



The string theory on AdS_3 as a marginal deformation of a linear dilaton background

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Abstract

We investigate N -point string scattering amplitudes in AdS_3 space. Based on recent observations on the solutions of KZ and BPZ-type differential equations, we discuss how to describe the string theory in AdS_3 as a marginal deformation of a (flat) linear dilaton background. This representation resembles the called “discrete light-cone Liouville” realization as well as the FZZ dual description in terms of the sine-Liouville field theory. Consequently, the connection and differences between those and this realization are discussed. The free field representation presented here permits to understand the relation between correlators in both Liouville and WZNW theories in a very simple way. Within this framework, we discuss the spectrum and interactions of strings in Lorentzian AdS_3 .

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1. Introduction

In this paper we study a representation of the N -point correlation functions describing string scattering amplitudes in Euclidean AdS_3 . The tree-level amplitudes of strings in this space were extensively studied in the literature and received renewed attention since the formulation of the AdS/CFT correspondence. The first exact computation of three and two-point functions was done in Refs. [1,2] and was subsequently extended and studied in rigorous detail in Refs. [3–5] by Teshner. The interaction processes of winding string states were studied latter [6,7], after Maldacena and Ooguri proposed the inclusion of those states in the spectrum of the theory [8]. Besides, several formalisms yielding consistent results were shown to be useful tools for studying

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the correlators in this non-compact CFT [9–14]; however, the most fruitful method for analyzing the functional properties of these observables was, so far, the employing of the analogies existing between this CFT and its relative: the Liouville field theory. Though our understanding of correlation functions in Euclidean and Lorentzian AdS_3 was substantially increasing in the last years, several features remain still as open questions: The factorization properties of the generic four-point function and the closeness of the operator product expansion of unitary states are, perhaps, the most important puzzles from the viewpoint of the applications to string theory. The motivation of this work is that of studying some aspects of the N -point functions in this CFT for generic N . The general expression for the case $N > 3$ is not known, but a new insight about its functional form appeared recently due to the discovery of a very direct relation between these and analogous correlators in Liouville field theory [15–18]. Although the general expression of the N -point function is not known even for the Liouville field theory, such a connection between correlators in both theories still represents a great opportunity for exploring some aspects of their analytic properties. We will show here how the existence of such a connection leads to find a way of representing the worldsheet theory of AdS_3 strings in terms of a tachyonic linear dilaton background. This is an important observation due to the fact that, precisely, the representation of the $SL(2, R)_k/U(1)$ WZNW in terms of a tachyonic background (that is believed to be dual of it) turned out to be a useful method for describing the physics of strings in the black hole background; see for instance [10,11,21,29]. Among those results obtained in the last years by means of the employing of such connection between curved and linear-dilaton tachyonic backgrounds, the one we find most important is the construction of a matrix model for the 2D black hole geometry [19,20] which clearly shows the relevance that the study of this kind of duality has within the context of string theory. Here, we are interested in establishing a precise connection between string theory in AdS_3 and a similar tachyonic linear dilaton background; and we describe it in the next section. To be precise, the paper is organized as follows: in Section 2 we discuss how the Ribault formula, connecting correlators in both WZNW and Liouville CFTs, leads to a free field realization of the worldsheet theory on AdS_3 . Then, it turns out that this realization is a Liouville-like model on the worldsheet and can be thought of as a dual description as well. We consequently analyze the relation with other approaches and the CFT formulation carefully. In Section 3 we comment on the spectrum and interactions of strings in Lorentzian AdS_3 and make some remarks on the integral representation of the free field formulation we propose. In particular, we study the factorization limit of four-point functions in both WZNW and sine-Liouville model and focus our attention on the scattering processes that violate the winding number conservation in AdS_3 . We show that intermediate states with both winding number $\omega = 0$ and $\omega = 1$ arise in the internal channels of four-point amplitudes. We also discuss how the upper bound for the violation of total winding conservation in N -point correlators naturally appears in this context.

2. The linear dilaton background

2.1. The Ribault–Teschner formula

Recently, a new formula connecting correlators in both $SL(2, C)_k/SU(2)$ WZNW and Liouville conformal theories was discovered by Teschner and Ribault [16], who studied an improved version of a previous result due to Stoyanovsky [15]. In Ref. [17], Ribault extended the result of [16] by achieving to write down a formula connecting the N -point correlation functions in Euclidean AdS_3 string theory and certain subset of M -point functions in Liouville field theory,

where the relation between N and M turns out to be determined by the winding number of the string states involved in the correlators. If $\Phi_{j,m,\bar{m}}^\omega(z)$ represents the vertex operators in the WZNW model creating string states belonging to the flowed representations of $\widehat{sl}(2)_k \times \widehat{sl}(2)_k$ affine algebra, and $V_\alpha(z)$ represents the vertex operators of the quantum Liouville theory with cosmological term $\mu e^{\sqrt{2}b\varphi(z)}$, then the Ribault formula reads

$$\begin{aligned} & \left\langle \prod_{i=1}^N \Phi_{j_i, m_i, \bar{m}_i}(z_i) \right\rangle_{SL(2)} \\ &= N_k(j_1, \dots, j_N; m_1, \dots, m_N) \prod_{r=1}^M \int d^2w_r F_k(z_1, \dots, z_N; w_1, \dots, w_M) \\ & \quad \times \left\langle \prod_{t=1}^N V_{\alpha_t}(z_t) \prod_{r=1}^M V_{-\frac{1}{2b}}(w_r) \right\rangle_L, \end{aligned} \tag{1}$$

where the normalization factor is given by

$$N_k(j_1, \dots, j_N; m_1, \dots, m_N) = \frac{2\pi^{3-2N} b}{M! c_k^{M+2}} (\pi^2 \mu b^{-2})^{-s} \prod_{i=1}^N \frac{c_k \Gamma(m_i - j_i)}{\Gamma(1 + j_i - \bar{m}_i)} \tag{2}$$

while the z -dependent function is

$$\begin{aligned} & F_k(z_1, \dots, z_N; w_1, \dots, w_M) \\ &= \frac{\prod_{1 \leq r < l \leq N} |z_r - z_l|^{k-2(m_r+m_l+\omega_r\omega_l k/2+\omega_l m_r+\omega_r m_l)}}{\prod_{1 < r < l \leq M} |w_r - w_l|^{-k} \prod_{t=1}^N \prod_{r=1}^M |w_r - z_t|^{k-2m_t}} \\ & \quad \times \frac{\prod_{1 \leq r < l \leq N} (\bar{z}_r - \bar{z}_l)^{m_r+m_l-\bar{m}_r-\bar{m}_l+\omega_l(m_r-\bar{m}_r)+\omega_r(m_l-\bar{m}_l)}}{\prod_{1 < r < l \leq M} (\bar{w}_r - \bar{w}_l)^{m_l-\bar{m}_l}}. \end{aligned} \tag{3}$$

The parameter b of the Liouville theory is related to the Kac–Moody level k through

$$b^{-2} = k - 2, \tag{4}$$

while the quantum numbers of the states of both conformal models are also related each other in a simple way; namely

$$\alpha_i = b j_i + b + b^{-2}/2, \quad i = \{1, 2, \dots, N\}. \tag{5}$$

The factor c_k in (2) is a k -dependent (j -independent) normalization; see [17]. Furthermore, the following constraints also hold

$$\sum_{i=1}^N m_i = \sum_{i=1}^N \bar{m}_i = \frac{k}{2}(N - M - 2), \tag{6}$$

$$\sum_{i=1}^N \omega_i = M + 2 - N, \tag{7}$$

$$s = -b^{-1} \sum_{i=1}^N \alpha_i + b^{-2} \frac{M}{2} + 1 + b^{-2}. \tag{8}$$

In (8), the integer number s refers to the amount of screening operators $V_b(z) = \mu e^{\sqrt{2}b\varphi(z)}$ to be included in Liouville correlators in order to get a non vanishing result. The whole amount of vertex operators involved in the r.h.s. of (1), as mentioned, is given in terms of the winding numbers of the AdS_3 strings; namely

$$\sum_{i=1}^N \omega_i = M + 2 - N. \quad (9)$$

This allows for the possibility of describing scattering amplitudes that, in principle, violate the winding number conservation.

