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# TCHAKALOFF'S THEOREM AND $K$-INTEGRAL POLYNOMIALS IN BANACH SPACES

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Abstract. Tchakaloff's theorem gives a quadrature formula for polynomials of a given degree with respect to a compactly supported positive measure which is absolutely continuous with respect to Lebesgue measure. We study the validity of two possible analogues of Tchakaloff's theorem in an infinite-dimensional Banach space $E$: a weak form valid when $E$ has a Schauder basis, and a stronger form requiring conditions on the support of the measure as well as on the space $E$.

## Introduction

Tchakaloff's theorem [14] asserts the existence of an exact quadrature formula with positive coefficients for polynomials of prescribed degree in $n$ real variables and with respect to a positive, compactly supported measure which is absolutely continuous with respect to Lebesgue $n$-volume measure. That is, if $\mu$ is such a measure whose support is $K \subset \mathbb{R}^n$, then there exist points $a_1, \ldots, a_m$ in $K$ and positive real numbers $c_1, \ldots, c_m$ such that

$$\int_K q(\gamma)\, d\mu(\gamma) = \sum_{i=1}^{m} c_i q(a_i)$$

for all polynomials $q \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $k$.

This theorem has been extended and generalised by several authors. In [10], Mysovskikh weakens the compactness requirement of $K$ (to closed), provided all moments of $\mu$ are convergent. Putinar [11] and Curto and Fialkow [4] give new proofs requiring convergent moments only up to degree $k+1$ and also give bounds on the number of nodes ($m$ in the formula above) required for the existence of such a quadrature formula. Other results regarding Tchakaloff's theorem may be found in the books [12] and [13].

Our main goal in this paper is to obtain an infinite-dimensional analogue of Tchakaloff's theorem. Throughout, $E$ will denote a real Banach space, and $E'$ its dual Banach space, with $\| \;\|_E$ and $\| \;\|_{E'}$ their norms, respectively. For simplicity of notation we let $w^*$ stand for the weak*-topology on $E'$. $K \subset E'$ will denote a $w^*$-compact subset of $E'$, and $B_{E'}$ the closed unit ball of $E'$. In this setting, a

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Tchakaloff-type theorem would say that given a positive measure $\mu$ on $K \subset E'$, the polynomial $P : E \longrightarrow \mathbb{R}$ given by

$$P(x) = \int_K \gamma(x)^k \, d\mu(\gamma)$$

may be written as an (infinite) sum $\sum_{i=1}^{\infty} c_i \gamma_i(x)^k$, with $c_i \in \mathbb{R}$ and $\gamma_i \in K$. The convergence of this series, however, may be interpreted in several ways. One is the pointwise convergence of the series; i.e. the series converges for each fixed $x \in E$. Another, stronger form is the convergence condition required in the definition of *nuclear* polynomial. We recall [7] that a polynomial $P : E \longrightarrow \mathbb{R}$ is said to be nuclear if

$$P(x) = \sum_{i=1}^{\infty} c_i \gamma_i(x)^k, \text{ with } c_i \in \mathbb{R} \text{ and } \gamma_i \in B_{E'},$$

with $\sum_{i=1}^{\infty} |c_i| \|\gamma_i\|_{E'}^k < \infty$.

In this second interpretation, Tchakaloff's theorem turns out to be false in general: indeed, when $K = B_{E'}$ polynomials given by measures in the sense that

$$P(x) = \int_{B_{E'}} \gamma(x)^k \, d\mu$$

are called *integral* polynomials, and it is well-known that not all integral polynomials over $\ell_1$ are nuclear [7, Chapter 2]. R. Alencar [1] gave a condition for equality of integral and nuclear polynomials, and the relationship between the two types of polynomials on different kinds of Banach spaces has been extensively studied [2], [3], [5].

In this article we consider both modes of convergence of the series $\sum_{i=1}^{\infty} c_i \gamma_i(x)^k$ and obtain conditions under which the respective analogues of Tchakaloff's theorem hold. We will call *weak form* the one corresponding to the pointwise convergence, and *strong form* the one requiring convergence as in the definition of nuclear polynomials.

In section 1 we study the validity of the weak form of the theorem and obtain such a theorem for Banach spaces $E$ with a Schauder basis. In section 2, we study the strong form of Tchakaloff's theorem. This leads us to define $K$-integral and $K$-nuclear polynomials and to seek conditions for their coincidence. All the results in section 2 are valid also in the complex case.

## 1. A weak form of Tchakaloff's theorem

Recall that given a Banach space $E$ with Schauder basis $\{e_j\}_{j \in \mathbb{N}}$, for any $x \in E$ there exists a unique sequence $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$ such that $x = \sum_{j=1}^{\infty} x_j \, e_j$. Also, there exists $C > 0$ such that the family of linear operators $\Pi_n : E \to E$ $(n \geq 1)$ defined by $\Pi_n(x) = \sum_{j=1}^{n} x_j \, e_j$ is uniformly bounded, i.e. $\|\Pi_n(x)\|_E \leq C \, \|x\|_E$, for all $x \in E$ and $n \in \mathbb{N}$. For convenience we consider $\Pi_0(x) = 0$.

