

Comment on “Sound velocity and multibranch Bogoliubov spectrum of an elongated Fermi superfluid in the BEC-BCS crossover”

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The work by T. K. Ghosh and K. Machida [Phys. Rev. A **73**, 013613 (2006) and cond-mat/0510160] on the sound velocity in a cylindrically confined Fermi superfluid obeying a power-law equation of state is shown to make use of an improper projection of the sound wave equation. This inaccuracy fully accounts for the difference between their results and those previously reported by Capuzzi *et al.* [Phys. Rev. A **73**, 021603(R) (2006) and cond-mat/0509323]. In this Comment, we show that both approaches lead exactly to the same result when the correct weight function is used in the projection. Plots of the correct behavior of the phonon and monopole-mode spectra in the BCS and unitary limits and in the BEC regime are also shown.

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In their recent study on sound propagation in an elongated Fermi superfluid in the BEC-BCS crossover, Ghosh and Machida [1] have reported a calculation of the sound velocity u_1 for the case of a power-law equation of state (EOS) as previously analyzed for bosons by Zaremba [2] and for fermions by Capuzzi *et al.* [3]. Their result is found to be incorrect due to the use of an improper projection of the sound wave equation.

The eigenvalue equation for small-amplitude density modes $\delta n_q(\mathbf{r}_\perp)e^{iqz}$ is obtained by linearization of the hydrodynamic equations around equilibrium and reads

$$M\omega^2\delta n_q = q^2(n_0\partial\mu/\partial n|_{n=n_0}\delta n_q) - \nabla_\perp \cdot [n_0\nabla_\perp(\partial\mu/\partial n|_{n=n_0}\delta n_q)], \quad (1)$$

where n_0 is the equilibrium density profile, ω the frequency of the perturbation, and q its wave vector along z . Equation (1) reduces to Eq. (8) in Ref. [1] for a power-law EOS $\mu(n) = Cn^\gamma$.

To obtain the dispersion relation $\omega(q)$ for any value of q , one must resort to the numerical solution of the eigenvalue equation (1). A possible method to solve such an equation consists of expanding the eigenmodes δn_q in a complete set of basis functions (see, e.g., Zaremba [2]) as

$$\delta n(\mathbf{r}_\perp) = \sum_\alpha b_\alpha \delta n_\alpha(\mathbf{r}_\perp), \quad (2)$$

where $\alpha = (n_r, m)$ labels the basis functions, with n_r the radial number and m the number for the azimuthal angular momentum. By inserting this expression into Eq. (1) and projecting the result onto an element of the basis, a matrix representation of the eigenvalue equation is found, which is suitable for a numerical solution. This procedure allows some freedom in the choice of the basis and of the projection, as long as these

satisfy the boundary conditions. However, in order to obtain a standard eigenvalue equation of the form

$$\lambda \mathbf{v} = \mathbf{A} \cdot \mathbf{v} \quad (3)$$

one must choose a projection in which the basis is orthogonal [4]. For the basis functions adopted in Ref. [1], cf. Eqs. (10) and (11), the projection must be performed with a weight function $w(r) \propto (1 - \tilde{r}^2)^{-\gamma_0}$ [5], where $\tilde{\mathbf{r}} = \mathbf{r}_\perp/R$ and $\gamma_0 = 1/\gamma - 1$, R being the radius of the density profile. The orthogonality condition thus reads

$$\int w(\tilde{r}) \delta n_\alpha^*(\tilde{\mathbf{r}}) \delta n_{\alpha'}(\tilde{\mathbf{r}}) d^2\tilde{r} \propto \delta_{\alpha\alpha'}. \quad (4)$$

Therefore, for Eq. (13) in Ref. [1] to be correct, the integrals defining the matrix $M_{\alpha\alpha'}$ must include the weight function $w(r)$ and thus read

$$M_{\alpha\alpha'} = A^2 \int d^2\tilde{r} (1 - \tilde{r}^2)^{\gamma_0\tilde{r}^2 + |m| + |m'|} e^{i(m-m')\phi} \times P_{n_r}^{(\gamma_0, |m'|)}(2\tilde{r}^2 - 1) P_{n_r}^{(\gamma_0, |m|)}(2\tilde{r}^2 - 1), \quad (5)$$

where $P_{n_r}^{(\gamma_0, m)}$ are Jacobi polynomials. Equation (5) is what should have been used in Ref. [1], instead of Eq. (14), where the weight function $w(r)$ is missing and the constant A takes a different value since $\delta n_\alpha(\mathbf{r}_\perp)$ has not been normalized with $w(r)$. A Fermi superfluid with $\gamma = 1$ corresponds to a Bose-Einstein condensate (BEC) of molecules and has been previously analyzed by Zaremba [2] for bosonic atoms. In this case $w(r) = 1$, and Eq. (5) reduces to Eq. (14) in [1].