Notice that, according to (4), the central charge of the Liouville theory in the r.h.s. of (1) is given by $c = 1 + 6Q^2$ with $Q = b + b^{-1}$, while the central charge of the level k WZNW model is $c = 3 + 6b^{-2} = 3 + \frac{6}{k-2}$. Then, if one is interested in interpreting the relation (1) as stating the equivalence between a pair of conformal models (and we do have such intention), both central charges should be made to coincide (presumably by considering an additional internal CFT supplementing the Liouville action) and, besides, the overall factor $F_k(z_1, \dots, z_N; w_1, \dots, w_M)$ should be suitable to be interpreted as arising from the Wick contractions in the OPE of such internal theory in the r.h.s. Indeed, the aim of this note is that of demonstrating that Ribault formula (1) can be actually thought of as a realization of the worldsheet conformal theory of AdS_3 strings in terms of free fields. To be more precise: in the next section we explicitly construct a free field representation of the $sl(2)_k$ WZNW model in terms of a product of CFTs with the form

$$\text{Liouville} \times U(1) \times \text{time}, \quad (10)$$

where the factor *time* refers to a time-like free boson while the $U(1)$ corresponds to a field with non-trivial background charge (see below for details). Within this framework, the Ribault formula is better understood and, since it is rigorously deduced from the correspondence between KZ and BPZ differential equations [15], enables us to prove the equivalence between the worldsheet CFT of AdS_3 strings and a (tachyonic) linear dilaton background. The realization we propose here turns out to be reminiscent of the quoted FZZ conjectured duality and, besides, is closely related to the discrete light-cone Liouville approach [22] as we will demonstrate in Section 3. We will also show the connection with the Wakimoto free field representation in detail and discuss the computation of WZNW correlation functions within this framework. Such correlators actually correspond to the scattering amplitudes of winding strings in the AdS_3 space.

In order to avoid redundant introductions, we will employ the standard notation for describing both WZNW and Liouville conformal models and refer to the bibliography for details of the nomenclature and conventions.

2.2. The conformal field theory

Here, we attempt to construct a CFT theory with the property of realizing the right-hand side of the relation (1). This CFT will take the form (10), with the Liouville CFT as a particular factor. The first step for constructing such realization is supplementing the Liouville action with a pair of additional bosonic fields $X^0(z)$ and $X^1(z)$ with time-like and space-like signature respectively. These fields have correlators

$$\langle \varphi(z_a) \varphi(z_b) \rangle = \langle X^1(z_a) X^1(z_b) \rangle = -\langle X^0(z_a) X^0(z_b) \rangle = -2 \log |z_a - z_b|.$$

Moreover, we will demand that field $X^1(z)$ couples to the worldsheet curvature generating a linear dilaton term in the $U(1)$ direction. In terms of the Liouville field and its new partners the “free part” of the action reads

$$S = \frac{1}{4\pi} \int d^2z \left(-\partial\varphi\bar{\partial}\varphi + QR\varphi + \mu e^{\sqrt{2}b\varphi} + \partial X^0\bar{\partial}X^0 - \partial X^1\bar{\partial}X^1 + i\alpha_0 R X^1 \right), \tag{11}$$

where the first three terms correspond to the Liouville action, with $Q = b + b^{-1}$, and the last term represents a second “background charge” with $\alpha_0 = -\sqrt{k}$. On the other hand, a marginal deformation has to be also included in order to fully represent the r.h.s. of (1). This is due to the presence of the M additional integrated vertex operators $V_{-\frac{1}{2b}}(w)$ which, in our description, turn out to correspond to “screening operators”. Besides, the arising of powers of $(w_r - z)$ in (3) has to be realized as well. Indeed, if we consider a deformation of action (11) given by the introduction of the marginal term

$$\int d^2z \Phi_{\text{aux}}(z),$$

$$\Phi_{\text{aux}}(z) = V_{-\frac{1}{2b}}(z) \times e^{i\sqrt{\frac{1}{2b^2}+1}X^1(z)} \times \text{h.c.} = e^{-\sqrt{\frac{k-2}{2}}\varphi(z)+i\sqrt{\frac{k}{2}}X^1(z)} \times \text{h.c.} \tag{12}$$

then both the factor $F_k(z_1, \dots, z_N; w_1, \dots, w_M)$ and the M insertions of operators

$$\int d^2w V_{-\frac{1}{2b}}(w), \quad V_\alpha(w) = e^{\sqrt{2}\alpha\varphi(w)} \times \text{h.c.},$$

naturally arise in (1). It is easy to verify that $\Phi_{\text{aux}}^+(z)$ is a $(1, 1)$ -operator with respect to the CFT described by action (11) and stands in the correlators (i.e., a bunch of them) to screen the background charge α_0 (and to help operators $e^{\sqrt{2}\varphi(z)}$ in the task of screening the charge Q). The factor $\frac{1}{M!}$ in (1) is also consistent with the Coulomb gas-like realization of the correlators due to the multiplicity of permutation of the M screening to be inserted. Furthermore, the conservation laws

$$\sum_{i=1}^N \alpha_i - \frac{1}{2b}M + bs = Q, \tag{13}$$

$$\sum_{i=1}^N \left(m_i + \frac{k}{2}\omega_i \right) = \sum_{i=1}^N \left(\bar{m}_i + \frac{k}{2}\omega_i \right) = 0, \tag{14}$$

$$\sum_{i=1}^N \omega_i = M + 2 - N, \tag{15}$$

that are necessary for the correlators to be non-vanishing, agree with those conditions coming from the integration over the zero-mode of fields $\varphi(z)$, $X^0(z)$ and $X^1(z)$. In terms of the quantum numbers j_i and k Eq. (13) reads

$$\sum_{i=1}^N j_i + N + s - 1 + \frac{k-2}{2}(N - M - 2) = 0. \tag{16}$$

From (15) we conclude that, in order to conserve the total winding number in AdS_3 space, we have to consider the particular case $M = N - 2$ and, consequently,

$$\sum_{i=1}^N j_i + N + s = 1. \tag{17}$$

This leads to the configuration studied in [16]. Operator $\Phi_{\text{aux}}(z)$ arises in the action of the CFT as being coupled by a constant μ/c_k , according to the KPZ scaling manifested in Eq. (1). Thus, c_k controls the scale where the Liouville potential $e^{\sqrt{2}b\varphi(z)}$ competes with the screening term $e^{-\sqrt{\frac{2}{k-2}}\varphi(z)+i\sqrt{\frac{k}{2}}X(z)}$.

By using the coordinates $\rho(z)$, $X^0(z)$ and $X^1(z)$, the vertex operators representing the winding string states in AdS_3 would be given by the following expression (cf. (48))

$$\Phi_{j,m,\bar{m}}^\omega(z, \bar{z}) = \frac{c_k \Gamma(m-j)}{\Gamma(j+1-\bar{m})} V_{j+k/2}^{\sqrt{k-2}}(z, \bar{z}) \times e^{i\sqrt{\frac{2}{k}}(m-\frac{k}{2})X^1(z)-i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)X^0(z)} \times \text{h.c.}, \tag{18}$$

where h.c. stands for the anti-holomorphic contribution. The normalization $\frac{\Gamma(m-j)}{\Gamma(1+j-\bar{m})}$ is required to generate the m -dependent overall factor $N_k(j_1, \dots, j_N; m_1, \dots, m_N)$. Besides, notice that operators (18) have conformal dimension

$$\begin{aligned} h &= \alpha(Q - \alpha) + \left(m - \frac{k}{2}\right) + \frac{1}{k} \left(m - \frac{k}{2}\right)^2 - \frac{1}{k} \left(m + \frac{k}{2}\omega\right)^2 \\ &= -\frac{j(j+1)}{k-2} - m\omega - \frac{k}{4}\omega^2 \end{aligned} \tag{19}$$

which precisely coincides with the mass-shell conditions for string states in AdS_3 , see [8]. Then, the full theory turns out to be realized by the action

$$\begin{aligned} S &= \frac{1}{4\pi} \int d^2z \left(\partial X^0 \bar{\partial} X^0 - \partial X^1 \bar{\partial} X^1 - \partial \varphi \bar{\partial} \varphi + \frac{k-1}{\sqrt{2(k-2)}} R\varphi - i\sqrt{\frac{k}{2}} R X^1 \right. \\ &\quad \left. + \frac{\mu}{c_k} e^{-\sqrt{\frac{k-2}{2}}\varphi+i\sqrt{\frac{k}{2}}X^1} + \mu e^{\sqrt{\frac{2}{k-2}}\varphi} \right) \end{aligned} \tag{20}$$

and corresponds to the following stress tensor

$$\begin{aligned} T(z) &= \frac{1}{2} (\partial X^0(z))^2 - \frac{1}{2} (\partial X^1(z))^2 - i\sqrt{\frac{k}{2}} \partial^2 X^1(z) - \frac{1}{2} (\partial \varphi(z))^2 \\ &\quad + \frac{k-1}{\sqrt{2(k-2)}} \partial^2 \varphi(z). \end{aligned} \tag{21}$$

