Quantum gate arrays can be programmed to evaluate the expectation value of any operator

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A programmable gate array is a circuit whose action is controlled by input data. In this paper we describe a special-purpose quantum circuit that can be programmed to evaluate the expectation value of any operator O acting on a space of states of N dimensions. The circuit has a program register whose state $|\Psi(O)\rangle_P$ encodes the operator O whose expectation value is to be evaluated. The method requires knowledge of the expansion of O in a basis of the space of operators. We discuss some applications of this circuit and its relation to known instances of quantum state tomography.

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I. INTRODUCTION

An important feature of classical computers is that they can be programmed. That is to say, a fixed universal device can perform different tasks depending on the state of some input registers. These registers define the program the device is executing. Quantum computers [1] have a rather different property. Thus, Nielsen and Chuang established in Ref. [2] that a general purpose programmable quantum computer does not exist. Such a device would have to have the following features. It should consist of a fixed gate array with a data register and a program register. The array should work in such a way that the state of the program register encodes the unitary operator $\mathcal{U}$ that is applied to the state of the data register. As shown in Ref. [2], such devices cannot be universal since different unitary operators require orthogonal states of the program register. However, some interesting examples of programmable devices could still be constructed. For example, nondeterministic programmable gate arrays were first considered in Ref. [2] and analyzed later in a variety of examples [3]. More recently, quantum "multimeters" were introduced and discussed in Ref. [4]. Such devices are fixed gate arrays acting on a data register and a program register, together with a final fixed projective measurement on the composite system. They are programmable quantum measurement devices [4] that act either nondeterministically or in an approximate way (see Ref. [5]).

In this paper we will describe a different kind of programmable quantum gate array that is useful to solve the following problem. Suppose that we are given an operator O acting on a N-dimensional Hilbert space and a quantum state ρ. By this we mean that someone supplies us with many copies of a quantum system prepared in the same state ρ and defines for us the operator O by specifying its expansion in a basis of the space of operators. Our task is to compute the expectation value of O in the state ρ. We will show that it is possible to construct a programmable circuit that evaluates such expectation value by measuring the polarization of a single qubit. The inputs of such circuit are a data register, a program register, and an auxiliary qubit. The circuit evaluates the expectation value of an operator O (specified by the program register) in the quantum state ρ of the data register. The expectation value $\mathrm{Tr}(\rho O)$ is obtained by performing a measurement of the polarization of the auxiliary qubit. We will describe how to construct these circuits and exhibit an interesting example: a programmable array to efficiently solve a class of quantum decision problems concerning properties of quantum phase space distributions. The paper is organized as follows. In Sec. II we review a tomographic scheme based on the use of the so-called scattering circuit, which is the basis of our method. In Sec. III we show how to build programmable tomographic devices and how to transform them into programmable multimeters, whose output is the expectation value of an arbitrary operator. In Sec. IV we present an example of the use of our method to determine averages of phase-space distributions over several phase space regions. Finally we present our conclusions in Sec. V.

II. STATE TOMOGRAPHY USING THE SCATTERING CIRCUIT

The quantum gate arrays discussed in this paper are designed using the "scattering circuit" shown in Fig. 1 as a simple primitive. In such circuit a system, initially in the state ρ, is brought in contact with an ancillary qubit prepared in the state $|0\rangle$. This ancilla acts as a probe particle in a scattering experiment. The algorithm consists of the following steps: (i) Apply a Hadamard transform H to the ancillary qubit. Since $H|0\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$ and $H|1\rangle=(|0\rangle-|1\rangle)/\sqrt{2}$, the new state of the qubit is $(|0\rangle+|1\rangle)/\sqrt{2}$. (ii) Apply a "controlled–A" operator, which does nothing if the

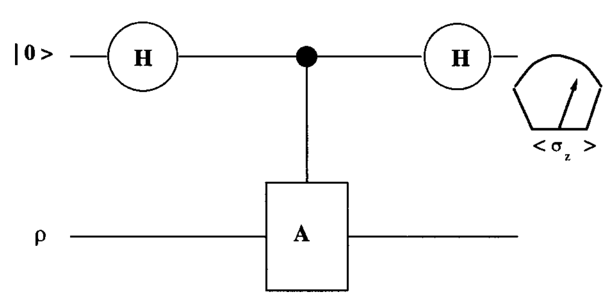

FIG. 1. The scattering circuit that can be used to evaluate real and imaginary parts of the expectation value $\mathrm{Tr}(\rho A)$ for a unitary operator A. H denotes a Hadamard transform. The "controlled-A" operation is such that $(\text{ctrl-}A)|q\rangle|\Psi\rangle=|q\rangle A^q|\Psi\rangle$.

