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# A LIOUVILLE THEOREM FOR INDEFINITE FRACTIONAL DIFFUSION EQUATIONS AND ITS APPLICATION TO EXISTENCE OF SOLUTIONS

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Abstract. In this work we obtain a Liouville theorem for positive, bounded solutions of the equation

$$(-\Delta)^s u = h(x_N) f(u) \quad \text{in } \mathbb{R}^N$$

where $(-\Delta)^s$ stands for the fractional Laplacian with $s \in (0,1)$, and the functions $h$ and $f$ are nondecreasing. The main feature is that the function $h$ changes sign in $\mathbb{R}$, therefore the problem is sometimes termed as indefinite. As an application we obtain a priori bounds for positive solutions of some boundary value problems, which give existence of such solutions by means of bifurcation methods.

## 1. Introduction and main results.

The objective of the present paper is to obtain a Liouville theorem for a nonlocal elliptic equation involving the fractional Laplacian. This operator is defined for sufficiently smooth functions by

$$(-\Delta)^s u(x) = c(N,s) \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where $0 < s < 1$, $c(N,s)$ is a normalization constant whose value will be of no importance for us and the integral is to be understood in the principal value sense.

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During the last years there has been an increasing amount of research on equations driven by $(-\Delta)^s$. The main interest is to test whether the known features for its local counterpart $-\Delta$, obtained by setting $s = 1$, remain valid for arbitrary $s \in (0, 1)$. In general, this has led to adaptation of the standard techniques and to the search for new tools. Moreover, sometimes even stronger results can be obtained in the nonlocal case as our main theorem below shows.

When it comes to Liouville theorems there is a more or less satisfactory understanding of the equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1}$$

The case $f(t) = t^p$ was considered in [17, 37], while in [18] some more general nonlinearities were analyzed. In these works, the authors obtained for arbitrary $s \in (0, 1)$ an analogue of the previously proved results in $s = 1$ (cf. [28, 13, 8, 31]).

However, the situation is fairly different when (1) is replaced by a *non-autonomous* equation. Let us mention the papers [22], [25], which deal with Hénon equation

$$(-\Delta)^s u = |x|^\alpha u^p \quad \text{in } \mathbb{R}^N,$$

where $\alpha > 0$ and $p > 1$. As for equations which involve weights with change sign, only the paper [19] is known to us. There, the authors consider the equation

$$(-\Delta)^s u = x_N u^p \quad \text{in } \mathbb{R}^N, \tag{2}$$

and show that there do not exist positive, bounded solutions for any $p > 1$, provided that $s \geq \frac{1}{2}$. This result is subsequently used to obtain a priori bounds for solutions of some related boundary value problems (see (4) below). The main technique in [19] is the reduction of the problem to a local one by means of the extension problem introduced in [11]. The notation in (2) is the usual one: for a point $x \in \mathbb{R}^N$ we write $x = (x', x_N)$, where $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$.

The local version of problems related to (2) has been also considered for instance in the works [6] and [32], but in our opinion perhaps the most general result in this regard was obtained in [23], where the problem

$$-\Delta u = h(x_N) f(u) \quad \text{in } \mathbb{R}^N$$

is studied. Here $h$ and $f$ are nondecreasing functions which verify some additional conditions, the main feature being that the function $h$ is assumed to be nonpositive for $x_N < 0$ and positive for $x_N > 0$. As for the nonlinearity $f$, the natural example is $f(t) = t^p$, with $p > 1$.

Our intention in this work is to obtain a similar Liouville theorem for the problem

$$(-\Delta)^s u = h(x_N) f(u) \quad \text{in } \mathbb{R}^N, \tag{3}$$

where both $h$ and $f$ are monotone and $h$ is allowed to change sign. We state below our precise hypotheses on $h$ and $f$, but it is interesting to remark that they are less stringent than in the case $s = 1$ considered in [23].

On the functions $h$ and $f$ we will assume the following, which will be termed altogether as hypotheses (H):

(H1) $h \in C^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$.
(H2) $h$ is nondecreasing in $\mathbb{R}$, with $h(0) = 0$ and $h(t) > 0$ for $t > 0$.
(H3) $\lim_{t \to +\infty} h(t) = +\infty$.
(H4) $f$ is locally Lipschitz and nondecreasing in $[0, +\infty)$, with $f(0) = 0$ and $f > 0$ in $(0, +\infty)$.

(H5) $\lim_{r,t\to 0} \dfrac{f(r)-f(t)}{r-t} = 0.$

Observe that condition (H2) could be stated with respect to another point different from zero, but this amounts only to a change of variables in (3). Natural examples for functions $h$ and $f$ are $h(t) = |t|^{\alpha-1}t$ for some $\alpha > 0$ and $f(t) = t^p$ for $p > 1$. The case $\alpha = 1$ then leads to (2).

Let us also mention here that, if $f \in C^1$ near the origin then condition (H5) is equivalent to $f'(0) = 0$. Nevertheless, the case $f'(0) > 0$ could also be included in our main theorem by arguing as in [4, Pag 13]. However, since the main application of the Liouville theorem presented in this work is concerned with the existence result established in Theorem 1.2 below, we have considered that this case could be omitted.

We now come to the statement of our Liouville theorem for (3). We will be dealing throughout with classical solutions, that is, functions $u \in C^{2s+\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$, verifying (3) at every point of $\mathbb{R}^N$. However, it is to be noted that by bootstrapping and the regularity theory developed in [12] and [36], solutions in the viscosity sense turn out to be classical.

**Theorem 1.1.** *Assume $h$ and $f$ verify hypotheses* (H)*. Then problem* (3) *does not admit any positive, bounded solution.*

A natural application of this Liouville theorem arises when considering boundary value problems with indefinite weights, for instance

$$\begin{cases} (-\Delta)^s u = \lambda u + a(x)u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4}$$

where $a \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Here $p > 1$, $\lambda \in \mathbb{R}$ is a parameter and $a$ is assumed to change sign in a "controlled" way. The local case $s = 1$ has been extensively studied, to mention a few, in [1, 2, 3, 6, 7, 9, 14, 15, 23] (see more references in [23]).

As for the fractional case $s \in (0,1)$, we refer to [30] and [27], where variational techniques were used. The use of variational techniques allows for somewhat relaxed hypotheses, however they only give existence of positive solutions of (4) for positive values of $\lambda$. On the other hand, the approach we follow here, based on a priori bounds and bifurcation theory, is suitable for generalization to a nonvariational setting. Indeed, the a priori bounds can be obtained as in [5], while the application of bifurcation theory requires only minor technical adjustments (which however go beyond the scope of this work).