Hence, including the contributions of both fields $X^0(z)$ and $X^1(z)$, the central charge now results

$$c = c_\varphi + c_{X^0, X^1} = 1 + 6Q^2 + 2 - 6k = 3 + \frac{6}{k-2}, \tag{22}$$

and this exactly coincides with the central charge of the $SL(2, R)_k$ WZNW model. Since the theory actually presents a $s\bar{l}(2)_k \times s\bar{l}(2)_k$ symmetry, we can represent that algebra by means of the fields introduced above; namely

$$J^+(z) = \left(i\sqrt{\frac{k}{2}} \partial X^1(z) + \sqrt{\frac{k-2}{2}} \partial \varphi(z) \right) e^{-i\sqrt{\frac{2}{k}}(X^0(z)-(k-1)X^1(z))+\sqrt{2(k-2)}\varphi(z)},$$

$$\begin{aligned}
 J^-(z) &= \left(i\sqrt{\frac{k}{2}}(k-3)\partial X^1(z) + \sqrt{\frac{k-2}{2}}(2k-1)\partial\varphi(z) \right) \\
 &\quad \times e^{i\sqrt{\frac{2}{k}}(X^0(z)-(k-1)X^1(z))-\sqrt{2(k-2)}\varphi(z)}, \\
 J^3(z) &= i\sqrt{\frac{k}{2}}\partial X^0(z)
 \end{aligned}
 \tag{23}$$

and their anti-holomorphic analogues. These currents realize the appropriate OPE; namely

$$J^3(z)J^\pm(w) = \pm \frac{1}{(z-w)}J^\pm(w) + \dots,
 \tag{24}$$

$$J^3(z)J^3(w) = -\frac{k/2}{(z-w)^2} + \dots,
 \tag{25}$$

$$J^-(z)J^+(w) = \frac{k}{(z-w)^2} - \frac{2}{(z-w)}J^3(w) + \dots,
 \tag{26}$$

where the dots “...” represent regular terms in the short distance expansion. This means that the Fourier modes J_n^a , defined as

$$J_n^a = \frac{1}{2\pi i} \oint dz z^{-1-n} J^a(z)
 \tag{27}$$

with $a = \{3, +, -\}$, satisfy the $sl(2)_k$ affine Kac–Moody algebra, namely

$$\begin{aligned}
 [J_n^-, J_m^+] &= -2J_{n+m}^3 + nk\delta_{n,-m}, \\
 [J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm, \\
 [J_n^3, J_m^3] &= -n\frac{k}{2}\delta_{n,-m}.
 \end{aligned}$$

Then, our free field realization is now complete, and it enables to understand the Ribault formula (1) in a natural way; namely: as a simple realization of AdS_3 string theory in terms of a linear dilaton background with tachyon-type interactions represented by $(1, 1)$ -exponential fields (12). Moreover, due to the closed relation between the non-linear σ -model in both AdS_3 and in 2D black hole, this realization turns out to be reminiscent of the FZZ duality. In the next section we discuss this feature as well as the relation with other approaches based on linear dilaton description. It is worth mentioning that other interesting relations between two-dimensional CFTs and the theory with $\widehat{sl}(2)_k$ symmetry can be found in the literature. One of these, which in our opinion deserves particular attention, is the one presented in Ref. [23], where it is shown how to add appropriate ghost fields to an arbitrary CFT and we use them to construct the $\widehat{sl}(2)_k$ currents. It would be interesting to understand whether there exists a relation between our free field realization and such kind of embedding.

2.3. Free field representation

Here, as mentioned, we address the problem of establishing a connection between the free field realization presented above and those formalisms that, following analogous lines, were employed in the literature to represent string theory in AdS_3 . For instance, a natural question arises as to whether is a clear connection between the realization we presented here and the quoted Wakimoto free field representation of the $\widehat{sl}(2)_k$ current algebra. But, before analyzing

such a relation, we find convenient to comment on a second approach, called the discrete light-cone Liouville theory.

In Ref. [22], Hikida, Hosomichi and Sugawara “demonstrate(d) that string theory (on AdS_3 background) can be reformulated as a string theory defined on a linear dilaton background”. Our aim here is to show that the “approach of free field realization” of [22] is related to ours in a very simple way which, basically, corresponds to perform a $U(1)$ transformation on the fields and replace the marginal deformation in the action. Namely, let us define new coordinates by means of the rotation

$$\rho(z) = (1 - k)\varphi(z) + i\sqrt{k(k - 2)}X^1(z), \tag{28}$$

$$Y^1(z) = i\sqrt{k(k - 2)}\varphi(z) - (1 - k)X^1(z) \tag{29}$$

and simply rename

$$Y^0(z) = X^0(z). \tag{30}$$

This rotates the background charge as

$$\frac{Q}{\sqrt{2}}\varphi(z) - i\sqrt{\frac{k}{2}}X^1(z) = -\frac{1}{\sqrt{2(k - 2)}}\rho(z). \tag{31}$$

Hence, the dilaton field only depends on the new coordinate $\rho(z)$ and the theory becomes strongly coupled in the large $\rho(z)$ region. In these coordinates, the currents (23) take the usual form that appears in the standard bosonization of parafermions; namely

$$J^\pm(z) = \left(-i\sqrt{\frac{k}{2}}\partial Y^1(z) \pm \sqrt{\frac{k - 2}{2}}\partial\rho(z)\right)e^{\mp i\sqrt{\frac{2}{k}}(Y^0(z) + Y^1(z))}, \tag{32}$$

$$J^3(z) = i\sqrt{\frac{k}{2}}\partial Y^0(z). \tag{33}$$

Then we find that the Sugawara stress tensor takes the form

$$T(z) = -\frac{1}{2}(\partial\rho(z))^2 - \frac{1}{\sqrt{2(k - 2)}}\partial^2\rho(z) - \frac{1}{2}(\partial Y^1(z))^2 + \frac{1}{2}(\partial Y^0(z))^2, \tag{34}$$

which precisely corresponds to the free field realization discussed in Ref. [22]. In that paper, the interaction term to be added to the CFT (34) in order to fully describe the non-linear σ -model on AdS_3 was given by the (1, 1)-operator

$$\partial Y_L^1(z)\bar{\partial} Y_R^1(\bar{z})e^{-\sqrt{\frac{2}{k-2}}\rho(z,\bar{z})} \tag{35}$$

instead of $\Phi_{\text{aux}}(z)$ in (12). It is important mentioning that the action containing the interaction term (35) and its duality to the $N = 2$ Liouville (and sine-Liouville eventually) was also observed in Ref. [24] and further clarified in Ref. [25], where its connection to the $SL(2, R)_k/U(1)$ WZNW model is discussed. This was also studied in [26] as a particular screening charge of the model.

Furthermore, the new coordinates (28)–(29) are also useful to discuss another realization, based on the FZZ duality. In fact, in terms of the fields $\rho(z)$, $Y^1(z)$ and $Y^0(z)$, the sine-Liouville

theory is obtained by replacing the graviton-like marginal deformation (35) by the operator

$$\begin{aligned}
 & e^{-\sqrt{\frac{k-2}{2}}\rho(z)} \left(e^{i\sqrt{\frac{k}{2}}(Y_L^1(z) - Y_R^1(\bar{z}))} + e^{i\sqrt{\frac{k}{2}}(Y_R^1(\bar{z}) - Y_L^1(z))} \right) \\
 & = 2e^{-\sqrt{\frac{k-2}{2}}\rho(z)} \cos(\sqrt{k/2}(Y_L^1(z) - Y_R^1(\bar{z}))),
 \end{aligned} \tag{36}$$

where $Y_L^1(z)$ and $Y_R^1(\bar{z})$ refer to the left and right modes of the field respectively (cf. (65)). It is worth emphasizing that (35) and (36) represent two different marginal deformations of the same linear dilaton backgrounds which, besides, preserve the $\widehat{sl}(2)_k \times \widehat{sl}(2)_k$ symmetry, as it can be verified from the OPE with currents (23).

