fig. 7 shows the values of the integral $V_{011}$ through contour lines for real and negative values of $\bar{s}$ (in this case, the integral becomes real). The values of this integral for complex values of $\bar{s}$ and $\bar{\rho} = 0.5$ are shown in Figs. 8–10, while Figs. 11–13 show the corresponding values for $\bar{\rho} = 1$. Graphs with the same characteristics to those already described but for the case of integral $V_{010}$ for $\rho \leq R$ are shown in Figs. 14–20.

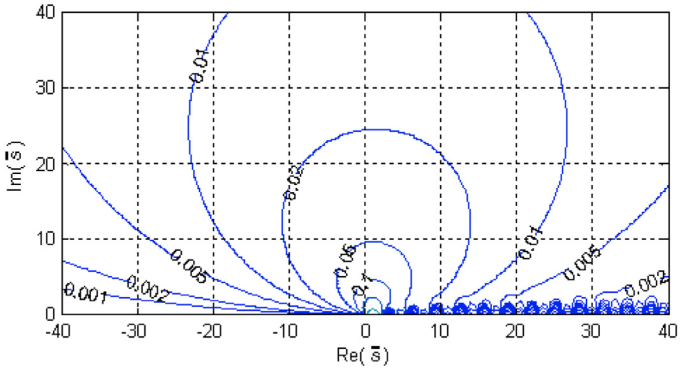


**Fig. 19.** Imaginary part of $V_{010}$ for $\bar{\rho} = 1$.

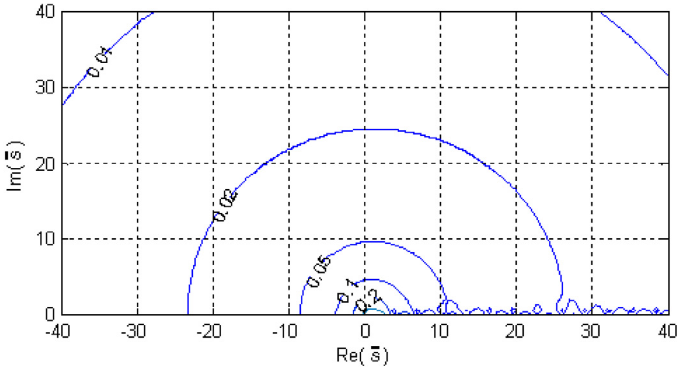


**Fig. 20.** Modulus of $V_{010}$ for $\bar{\rho} = 1$.

## 9. Conclusions

The basic integrals that are solved in this paper involve the product of two Bessel functions as argument. The representation of this product through definite integrals, together with the change of the integration order, provide an efficient way of solving these integrals that are essential for the implementation of a first order thin layers formulation. This formulation can be used both for the study of the dynamic response of systems with soil–structure interaction and for the site characterization by wave propagation methods.

## Acknowledgments

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