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state of the ancilla is $|0\rangle$ and applies the unitary operator A to the system if the ancilla is in state $|1\rangle$. (iii) Apply another Hadamard gate to the ancilla and perform measurements of its spin polarizations along the z and y axes. Given sufficiently many independent instances of the experiment, the measurements yield the expectation values $\langle\sigma_z\rangle$ and $\langle\sigma_y\rangle$ of the Pauli spin operators σ_z and σ_y. This algorithm has the following remarkable property:

$$\langle\sigma_z\rangle=\mathrm{Re}[\mathrm{Tr}(A\rho)], \quad \langle\sigma_y\rangle=\mathrm{Im}[\mathrm{Tr}(A\rho)]. \tag{1}$$

Different versions of this circuit play an important role in many quantum algorithms [6–9]. In particular, the scattering circuit was recently used as a basic tool to interpret tomography and spectroscopy as two dual forms of the same quantum computation [10].

For our purpose it is useful to review how to use this scattering circuit as a primitive to design a tomographer (i.e., a device that after a number of experiments determines the quantum state of the system). As a consequence of Eq. (1), we see that every time we run the algorithm for a known operator A, we extract information about the state ρ. Doing so for a complete basis of operators $\{A(\alpha)\}$, one gets complete information and determines the full density matrix. Different tomographic schemes are characterized by the basis of operators $A(\alpha)$ they use. Of course, completely determining the quantum state requires an exponential amount of resources. In fact, if the dimensionality of the Hilbert space of the system is N, then the complete determination of the quantum state involves running the scattering circuit for a complete basis of N^2 operators $A(\alpha)$. However, evaluating any coefficient of the decomposition of ρ in a given basis can be done efficiently provided that the operators $A(\alpha)$ can be implemented by efficient networks. A convenient basis set is defined as (see, for example, Refs. [10,11])

$$A(\alpha)=A(q,p)=U^qRV^{-p}\exp(i\pi pq/N). \tag{2}$$

Here, both q and p are integers between 0 and $N-1$, U is a cyclic shift operator in the computational basis ($U|n\rangle=|n+1\rangle$), V is the cyclic shift operator in the basis related to the computational one via the discrete Fourier transform, and R is the reflection operator ($R|n\rangle=|N-n\rangle$). It is straightforward to show that the operators $A(q,p)$ are Hermitian, unitary, and form a complete orthonormal basis of the space of operators satisfying

$$\mathrm{Tr}[A(\alpha)A(\alpha')]=N\,\delta_N(q'-q)\,\delta_N(p'-p), \tag{3}$$

where $\delta_N(x)$ is the periodic Kronecker delta function that is equal to one if $x=0$ (modulo N) and vanishes otherwise (the above operators form a "quorum," as defined in Ref. [12]). With this choice for $A(q,p)$, the scattering circuit directly evaluates the discrete Wigner function [10,11,13,14].

III. PROGRAMMABLE TOMOGRAPHERS AND PROGRAMMABLE MULTIMETERS

We will now show how to design a programmable gate array to evaluate the expectation value of any operator O. We will assume that we know how to expand O in a basis such as the one used above: $O=\Sigma_{q,p}o(q,p)A(q,p)$. As the operators $A(q,p)$ are not only unitary but also Hermitian, the real and imaginary parts of the complex coefficients $o(q,p)$ define the expansion of the Hermitian and anti-Hermitian pieces of O in the basis $A(q,p)$. The expectation value of these two pieces can be evaluated separately using the procedure described below (the results can then be combined to get the expectation of O). So, in what follows we will assume that the operator at hand is Hermitian and that the coefficients of its expansion in the basis $A(q,p)$ are real numbers. To introduce our method, it is convenient to note first that the evaluation of the expectation value of the operators $A(q,p)$ can be done using a programmable circuit that is independent of q and p. Such circuit is illustrated in Fig. 2. This is an application of the scattering circuit shown in Fig. 1 with two program registers used to encode the values of q and p. When the quantum state of the program is $|\Psi\rangle_P=|q\rangle|p\rangle$ the circuit evaluates the expectation value of $A(q,p)$. This is accomplished by letting the program registers to act as controls of the operators U and V (which, as mentioned, generate cyclic shifts either in the computational or in the conjugated basis). Thus, the action of the circuit is such that when the auxiliary qubit is in state $|\sigma\rangle$ ($\sigma=0,1$) and the state of the program is $|q\rangle|p\rangle$, the operator $A^{\sigma}(q,p)$ is applied to the system register. The network in Fig. 2 is efficient since it can be built using a number of elementary gates which scales polynomially with $\log_2(N)$ [11].