In the present situation, we will be assuming that $a$ verifies the structural conditions, termed henceforth as hypotheses (A):

(A1) The set $\Gamma := \{x \in \overline{\Omega} : \ a(x) = 0\}$ is a smooth manifold of dimension $N-1$ contained in $\Omega$.

(A2) There exist $\gamma > 0$ and positive, continuous functions $b_1$, $b_2$ defined in $\Omega$ such that in a neighborhood of $\Gamma$

$$a(x) = \begin{cases} b_1(x)d(x)^{\gamma} & x \in \Omega^+ \\ -b_2(x)d(x)^{\gamma} & x \in \Omega^- \end{cases}$$

where $d(x) := \operatorname{dist}(x,\Gamma)$, $\Omega^+ := \{x \in \Omega : \ a(x) > 0\}$, $\Omega^- := \{x \in \Omega : \ a(x) < 0\}$.

Observe that hypotheses (A) imply that the set $\Gamma = \{a = 0\}$ is contained in $\Omega$ and has empty interior. Moreover, $a$ has different signs on "both sides" of $\Gamma$. This is equivalent to saying that $\overline{\Omega^+} \cap \overline{\Omega^-} = \Gamma$ (see Figure 1).

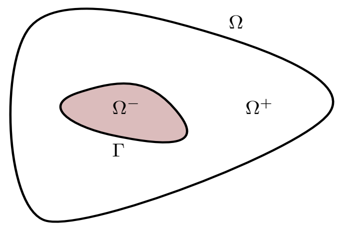


Figure 1. A possible configuration for $\Omega^+$ and $\Omega^-$ in hypotheses (A).

With these assumptions on the weight $a$, and assuming in addition that $p$ is subcritical, we can obtain a priori bounds for all positive solutions of (4) in bounded $\lambda$-intervals. And with the aid of bifurcation theory, these a priori bounds lead to an existence result. We denote by $\lambda_1(\Omega)$ the first eigenvalue of $(-\Delta)^s$ in $\Omega$.

**Theorem 1.2.** *Assume $s \in (0,1)$, $p > 1$ and let $a \in C^\alpha(\overline{\Omega})$ verify hypotheses* (A). *If $N \leq 2s$ or $N > 2s$ and $p$ is such that*

$$p < \frac{N+2s}{N-2s},$$

*then problem* (4) *admits at least a positive classical solution for every $\lambda < \lambda_1(\Omega)$. Moreover, there exists $\Lambda \geq \lambda_1(\Omega)$ such that there are no such solutions if $\lambda > \Lambda$.*

It is worthy of mention that in some cases one can guarantee the existence of positive solutions of (4) also for values $\lambda > \lambda_1(\Omega)$. Indeed, a more precise description of the bifurcation diagram near the point $(\lambda_1(\Omega), 0)$ can be performed by using for instance the Crandall-Rabinowitz theorem (cf. [20]), but this is definitely out of the scope of this article. See also [30] and [27].

Let us briefly mention our method of proof. For the Liouville theorem we will use the moving planes method to establish monotonicity of the solution in the direction $x_N$. Nevertheless, the application of the method is by no means standard, since in spite of the problem being posed in $\mathbb{R}^N$, at some point we can deal with problems in half-spaces, which allows us to introduce the Green's function obtained in [24], as was done in our previous work [4]. The use of the Green's function is what definitely distinguishes the case $s \in (0,1)$ from $s = 1$, thus allowing the hypotheses to be less restrictive.

As for the a priori bounds, we follow the approach in [5] (but see also [16] for a related approach). The blow-up method introduced in [29] is used, but we need to resort to the barriers introduced in [5] when the limit problem is posed in a half-space. Finally, Theorem 1.2 can be achieved with an application of the global bifurcation theorem of Rabinowitz (cf. [34]), and an analysis along the lines of the one made in [21].

The rest of the paper is organized as follows: in Section 2 we will prove our Liouville theorem. Section 3 is dedicated to obtaining the a priori bounds, while in Section 4 we will consider the question of existence of solutions.

2. **The Liouville theorem.** This section is dedicated to the proof of Theorem 1.1. The main step in the proof is to show that any bounded, positive solution $u$ of (3) has to be increasing in the $x_N$ direction. For this we will use the moving planes method as in [4] (see also [24]).

We begin by introducing some notation, which is for the most part rather standard in this context. We denote $x = (x', x_N)$ for points $x \in \mathbb{R}^N$ and for $\lambda \in \mathbb{R}$, let

$$\Sigma_\lambda := \{x \in \mathbb{R}^N : \ x_N < \lambda\},$$

$$T_\lambda := \{x \in \mathbb{R}^N : \ x_N = \lambda\},$$

$$x^\lambda := (x', 2\lambda - x_N) \text{ (the reflection of } x \text{ with respect to } T_\lambda).$$

For a positive, bounded, classical solution $u$ of problem (3), we also set

$$\begin{matrix} u_\lambda(x) = u(x^\lambda) \\ w_\lambda(x) = u_\lambda(x) - u(x) \end{matrix} \qquad x \in \mathbb{R}^N.$$

*Proof of Theorem 1.1.* Assume there exists a positive, bounded, classical solution $u$ of (3). The proof proceeds in two main stages: first we show that $u$ is increasing in the $x_N$ direction, that is, $w_\lambda \geq 0$ in $\Sigma_\lambda$ for every $\lambda \in \mathbb{R}$. Then we will prove that this is impossible by a simple principal eigenvalue argument. All this will be accomplished in a series of steps.

**Step 1**. $w_\lambda \geq 0$ in $\Sigma_\lambda$ when $\lambda \leq 0$.

By contradiction let us suppose that there exists $\lambda \leq 0$ such that $w_\lambda < 0$ somewhere in $\Sigma_\lambda$. Then we can define the nonempty open set

$$D_\lambda = \{x \in \Sigma_\lambda : \ w_\lambda(x) < 0\}, \tag{5}$$

and the function

$$v_\lambda = w_\lambda \chi_{D_\lambda} \leq 0. \tag{6}$$

Observe that $x_N^\lambda > x_N$ when $x \in \Sigma_\lambda$, so that the monotonicity of both $h$ and $f$ and the nonnegativity of $f$ give, for $x \in D_\lambda$:

$$(-\Delta)^s w_\lambda(x) \geq h(x_N)(f(u^\lambda(x)) - f(u(x))) \geq 0, \tag{7}$$

since $h(x_N) \leq 0$ for $x \in \Sigma_\lambda$ when $\lambda \leq 0$.