At this point, we are able to show the explicit connection with the Wakimoto free field representation as well. It follows from a (de)bosonization procedure. In fact, the Wakimoto free fields appear once we define

$$\beta(z) = J^-(z), \quad \gamma(z) = e^{-i\sqrt{\frac{2}{k}}(Y^0(z) + Y^1(z))}, \tag{37}$$

$$\phi(z) = \rho(z) + i\sqrt{\frac{k-2}{k}}(Y^0(z) + Y^1(z)). \tag{38}$$

This leads to find the standard representation for the stress-tensor, namely

$$T(z) = -\frac{1}{2}(\partial\phi(z))^2 - \frac{1}{\sqrt{2(k-2)}}\partial^2\phi(z) - \beta(z)\partial\gamma(z) \tag{39}$$

which, of course, corresponds to the Sugawara construction by employing (23). Regarding the interaction term (written in this language) the authors of [22] showed that (35) agrees with the usual form

$$\beta(z)\bar{\beta}(\bar{z})e^{-\sqrt{\frac{2}{k-2}}\phi(z)}$$

for the screening operator of the non-linear σ -model on AdS_3 [1,6], up to a total derivative of the zero-dimension operator $e^{-\sqrt{\frac{2}{k-2}}\rho(z)}$.

It is also worth pointing out that the relation between both marginal deformations (35) and (36) is similar to the one existing between the “standard” and the “conjugate” representations of the $SL(2, R)_k/U(1) \times U(1)$ WZNW model discussed in Refs. [6,27,28]. While (36) is associated to the representations $\Phi_{j,m,\bar{m}}^\omega(z)$ employed in [2], the “screening” of the form (35) corresponds to the conjugate representations $\tilde{\Phi}_{j,m,\bar{m}}^\omega(z)$ which are associated to the discrete states of the 2D black hole [27].

3. String theory on $AdS_3 \times \mathcal{N}$

3.1. The spectrum of the free theory

The string theory on Euclidean AdS_3 is described by the $SL(2, C)/SU(2)$ WZNW model, while the Lorentzian version is constructed in terms of its analytic extension to the $SL(2, R)_k$ WZNW, which is certainly less understood in what respects to the formal aspects. Within the context of string theory, the Kac–Moody level k of the WZNW theory turns out to correspond to the quotient between the typical string length scale l_s and the “AdS radius” $l \sim (-\Lambda)^{-1/2}$, namely $k \sim l^2/l_s^2$. Consequently, k controls the coupling of the theory leading, in the large k limit, to both classical and flat limit.

The Hilbert space of this theory is then given in terms of certain representations of $SL(2, R)_k \times SL(2, R)_k$. Then, the string states are described by vectors $|\Phi_{j,m,\bar{m}}^\omega\rangle$ which are defined by acting with the vertex operators $\Phi_{j,m,\bar{m}}^\omega(z)$ on the $SL(2, R)_k$ -invariant vacuum $|0\rangle$; namely

$$\lim_{z \rightarrow 0} \Phi_{j,m,\bar{m}}^\omega(z)|0\rangle = |\Phi_{j,m,\bar{m}}^\omega\rangle.$$

In order to define the string theory precisely, it is necessary to identify which is the subset of representations that have to be taken into account. Such a subset has to satisfy several requirements: In the case of the free theory these requirements are associated to the normalizability and unitarity of the string states. At the level of the interacting theory, additional properties are requested, as the closeness of the fusion rules, the factorization properties of N -point functions, etc.

Even in the case of the free string theory, the fact of considering non-compact curved backgrounds is not trivial at all. The main obstacle in constructing the space of states is the fact that, unlike what happens in flat space, in curved space the Virasoro constraints are not enough to decouple the negative-norm string states. Then, in the early attempts for constructing a consistent string theory in AdS_3 , additional ad hoc constraints were imposed on the vectors of the $SL(2, R)_k$ representations. Usually, the vectors of $SL(2, R)$ representations are labeled by a pair of indices j and m , and such additional constraints, demanded as sufficient conditions for unitarity, implied an upper bound for the index j of certain representations (namely, the discrete ones and consequently for the mass spectrum). The modern approaches to the “negative norm states problem” include such a kind of constraint as well, although it does not implies a bound on the mass spectrum as the old versions do; see [7,39–42] for details. The mentioned upper bound for the index j of discrete representations, often called “unitarity bound”, reads

$$\frac{1-k}{2} < j < -\frac{1}{2}. \tag{40}$$

In the case of Euclidean AdS_3 , the spectrum of string theory is given by the continuous series of $SL(2, C)$, parameterized by the values $j = -\frac{1}{2} + i\lambda$ with $\lambda \in R$ and by real m . Besides, the case of string theory in Lorentzian AdS_3 is richer and its spectrum is composed by states belonging to both continuous $C_\lambda^{\alpha,\omega}$ and discrete $D_j^{\omega,\pm}$ series. The continuous series $C_\lambda^{\alpha,\omega}$ have states with $j = -\frac{1}{2} + i\lambda$ with $\lambda \in R$ and $m - \alpha \in Z$, with $\alpha \in [0, 1) \in R$ (as in $SL(2, C)$, obviously). On the other hand, the states of discrete representations $D_j^{\pm,\omega}$ satisfy $j = \pm m - n$ with $n \in Z_{\geq 0}$. So, the next step is explaining the index ω : In order to fully parameterize the spectrum in AdS_3 we have to introduce the “flowed” operators \tilde{J}_n^a ($a \in \{3, -, +\}$), which are defined through the spectral flow automorphism

$$J_n^3 \rightarrow \tilde{J}_n^3 = J_n^3 + \frac{k}{2}\omega\delta_{n,0}, \tag{41}$$

$$J_n^\pm \rightarrow \tilde{J}_n^\pm = J_{n\pm\omega}^\pm \tag{42}$$

acting of the original $\widehat{sl}(2)_k$ generators J_n^a . Then, states $|\Phi_{j,m,\bar{m}}^\omega\rangle$ belonging to the discrete representations $D_j^{\pm,\omega}$ are those obeying

$$\tilde{J}_0^\pm |\Phi_{j,m,\bar{m}}^\omega\rangle = (\pm j - m) |\Phi_{j,m\pm 1,\bar{m}}^\omega\rangle, \tag{43}$$

$$\tilde{J}_0^3 |\Phi_{j,m,\bar{m}}^\omega\rangle = m |\Phi_{j,m,\bar{m}}^\omega\rangle \tag{44}$$

and being annihilated by the positive modes, namely

$$\tilde{J}_n^a |\Phi_{j,m,\tilde{m}}^\omega\rangle = 0, \quad n > 0; \tag{45}$$

and analogously for the anti-holomorphic modes. Notice that states with $m = \pm j$ represent highest (respectively lowest) weight states. On the other hand, primary states of the continuous representations $C_\lambda^{\alpha,\omega}$ turn out to be annihilated by all the positive modes, as in (45), though for any zero-mode J_0^a . The excited states in the spectrum are defined by acting with the negative modes J_{-n}^a ($n \in \mathbb{Z}_{>0}$) on the Kac–Moody primaries $|\Phi_{j,m,\tilde{m}}^\omega\rangle$; these negative modes play the role of creation operators (i.e., creating the string excitation). The “flowed states” (namely those being primary vectors with respect to the \tilde{J}_n^a defined with $|\omega| > 1$) are not primary with respect to the $\widehat{sl}(2)_k$ algebra generated by J_n^a , and this is clearly because of (42). However, highest weight states in the series $\mathcal{D}_j^{\pm,\omega}$ are identified with lowest weight states of $\mathcal{D}_{-k/2-j}^{-,\omega}$, which means that spectral flow with $|\omega| = 1$ is closed among certain subset of Kac–Moody primaries.

The states belonging to discrete representations have a discrete energy spectrum and represent the quantum version of those string states that are confined in the centre of AdS_3 space; these are called “short strings” and are closely related to the states that are confined close to the tip in the 2D Euclidean black hole geometry. On the other hand, the states of the continuous representations describe massive “long strings” that can escape to the infinity because of the coupling with the NS–NS field. For the long strings, the quantum number ω can be actually thought of as a winding number asymptotically.