**Lemma 1.1.** *Let $E$ be a Banach space with Schauder basis $\{e_j\}_{j \in \mathbb{N}}$. Given a $w^*$-compact set $K \subset E'$ and a positive Borel measure $\mu$ defined on $K$, then for all*

*$m,k\in\mathbb{N}$, there exist finite sets $\{b_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset\mathbb{R}^+$ and $\{\gamma_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset K$, for some $M(m,k)\leq\binom{m+k}{k}$, such that*

$$\int_K \gamma^r(\Pi_n(x))d\mu(\gamma)=\sum_{i=1}^{M(m,k)} b_i^{(m,k)}\ \left(\gamma_i^{(m,k)}(\Pi_n(x))\right)^r$$

*for all $n\leq m,\ r\leq k$ and $x\in E$.*

*Proof.* Given $x\in E$, let $\hat{x}$ be the canonical inclusion of $x$ into the bidual space $E''$. As usual, $\hat{x}(\gamma)=\gamma(x)$ for all $\gamma\in E'$. Let us define $\Phi_m:K\to\mathbb{R}^m$ by

$$\Phi_m(\gamma)=(\hat{e}_1(\gamma),\ldots,\hat{e}_m(\gamma))=(\gamma(e_1),\ldots,\gamma(e_m)).$$

Since the functionals $\{\hat{e}_j\}_{j\in\mathbb{N}}$ are $w^*$-continuous, $\Phi_m(K)$ is a compact set w.r.t. the Euclidean topology on $\mathbb{R}^m$. We consider the push-forward measure $\Phi_{m*}(\mu)$ defined on $\Phi_m(K)$ by $\Phi_{m*}(\mu)(\Delta)=\mu(\Phi_m^{-1}(\Delta))$. Note that

$$\int_K \gamma^r(\Pi_n(x))d\mu(\gamma)=\int_K\left[\gamma\left(\sum_{j=1}^n x_j\,e_j\right)\right]^r d\mu(\gamma)$$

$$=\int_K\left(\sum_{j=1}^n x_j\,\gamma(e_j)\right)^r d\mu(\gamma)$$

$$=\int_{\Phi_m(K)}\left(\sum_{j=1}^n x_j\,z_j\right)^r d\Phi_{m*}(\mu)(z_1,\ldots,z_m).$$

Since $\Phi_m(K)\subset\mathbb{R}^m$ is a compact set, by Tchakaloff's theorem [4], there exist $\{b_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset\mathbb{R}^+$ and $\{\mathrm{z}_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset\Phi_m(K)$ such that

$$\int_{\Phi_m(K)}\left(\sum_{j=1}^n x_j\,z_j\right)^r d\Phi_{m*}(\mu)=\sum_{i=1}^{M(m,k)} b_i^{(m,k)}\ \left(\sum_{j=1}^n(\mathrm{z}_i^{(m,k)})_j x_j\right)^r.$$

Since $\{\mathrm{z}_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset\Phi_m(K)$, we conclude the existence of linear functionals $\{\gamma_i^{(m,k)}\}_{i=1}^{M(m,k)}\subset K$ such that $\gamma_i^{(m,k)}(e_j)=(\mathrm{z}_i^{(m,k)})_j$. Hence,

$$\int_K \gamma^r(\Pi_n(x))d\mu(\gamma)=\sum_{i=1}^{M(m,k)} b_i^{(m,k)}\ \left(\sum_{j=1}^n\gamma_i^{(m,k)}(e_j)x_j\right)^r$$

$$=\sum_{i=1}^{M(m,k)} b_i^{(m,k)}\ \left(\gamma_i^{(m,k)}\left(\sum_{j=1}^n x_j e_j\right)\right)^r$$

$$=\sum_{i=1}^{M(m,k)} b_i^{(m,k)}\ \left(\gamma_i^{(m,k)}(\Pi_n(x))\right)^r.\qquad\square$$

*Remark* 1.2. Note that $\sum_{i=1}^{M(m,k)} b_i^{(m,k)} = \mu(K)$ and thus, $b_i^{(m,k)} \leq \mu(K)$. So it is possible to enlarge the set of indices, and, repeating each one of the linear functionals at most $m$ times, we can assume that $b_i^{(m,k)} \leq \mu(K)/m$. Given $m, k \in \mathbb{N}$, we will denote by $N(m,k) \leq m\, M(m,k)$ the cardinality of a set of indices such that the sequences $\{b_i^{(m,k)}\}_{i=1}^{N(m,k)} \subset \mathbb{R}^+$ and $\{\mathrm{z}_i^{(m,k)}\}_{i=1}^{N(m,k)} \subset \Phi_m(K)$ verify

$$\int_K \gamma^r(\Pi_n(x))d\mu = \sum_{i=1}^{N(m,k)} b_i^{(m,k)} \ \left(\gamma_i^{(m,k)}(\Pi_n(x))\right)^r$$

for all $n \leq m,\ r \leq k,\ x \in E$, and $b_i^{(m,k)} \leq \mu(K)/m \to 0$ as $m \to \infty$.

We are now ready to prove the following Tchakaloff-type theorem.

**Theorem 1.3.** *Let $E$ be a Banach space with Schauder basis $\{e_j\}_{j\in\mathbb{N}}$. Given a $w^*$-compact set $K \subset E'$ and a positive Borel measure $\mu$ defined on $K$, for each $k \in \mathbb{N}$ there exist sequences $\{c_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{\varphi_i\}_{i\in\mathbb{N}} \subset E'$ such that $c_i \underset{i\to\infty}{\longrightarrow} 0$, and*

$$\int_K \gamma^k(x)\, d\mu(\gamma) = \sum_{i=1}^{+\infty} c_i \varphi_i^k(x) \qquad \text{for all } x \in E.$$

*Proof.* We have that $\lim_m \gamma(\Pi_m(x)) = \gamma(x)$ for all $x \in E$ and $\gamma \in E'$, and since $K$ is bounded, by the Dominated Convergence Theorem,

$$\int_K \gamma^k(x)\, d\mu(\gamma) = \lim_m \int_K \gamma^k(\Pi_m(x))\, d\mu(\gamma).$$