Arguing as in Lemma 5 of [4], we see that

$$(-\Delta)^s v_\lambda \geq (-\Delta)^s w_\lambda \geq 0 \quad \text{in } D_\lambda. \tag{8}$$

Since $v_\lambda = 0$ in $\mathbb{R}^N \setminus D_\lambda$, we may apply the maximum principle for open sets contained in a half-space (Lemma 4 in [4]) to obtain that $w_\lambda \geq 0$ in $D_\lambda$, a contradiction. This contradiction shows that $D_\lambda$ is empty, so that $w_\lambda \geq 0$ in $\Sigma_\lambda$ when $\lambda \leq 0$.

**Step 2**. Setting

$$\lambda_* := \sup\{\lambda \in \mathbb{R} : \ w_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for every } \mu < \lambda\},$$

we have $\lambda_* = +\infty$.

Assume for a contradiction that $\lambda_* < +\infty$. We first observe that, by the definition of $\lambda_*$, there exists a sequence of positive numbers $\{\lambda_n\}$ such that $\lambda_n \downarrow \lambda_* \geq 0$ and points $x_n \in \Sigma_{\lambda_n}$ such that $w_{\lambda_n}(x_n) < 0$. From now on, we will use the notation $w_n$, $v_n$ and $D_n$ instead of $w_{\lambda_n}$, $v_{\lambda_n}$ and $D_{\lambda_n}$, where all these functions are obtained by just setting $\lambda = \lambda_n$ in the previous definitions. Let us define

$$W_n := D_n \cap \mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : \ x_N > 0\}$. We claim that $W_n \neq \emptyset$. To prove this, assume on the contrary that

$$D_n \subseteq \overline{\mathbb{R}^N_-},$$

where $\mathbb{R}^N_- = \{x \in \mathbb{R}^N : \ x_N < 0\}$. Arguing as in Step 1 we see that both (7) and (8) hold. Thus by the maximum principle in [4] we arrive at $v_n \geq 0$ in $D_n$, which is a contradiction. The contradiction shows that $W_n \neq \emptyset$.

To proceed further, choose points $x_n \in W_n$ such that

$$-v_n(x_n) \geq \frac{1}{2}\|v_n\|_{L^\infty(W_n)}. \tag{9}$$

Notice that, by definition, $0 < x_{n,N} \leq \lambda_n$, and we may assume by passing to a subsequence that

$$x_{n,N} \to a \in [0, \lambda_*]. \tag{10}$$

We define next the functions

$$\widetilde{u}_n(x) := u(x' + x'_n, x_N), \quad x \in \mathbb{R}^N,$$

which are positive solutions of problem (3). In addition, they verify $\|\widetilde{u}_n\|_{L^\infty(\mathbb{R}^N)} = \|u\|_{L^\infty(\mathbb{R}^N)}$. We also set, for $x \in \mathbb{R}^N$,

$$\begin{aligned} \widetilde{w}_n(x) &:= w_n(x' + x'_n, x_N) \\ \widetilde{v}_n(x) &:= v_n(x' + x'_n, x_N). \end{aligned}$$

Observe that $\widetilde{v}_n = \widetilde{w}_n \chi_{\widetilde{D}_n}$, where $\widetilde{D}_n = \{x \in \Sigma_{\lambda_n} : \widetilde{w}_n(x) < 0\}$.

Since $x_n \in W_n$, it is easily seen that

$$z_n := (0, x_{n,N}) \in \widetilde{W}_n := \widetilde{D}_n \cap \mathbb{R}^N_+. \tag{11}$$

Moreover, by (9), we also have

$$-\widetilde{v}_n(z_n) \geq \frac{1}{2}\|\widetilde{v}_n\|_{L^\infty(\widetilde{W}_n)}, \tag{12}$$

and

$$z_n \to z_0 = (0, a) \in \Sigma_{\lambda_*} \setminus \mathbb{R}^N_-, \tag{13}$$

owing to (10). Our next intention is to obtain an integral inequality involving the $L^\infty$ norm of $\widetilde{v}_n$ in $\widetilde{W}_n$, in the spirit of [24]. Arguing as in Lemma 5 in [4] we deduce that

$$\begin{aligned} (-\Delta)^s \widetilde{v}_n &\geq h(x_N)(f(\widetilde{u}_n^{\lambda_n}) - f(\widetilde{u}))\chi_{\widetilde{W_n}} \\ &\geq a_n \chi_{\widetilde{W_n}} \widetilde{v}_n \end{aligned} \qquad \text{in } \Sigma_{\lambda_n}$$

in the viscosity sense, where, since $x_N \leq \lambda_* + 1$,

$$a_n(x) := h(\lambda_* + 1)\frac{f(\widetilde{u}_n^{\lambda_n}) - f(\widetilde{u}_n)}{\widetilde{u}_n^{\lambda_n} - \widetilde{u}_n}, \quad x \in \mathbb{R}^N. \tag{14}$$

Proceeding now as in Lemma 6 in [4], we obtain

$$\widetilde{v}_n(x) \geq \int_{\widetilde{W}_n} G_n(x,y) a_n(y) \widetilde{v}_n(y)\, dy, \quad x \in \Sigma_{\lambda_n}. \tag{15}$$

Here $G_n(x,y)$ stands for the Green's function in the half-space $\Sigma_{\lambda_n}$. Notice that, by means of a change of variables it is easily seen that

$$G_n(x,y) = G^+_\infty(x', \lambda_n - x_N, y', \lambda_n - y_N), \quad \text{for } x, y \in \Sigma_{\lambda_n}, \tag{16}$$

where $G^+_\infty$ stands for the Green's function in the "standard" half-space $\mathbb{R}^N_+$ (cf. [24]). Here and in what follows we are taking the liberty of expressing all Green's functions as depending on two variables $(x,y)$ or four variables $(x',x_N,y',y_N)$, hoping that no confusion arises. Taking $x=z_n\in\Sigma_{\lambda_n}$ in (15) and using (12) we arrive at

$$\frac{1}{2}\|\widetilde{v}_n\|_{L^\infty(\widetilde{W}_n)} \le \|\widetilde{v}_n\|_{L^\infty(\widetilde{W}_n)} \int_{\widetilde{W}_n} G_n(z_n,y)a_n(y)\,dy.$$

Since we are assuming that the norms $\|\widetilde{v}_n\|_{L^\infty(\widetilde{W}_n)}$ are nonzero, we deduce that

$$\frac{1}{2} \le \int_{\widetilde{W}_n} G_n(z_n,y)a_n(y)\,dy. \tag{17}$$

Our ultimate aim is to show that this inequality is impossible.