The vertex operators of the theory on $AdS_3 \times \mathcal{N}$ are then given in terms of the $SL(2, R)_k$ representations and take the form

$$\Phi_{j,m,\tilde{m},q}^\omega(z) = \Phi_{j,m,\tilde{m}}^\omega(z) \times V_q(z), \tag{46}$$

where $V_q(z)$ represents the vertex operator on the “internal” CFT defined on \mathcal{N} . The Virasoro constraint $L_0 = 1$ then implies that the conformal dimension of the fields satisfies

$$h_{(j,m,q)} = -\frac{j(j+1)}{k-2} - m\omega - \frac{k}{4}\omega^2 + h_{\mathcal{N}}(q) = 1, \tag{47}$$

where $h_{\mathcal{N}}(q)$ is the conformal dimension of $V_q(z)$. This agrees with the functional form we obtained in Section 2. Regarding the value of k , this is given by $k = 2(26 - c_{\mathcal{N}})/(23 - c_{\mathcal{N}})$, where $c_{\mathcal{N}}$ is the central charge on \mathcal{N} .

Besides those we commented above, there are more unitary representations which, in principle, could be also included in the definition of the Hilbert space: For instance, we have the complementary series \mathcal{E}_α^j with $-1 < j < -\frac{1}{2}$, $-\frac{1}{2} - j < |\alpha - \frac{1}{2}|$ and $m - \alpha \in \mathbb{Z}$. It is usually assumed that these representations are not explicitly (but implicitly) present in the space of states since the continuous and discrete series form a complete basis of the square integrable functions on AdS_3 (and thus all additional states would be already taken into account as certain linear combinations of states in $\mathcal{D}_j^{\pm,\omega}$ and $C_\lambda^{\alpha,\omega}$). However, this argument is valid in the large k regime, where the semi-classical arguments based on the differential functions on the space make sense. For instance, in order to conclude that the complementary series are to be excluded it would be necessary to show that, in addition, such states do not decouple when taking the large k limit in the operator product expansion: If those state do not decouple then it would be necessary to exclude them in the definition of the Hilbert space. Conversely, if their do decouple in the semi-classical limit, then the argumentation in terms of the composition of the Hilbert space in the large k limit is not enough for excluding them ab initio and they could be actually necessary. However, there exists more evidence in favour of the idea that the whole spectrum of the theory

is constructed just by the continuous and discrete series; this is suggested by the modular invariance of the one-loop partition function in thermal AdS_3 (studied in Ref. [36]) and the fact that only long strings and short strings do appear in the spectrum.

Besides, other states, having momentum $j = -((n + 1)(k - 2) + 1)/2$ with $n \in \mathbb{Z}_{\geq 0}$, also appear in the theory. These are certainly non-perturbative states since, for heavy states, they have masses of the order j and, hence, of the order of the string tension. Since k controls the semi-classical limit, these states would decouple from the perturbative spectrum when k goes to infinity. These states are associate to worldsheet instantons and the semi-classical interpretation of them relates their presence with the emergence of non-local effects in the dual CFT on the boundary (according to the AdS/CFT correspondence).

3.2. Discrete light-cone Liouville theory

Now, we discuss the indices j, m and ω of $SL(2, R)_k$ representation in the context of the free field description.

In our realization we were considering operators of the form

$$\Phi_{j,m,\bar{m}}^\omega(z) \propto e^{\sqrt{\frac{2}{k-2}}(j+\frac{k}{2})\varphi(z)+i\sqrt{\frac{2}{k}}(m-\frac{k}{2})X^1(z)-i\sqrt{\frac{2}{k}}(m+\frac{k}{2})\omega X^0(z)}, \tag{48}$$

which can be actually thought of as operators on the product $\frac{SL(2,R)_k}{U(1)} \times \text{time}$, where the time-like $U(1)$ is parameterized by the bosonic field $X^0(z)$. The part of the vertex describing the theory on the coset has conformal dimension

$$h_{(j,m)} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}. \tag{49}$$

Then, we notice that this formula remains invariant under the involution

$$j \rightarrow -j(k-1) - m(k-2) - k/2, \tag{50}$$

$$m \rightarrow jk + m(k-1) + k/2. \tag{51}$$

In order to extend this symmetry to the whole theory (i.e., to the theory on the product $\frac{SL(2,R)_k}{U(1)} \times U(1)$) we have to consider a transformation on the spectral flow parameter ω as well; namely

$$\omega \rightarrow -\omega - 1 - 2(j+m). \tag{52}$$

Performing transformation (50)–(52) on (48), we get

$$V_{j,m,\bar{m},\omega}(z) = e^{-\sqrt{\frac{2}{k-2}}(j(k-1)+m(k-2))\varphi(z)+i\sqrt{\frac{2}{k}}(j+m(k-1))X^1(z)-i\sqrt{\frac{2}{k}}(m+\frac{k}{2})\omega X^0(z)} \tag{53}$$

and, expressed in terms of fields $\rho(z), Y^0(z)$ and $Y^1(z)$, these operators become

$$V_{j,m,\bar{m},\omega} = e^{\sqrt{\frac{2}{k-2}}\rho(z)-i\sqrt{\frac{2}{k}}mY^1(z)-i\sqrt{\frac{2}{k}}(m+\frac{k}{2})Y^0(z)}, \tag{54}$$

which are precisely those that were considered in the discrete light-cone Liouville approach to string theory in AdS_3 discussed in [22] (up to time-reversion $Y^0(z) \rightarrow -Y^0(z)$). In Ref. [22] the authors proposed that the winding number of strings in AdS_3 can be seen as a quantized momentum in the direction $(Y^1 - Y^0)/\sqrt{2}$ which, according to the discrete light-cone prescription, turns out to be compactified with period $2\pi\sqrt{2/k}$; namely

$$Y^0(z) - Y^1(z) = Y^0(z) - Y^1(z) + \frac{4\pi n}{\sqrt{k}}, \quad n \in \mathbb{Z}. \tag{55}$$

Within this framework, the index ω corresponds to the topological winding along the compact light-like direction, and in terms of the Liouville field $\varphi(z)$ this direction reads

$$X^0(z) - X^1(z) - k \left(i \sqrt{\frac{k-2}{k}} \varphi(z) - X^1(z) \right). \quad (56)$$

Now, the question is: what do the quantum numbers j and m mean in terms of this free field representation? To understand this, let us comment on the $SL(2, R)_k$ representations again: as already mentioned, the principal continuous series $\mathcal{C}_\lambda^{\alpha, \omega}$ correspond to $j = -\frac{1}{2} + i\lambda$ with $\lambda \in R$ and, through (50), this results in the new values $j = -\frac{1}{2} + i\tilde{\lambda} - m(k-2)$ with $\tilde{\lambda} \in R$, which only seem to belong to the continuous series if $m = 0$. Besides, (51) implies that, after performing the change (51), m turns out to be a non-real number. Then, the relation between (48) and (53) cannot be thought of as a simple identification between states of different continuous representations but it does correspond to different free field realizations (at least in what respects to the continuous series $\mathcal{C}_\lambda^{\alpha, \omega}$). On the other hand, regarding the discrete representations, it is worth mentioning that the quantity $j + m$ remains invariant under the involution (50)–(51); though it is not the case for the difference $j - m$. Unlike the states of continuous representations, transformation (50)–(52) is closed among certain subset of states of discrete representations. This is because such transformation maps states of the discrete series with $2(j + m) \in Z$ among themselves. In particular, the case $m + j = 0$ corresponds to the well-known identification between discrete series $\mathcal{D}_j^{\pm, \omega=0}$ and $\mathcal{D}_{-k/2-j}^{\mp, \omega=\pm 1}$ since in that case (50)–(52) reduce to $j \rightarrow -k/2 - j$, $m \rightarrow k/2 - m = k/2 - j$, $\omega \rightarrow -1 - \omega$. The relation between quantum numbers manifested by (50) and (51) permits to visualize the relation between the vertex considered in our construction and those of Ref. [22] and consequently relate the free field realization (20) with the one of the discrete light-cone approach. The relation between both is a kind of “twisting” and is closely related to the representations studied in [27]. Other difference regards the interaction: unlike the free field representation based on action (20), the one of [22] comes from a field redefinition (through a bosonization and a $SO(1, 2)$ transformation after of it) of the non-linear σ -model on AdS_3 . Thus, the action employed in the discrete light-cone realization of AdS_3 strings involves the interaction term (35). This makes an important difference when trying to compute correlation functions. In [22] it was discussed how the interaction term (35) plays an important role for the emergence of short string states in the spectrum. Within such framework, such states are an effect due to the interaction term.