Using Lemma 1.1, we have

$$\int_K \gamma(x)^k\, d\mu(\gamma) = \lim_m \sum_{n=1}^{m} \int_K \gamma^k(\Pi_n(x)) - \gamma^k(\Pi_{n-1}(x))\, d\mu(\gamma)$$

$$= \lim_m \sum_{n=1}^{m} \sum_{i=1}^{N(n,k)} b_i^{(n,k)} \left[\left(\gamma_i^{(n,k)}(\Pi_n(x))\right)^k - \left(\gamma_i^{(n,k)}(\Pi_{n-1}(x))\right)^k\right]$$

$$= \lim_m \sum_{n=1}^{m} \sum_{i=1}^{N(n,k)} b_i^{(n,k)} \left[\left(\Pi_n^*\gamma_i^{(n,k)}(x)\right)^k - \left(\Pi_{n-1}^*\gamma_i^{(n,k)}(x)\right)^k\right],$$

where $\Pi_n^*$ denotes the adjoint operator of $\Pi_n$. Since $k \in \mathbb{N}$ is fixed, from now on we will modify the previous notation and write $(n)$ instead of $(n,k)$ because it is clear the number of terms, the coefficients and the linear functionals depend on $k$.

For each $b_i^{(n)}$ we have a term that is added and another that is subtracted. We now duplicate the set of indices of these sums in such a way that for any $n \in \mathbb{N}$ and $i \in \{1, \ldots, N(n)\}$, we introduce the signs $(+)$ or $(-)$ to distinguish the terms $b_i^{(n)} \ \left(\Pi_n^*\gamma_i^{(n)}(x)\right)^k$ and $-b_i^{(n)} \ \left(\Pi_{n-1}^*\gamma_i^{(n)}(x)\right)^k$. In order to do this we define the (total) order relation $(n, i, \text{sign})$ for $n_1, n_2 \in \mathbb{N}$ and $i_1 \in \{1, \ldots, N(n_1)\}$, $i_2 \in \{1, \ldots, N(n_2)\}$ by setting $(n_1, i_1, \text{sign}_1) \leq (n_2, i_2, \text{sign}_2)$ if and only if:

- $n_1 < n_2$, or
- $n_1 = n_2$ and $i_1 < i_2$, or
- $n_1 = n_2,\ i_1 = i_2$ and $\text{sign}_1 = (+)$, or
- $n_1 = n_2,\ i_1 = i_2$ and $\text{sign}_1 = \text{sign}_2 = (-)$.

In this way we can define $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, an order preserving bijection from $\mathbb{N}$ onto the set $\{(n, i, \text{sign})\}$ where $n \in \mathbb{N},\ i \in \{1, \ldots, N(n)\}$ and $\text{sign} \in \{+, -\}$. Then we have

$$\int_K \gamma(x)^k \, d\mu(\gamma) = \lim_m \sum_{(n,i,\text{sign}) \leq (m, N(m), (-))} b_i^{(n)} \left[ \left( \Pi_n^* \gamma_i^{(n)}(x) \right)^k - \left( \Pi_{n-1}^* \gamma_i^{(n)}(x) \right)^k \right],$$

which with our new indices is

$$\int_K \gamma(x)^k \, d\mu(\gamma) = \lim_m \sum_{i \leq \Lambda^{-1}((m, N(m), (-)))} c_i \varphi_i^k(x) \tag{1.1}$$

where $c_i = \Lambda_3(i) b_{\Lambda_2(i)}^{(\Lambda_1(i))}$ and

$$\varphi_i = \begin{cases} \Pi_{\Lambda_1(i)}^* \gamma_{\Lambda_2(i)}^{(\Lambda_1(i))} & \text{if } \Lambda_3(i) = (+), \\ \\ \Pi_{\Lambda_1(i)-1}^* \gamma_{\Lambda_2(i)}^{(\Lambda_1(i))} & \text{if } \Lambda_3(i) = (-). \end{cases}$$

In order to study the series in the right side of (1.1), we may introduce the notion of *blocks*. By a block we mean the difference of two generic partial sums of the series under study. Thus, a block is a finite sum

$$\sum_{i=\Lambda^{-1}((s_1, r_1, \text{sign}_1))}^{\Lambda^{-1}((s_2, r_2, \text{sign}_2))} c_i \varphi_i^k(x),$$

where $s_1, s_2 \in \mathbb{N},\ r_1 \in \{1, \ldots, N(s_1)\},\ r_2 \in \{1, \ldots, N(s_2)\},\ \text{sign}_1 \in \{+, -\}$, $\text{sign}_2 \in \{+, -\}$ and $(s_1, r_1, \text{sign}_1) \leq (s_2, r_2, \text{sign}_2)$. In particular, for $m \geq 1$, let us denote by $\mathfrak{B}_m$ the finite sum

$$\mathfrak{B}_m(x) = \sum_{i=\Lambda^{-1}((m,1,(+)))}^{\Lambda^{-1}((m, N(m), (-)))} c_i \varphi_i^k(x),$$

and let us refer to $\mathfrak{B}_m$ as a *complete block*. So, what we have in (1.1) is

$$\int_K \gamma(x)^k \, d\mu(\gamma) = \lim_m \sum_{j=1}^m \mathfrak{B}_j(x). \tag{1.2}$$

Note that we then have convergence of $\sum_i c_i \varphi_i^k(x)$ when summing by complete blocks, but we need to prove

$$\int_K \gamma(x)^k \, d\mu(\gamma) = \sum_{i=1}^\infty c_i \varphi_i^k(x).$$

We will prove that the sequence of partial sums is a Cauchy sequence. However, it is clear that in many cases we have to consider differences of generic partial sums involving not only complete blocks. For this, we have to distinguish the following two different situations.

• *Symmetric incomplete blocks* are those which, for some $s \in \mathbb{N}$, are of the form

$$\sum_{i=\Lambda^{-1}((s,r,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x)$$

where $1 < r$ or $t < N(s)$.