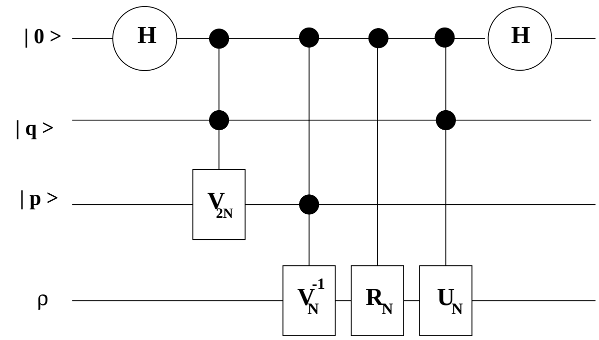

FIG. 2. Programmable gate array evaluating $\mathrm{Tr}(\rho A(q,p))$ from the polarization of the first qubit. The program state is $|\Psi\rangle_P=|q\rangle|p\rangle$. All "controlled–$O$" operators act as: (ctrl-O)$|n\rangle|\Psi\rangle=|n\rangle O^n|\Psi\rangle$. U_K (V_K) are cyclic shift operators in the computational (conjugated) basis of a K-dimensional space. A subscript in an operator denotes the dimensionality of the space in which it acts.

The circuit has an obvious property. Different states $|q\rangle|p\rangle$ are used to program the evaluation of the expectation value of orthogonal operators $A(q,p)$. It is clear that by restricting to such program states one has no real advantage with respect to the case in which q and p are stored as classical information. However, we can use more general program states. If the program register is in the state $|\Psi\rangle_P=\Sigma_{q,p}c(q,p)|q\rangle|p\rangle$, then the same circuit evaluates the expectation value of a linear combination of the operators $A(q,p)$ since the final polarization is

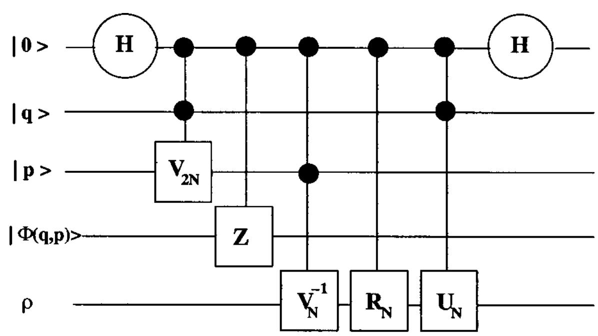

FIG. 3. Programmable gate array evaluating $\mathrm{Tr}(\rho O)$ from the polarization of the first qubit. The program state is $|\Psi\rangle_P = \Sigma_{q,p} c(q,p)|q\rangle|p\rangle|\Phi(q,p)\rangle$, where $c(q,p)$ and $\Phi(q,p)$ define the polar decomposition of the coefficients $o(q,p) = \mathrm{Tr}(OA(q,p))/N$.

$$\langle \sigma_z \rangle = \mathrm{Tr}\left(\rho \sum_{q,p} |c(q,p)|^2 A(q,p) \right). \tag{4}$$

Equation (4) shows that this algorithm can be used to evaluate the expectation value of any operator that can be written as a convex sum of the basis set $A(q,p)$. This is not general enough since the expansion of a Hermitian operator can include negative coefficients. For this purpose the method can be extended as follows: The most general Hermitian operator can be expressed in the basis $A(q,p)$ as $O = \Sigma_{q,p} c^2(q,p)\exp(i\pi\phi(q,p))A(q,p)$, where $c(q,p)$ is a real number and $\phi(q,p)$ is either 0 or 1 [$\phi(q,p)$ simply stores the information about the sign of each coefficient]. We will assume that $\Sigma_{q,p} c^2(q,p) = 1$ (if this is not the case we can always renormalize the coefficients). For operators with some negative coefficients we can use a third register consisting of a single qubit to store $\phi(q,p)$. The circuit evaluating the expectation value of O is shown in Fig. 3. The first two program registers store q and p and are used exactly in the same way as above. The third one, storing $\phi(q,p)$, is acted upon with a σ_z operator, introducing the required phase $\exp(i\pi\phi(q,p))$. Then, if the state of the program register is $|\Psi\rangle_P = \Sigma_{q,p} c(q,p)|q\rangle|p\rangle|\phi(q,p)\rangle$, the final polarization measurement turns out to be

$$\langle \sigma_z \rangle = \mathrm{Tr}\left(\rho \sum_{q,p} c^2(q,p) e^{i\pi\phi(q,p)} A(q,p) \right) = \mathrm{Tr}(\rho O). \tag{5}$$