In the following subsections we study what happens when the interactions are taken into account.

3.3. The spectrum of the interacting theory

When interactions are considered, additional restrictions to those imposed for unitarity on the free spectrum may appear. This is because, besides the requirement of normalizability and unitarity, it is necessary to guarantee that the fusion rules result closed among the unitary spectrum. Otherwise, it could be possible to produce negative norm states by scattering processes involving unitarity incoming states. This problem was addressed in [7,36–38]. After integrating the N -point correlators over the worldsheet insertions z_a ($N \in \{1, 2, \dots, N\}$), the pole structure of scattering amplitudes and the factorization properties of them could also result in additional constraints on the external (incoming and outgoing) states, and in some examples such constraints can be even more restrictive than those required for unitarity. For instance, in the case of string theory in AdS_3 the N -point scattering amplitudes are known to be well defined only if the external momenta of

the states involved in the process satisfy

$$\sum_{a=1}^N j_a > 3 - N - k. \tag{57}$$

This restriction is stronger than the unitarity bound (40) imposed on each state. In Ref. [7], the condition (57) was analyzed from the viewpoint of the AdS/CFT correspondence and it was concluded that only those correlators satisfying such constraint would have a clear interpretation in terms of a local CFT on the boundary.

We explicitly compute the interactions in the next subsection, where we pay particular attention to the integral representation of correlation functions and its factorization properties.

3.4. Four-point function

Now, we move to the study of the factorization with the attention focused on the role played by the winding states. Let us begin by considering the integral representation of the four-point function; namely

$$\begin{aligned} & \left\langle \prod_{a=1}^4 \Phi_{j_a, m_a, \bar{m}_a}^{\omega_a=0}(z_a) \right\rangle \\ &= \frac{b}{\pi^5} (\pi^2 \mu b^{-2})^{-s} \prod_{a=1}^4 \frac{\Gamma(m_a - j_a)}{\Gamma(j_a + \bar{m}_a + 1)} \prod_{i=1}^2 \int d^2 w_i \prod_{r=1}^s \int d^2 v_r |w_1 - w_2|^2 \\ & \times \prod_{a < b}^4 |z_a - z_b|^{-2(m_a + m_b) - \frac{4}{k-2}(j_a j_b + \frac{k}{2}(j_a + j_b + 1))} \prod_{a=1}^4 \prod_{r=1}^s |z_a - v_r|^{-2 - \frac{2}{k-2}(j_a + 1)} \\ & \times \prod_{a=1}^4 \prod_{i=1}^2 |z_a - w_i|^{2(m_a + j_a)} \prod_{r=1}^s \prod_{i=1}^2 |v_r - w_i|^2 \prod_{r < t}^s |v_r - v_t|^{\frac{4}{2-k}}. \end{aligned} \tag{58}$$

Actually, this is the Coulomb gas-like representation of the Tschner–Ribault where, for simplicity, we considered the case $m_a = \bar{m}_a$ ($a = 1, 2, 3, 4$), while the general case with $m - \bar{m} \in \mathbb{Z}$ and generic N is discussed in the following subsection for the case of sine-Liouville theory in more detail. Besides, we will consider the case where k is large enough (though finite) to allow configurations of the form $-2j_a \in N$ within the unitarity bound (40). This case allows for the correlators to be “resonant” in the sense that can be realized by using an integer numbers of screening charges since we assume here the particular cases $-\sum_{a=1}^4 j_a \in \mathbb{Z}_{>0}$. In (58), the inserting points w_i refer to those where the screenings of the kind $\Phi_{\text{aux}}(w)$ are inserted, while the points v_r are the locations of the screenings $V_b(v)$.

Now, let us consider the coincidence limit of a pair of external states (let us say $z_2 \rightarrow z_1$ renaming $z_1 = 0$ and $z_2 = \epsilon$). This enables to study the limit where the four-point amplitude factorizes into a pair of three-point sub-amplitudes, and the mass-shell conditions of the intermediate states that are interchanged between both subprocesses arise as poles developed in the coincidence limit $\epsilon \rightarrow 0$. In order to analyze this factorization limit, we should also sum over the different ways of taking the limit and, in principle, take \tilde{s} operators of the form $V_b(v)$ and p operators of the form $\Phi_{\text{aux}}(w)$ going to $z_1 = 0$ as well (with all the possible \tilde{s} and p ; $0 \leq \tilde{s} \leq s$ and $0 \leq p \leq 2$ since those are the total amount of screening operators that are available). To parameterize the limit, it is convenient to rename the inserting points as $v_r = v_r/z_2$ for those with $r \leq \tilde{s}$

and $w_i = w_i/z_2$ for those with $i \leq p$. By explicitly taking the limit $z_2 \rightarrow z_1 = 0$ and collecting the powers of $|z_2|$ we find that the four-point correlator (58) factorizes into two pieces: a three-point correlator in the left-hand side containing p screenings $\Phi_{\text{aux}}(w)$ and \tilde{s} screenings $V_b(v)$, and a second three-point function in the right-hand side which is composed by $2 - p$ screenings $\Phi_{\text{aux}}(w)$ and $s - \tilde{s}$ screenings of the kind $V_b(v)$. The intermediate states, which are interchanged in the channels of the four-point string scattering process, satisfy the mass-shell conditions that can be read of as pole conditions of the propagator; these pole conditions are extracted from the divergences after integrating over $\int d^2z_2$. Hence, as mentioned, such mass-shell conditions can be simply read from the exact value of the exponent η_p of $|z_2|^{2\eta_p}$; and this power turns out to be

$$\eta_p = (p - 1)(m_1 + m_2) + p(j_1 + j_2) + \delta_{p,2} + (p - 1)\tilde{s} + p + \frac{2}{k - 2} \left(j_1 j_2 + \frac{k}{2}(j_1 + j_1 + 1) + \tilde{s}(j_1 + j_2 + 2) - \frac{1}{2}\tilde{s}(\tilde{s} - 1) \right). \tag{59}$$

The case $p = 1$ corresponds to the factorization of (58) in terms of a pair of three-point sub-amplitudes conserving (each of them and the total one) the winding number. This is because the N -point Liouville amplitudes with $N - 2$ additional operators $\Phi_{\text{aux}}(w)$ represent WZNW amplitudes conserving $\sum_{a=1}^N \omega_a$. On the other hand, the case $p = 0$ corresponds to the case where the conserving winding correlator (58) splits into the sum of a pair of violating winding correlators. This is because the three-point function in the left-hand side has no screening of the kind $\Phi_{\text{aux}}(v)$ (a deficit) while the three-point function of the right-hand side has two of them (an excess).

In the case $p = 1$, the factor $|z_2|^{2\eta_p}$ standing in the limit $z_2 = \epsilon \rightarrow 0$ develops poles located at $\eta_1 = -n$ with $n \in Z_{\geq 0}$. This conditions can be written in the following convenient way

$$h_j = -\frac{j(j + 1)}{k - 2} + n = 1, \quad j = j_1 + j_2 + \tilde{s} + 1 = 0, \tag{60}$$

where $\tilde{s} = \{0, 1, 2, \dots, s\}$. This precisely agrees with the mass-shell condition of an intermediate state belonging to the $SL(2, R)_k$ representation with quantum numbers

$$m = m_1 + m_2, \quad \omega = 0.$$

Analogously, the poles arising in the case $p = 0$ are located at $\eta_0 = -n$ and this can also be written in a similar way; namely

$$h_j = -\frac{j(j + 1)}{k - 2} - m - \frac{k}{4} + n = 1, \quad j = j_1 + j_2 + \tilde{s} + \frac{k}{2} = 0, \tag{61}$$

where m and ω are now given by

$$m = m_1 + m_2 - \frac{k}{2}, \quad \omega = 1.$$

Again, Eq. (61) represents the mass-shell condition for the intermediate state belonging to the “flowed” $SL(2, R)_k$ representations. The case $p = 2$ is similar and shows that the amplitude (58) factorizes in such a way that the poles arising in the limit where two states of the four coincide can be naturally interpreted as mass-shell conditions of states of “flowed” $SL(2, R)_k$ representations with $\omega = 0$ and $\omega = 1$.