• *Non-symmetric incomplete blocks* are those where the first term is of the form $(s, j, (-))$ or the last term is of the form $(s, j, (+))$. The following are the three different examples of non-symmetric incomplete blocks we can find:

$$\sum_{i=\Lambda^{-1}((s,j,(-)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x), \qquad \sum_{i=\Lambda^{-1}((s,j,(+)))}^{\Lambda^{-1}((s,t,(+)))} c_i \varphi_i^k(x) \quad \text{or} \quad \sum_{i=\Lambda^{-1}((s,j,(-)))}^{\Lambda^{-1}((s,t,(+)))} c_i \varphi_i^k(x),$$

where $s \in \mathbb{N}$ and $1 \leq j \leq t \leq N(s)$ in the first and second sums, and

$$1 \leq j < t \leq N(s)$$

in the last.

Now, let us fix $\varepsilon > 0$, and note the following facts.

a) From the convergence (1.2), it is possible to find $n_1$ such that for $s_1, s_2 > n_1$,

$$\left| \sum_{i=\Lambda^{-1}((s_1,1,(+)))}^{\Lambda^{-1}((s_2,N(s_2),(-)))} c_i \varphi_i^k(x) \right| = \left| \sum_{j=s_1}^{s_2} \mathfrak{B}_j(x) \right| < \varepsilon/3.$$

That is, sums involving only complete blocks are smaller than $\varepsilon/3$ for $s_1$ and $s_2$ large enough.

b) Given $s \in \mathbb{N}$, we can find an upper bound for the absolute value of a symmetric incomplete block

$$\sum_{i=\Lambda^{-1}((s,r,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x) \quad \text{where } 1 < r \text{ or } t < N(s).$$

In this case, we have

$$\begin{aligned}
\left| \sum_{i=\Lambda^{-1}((s,r,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x) \right| &= \left| \sum_{i=r}^{t} b_i^{(s)} \left[ \left( \Pi_s^* \gamma_i^{(s)}(x) \right)^k - \left( \Pi_{s-1}^* \gamma_i^{(s)}(x) \right)^k \right] \right| \\
&\leq \sum_{i=r}^{t} \left| b_i^{(s)} \right| \left| \left( \gamma_i^{(s)}(\Pi_s x) \right)^k - \left( \gamma_i^{(s)}(\Pi_{s-1} x) \right)^k \right| \\
&\leq \left[ \sum_{i=r}^{t} \left| b_i^{(s)} \right| \right] \max_{r \leq i \leq t} \left| \left( \gamma_i^{(s)}(\Pi_s x) \right)^k - \left( \gamma_i^{(s)}(\Pi_{s-1} x) \right)^k \right| \\
&\leq \mu(K) \max_{r \leq i \leq t} \left| \left( \gamma_i^{(s)}(\Pi_s x) \right)^k - \left( \gamma_i^{(s)}(\Pi_{s-1} x) \right)^k \right|.
\end{aligned}$$

Note that for $u, v \in E$,

$$|\gamma^k(u) - \gamma^k(v)| = \left| \sum_{j=0}^{k-1} \left[ \gamma^{k-j}(u)\gamma^j(v) - \gamma^{k-j-1}(u)\gamma^{j+1}(v) \right] \right|$$

$$\leq \sum_{j=0}^{k-1} \left| \gamma^{k-j-1}(u)\gamma^j(v) \right| \; |\gamma(u) - \gamma(v)|$$

$$\leq \sum_{j=0}^{k-1} \|\gamma\|_{E'}^k \; \|u\|_E^{k-j-1} \; \|v\|_E^j \; \|u - v\|_E$$

$$\leq k \, \|\gamma\|_{E'}^k \max\{\|u\|_E^{k-1}, \|v\|_E^{k-1}\} \, \|u - v\|_E.$$

Then, we have

$$\left| \sum_{i=\Lambda^{-1}((s,r,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x) \right| \leq k \, \mu(K) \left( \max_{r \leq i \leq t} \|\gamma_i^{(s)}\|_{E'}^k \right)$$

$$\times \max\{\|\Pi_s x\|_E^{k-1}, \|\Pi_{s-1} x\|_E^{k-1}\} \, \|\Pi_s x - \Pi_{s-1} x\|_E.$$

Since $\gamma_i^{(s)} \in K$ for all $s$ and $i$, we have that $\|\gamma_i^{(s)}\|_{E'}$ is bounded independently of $i$ and $s$. Also, since the projections were defined using the Schauder basis and $\Pi_n x \underset{n\to\infty}{\to} x$, we deduce that $\|\Pi_n x\|_E \leq C\|x\|_E$ for all $x \in E, n \geq 1$, and $\|\Pi_s x - \Pi_{s-1} x\|_E \underset{s\to\infty}{\to} 0$. Hence, there exists $n_2$ such that, for all $s > n_2$, we have

$$\left| \sum_{i=\Lambda^{-1}((s,r,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x) \right| < \varepsilon/3.$$

c) In case we have to find an upper bound for a non-symmetric incomplete block, we can split the sum in at most three terms. Namely, for the different kinds of non-symmetric incomplete blocks, we have

- $$\sum_{i=\Lambda^{-1}((s,j,(-)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x) = c_{\Lambda^{-1}((s,j,(-)))} \varphi^k_{\Lambda^{-1}((s,j,(-)))}(x) + \sum_{i=\Lambda^{-1}((s,j+1,(+)))}^{\Lambda^{-1}((s,t,(-)))} c_i \varphi_i^k(x),$$

where $s \in \mathbb{N}$ and $1 \leq j \leq t \leq N(s)$.

- $$\sum_{i=\Lambda^{-1}((s,j,(+)))}^{\Lambda^{-1}((s,t,(+)))} c_i \varphi_i^k(x) = \sum_{i=\Lambda^{-1}((s,j,(+)))}^{\Lambda^{-1}((s,t-1,(-)))} c_i \varphi_i^k(x) + c_{\Lambda^{-1}((s,t,(+)))} \varphi^k_{\Lambda^{-1}((s,t,(+)))}(x),$$

where $s \in \mathbb{N}$ and $1 \le j \le t \le N(s)$.