Summarizing, we showed that the measurement of the expectation value of any operator O can be done using a programmable gate array. The hardware architecture is associated with the particular choice of basis $A(q,p)$, which is just a matter of convenience, and is independent of O. The software used to program the array is obviously determined by the choice of hardware. The expectation values of the Hermitian and anti-Hermitian parts of the operator O are computed separately using a method that requires knowledge of the expansion of these operators in the basis $A(q,p)$. If the coefficients in the expansion are written as $o(q,p) = \mathrm{Tr}(OA(q,p))/N = c^2(q,p)\exp(i\pi\phi(q,p))$, the program state that needs to be prepared is $|\Psi\rangle_P = \Sigma_{q,p} c(q,p)|q\rangle|p\rangle|\phi(q,p)\rangle$ [where $\phi(q,p)=0$ or 1]. Thus, the coefficients of the expansion of O in the basis $A(q,p)$ define the program state $|\Psi\rangle_P$ required to measure its expectation value. It is clear that in most cases the method will not be efficient. For example, both the task of defining the operator by specifying the coefficients $o(q,p)$ as well as the preparation of the program state $|\Psi\rangle_P$ are likely to be inefficient. The existence of efficient networks to implement "controlled–$A(q,p)$" operations is a less stringent conditions that is fulfilled by the basis defined in Eq. (2) [11]. Having said this, it is worth noting that there are sets of problems that can be efficiently solved using this method. We will now describe one such example.

IV. AN APPLICATION: EVALUATING AVERAGES OF PHASE-SPACE DISTRIBUTIONS

We will show that the circuits of Figs. 2 and 3 can be easily adapted to evaluate the sum of values of the Wigner function over various phase-space domains. The program register is used to define the domain over which the Wigner function is averaged. If the domain is a line, the algorithm just evaluates the probability for the occurrences of the results of the measurement of a family of observables (see below). However, for more general domains (such as line segments, parallelograms, etc.) the circuit evaluates properties that characterize a quantum state that cannot be simply casted in terms of probabilities. In this sense the circuit is as a programmable tomographer measuring various features of phase-space distributions. Before going into more details let us briefly review some properties of Wigner functions. Discrete Wigner functions can be used to represent the quantum state of a system in phase space [10,11,13,14]. For a system with an N-dimensional space of states such function is defined on a lattice of $2N\times 2N$ points (q,p) where both q and p take values between 0 and $2N-1$ (only $N\times N$ of these values are independent). At each phase-space point the Wigner function is defined in terms of the operators $A(q,p)$ given in Eq. (2) as

$$W(q,p) = \frac{1}{2N}\mathrm{Tr}(A(q,p)\rho). \tag{6}$$

As mentioned above, the measurement of this function can be done by using the scattering circuit [10] or the programmable gate array of Fig. 2. The program register encodes the value of q and p. In general, if the program state is $|\Psi\rangle_P = \Sigma_{q,p} c(q,p)|q\rangle|p\rangle$, the final polarization measurement is $\langle\sigma_z\rangle = 2N\Sigma_{qp}|c(q,p)|^2 W(q,p)$. Thus, the program defines the region over which we average the value of the Wigner function.

In general, preparing the program state associated with a general phase-space region can be complicated. However, there are simple procedures to prepare the program states corresponding to general lines, segments, and parallelograms. Let us begin with the simplest case: For the program state $|\Psi\rangle_P = \Sigma_q |q\rangle|p_0\rangle/\sqrt{2N}$, it is clear that the final polarization measurement reveals the sum of values of the Wigner function along the horizontal line defined as $p=p_0$, i.e., $\langle\sigma_z\rangle=\Sigma_q W(q,p_0)$. Other program states are easy to prepare. For example, the state programming the evaluation of the average Wigner function along the line defined by the equation $q-p=0$, mod$(2N)$, is the generalized Bell state $|\Psi\rangle_P=\Sigma_q|q\rangle|q\rangle/\sqrt{2N}$, which can be efficiently prepared. A natural question then arises: How hard is to program the evaluation of the average Wigner function along an arbitrary line? To answer this question it turns out to be useful to apply a method which is based on a rather interesting property of discrete Wigner functions. In fact, the discrete Wigner functions transform ''classically'' when the quantum state evolves under a special class of unitary transformations. These transformations are known as ''quantum cat maps'' and correspond to the quantization of linear transformations on the torus. For example, consider the following linear transformation of the phase space grid:

$$q=bq'+p', \quad p=(ab-1)q'+ap', \tag{7}$$

where a and b are integers. To this linear, area-preserving, transformation we can associate a unitary operator (such unitary operator is known as a quantum cat map and is parametrized by the integers a and b, see Ref. [11] for details). The unitary operator is such that the state evolves in such a way that the discrete Wigner function ''flows'' according to the rule given in Eq. (7), i.e.,

$$W(q',p',t+1)=W(q,p,t). \tag{8}$$

The linear transformation (7) maps lines into lines. Therefore, if we want to evaluate the average Wigner function along an arbitrary line L we can use the following strategy: One can first transform the state with the appropriate unitary operator that maps the line L into a line for which the program state is easy to prepare (for example, a vertical line, or a line defined by the equation $q-p=c$). Then, one can evaluate the average of the Wigner function along the ''easy'' line using the method discussed above (with a program state that is easy to prepare). For the method to be practical we still need to show that one can efficiently find the linear transformation (7) mapping an arbitrary line L, defined by the equation $n_1q+n_2p=n_3$, into the line $q+p=n_3$ (whose program state is easy to prepare). This can be done as follows: If either n_1 or n_2 are odd numbers the line L has $2N$ points and the parameters of the linear transformation (7) mapping L to the line defined by $q+p=n_3$ are $b=1+n_2$, $n_2a=1-n_1$ (similar results can be obtained when both n_1 and n_2 are even). Finally, we should mention that the method can be made fully programmable by adding extra registers to store the integers b and c parametrizing any cat map.

Evaluating sums of Wigner functions over phase-space lines is particularly interesting because of a crucial property of such functions. Thus, adding $W(q,p)$ along the line $ap-bq=c$ one obtains the probability to detect the eigenstate of the translation operator $T(b,a)=U^aV^b\exp(i\pi ab/N)$ with eigenvalue $\exp(i\pi c/N)$. As a consequence, in this case the programmable gate array evaluates probabilities for the possible results of a set of measurements.

Lines are a special case since more general phase-space domains cannot be associated with projection operators. However, it is clear that the method described above can be applied to efficiently prepare program states for other phase-space domains such as general (tilted) parallelograms. One first trivially prepare the program state for a parallelogram limited by vertical and horizontal segments and later tilt it applying the strategy based on the use of cat maps that was described above. Other simple phase-space regions can also be programmed using variations of this method. It is also interesting to note that using variations of the circuit shown in Fig. 3 we can also subtract values of the Wigner function in different phase-space regions (which could be useful if one is interested in comparing their values).

V. CONCLUSIONS

In this paper we established the existence of a gate array that can be programmed to evaluate the expectation value of any operator acting on an N-dimensional Hilbert space. The expectation value is obtained by measuring the polarization of a single auxiliary qubit. As an example, we showed how to program the evaluation of sums of values of the discrete Wigner function over various simple phase-space domains. It is important to mention that our method is only efficient to determine if the sum of the Wigner function in a phase space domain [with up to $o(N)$ points] is greater than a fixed, N-independent, threshold (since this does not require exponential precision). This is a ''quantum decision problem'' whose input data (encoded in the system's state ρ) is inherently quantum. Due to the nature of the input data, this problem cannot even be formulated on a classical computer. Interest in problems with quantum input data have recently increased, partly due to their significance in connection with the potential detection of entanglement [15–17] as well as their relation with tomographic problems like the one described here. The problem of evaluating the expectation value of a Hermitian operator with a quantum circuit of fixed architecture was addressed in Ref. [16]. However, for the method to be applied one requires knowledge of the spectrum of the operator (or at least of its lowest eigenvalue) and the ability to prepare a state which, up to a rescaling, is proportional to the operator O. As mentioned above, the method we propose would apply in a slightly different situation where the information about the operator is restricted to the knowledge of the coefficients of its expansion in a given basis (of course, given such information one could compute the spectrum with an exponentially large overhead). The extension of some of the above results to continuous variables is still under investigation [17]. After completing this work we became aware of the related approach to the construction of quantum universal detectors presented in Ref. [18].

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