Observe that the sequence $\{u_n\}$ is uniformly bounded and every $u_n$ is a positive solution of (3). Then, with the use of standard regularity (cf. [12, 36]) and by a diagonal argument, we may assume by passing to a subsequence that, for every $\beta\in(0,1)$,

$$\widetilde{u}_n \to \widetilde{u} \text{ in } \mathcal{C}^{2s+\beta}_{\mathrm{loc}}(\mathbb{R}^N),$$

where $\widetilde{u}$ is a nonnegative solution of (3). By the strong maximum principle, we may ensure that either $\widetilde{u}>0$ or $\widetilde{u}\equiv 0$ in $\mathbb{R}^N$. We have to analyze these two cases separately.

**Case (a).** $\widetilde{u}>0$ in $\mathbb{R}^N$. We claim that

$$\widetilde{w}_{\lambda_*} := \widetilde{u}^{\lambda_*} - \widetilde{u} > 0, \quad \text{in } \Sigma_{\lambda_*}. \tag{18}$$

Indeed, it is clear that $\widetilde{w}_{\lambda_*}\ge 0$ in $\Sigma_{\lambda_*}$. So, suppose that there exists $x_0\in\Sigma_{\lambda_*}$ such that $\widetilde{w}_{\lambda_*}(x_0)=0$. Then, as in (7),

$$(-\Delta)^s\widetilde{w}_{\lambda_*}(x_0) \ge h(x_{0,N})(f(\widetilde{u}^{\lambda_*}(x_0)) - f(\widetilde{u}(x_0))) = 0,$$

where $x_{0,N}$ is the last component of the point $x_0$. On the other hand, using the fact that $\widetilde{w}_{\lambda_*}$ is antisymmetric, it follows that

$$\begin{aligned} 0 \le (-\Delta)^s\widetilde{w}_{\lambda_*}(x_0) &= -\left(\int_{\Sigma_{\lambda_*}} + \int_{\Sigma^c_{\lambda_*}}\right)\frac{\widetilde{w}_{\lambda_*}(y)}{|x_0-y|^{N+2s}}dy \\ &= -\int_{\Sigma_{\lambda_*}} \widetilde{w}_{\lambda_*}(y)\left(\frac{1}{|x_0-y|^{N+2s}} - \frac{1}{|x_0-y^{\lambda_*}|^{N+2s}}\right)dy, \end{aligned}$$

where we have made the change of variables $y\to y^{\lambda_*}$ in the integral taken in $\Sigma^c_{\lambda_*}$. Now, for $y\in\Sigma_{\lambda_*}$ we always have $|x_0-y|\le|x_0-y^{\lambda_*}|$, so that the integrand above is nonnegative, and we deduce that $\widetilde{w}_{\lambda_*}=0$ in $\mathbb{R}^N$ (that is, $\widetilde{u}$ is symmetric with respect to the hyperplane $T_{\lambda_*}$). This is impossible since, taking $\widetilde{x}\in\Sigma_{\lambda_*}$ with $\widetilde{x}_N<0<2\lambda_*-\widetilde{x}_N$ and $h(2\lambda_*-\widetilde{x}_N)>0$ we obtain

$$0 = (-\Delta)^s\widetilde{w}_{\lambda_*}(\widetilde{x}) = f(\widetilde{u}(\widetilde{x}))(h(2\lambda_*-\widetilde{x}_N) - h(\widetilde{x}_N)) > 0,$$

which is a contradiction. This contradiction shows (18). Notice that, for the problem at hand, to prove that there cannot exist symmetric solutions with respect to hyperplanes $T_\lambda$, the presence of the function $h$ is crucial, contrary to what happens in the case of a half-space with zero Dirichlet condition, treated in [4].

Next we observe that by our choice of $z_n$, it follows that $\widetilde{w}^{\lambda_*}(z_0)=0$. Since we have just shown the positivity of $\widetilde{w}^{\lambda_*}$ in $\Sigma_{\lambda_*}$, we have $z_0\in T_{\lambda_*}$, that is, $a=\lambda_*$ in (13). Let us show that this contradicts (17). Indeed, using that the coefficients $a_n$

are uniformly bounded by the Lipschitz constant $L$ of $f$, and $\widetilde{W}_n \subset \Sigma_{\lambda_n} \cap \mathbb{R}^N_+$, we get from (17) that:

$$\frac{1}{2} \leq L \int_{\Sigma_{\lambda_n} \cap \mathbb{R}^N_+} G_n(z_n, y) dy. \tag{19}$$

Taking into account the characterization (16) of $G_n$ and performing a change of variables in (19), it follows that

$$\begin{aligned} \frac{1}{2} &\leq \int_{\Sigma_{\lambda_n} \cap \mathbb{R}^N_+} G^+_\infty(0, \lambda_n - z_{n,N}, y) dy \\ &\leq \int_{\Sigma_{\lambda_*+1} \cap \mathbb{R}^N_+} G^+_\infty(0, \lambda_n - z_{n,N}, y) dy. \end{aligned} \tag{20}$$

However, using part (c) of Lemma 7 in [4], we deduce that the last integral converges to zero, since $\lambda_n - z_{n,N} \to 0$. This contradiction rules out the case $\widetilde{u} > 0$ in $\mathbb{R}^N$.

**Case (b).** $\widetilde{u} \equiv 0$ in $\mathbb{R}^N$. Thanks to our hypotheses on $f$ and using that $u_n \to 0$ uniformly on compact sets of $\mathbb{R}^N$, we deduce that $a_n \to 0$ as $n \to \infty$, uniformly on compact sets of $\mathbb{R}^N$. We will prove that this entails the convergence of the right hand side of (17) to zero when $n \to \infty$, obtaining the same type of contradiction as before.

In fact, using (17) and (16), we see that

$$\begin{aligned} \frac{1}{2} &\leq \int_{\Sigma_{\lambda_n} \cap \mathbb{R}^N_+} G_n(z_n, y) a_n(y) dy \\ &= \int_{\Sigma_{\lambda_n} \cap \mathbb{R}^N_+} G^+_\infty(0, \lambda_n - z_{n,N}, y', \lambda_n - y_N) a_n(y', \lambda_n - y_N) dy \\ &\leq \|a_n\|_{L^\infty(B^+_R)} \int_{\Sigma_{\lambda_*+1} \cap B^+_R} G^+_\infty(0, \lambda_n - z_{n,N}, y', \lambda_n - y_N) dy \\ &+ L \int_{\Sigma_{\lambda_*+1} \cap (B^+_R)^c} G^+_\infty(0, \lambda_n - z_{n,N}, y', \lambda_n - y_N) dy, \end{aligned} \tag{21}$$

where $B^+_R = B_R \cap \mathbb{R}^N_+$. A minor variation of parts (a) and (b) in Lemma 7 of [4] gives

$$\lim_{R \to +\infty} \int_{\Sigma_{\lambda_*+1} \cap (B^+_R)^c} G^+_\infty(0, \lambda_n - z_{n,N}, y', \lambda_n - y_N) dy = 0,$$

uniformly in $n \in \mathbb{N}$, while

$$\int_{\Sigma_{\lambda_*+1} \cap B^+_R} G^+_\infty(0, \lambda_n - z_{n,N}, y', \lambda_n - y_N) dy \leq C$$

for fixed $R$. Thus we can fix $R > 0$ such that the last term in (21) is less than $\frac{1}{4}$, say, to get

$$\frac{1}{4} \leq C \|a_n\|_{L^\infty(B^+_R)},$$

which is a contradiction since $a_n \to 0$ uniformly on compact sets.