- $$\sum_{i=\Lambda^{-1}((s,j,(-)))}^{\Lambda^{-1}((s,t,(+)))} c_i\varphi_i^k(x) = c_{\Lambda^{-1}((s,j,(-)))}\varphi^k_{\Lambda^{-1}((s,j,(-)))}(x) + \sum_{i=\Lambda^{-1}((s,j+1,(+)))}^{\Lambda^{-1}((s,t-1,(-)))} c_i\varphi_i^k(x) + c_{\Lambda^{-1}((s,t,(+)))}\varphi^k_{\Lambda^{-1}((s,t,(+)))}(x),$$

where $s \in \mathbb{N}$ and $1 \le j < t \le N(s)$.

Now, we can use the previous bound for symmetric incomplete blocks and Remark 1.2 to conclude the existence of $n_3 \in \mathbb{N}$ such that, for $s > n_3$, we have

$$\left| \sum_{i=\Lambda^{-1}((s,r,(-)))}^{\Lambda^{-1}((s,t,(+)))} c_i\varphi_i^k(x) \right| < \varepsilon/3.$$

In any case, given two partial sums of the series, their difference can be split in at most three terms: two symmetric or non-symmetric incomplete blocks (the first and the last) and a finite sum of complete blocks. To be more precise, given a block defined by a difference of two generic partial sums of the series:

$$\sum_{i=\Lambda^{-1}((s_1,r_1,\mathrm{sign}_1))}^{\Lambda^{-1}((s_2,r_2,\mathrm{sign}_2))} c_i\varphi_i^k(x)$$

where $s_1, s_2 \in \mathbb{N}$, $r_1 \in \{1,\dots,N(s_1)\}$, $r_2 \in \{1,\dots,N(s_2)\}$, $\mathrm{sign}_1 \in \{+,-\}$, $\mathrm{sign}_2 \in \{+,-\}$ and $(s_1, r_1, \mathrm{sign}_1) \le (s_2, r_2, \mathrm{sign}_2)$, we have many different situations.

- If $s_1 = s_2$, then the block is just one complete or incomplete block, and then, for $s_2 = s_1 > n_0 \ge \max\{n_1, n_2, n_3\}$, its absolute value is bounded by $\varepsilon/3$.
- If $s_1 + 1 = s_2$, then the block is the sum of two complete or incomplete blocks, and so, for $s_2 = s_1 + 1 > n_0 \ge \max\{n_1, n_2, n_3\}$, its absolute value is bounded by $2\varepsilon/3$.
- If $s_1 + 1 < s_2$, then the block can be written as

$$\sum_{i=\Lambda^{-1}((s_1,r_1,\mathrm{sign}_1))}^{\Lambda^{-1}((s_1,N(s_1),(-)))} c_i\varphi_i^k(x) + \sum_{j=s_1+1}^{s_2-1} \mathfrak{B}_j(x) + \sum_{i=\Lambda^{-1}((s_2,1,(+)))}^{\Lambda^{-1}((s_2,r_2,\mathrm{sign}_2))} c_i\varphi_i^k(x).$$

In this case, for $s_2 > s_1 > n_0 \ge \max\{n_1, n_2, n_3\}$ each one of them is smaller than $\varepsilon/3$, hence

$$\left| \sum_{i=\Lambda^{-1}((s_1,r,\mathrm{sign}_1))}^{\Lambda^{-1}((s_2,t,\mathrm{sign}_2))} c_i\varphi_i^k(x) \right| < \varepsilon. \qquad \square$$

**Corollary 1.4.** *Let $K$ be a $w^*$-compact subset of $E'$, and $\mu$ a positive Borel measure defined on $K$. If there exists a Schauder basis $\{e_j\}_{j\in\mathbb{N}}$ of $E$ and a subsequence $\{n_s\}_{s\in\mathbb{N}}$ of the natural numbers such that $\Pi^*_{n_s}(K) \subset \lambda K$ for some $\lambda \ge 1$, for all $s \in \mathbb{N}$, then for each $k \in \mathbb{N}$ there exist sequences $\{c_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{\varphi_i\}_{i\in\mathbb{N}} \subset K$ such that $c_i \underset{i\to\infty}{\longrightarrow} 0$, and*

$$\int_K \gamma^k(x)\, d\mu(\gamma) = \sum_{i=1}^{+\infty} c_i\varphi_i^k(x).$$

Applying the the Hahn decomposition theorem, given a signed measure $\mu$, we obtain the following.

**Corollary 1.5.** *Let $K$ be a $w^*$-compact subset of $E'$, and $\mu$ a Borel measure defined on $K$. If there exists a Schauder basis $\{e_j\}_{j\in\mathbb{N}}$ of $E$ and a subsequence $\{n_s\}_{s\in\mathbb{N}}$ of the natural numbers such that $\Pi^*_{n_s}(K) \subset \lambda K$ for some $\lambda \geq 1$, for all $s \in \mathbb{N}$, then for each $k \in \mathbb{N}$ there exist sequences $\{c_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ and $\{\varphi_i\}_{i\in\mathbb{N}} \subset K$ such that $c_i \underset{i\to\infty}{\longrightarrow} 0$, and*

$$\int_K \gamma^k(x)\, d\mu(\gamma) = \sum_{i=1}^{+\infty} c_i \varphi_i^k(x).$$

The following example shows that the condition $\Pi^*_m(K) \subset \lambda K$ is not automatically satisfied for a given Schauder basis of $E$ and may hold for one basis and not for another.