To further analyze how the correct orthogonality condition affects the results, we have numerically solved the eigenvalue equation for sound propagation in a superfluid Fermi gas in the BCS and unitary limits, and in the BEC regime with $y = 1/k_F a = 0.25$. Our results for the two lowest frequency modes as functions of q are shown in Figs. 1 and 2. The lowest mode, shown in Fig. 1, is soundlike and has a phononic dispersion relation at long wavelengths. We observe that the slope of the dispersion relation, i.e., the sound velocity, is lower than that found in Ref. [1] and bends down as q increases. The first excited state, displayed in Fig. 2,

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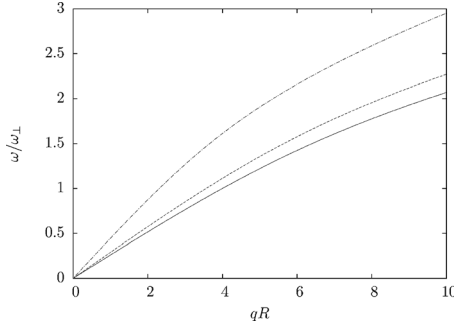


FIG. 1. Dispersion relation $\omega(q)$ (in units of ω_{\perp}) for the lowest-frequency (sound) mode as a function of qR with R the radius of the fermion density profile in the BCS limit. The dash-dotted, dashed, and solid lines correspond to fermions in the BCS and unitary limits, and in the BEC regime with $\gamma=0.25$, respectively.

corresponds to a monopolar compressional mode that for $q=0$ is purely radial. Furthermore, its frequency is known analytically at $q=0$ [6] as $\omega_0=\sqrt{10/3}\omega_{\perp}$ in the BCS and unitary limits and as $\omega_0=2\omega_{\perp}$ in the BEC limit ($\gamma\rightarrow+\infty$). Figure 2 in the present work is to be compared with Fig. 3 in Ref. [1], where the monopolar mode is not correctly depicted. We also note that the effective mass associated with this mode is also different from that found by Ghosh and Machida.

To obtain an analytical expression for the sound velocity $u_1\equiv d\omega(q)/dq|_{q=0}$, one can use Eq. (13) in Ref. [1] for the lowest-frequency mode and expand it to first order in q^2 . This demonstrates that the off-diagonal terms in $M_{\alpha\alpha'}$ do not enter the calculation of the sound velocity, as pointed out by Zaremba [2] for bosons. From the definition (5), one obtains $M_{00}=\gamma/(\gamma+1)$ and thus

$$u_1 = \sqrt{\frac{\gamma}{2\gamma+2}}v_F, \quad (6)$$

with $v_F=\sqrt{2\bar{\mu}/M}$ and $\bar{\mu}$ the chemical potential, in agreement with our result previously obtained in [3]. This is the sound velocity that also Ghosh and Machida should have obtained if they had taken into account the weight function $w(r)$. Their

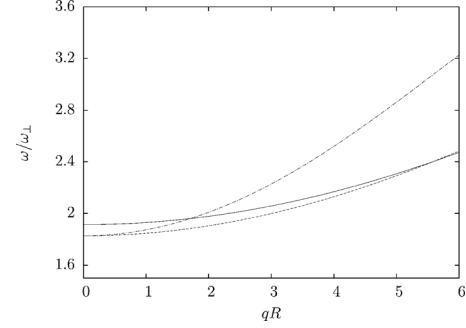


FIG. 2. The same as in Fig. 1 for the monopole mode.

improper projection of the eigenvalue equation leads them to the pathological expression $u_1=\sqrt{(2-\gamma)\gamma/4}v_F$, which predicts no sound propagation for $\gamma\geq 2$. The same problem affects the calculation of the effective mass m_b for the monopole mode [cf. Eq. (27) in Ref. [1]], which once corrected is

$$m_b = \frac{M\hbar\omega_{\perp}}{2\mu} \frac{(2+2\gamma)^{3/2}(1+3\gamma)}{\gamma(1+\gamma+2\gamma^2)}. \quad (7)$$

An alternative and more direct procedure to evaluate the sound velocity is the one that we have outlined in [3]. The spatial dependence is eliminated from Eq. (1) above by integrating in the (x,y) plane. This yields the dispersion relation

$$\omega(q) = q \left(\frac{1}{M} \int n_0 \partial\mu/\partial n|_{n=n_0} \delta n_q d^2r_{\perp} \bigg/ \int \delta n_q d^2r_{\perp} \right)^{1/2} \quad (8)$$

for a perturbation with $\int \delta n_q d^2r_{\perp} \neq 0$. Hence, the calculation of the sound velocity requires only the expression of δn_q calculated at $q=0$. By using $\delta n_{q=0}=(\partial\mu/\partial n|_{n=n_0})^{-1}$, we obtain

$$u_1 = \left(\frac{1}{M} \int n_0 d^2r_{\perp} \bigg/ \int (\partial\mu/\partial n|_{n=n_0})^{-1} d^2r_{\perp} \right)^{1/2}. \quad (9)$$

This expression provides the exact velocity of sound propagation in cylindrically confined hydrodynamic gases with EOS $\mu(n)$, and for $\mu(n)\propto n^{\gamma}$ leads to Eq. (6).

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