**Step 3.** Completion of the proof.

By Step 2, we deduce that $u$ is nondecreasing in the $x_N$ direction. Next, for every $k \in \mathbb{N}$, let $B_k = B_1(ke_N)$ be the ball of center $ke_N$ and radius 1. By the monotonicity of $u$ shown above, we obtain, for every $k \geq 1$:

$$u(x) \geq u_1 = \min_{B_1} u > 0, \quad x \in B_k.$$

Setting $m_0 = \inf_{u_1 \le t \le \|u\|_{L^\infty(\mathbb{R}^N)}} \frac{f(t)}{t} > 0$, we see that

$$(-\Delta)^s u \ge h(k-1) m_0 u \quad \text{in } B_k.$$

According to well-known properties of the principal eigenvalue (see for instance Theorem 1.1 of [33]), we deduce that

$$h(k-1)m_0 \le \lambda_1(B_k) = \lambda_1(B_1),$$

and we arrive at a contradiction by letting $k \to +\infty$, since we are assuming that $h$ goes to infinity at infinity. The proof is concluded. □

3. **A priori bounds.** In this section we will show that all solutions of (4) are a priori bounded, provided that $p$ is subcritical and $a$ verifies hypotheses (A). The technique is the standard one introduced in [29], with the adaptations to the non-local setting provided by [5]. It relies in the Liouville theorems obtained in [37] and [17], the monotonicity of solutions in half-spaces proved in [4] and our new Theorem 1.1.

**Theorem 3.1.** *Assume $s \in (0,1)$, $p > 1$ and let $a \in C^\alpha(\overline{\Omega})$ verify hypotheses* (A). *If $N \le 2s$ or $N > 2s$ and $p$ is such that*

$$p < \frac{N+2s}{N-2s},$$

*then for every $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_2 > \lambda_1$ there exists $M = M(\lambda_1, \lambda_2)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le M$$

*for every positive, classical solution $u$ of* (4) *with $\lambda \in [\lambda_1, \lambda_2]$.*

*Proof.* Assume on the contrary the existence of an interval $[\lambda_1, \lambda_2]$ and sequences $\{\lambda_k\} \subset [\lambda_1, \lambda_2]$ and $u_k$ such that $u_k$ is a positive, classical solution of (4) with $\lambda = \lambda_k$ and

$$M_k := \|u_k\|_{L^\infty(\Omega)} \to +\infty.$$

For every $k$, take a point $x_k \in \Omega$ where $u_k$ achieves its maximum. We may assume $x_k \to x_0 \in \overline{\Omega}$. Now, three cases are possible:

(a) $x_0 \in \Omega \setminus \Gamma$;

(b) $x_0 \in \Gamma$;

(c) $x_0 \in \partial\Omega$.

The cases (a) and (c) are to some extent standard, and only the remaining case (b) deserves special attention. Let us see that we reach a contradiction assuming each one of them.

(a) Let $\mu_k = M_k^{-\frac{p-1}{2s}} \to 0$ and introduce the functions

$$v_k(y) = \frac{u_k(x_k + \mu_k y)}{M_k}, \quad y \in \Omega_k,$$

where

$$\Omega_k := \{y \in \mathbb{R}^N : \ x_k + \mu_k y \in \Omega\}. \tag{22}$$

It can be easily seen that $\Omega_k \to \mathbb{R}^N$ as $k \to +\infty$. On the other hand, it is clear that $v_k$ verifies $0 < v_k \le 1$ and $v_k(0) = 1$. Moreover, a short calculation shows that $v_k$ is a solution of the problem

$$\begin{cases} (-\Delta)^s v_k = \lambda_k \mu_k^{2s} v_k + a_k(y) v_k^p & \text{in } \Omega_k \\ v_k = 0 & \text{in } \mathbb{R}^N \setminus \Omega_k, \end{cases}$$

where $a_k(y) = a(x_k + \mu_k y)$, $y \in \Omega_k$. Thus we may use standard regularity (cf. [12, 36]) to obtain that, through a subsequence, $v_k \to v$ uniformly in compact sets of $\mathbb{R}^N$, where $v$ is a nonnegative, bounded, viscosity solution of

$$(-\Delta)^s v = a(x_0) v^p \quad \text{in } \mathbb{R}^N.$$

By the strong maximum principle and regularity theory we actually have that $v$ is a positive, classical solution.

By our hypotheses in this case, we know that $a(x_0) \neq 0$. If $a(x_0) < 0$, we observe that $v(0) = 1$ while $v \leq 1$. Thus $v$ attains a global maximum at $y = 0$ and $(-\Delta)^s v(0) = a(x_0) < 0$, which is impossible. If, on the contrary, $a(x_0) > 0$, we reach a contradiction when $N \leq 2s$ by Theorem 1.2 in [26], and when $N > 2s$ and $p$ is subcritical by Theorem 4 in [37] (see also [17]).

(b) We may assume with no loss of generality that the outward normal to $\partial\Omega^+$ at $x_0$ is $\nu(x_0) = -e_N$. Let $d_k = d(x_k)$ and recall that we are denoting $d(x) = \operatorname{dist}(x, \Gamma)$. Define

$$\eta_k = M_k^{-\frac{p-1}{2s+\gamma}} \to 0$$

and

$$\widetilde{v}_k(y) = \frac{u_k(x_k + \eta_k y)}{M_k}, \quad y \in \widetilde{\Omega}_k,$$

where $\widetilde{\Omega}_k := \{y \in \mathbb{R}^N : \ x_k + \eta_k y \in \Omega\}$. Observe that in the present situation $\widetilde{\Omega}_k \to \mathbb{R}^N$ as $k \to +\infty$. It is not hard to see that $\widetilde{v}_k$ verifies the equation

$$(-\Delta)^s \widetilde{v}_k = \lambda_k \eta_k^{2s} \widetilde{v}_k + \widetilde{a}_k(y) \widetilde{v}_k^p \quad \text{in } \widetilde{\Omega}_k, \tag{23}$$

where $\tilde{a}_k(y) = \eta_k^{-\gamma} a(x_k + \eta_k y)$. It is also immediate that $0 < \widetilde{v_k} \leq 1$ and $\widetilde{v}_k(0) = 1$.