**Example 1.6.** We can consider in $E = c_0$ the canonical basis $\{e_n\}_{n\in\mathbb{N}}$ and the basis $\{s_n\}_{n\in\mathbb{N}}$, where

$$s_n = (1, 1, \ldots, 1, 0, \ldots) = \sum_{j=1}^{n} e_j.$$

It is easy to verify that the operators $\Pi_n : c_0 \longrightarrow c_0$ are given by

$$\Pi_n(x) = (x_1 - x_{n+1}, x_2 - x_{n+1}, \ldots, x_n - x_{n+1}, 0, \ldots),$$

where $x = (x_j)_{j\in\mathbb{N}}$, and that their adjoints $\Pi^*_n : \ell_1 \longrightarrow \ell_1$, for $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, are

$$\Pi^*_n(\gamma) = (\gamma_1, \gamma_2, \ldots, \gamma_n, -\sum_{j=1}^{n} \gamma_j, 0, \ldots).$$

Now consider the $w^*$-compact subset of $\ell_1$:

$$K = \left\{\gamma \in \ell_1 : |\gamma_k| \leq \frac{1}{k^2} \text{ for all } k \in \mathbb{N}\right\}.$$

Then given any $\lambda > 1$, $\Pi^*_m(K) \not\subset \lambda K$ for large $m$. Indeed, consider $n$ large enough so that

$$\frac{\pi^2}{7} < \sum_{j=1}^{n} \frac{1}{j^2} < \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Fix $a = (1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, 0, \ldots) \in K$. Now, for any $m$ larger than $n$ and $\frac{7\lambda}{\pi^2}$, note that $\sum_{j=1}^{m} a_j = \sum_{j=1}^{n} \frac{1}{j^2}$, so

$$\Pi^*_m(a) = \left(1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, 0, \ldots, 0, -\sum_{j=1}^{n} \frac{1}{j^2}, 0, \ldots\right)$$

and $|\Pi^*_m(a)_{m+1}| = \sum_{j=1}^{n} \frac{1}{j^2} > \frac{\pi^2}{7} > \frac{\lambda}{m}$, so $\Pi^*_m(a) \notin \lambda K$. Note, however, that if one uses the canonical basis $\{e_n\}$ in $c_0$, then $\Pi^*_m(\gamma) = (\gamma_1, \gamma_2, \ldots, \gamma_m, 0, \ldots)$ and $\Pi^*_m(K) \subset K$ for all $m$.

## 2. A strong form of Tchakaloff's theorem

We now shift the focus from the Banach space $E$ to the support $K$ of the measure. For the whole section $K$ is assumed to be a balanced convex $w^*$-compact subset of $E'$. We define:

**Definition 2.1.** A $k$-homogeneous polynomial $P : E \longrightarrow \mathbb{R}$ is *$K$-integral* if

$$P(x) = \int_K \gamma(x)^k \, d\mu(\gamma)$$

for a regular Borel measure $\mu$ on $K$. Its $K$-integral norm is the infimum of the total variations of all measures satisfying the equality. We denote with $\mathcal{P}^k_{KI}(E)$ the Banach space of all $k$-homogeneous $K$-integral polynomials on $E$.

A $k$-homogeneous polynomial $P : E \longrightarrow \mathbb{R}$ is *$K$-nuclear* if

$$P(x) = \sum_{i=1}^{\infty} c_i \gamma_i(x)^k$$

with $(c_i)_{i\in\mathbb{N}} \in \ell_1$ and $(\gamma_i)_{i\in\mathbb{N}} \subset K$. The $K$-nuclear norm $\|P\|_{KN}$ is the infimum of $\sum_{i=1}^{\infty} |c_i| \|\gamma_i\|^k_{E'}$ over all possible representations of this type for $P$. We denote with $\mathcal{P}^k_{KN}(E)$ the Banach space of all $k$-homogeneous $K$-nuclear polynomials on $E$.

Note that all $K$-nuclear polynomials are $K$-integral. These polynomials are well-known when $K$ is the closed unit ball of $E'$, in which case, as mentioned in the Introduction, they are called *integral* and *nuclear* respectively, and the subscript $K$ is dropped from the notation.

Recall from the Introduction that we seek here to prove a strong form of Tchakaloff's theorem, i.e. the coincidence of $K$-nuclear and $K$-integral polynomials. This will not always be the case, but we find conditions on $K$ and $E$ under which this holds.

Note that $K$-integral polynomials are $K$-bounded in the sense of [6]: $|P(x)| \leq c\|x\|^k_K$, where $\|x\|_K = \max_K |\gamma(x)|$. The smallest constant $c$ for which such an inequality holds is called the $K$-bounded norm of the polynomial. The space of $K$-bounded polynomials with this norm is a Banach space which we will denote $\mathcal{P}^k_K(E)$. This space has been extensively studied in [6], where it was shown to be isometrically isomorphic to the space of continuous $k$-homogeneous polynomials on $E_K$, a Banach space constructed from $E$ and $K$. We undertake an analogous study of $K$-integral and $K$-nuclear polynomials and will make use of the same space $E_K$ (defined below).

We show in Proposition 2.5 that the space of $K$-integral (respectively, $K$-nuclear) polynomials on $E$ is isometrically isomorphic to the space of integral (respectively, nuclear) polynomials on $E_K$. This will allow us to apply the extensive literature regarding integral and nuclear polynomials by studying $E_K$ and to obtain sufficient conditions for the strong form of Tchakaloff's theorem in Corollary 2.6.

We follow [6] in the construction of $E_K$ and $E'_K$. Given a balanced convex $w^*$-compact subset $K$ of $E'$, define on $E$ the seminorm

$$\|x\|_K = \max_{\gamma\in K} |\gamma(x)|.$$

Since $w^*$-compact sets are norm-bounded, this seminorm is norm-continuous, and its kernel ${}^\circ K = \bigcap_{\gamma\in K} \ker\gamma$ is a closed subspace of $E$. Consider $E/{}^\circ K$ with the, here, norm $\|\cdot\|_K$. This space need not be complete; denote its completion by $E_K$, and by $\pi$ the map $\pi : E \longrightarrow E_K$. Note that $\pi$ need not be surjective and that $E_K$ is not a quotient space, as the following example shows.