The factorization of four-point function was also studied in Ref. [7] by using the usual x -basis (by Fourier transforming in m). There, it was shown that the internal channels of the amplitudes contain contributions of both $\omega = 0$ and $\omega = 1$ winding states for certain processes and this is

consistent with our result here (see Section 4 of [7] for details). Moreover, it was claimed in [7] that only the four-point functions satisfying the constraint

$$j_1 + j_2 > -\frac{k+1}{2}, \quad j_3 + j_4 > -\frac{k+1}{2} \tag{62}$$

present a well behaved factorization limit enabling to interpret the pole structure as corresponding to physical state contributions in the intermediate channels. In fact, those correlation functions which do not satisfy (62) receive contributions from poles located at

$$j + j_1 + j_2 + k = -n, \quad n \in \mathbb{Z}_{\leq 0}, \tag{63}$$

and these are not suitable for natural interpretation. A deeper understanding of the additional conditions for the factorization in AdS_3 is a non-trivial aspect and requires further study indeed.

Now, we will compare the features of the free field representation discussed above with their analogues for the case of sine-Liouville field theory.

3.5. Relation with sine-Liouville field theory

Now, let us return to the tachyonic-type interaction term (36) (see Refs. [30–35] for related works on the exact tachyonic backgrounds). One of the most interesting aspect of the relation between AdS_3 string theory and the CFT on linear dilaton background (20) is the fact that it looks very much like the FZZ duality, which associates the 2D black hole (i.e., the $SL(2, R)_k/U(1)$ WZNW model) to the sine-Liouville field theory. Indeed, the realization (21) turns out to be reminiscent of such duality due to the presence of the exponential (tachyonic) term $\Phi_{aux}(z)$, being

$$\Phi_{aux}(z) = e^{-\sqrt{\frac{k-2}{2}}\rho(z) + i\sqrt{\frac{k}{2}}Y^1(z)} \times \text{h.c.} \tag{64}$$

Operator $\Phi_{aux}(z)$ is actually “a half” of the sine-Liouville interaction term (see (36) and (65) below and notice that it has the appropriate exponent). On the other hand, instead of $\Phi_{aux}(z)$, we could indistinctly include in (20) a slightly different interaction term; namely: its complex conjugate $\Phi_{aux}^*(z)$. Since the action is quadratic in the field $Y^1(z)$ then there is no reason for favoring a particular sign for $\pm Y^1(z)$. The conjugate interaction $\Phi_{aux}^*(z)$ would be “the other half” of the sine-Liouville interaction and differs from $\Phi_{aux}(z)$ only in the chirality of the $U(1)$ charge $i\sqrt{\frac{k}{2}}\partial X^1(z)$. Moreover, such a term would actually be required in order to allow for the violation of the winding number conservation of the coset theory in a positive amount $\sum_{i=1}^N \omega_i > 0$. Considering both terms together in the action leads to propose the correspondence between the WZNW model and the sine-Liouville (\times time) CFT through (1) since we could write

$$\Phi_{aux}(z) + \Phi_{aux}^*(z) = 2e^{-\sqrt{\frac{k-2}{2}}\rho(z)} \cos(\sqrt{k/2}(Y_L^1(z) + Y_R^1(\bar{z}))), \tag{65}$$

where, again, $Y_L^1(z)$ and $Y_R^1(\bar{z})$ refer to the left and right modes of the field respectively (cf. (36)). Besides, notice that if both $\Phi_{aux}^*(z)$ and $\Phi_{aux}(z)$ are included in the correlators, then the “cosmological term” $e^{\sqrt{2}b\varphi(z)}$ is not required anymore and can be excluded without restricting the generic correlators (i.e., if an analytic extension in the amount of screening charges M_{\pm} is considered; which is usual in the Coulomb gas-type realization, cf. [2,6]). This is because, in that case, the conservations laws coming from the integration over the zero-modes of the fields would

be the following ones

$$\sum_{i=1}^N j_i + N - 1 + \frac{k-2}{2}(M_- + M_+) = 0 \quad (s = 0), \tag{66}$$

$$\sum_{i=1}^N \left(m_i + \frac{k}{2} \omega_i \right) = \sum_{i=1}^N \left(\bar{m}_i + \frac{k}{2} \omega_i \right) = 0, \tag{67}$$

$$\sum_{i=1}^N \omega_i = (M_- - M_+) + 2 - N, \tag{68}$$

where M_+ and M_- refer to the amount of screening operators of the kind $\Phi_{\text{aux}}(z)$ and $\Phi_{\text{aux}}^*(z)$ to be included when realizing the correlators following the Coulomb gas-like prescription; (cf. [44] after replacing the notation as $j_i \rightarrow -1 - j_i$).

As it is well known, a similar relation between WZNW and sine-Liouville conformal models was originally conjectured by Fateev, Zamolodchikov and Zamolodchikov in their quoted unpublished paper, and this is the reason because it is known as the FZZ duality (see [43] for interesting discussions on related subjects). In the context we discussed here, a relation between the non-linear σ -model on AdS_3 and a sine-Liouville action seems to arise by a constructive procedure which simply follows from the identity (1), rigorously proven in [16,17]. It is worth emphasizing that, although it seems to be natural from the viewpoint of the symmetry under $\omega \rightarrow -\omega$, the inclusion of the term $\Phi_{\text{aux}}^*(z)$ (i.e., the “other half” of sine-Liouville) in the action (20) has to be assumed in order to eventually state such connection with the sine-Liouville field theory. However, the fact that both $\Phi_{\text{aux}}(z)$ and $\Phi_{\text{aux}}^*(z)$ contain the appropriate exponents turns out to be a suggestive fact. Hence, within this framework, the sine-Liouville field theory can be seen to arise as a relatively natural free field realization of the associated WZNW theory. In the following paragraphs we will analyze the correlation functions in sine-Liouville theory in order to emphasize the similarities existing without free field realization (58).

Let us begin by considering sine-Liouville model coupled to matter. The action of this CFT is

$$S = \frac{1}{2\pi} \int d^2z \left((\partial\rho)^2 - \frac{R\rho}{\sqrt{2(k-2)}} + 2e^{-\sqrt{\frac{k-2}{2}}\rho} \cos(\sqrt{k/2}(Y_L^1 - Y_R^1)) + \partial Y^\mu \bar{\partial} Y_\mu \right). \tag{69}$$

Here, the matter sector is represented by the bosons $Y^\mu(z)$, where $\mu \in \{0, 1, 2, \dots, d\}$. In Ref. [44] Fukuda and Hosomichi proposed a free field realization of the correlation functions in this theory by means of the insertion of two different screening operators¹ and by evaluating a Dotsenko–Fateev type integral representation (we describe this representation in detail below). The three-point correlators representing processes violating winding number were explicitly computed by the authors; this was achieved by the insertion of a different number of both screening charges. Similar integral representations have been discussed in the free field realization of string theory on Euclidean AdS_3 and in the case of two-dimensional black hole background. However, several differences can be noticed if a detailed comparison of these models is performed; the role played by the screening charges and the conjugated representations of the identity operator are examples of these distinctions.

¹ The screening operators realize the interaction term of sine-Liouville action; see (36).

The vertex operators creating primary states from the vacuum of the theory can be written in terms of the bosonic fields $\rho(z)$ and $Y^\mu(z)$ as follows²

$$\Phi_{j,m,\bar{m};q_i}(z) = e^{j\sqrt{\frac{2}{k-2}}\rho(z)+i\sqrt{\frac{2}{k}}(mY_L^1(z)-\bar{m}Y_R^1(\bar{z}))} e^{iq_i Y^i(z)} \tag{70}$$

with $i \in \{0, 2, 3, 4, \dots, d\}$. As before, the correlators of the fields are those of $U(1)$ bosons in two-dimensions; namely

$$\langle \rho(z_a)\rho(z_b) \rangle = -2 \log |z_a - z_b|, \quad \langle Y^\mu(z_a)Y^\nu(z_b) \rangle = -2\eta^{\mu\nu} \log |z_a - z_b| \tag{71}$$

with $\eta^{\mu\nu} = \text{diag}\{-1, +1, +1, \dots, +1\}$ and $\mu, \nu \in \{0, 1, 2, \dots, d\}$. The conformal dimension of (70) takes the form

$$h_{(j,m,q)} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k} + \frac{q_i q^i}{2} \tag{72}$$

satisfying the Virasoro constraint $h_{(j,m,q)} = \bar{h}_{(j,\bar{m},q)} = 1$ as requirement for the string theory. The quantum numbers parameterizing the spectrum fall within a lattice in the following form

$$p = m + \bar{m} \in Z, \tag{73}$$

$$\omega = \frac{m - \bar{m}}{k} \in Z \tag{74}$$

and the central charge of the model is given by

$$c = d + 2 + \frac{6}{k-2}. \tag{75}$$

Restriction $c = 26$ implies $k = 2(27 - d)/(24 - d)$ and is required to define the string theory as well.