Observe next that, by the smoothness assumption on $\Gamma$, we can write

$$d(x_k + \eta_k y) = d_k + \eta_k y \cdot \nu(\xi_k) + o(\eta_k),$$

where $\xi_k$ is the projection of $x_k$ onto $\Gamma$. Therefore, by hypotheses (A),

$$\widetilde{a}_k(y) = \begin{cases} b_1(x_k + \eta_k y) \left| \frac{d_k}{\eta_k} + \nu(\xi_k) \cdot y + o(1) \right|^\gamma, & \text{if } x_k + \eta_k y \in \Omega^+ \\ -b_2(x_k + \eta_k y) \left| \frac{d_k}{\eta_k} + \nu(\xi_k) \cdot y + o(1) \right|^\gamma, & \text{if } x_k + \eta_k y \in \Omega^-. \end{cases}$$

There are two further possibilities to consider:

(b1) Passing to a subsequence, $d_k/\eta_k \to d \geq 0$. Then

$$\widetilde{a}_k(y) \to b_1(x_0)(y_N - d)_+^\gamma - b_2(x_0)(y_N - d)_-^\gamma,$$

and we are denoting, for real $t$, $t_+ = \max\{t, 0\}$, $t_- = \max\{-t, 0\}$. Since the sequence $\widetilde{v}_k$ is bounded, we may pass to the limit as before to obtain that $\widetilde{v}_k \to \widetilde{v}$ locally uniformly in $\mathbb{R}^N$, where $\widetilde{v}$ is nonnegative, bounded and verifies $\widetilde{v}(0) = 1$. Moreover, passing to the limit in (23) we see that

$$(-\Delta)^s \widetilde{v} = h(y_N) \widetilde{v}^p \quad \text{in } \mathbb{R}^N, \tag{24}$$

in the viscosity sense, with $h(t) = b_1(x_0)(t-d)_+^\gamma - b_2(x_0)(t-d)_-^\gamma$, $t \in \mathbb{R}$. With a further translation in the $y_N$ direction we may suppose $d = 0$. Moreover, by regularity we actually find that $\widetilde{v}$ is a classical solution of (24), which contradicts Theorem 1.1.

(b2) Passing to a subsequence, $d_k/\eta_k \to +\infty$. This assumption implies that for large $k$ all points $x_k$ remain inside $\Omega^+$ or $\Omega^-$. Assume first that they all lie in $\Omega^+$. Let $\beta_k = (\eta_k/d_k)^{\frac{\gamma}{2s}} \to 0$ and introduce the functions

$$\widetilde{w}_k(y) = \widetilde{v}_k(\beta_k y),$$

which are easily seen to verify

$$(-\Delta)^s \widetilde{w}_k = \lambda_k(\beta_k\eta_k)^{2s}\widetilde{w}_k + \beta_k^{2s}\widetilde{a}_k(\beta_k y)\widetilde{w}_k^p$$

in $\{y \in \mathbb{R}^N : \ \beta_k y \in \widetilde{\Omega}_k\}$. Moreover:

$$\beta_k^{2s}\widetilde{a}_k(\beta_k y) = b_1(x_k + \beta_k\eta_k y)(1 + \beta_k^{1+\frac{2s}{\gamma}}\nu(\xi_k)\cdot y + o(\beta_k^{\frac{2s}{\gamma}}))^{\gamma}.$$

Thus we see that $\widetilde{w}_k \to \widetilde{w}$, which is a bounded, nontrivial solution of

$$(-\Delta)^s\widetilde{w} = b_1(x_0)\widetilde{w}^p \quad \text{in } \mathbb{R}^N,$$

a contradiction to our hypotheses as in case (a), since $b_1(x_0) > 0$. When the points $x_k$ lie in $\Omega^-$ we argue exactly in the same way and we obtain a solution of the same equation with $b_1(x_0)$ replaced by $-b_2(x_0)$. The contradiction follows also as in case (a).

(c) Assume again with no loss of generality $\nu(x_0) = -e_N$, where this time $\nu$ stands for the outward unit normal to $\partial\Omega$. Denote also $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. There are two cases to consider: by passing to a further subsequence, either $d_\Omega(x_k)\mu_k^{-1} \to +\infty$ or $d_\Omega(x_k)\mu_k^{-1} \to d \geq 0$, where $\mu_k = M_k^{-\frac{p-1}{2s}} \to 0$, as in case (a). In the former one we argue exactly as in case (a) to reach a contradiction. It is to be noted that the set $\Omega_k$ given in (22) verifies $\Omega_k \to \mathbb{R}^N$ with this assumption.

In the latter we argue as in [5]. Consider the projections $\tau_k$ of $x_k$ onto $\partial\Omega$, and introduce the functions

$$w_k(y) = \frac{u_k(\tau_k + \mu_k y)}{M_k}, \quad y \in D_k,$$

where

$$D_k = \{y \in \mathbb{R}^N : \ \tau_k + \mu_k y \in \Omega\}.$$

It is immediate that $0 \in \partial D_k$ and $D_k \to \mathbb{R}^N_+$ as $k \to +\infty$. Moreover, $w_k$ solves the equation

$$\begin{cases} (-\Delta)^s w_k = \lambda_k\mu_k^{2s}w_k + \bar{a}_k(y)w_k^p & \text{in } D_k \\ w_k = 0 & \text{in } \mathbb{R}^N \setminus D_k, \end{cases} \tag{25}$$

where now $\bar{a}_k(y) = a(\tau_k + \mu_k y)$.

Next consider the point $y_k = (x_k - \tau_k)/\mu_k \in D_k$. It is clear that $w_k(y_k) = 1$, and that $|y_k| = d_\Omega(x_k)\mu_k^{-1} \to d$. We claim that $d > 0$.

Indeed, by Lemma 6 in [5], we can choose $\theta \in (s, 2s)$ and $C > 0$, $\delta > 0$ such that

$$w_k(y) \leq C d_k(y)^{2s-\theta}, \quad \text{when } d_k(y) \leq \delta,$$

where $d_k(y) := \operatorname{dist}(y, \partial D_k)$. Taking into account that $|y_k| \geq d_k(y_k)$ because $0 \in \partial D_k$, we obtain

$$1 = w_k(y_k) \leq C|y_k|^{2s-\theta},$$

provided that $d_k(y_k) \leq \delta$, which shows that $|y_k|$ is bounded from below. This entails $d > 0$. Passing to another subsequence, we have $y_k \to y_0$ with $|y_0| = d > 0$, therefore $y_0 \in \mathbb{R}^N_+$.