**Example 2.2.** Let $E = \ell_1$ and let $K \subset \ell_\infty$ be the unit ball of $\ell_2$:

$$K = \{\gamma \in \ell_\infty : \sum_{k=1}^{\infty} |\gamma_k|^2 \leq 1\}.$$

Then for any $x \in E = \ell_1$,

$$\|x\|_K = \max_{\gamma \in B_{\ell_2}} |\gamma(x)| = \|x\|_2,$$

for $\ell_2$ is its own dual space. Also, ${}^\circ K = \{0\}$, and $E_K$ is the completion of $\ell_1$ in the 2-norm; that is, $E_K = \ell_2$, and $\pi : E \longrightarrow E_K$ is the inclusion of $\ell_1$ in $\ell_2$.

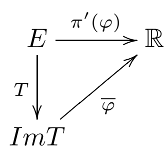
We will need the following characterization of the dual space $E'_K$ and its norm.

**Proposition 2.3.** *The transpose $\pi' : E'_K \longrightarrow E'$ of $\pi$ maps the unit ball of $E'_K$ to $K$. Thus $E'_K$ identifies with $\mathbb{R}K$ (the scalar multiples of $K$) and $\|\cdot\|_{E'_K}$ with the Minkowski seminorm $p_K$ of $K$.*

*Proof.* Since $Im\pi$ is dense in $E_K$, $\pi'$ is one-to-one. To see $\pi'(B_{E'_K}) = K$:

$\supset$: If $\gamma \in K$, $|\gamma(x)| \leq \|x\|_K$ for all $x$, so $\gamma$ factors continuously through $E/{}^\circ K$ and has norm bounded by one.

$\subset$: Define $T : E \longrightarrow C(K)$ (the space of continuous functions on $K$ with the sup-norm) by $Tx(\gamma) = \gamma(x)$ for all $\gamma \in K$. This is a continuous linear operator. If $\varphi \in B_{E'_K}$, then $\ker T \subset \ker(\pi'(\varphi))$, so $\pi'(\varphi)$ factors through $ImT$:




and $\|\overline{\varphi}\| \leq 1$: $|\overline{\varphi}(Tx)| = |\pi'(\varphi)(x)| = |\varphi(\pi(x))| \leq \|\pi(x)\|_K = \|Tx\|$. Thus $\overline{\varphi}$ extends to $C(K)$ and by the Riesz-Markov representation theorem can be represented by a regular Borel measure $\mu$ on $K$ with total variation $\|\mu\| = 1$. Hence

$$\pi'(\varphi)(x) = \overline{\varphi}(Tx) = \int_K \gamma(x)\, d\mu(\gamma).$$

Thus, since $K$ is balanced, convex, and $w^*$-compact, $\pi'(\varphi) \in K$. The other assertions follow immediately from this. $\square$

Note that $\mathbb{R}K$ is the union of $w^*$-compact sets, $\mathbb{R}K = \bigcup_r rK$, but need not be weak*-closed, as the following example shows.

**Example 2.4.** Let $K = \{\gamma \in \ell_\infty : |\gamma_k| \leq \frac{1}{k}\} \subset \ell_\infty$. Then $\mathbf{1} = (1, 1, \ldots) \in \overline{\mathbb{R}K}^{w^*}$. Indeed, let $I_n = (1, 1, \ldots, 1, 0, 0, \ldots) \in nK$; for all $a \in \ell_1$,

$$\mathbf{1}(a) = \sum_{k=1}^{\infty} a_k = \lim_n \sum_{k=1}^{n} a_k = \lim_n I_n(a).$$

But $\mathbf{1} \notin \mathbb{R}K$ and is not a $K$-continuous linear form on $\ell_1$.

We have the following proposition.

**Proposition 2.5.** *Let $K$ be a balanced, convex $w^*-$compact subset of $E'$, then*

$$\mathcal{P}^k_{KI}(E) \simeq_1 \mathcal{P}^k_I(E_K) \quad \text{and} \quad \mathcal{P}^k_{KN}(E) \simeq_1 \mathcal{P}^k_N(E_K).$$

*Proof.* In [6] it is proved that $\mathcal{P}^k_K(E) \simeq_1 \mathcal{P}^k(E_K)$: given $P \in \mathcal{P}^k_K(E)$, define $Q : E/{}^\circ K \longrightarrow \mathbb{R}$ by $Q(\pi(x)) = P(x)$ for all $x$. $Q$ is continuous and extends to $Q : E_K \longrightarrow \mathbb{R}$. The mapping $P \mapsto Q$ identifies $\mathcal{P}^k_K(E)$ with $\mathcal{P}^k(E_K)$. We need only check that a $K$-integral polynomial $P$ on $E$ corresponds to an integral polynomial $Q$ on $E_K$.

Note that if $\varphi_\alpha \longrightarrow^{w^*_{E_K}} \varphi$, then for any $x \in E$, $(\varphi_\alpha \circ \pi)(x) \longrightarrow (\varphi \circ \pi)(x)$. Thus by compactness of both spaces, $\pi'$ is a homeomorphism

$$(B_{E'_K}, w^*_{E_K}) \approx (K, w^*_E).$$

We identify regular Borel measures in both spaces via $\pi'$: if $C \subset B_{E'_K}$ and $A \subset K$, $\nu(A) = \mu(\pi'^{-1}(A))$ and $\mu(C) = \nu(\pi'(C))$. Note that the total variations of $\nu$ and $\mu$ are equal. Thus, if $P : E \longrightarrow \mathbb{R}$ is $K$-integral,

$$P(x) = \int_K \gamma(x)^k \, d\nu(\gamma) = \int_{B_{E'_K}} \varphi(\pi(x))^k \, d\mu(\varphi) = Q(\pi(x)),$$

and $P$ is identified with $Q$, an integral polynomial on $E_K$. Any $\nu$ representing $P$ identifies with a $\mu$ representing $Q$ and vice-versa, so the $K$-integral norm of $P$ is the integral norm of $Q$.