Arguing as in part (a) we obtain that $w_k \to w$ locally uniformly on compact sets of $\mathbb{R}^N_+$, where $w$ verifies $0 \leq w \leq 1$, $w(y_0) = 1$ and $w(y) \leq C y_N^{2s-\theta}$ for $y_N \leq \delta$. Thus

$w$ is continuous in $\mathbb{R}^N$ and vanishes in $\mathbb{R}^N \setminus \mathbb{R}^N_+$. Then, we can pass to the limit in (25) to obtain that $w$ is a nonnegative, bounded, viscosity solution of the problem

$$\begin{cases} (-\Delta)^s w = a(x_0) w^p & \text{in } \mathbb{R}^N_+ \\ w = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases}$$

By regularity theory and the maximum principle it actually follows that $w \in C^\infty(\mathbb{R}^N_+)$ is positive in $\mathbb{R}^N_+$. Moreover, $w$ attains a global maximum at $y_0$, thus $\nabla w(y_0) = 0$. This contradicts Theorem 1 in [4], which ensures that $\frac{\partial w}{\partial y_N} > 0$ in $\mathbb{R}^N_+$.

Summing up, our assumption $M_k \to +\infty$ leads to a contradiction in all cases, and this concludes the proof of the theorem. □

4. **Proof of Theorem 1.2.** In this section we will give the proof of our existence result, Theorem 1.2. The main tool is bifurcation theory, with the use of the a priori bounds given in Theorem 3.1.

Instead of working with (4), it is more convenient to deal with its odd extension, namely

$$\begin{cases} (-\Delta)^s u = \lambda u + a(x)|u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{26}$$

From now on, by a solution of (26) we will mean a pair $(\lambda, u)$. Also, it will slightly simplify our proofs to consider solutions in the viscosity sense. As we have already remarked, viscosity solutions are indeed classical, so this will not suppose any loss in generality. Thus our natural space will be $\mathbb{R} \times C(\overline{\Omega})$.

Problem (26) always possesses the branch of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$, but we are only interested in nontrivial solutions. Our purpose is to obtain positive solutions which bifurcate from the branch of trivial solutions at the value $(\lambda_1(\Omega), 0)$, where $\lambda_1(\Omega)$ stands for the first eigenvalue of $(-\Delta)^s$ in $\Omega$.

To make this more precise, we recall that a continuum $\mathcal{C} \subset \mathbb{R} \times C(\overline{\Omega})$ is a closed connected set. We will say that $\mathcal{C}$ is a continuum of positive solutions which bifurcates from $(\lambda_1(\Omega), 0)$ if $(\lambda_1(\Omega), 0) \in \mathcal{C}$ and $\mathcal{C} \setminus \{(\lambda_1(\Omega), 0)\}$ consists of positive solutions only. The next result is a consequence of the celebrated global bifurcation theorem of Rabinowitz.

**Lemma 4.1.** *Assume $p > 1$ and $a \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Then there exists an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times C(\overline{\Omega})$ of positive solutions of (26) bifurcating from $(\lambda_1(\Omega), 0)$.*

*Proof.* For $h \in C(\overline{\Omega})$, consider the boundary value problem

$$\begin{cases} (-\Delta)^s v = h(x) & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{27}$$

It is well-known that there exists a unique viscosity solution $v \in C(\overline{\Omega})$ of (27). By Proposition 1.2 in [35] we also have $v \in C^s(\overline{\Omega})$ and

$$\|v\|_{C^s(\overline{\Omega})} \le C \|h\|_{L^\infty(\Omega)},$$

for some positive $C$ independent of $h$. In this way, setting $v = Kh$, we define a compact, linear operator $K : C(\overline{\Omega}) \to C(\overline{\Omega})$.

Problem (26) is equivalent to the fixed point equation

$$u = \lambda K u + K(a|u|^{p-1}u) \tag{28}$$

in $C(\overline{\Omega})$ (with a slight abuse of notation, we are still denoting by $a|u|^{p-1}u$ the Nemytskii operator of the function $a(x)|u|^{p-1}u$ defined from $C(\overline{\Omega})$ to $C(\overline{\Omega})$). Denote

$$\mathcal{S} = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : \ u \text{ is a nontrivial solution of } (28)\}.$$

We can apply Theorem 1.3 in [34] to deduce that $\mathcal{S}$ contains a continuum $\mathcal{C}$ such that $(\lambda_1(\Omega), 0) \in \mathcal{C}$ and it is either unbounded or contains a point $(\mu, 0)$ with $\mu \neq \lambda_1(\Omega)$.

Next we argue as in [21]. Denote by $\mathcal{P}$ the set of functions in $C(\overline{\Omega})$ which do not change sign (observe that this is a closed set). We claim that:

$$\mathcal{C} \subset \mathbb{R} \times \mathcal{P}. \tag{29}$$

We begin by proving the existence of a small $\varepsilon > 0$ such that all solutions $(\lambda, u) \in \mathcal{C}$ with $\lambda \in B_\varepsilon(\lambda_1(\Omega)) \subset \mathbb{R}$ and $u \in B_\varepsilon(0) \subset C(\overline{\Omega})$ belong to $\mathbb{R} \times \mathcal{P}$. Indeed, suppose on the contrary that there exist sequences $\lambda_n \to \lambda_1(\Omega)$, $u_n \to 0$ such that $u_n$ is a changing sign solution of (28) with $\lambda = \lambda_n$. Let

$$v_n = \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}}.$$

It is easily seen that

$$v_n = \lambda_n K v_n + \|u_n\|_{L^\infty(\Omega)}^{p-1} K(a|v_n|^{p-1}v_n), \tag{30}$$

and hence by the compactness of $K$ we see that there exists $v \in C(\overline{\Omega})$ such that $v_n \to v$ uniformly in $\overline{\Omega}$ and $\|v\|_{L^\infty(\Omega)} = 1$. Passing to the limit in (30), it is clear that $v$ verifies $v = \lambda_1(\Omega) K v$, that is, $v$ is an eigenfunction of $(-\Delta)^s$ associated to $\lambda_1(\Omega)$. Hence we may assume with no loss of generality that $v \geq 0$ in $\Omega$ and the strong maximum principle and Hopf's principle imply then that $v > 0$ in $\Omega$ and there exists $c > 0$ such that

$$v(x) \geq c d_\Omega(x)^s \quad \text{for } x \in \Omega, \tag{31}$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ (cf. Proposition 2.7 in [10]). Moreover, it is easily seen that

$$\begin{cases} (-\Delta)^s(v_n - v) = \lambda_n v_n - \lambda_1(\Omega) v + a(x)|u_n|^{p-1} v_n & \text{in } \Omega \\ v_n - v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{32}$$

Since the right-hand side of the equation in (32) converges uniformly to zero in $\overline{\Omega}$, we may employ Theorem 1.2 in [35] to deduce that

$$\left\| \frac{v_n - v}{d_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \to 0$$

for some $\alpha \in (0,1)$. It then follows from (31) that $v_n > 0$ in $\Omega$ if $n$ is large, against the assumption. This shows that $\mathcal{C} \cap (B_\varepsilon(\lambda_1(\Omega)) \times B_\varepsilon(0)) \subset \mathbb{R} \times \mathcal{P}$ for some small $\varepsilon > 0$.

To prove that $\mathcal{C} \subset \mathbb{R} \times \mathcal{P}$, it is enough to show that $\mathcal{C} \cap (\mathbb{R} \times \mathcal{P}) \cap (\mathbb{R} \times \overline{\mathcal{P}^c}) = \emptyset$. If this were proved, we would have $\mathcal{C} = (\mathcal{C} \cap (\mathbb{R} \times \mathcal{P})) \cup (\mathcal{C} \cap (\mathbb{R} \times \overline{\mathcal{P}^c}))$, where these sets are disjoint and closed. Since we have shown that the first one is nonempty, the connectedness of $\mathcal{C}$ implies that the second one is empty, showing that $\mathcal{C} \subset \mathbb{R} \times \mathcal{P}$.

Thus let $(\lambda_0, u_0) \in \mathcal{C} \cap (\mathbb{R} \times \mathcal{P}) \cap (\mathbb{R} \times \overline{\mathcal{P}^c})$. By the first part of the proof, we have $(\lambda_0, u_0) \neq (\lambda_1(\Omega), 0)$. Since $u_0 \in \mathcal{P}$, we may assume without loss of generality that $u_0 \geq 0$ in $\overline{\Omega}$. By the strong maximum principle, either $u_0 \equiv 0$ or $u_0 > 0$ in $\Omega$. The first possibility leads, reasoning as above, to $\lambda_0 = \lambda_1(\Omega)$, which is impossible. Thus the second possibility holds and Hopf's principle gives in addition $u_0(x) \geq c d_\Omega(x)^s$ in $\Omega$ for some $c > 0$.

On the other hand, since $(\lambda_0, u_0) \in \mathbb{R} \times \overline{\mathcal{P}^c}$, there exists a sequence $(\lambda_n, u_n) \subset \mathcal{C}$ such that $\lambda_n \to \lambda_0$, $u_n \to u_0$ and $u_n$ changes sign. We can argue as before to obtain that actually

$$\left\| \frac{u_n - u_0}{d_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \to 0$$

as $n \to +\infty$, which implies that $u_n > 0$ in $\Omega$ for large $n$, a contradiction. The contradiction shows that $\mathcal{C} \cap (\mathbb{R} \times \mathcal{P}) \cap (\mathbb{R} \times \overline{\mathcal{P}^c}) = \emptyset$, thus establishing (29).

As a consequence of (29), we obtain that $\mathcal{C}$ is unbounded. Otherwise, we would have $(\mu, 0) \in \mathcal{C}$ for some $\mu \neq \lambda_1(\Omega)$. It is then seen much as before that $\mu$ is an eigenvalue of $(-\Delta)^s$ associated to a one-signed eigenfunction, which is impossible since $\mu \neq \lambda_1(\Omega)$. Thus $\mathcal{C}$ is unbounded in $\mathbb{R} \times C(\overline{\Omega})$.

Finally, let $\mathcal{C}^\pm = \{(\lambda, u) \in \mathcal{C} : \pm u > 0 \text{ in } \Omega\}$. It is clear that $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint, connected sets and $\mathcal{C} = \mathcal{C}^+ \cup \{(\lambda_1(\Omega), 0\} \cup \mathcal{C}^-$. Moreover, one of them has to be unbounded. If $\mathcal{C}^+$ is unbounded, we just set $\mathcal{C}_0 = \mathcal{C}^+ \cup \{(\lambda_1(\Omega), 0)\}$. Otherwise, we take $\mathcal{C}_0 = \{(\lambda, -u) : (\lambda, u) \in \mathcal{C}^-\} \cup \{(\lambda_1(\Omega), 0)\}$. In either case, $\mathcal{C}_0$ has the desired properties. This concludes the proof. □

Now we can proceed to the proof of Theorem 1.2.

*Proof of Theorem 1.2.* We begin by showing the nonexistence of positive solutions of (4) when $\lambda$ is large. This easily follows by noticing that, if $u$ is a positive solution of (4), then

$$(-\Delta)^s u \geq \lambda u \quad \text{in } \Omega^+.$$

It is well known, since $u > 0$ in $\Omega^+$ and $u \geq 0$ in $\mathbb{R}^N$, that this implies

$$\lambda < \lambda_1(\Omega^+).$$

Thus we may define

$$\Lambda = \sup\{\lambda \in \mathbb{R} : \text{ there exists a positive solution of (4)}\}.$$

By definition there are no positive solutions of (4) when $\lambda \geq \Lambda$.

Next let us show that there exists a positive solution of (4) for every $\lambda < \lambda_1(\Omega)$. By Lemma 4.1, there exists an unbounded continuum of positive solutions $\mathcal{C}_0$ of (4) bifurcating from $(\lambda_1(\Omega), 0)$. Let

$$\mu = \sup\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0 \text{ for some } u\}.$$

Since $\mathcal{C}_0$ bifurcates from $(\lambda_1(\Omega), 0)$, it is clear that $\mu \geq \lambda_1(\Omega)$. Now, we claim that there exists a positive solution of (4) for every $\lambda < \lambda_1(\Omega)$, which will conclude the proof of the theorem. It is here where our a priori bounds are handy.

Indeed, assume that for some $\lambda_0 < \lambda_1(\Omega)$ problem (4) does not admit any such solution. Applying again Theorem 3.1 we deduce the existence of $M_0$ such that every positive solution of (4) with $\lambda_0 \leq \lambda \leq \mu$ verifies $\|u\|_{L^\infty(\Omega)} \leq M_0$. Since $\mathcal{C}_0$ is connected, it follows that

$$\mathcal{C}_0 \subset [\lambda_0, \mu] \times B_{M_0}(0),$$

which is impossible, since $\mathcal{C}_0$ is unbounded. This shows the claim. □

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