Analogously, a $K$-nuclear $P$ identifies with a nuclear $Q$. □

Recall that a Banach space is Asplund if all its separable subspaces have separable duals. Asplundness has many equivalent formulations (for a presentation of Asplund spaces, see [15]). For example, strengthening the hypothesis of Theorem 1.3 by asking for the Schauder basis on $E$ to be shrinking, we would immediately obtain that $E$ is an Asplund space and thus, by Corollary 2.6 below, better conditions on the convergence of the series. We note that $E'$ having a $w^*$-Schauder basis is equivalent to the hypothesis of Theorem 1.3.

As an immediate consequence of [2] and [3], under any of the conditions in the following corollary, $K$-integral and $K$-nuclear polynomials coincide, and therefore the strong form of Tchakaloff's theorem is valid.

**Corollary 2.6.** *For a measure supported on $K \subset E'$, the strong form of Tchakaloff's theorem holds in any of the following cases:*

*–$E_K$ is Asplund.*

*–$B_{E'_K}$ is separable.*

*–$B_{E'_K}$ is weakly compact.*

We now address the question of the dependence of the space $\mathcal{P}^k_{KI}(E)$ on the compact set $K$ and the Banach space $E$. The following example shows that subsets $K_1$ and $K_2$ of $E'$ may be homeomorphic and produce different $K$-integral polynomials.

**Example 2.7.** Consider $E = \ell_1$ and the $w^*$-compact subsets of $E' = \ell_\infty$, $K_1 = B_{\ell_p}$ and $K_2 = B_{\ell_q}$, the unit balls of $\ell_p$ and $\ell_q$, where $1 \leq p < \infty$ and $1 \leq q < \infty$. It is well-known that the Mazur map [8] $\ell_p \longrightarrow \ell_q$ defined by

$$(\ldots, x_i, \ldots) \mapsto (\ldots, \operatorname{sgn}(x_i)|x_i|^{\frac{p}{q}}, \ldots)$$

is a (non-linear) homeomorphism between $K_1$ and $K_2$ in their corresponding norm topologies of $\ell_p$ and $\ell_q$. It is not hard to check that the same is true in their $w^*$-topologies, as well as in the norm topology of $\ell_\infty$. As in Example 2.2, we obtain $E_{K_1} = \ell_p$ and $E_{K_2} = \ell_q$. Thus $E'_{K_1} \neq E'_{K_2}$. But these are, respectively, the degree 1 $K_1$-integral and $K_2$-integral polynomials, as was shown in Proposition 2.3.

We have, however, the following proposition:

**Proposition 2.8.** *Let $E$ and $F$ be Banach spaces, and let $K_1$ and $K_2$ be $w^*$-compact subsets of their dual spaces $E'$ and $F'$, respectively. Then the following are equivalent.*

*a) The spaces $\mathcal{P}^k_{K_1 I}(E)$ and $\mathcal{P}^k_{K_2 I}(F)$ are isometrically isomorphic for all $k \geq 1$.*

*b) The spaces $E'_{K_1}$ and $F'_{K_2}$ are isometrically isomorphic.*

*c) There exists a homeomorphism $\alpha : K_1 \longrightarrow K_2$ such that for all $\varphi$ and $\psi$ in $K_1$,*

$$p_2\left(\frac{\alpha(\varphi)+\alpha(\psi)}{2}\right) = p_1\left(\frac{\varphi+\psi}{2}\right),$$

*where $p_i$ is the Minkowski norm defined by $K_i$.*

*Proof.* The implications a)$\Rightarrow$ b)$\Rightarrow$ c) are trivial. Now suppose c). We note first that $\alpha(0) = 0$. Indeed, if $\varphi = \psi = 0$, then we have $p_2(\alpha(0)) = p_1(0) = 0$, and thus $\alpha(0) = 0$. Also, we have $\alpha(-\psi) = -\alpha(\psi)$, since

$$p_2(\alpha(\psi) + \alpha(-\psi)) = p_1(0) = 0.$$

Hence $\alpha$ is an isometry:

$$p_2(\alpha(\varphi) - \alpha(\psi)) = p_2(\alpha(\varphi) + \alpha(-\psi)) = p_1(\varphi - \psi).$$

Thus $\alpha$, being surjective, extends by the Mazur-Ulam theorem [9] to a linear isometry $E'_{K_1} = \mathbb{R}K_1 \longrightarrow \mathbb{R}K_2 = F'_{K_2}$. Now consider $\alpha' : C(K_2) \longrightarrow C(K_1)$ such that $\alpha'(f) = f \circ \alpha$. For all $y \in F$, $\hat{y} \circ \alpha$ is $w^*$-continuous and a $K_1$-continuous linear form. Thus $\hat{y} \circ \alpha = \alpha'(\hat{y}) \in \hat{E}$, that is, $\alpha'(\hat{F}) \subset \hat{E}$. Analogously $(\alpha^{-1})'(\hat{E}) \subset \hat{F}$. For any $k \geq 1$,

$$\alpha' : [\hat{y}^k : y \in F] \longrightarrow [\hat{x}^k : x \in E]$$

is an isometric isomorphism. Now taking duals,

$$\mathcal{P}^k_{K_1 I}(E) \simeq^1 \mathcal{P}^k_{K_2 I}(F). \qquad \square$$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