Now, let us move to the correlation functions and the factorization procedure. As a first step in our analysis, it is convenient to start by reviewing the integral construction of Ref. [44] where a Coulomb gas-type realization was proposed. Actually, the treatment of the sine-Liouville interaction term as a perturbation is as usual in the Feigin–Fuchs realization. The introduction of the interaction effects can be viewed as the insertion of screening charges into the correlation functions. In this case, the screening operators are given by

$$\Phi_{\pm}(z) = \Phi_{1-\frac{k}{2}, \pm\frac{k}{2}, \pm\frac{k}{2}, 0}(z) = e^{\pm i\sqrt{\frac{k}{2}}(Y_L^1(z)-Y_R^1(\bar{z}))-\sqrt{\frac{k-2}{2}}\rho(z)}. \tag{76}$$

Then, the interaction term can be written as in (36); namely

$$\frac{1}{4\pi} \int d^2z (\Phi_+(z) + \Phi_-(z)). \tag{77}$$

Roughly speaking, $\Phi_+(z)$ and $\Phi_-(z)$ play the role of $\Phi_{\text{aux}}(z)$ and respectively $\Phi_{\text{aux}}^*(z)$ in (65). Accordingly, the Coulomb-gas like prescription, applied to the case of sine-Liouville theory, implies that the generic N -point function turns out to be proportional to

$$\sim \prod_{a=1}^N \int d^2z_a \prod_{r=1}^{s_+} \int d^2u_r \prod_{l=1}^{s_-} \int d^2v_l \left\langle \prod_{a=1}^N \Phi_{j_a m_a, \bar{m}_a, q_a}(z_a) \prod_{r=1}^{s_+} \Phi_+(u_r) \prod_{l=1}^{s_-} \Phi_-(v_l) \right\rangle \tag{78}$$

² Notice that we are indistinctly using the notation Φ to refer to both WZNW and sine-Liouville primaries.

which is the standard representation for the perturbative analysis of conformal models, where the expectation value is defined with respect to the free theory (i.e., without the sine-Liouville interaction term since this is already taken into account due to the presence of the screenings Φ_{\pm}). Then, by expanding the correlator, we find that this goes like

$$\begin{aligned}
 & \prod_{a=1}^N \int d^2 z_a \prod_{a < b}^{N-1, N} |z_a - z_b|^{-\frac{4j_a j_b}{k-2} + 2q_a^i q_{bi}} (z_a - z_b)^{\frac{2}{k} m_a m_b} (\bar{z}_a - \bar{z}_b)^{\frac{2}{k} \bar{m}_a \bar{m}_b} \\
 & \times \prod_{r=1}^{s_+} \int d^2 u_r \prod_{l=1}^{s_-} \int d^2 v_l \prod_{a=1}^N \left(\prod_{r=1}^{s_+} |z_a - u_r|^{2(j_a + m_a)} (\bar{z}_a - \bar{u}_r)^{-p_a} \right. \\
 & \times \left. \prod_{l=1}^{s_-} |z_a - v_l|^{2(j_a - m_a)} (\bar{z}_a - \bar{v}_l)^{p_a} \right) \\
 & \times \left(\prod_{r < t}^{s_+ - 1, s_+} |u_r - u_t|^2 \prod_{l < t}^{s_- - 1, s_-} |v_l - v_s|^2 \prod_{l=1}^{s_-} \prod_{r=1}^{s_+} |v_l - u_r|^{2-2k} \right). \tag{79}
 \end{aligned}$$

The charge symmetry conditions, yielding from the integration over the zero-modes of the fields, take in this case the form

$$\sum_{a=1}^N j_a + 1 = \frac{k-2}{2} (s_+ + s_-), \tag{80}$$

$$\sum_{a=1}^N m_a = \frac{k}{2} (s_+ - s_-), \tag{81}$$

$$\sum_{a=1}^N \bar{m}_a = \frac{k}{2} (s_- - s_+), \tag{82}$$

$$\sum_{a=1}^N q_a^i = 0. \tag{83}$$

It is instructive comparing (80) with (16) (after performing the Weyl reflection $j \rightarrow -1 - j$). This shows the parallelism between our realization of WZNW correlators and those quantities in sine-Liouville model.

Now, we are ready to see how the physical state conditions for the intermediate string states arise from the pole structure of the N -point functions in the factorization limit (in a similar way as we did it in the previous subsection for the WZNW correlators). In order to realize the factorization, we take again the coincidence limit for the inserting points of a pair of vertex operators, e.g., $z_1 \rightarrow z_2$. As we did it before, we take the limit of \tilde{s} screening operators going to z_1 (being $0 \leq \tilde{s} \leq s$), i.e., by splitting the set of screening operators in two different sectors of integration: those whose inserting points are taken as going to z_1 and those whose inserting points remain fixed in the worldsheet. Then, in order to take this limit conveniently, let us define the following change of variables

$$\varepsilon \equiv z_1 - z_2, \tag{84}$$

$$\varepsilon x_r \equiv z_1 - u_r, \quad r \leq \tilde{s}_+, \tag{85}$$

$$\varepsilon y_l \equiv z_1 - v_l, \quad l \leq \tilde{s}_-. \tag{86}$$

By replacing this in expression (79), we obtain that the (integrated) N -point correlators turns out to be proportional to

$$\begin{aligned} & \prod_{a=1}^N \int d^2 z_a \prod_{a < b}^{N-1, N} |z_a - z_b|^{-\frac{4j_a j_b}{k-2} + \frac{4}{k} m_a m_b + 2q_a^i q_{bi}} (\bar{z}_a - \bar{z}_b)^{\frac{2}{k} (\tilde{m}_a \tilde{m}_b - m_a m_b)} \\ & \times \sum_{\tilde{s}_+ = 0}^{s_+} \sum_{\tilde{s}_- = 0}^{s_-} |\varepsilon|^{\tilde{s}_+ (\tilde{s}_+ + 1) + \tilde{s}_- (\tilde{s}_- + 1) + (2-2k)\tilde{s}_+ \tilde{s}_-} C(s_+, \tilde{s}_+) C(s_-, \tilde{s}_-) \\ & \times \prod_{r=1}^{\tilde{s}_+} \int d^2 x_r \prod_{l=1}^{\tilde{s}_-} \int d^2 y_l \prod_{r=\tilde{s}_+ + 1}^{s_+} \int d^2 u_r \prod_{l=\tilde{s}_- + 1}^{s_-} \int d^2 v_l |W(\varepsilon; z_a, u_r, v_l)|^2 \\ & \times \prod_{a=1}^n \left(\prod_{r=\tilde{s}_+ + 1}^{s_+} |z_a - u_r|^{2(j_a + m_a)} (\bar{z}_a - \bar{u}_r)^{-p_a} \prod_{l=\tilde{s}_- + 1}^{s_-} |z_a - v_l|^{2(j_a - m_a)} (\bar{z}_a - \bar{v}_l)^{p_a} \right) \\ & \times \prod_{r < t}^{\tilde{s}_+ - 1, \tilde{s}_+} |x_r - x_t|^2 \prod_{l < t}^{\tilde{s}_- - 1, \tilde{s}_-} |y_l - y_t|^2 \prod_{l=1}^{\tilde{s}_-} \prod_{r=1}^{\tilde{s}_+} |y_l - x_r|^{2-2k} \\ & \times \prod_{t > r = \tilde{s}_+ + 1}^{s_+, s_+ - 1} |u_t - u_r|^2 \prod_{t > l = \tilde{s}_- + 1}^{s_-, s_- - 1} |v_t - v_l|^2 \prod_{l = \tilde{s}_- + 1}^{s_-} \prod_{r = \tilde{s}_+ + 1}^{s_+} |v_l - u_r|^{2-2k}, \end{aligned} \tag{87}$$

where we defined the function

$$\begin{aligned} & |W(\varepsilon; z_a, u_r, v_l)|^2 \\ & = \prod_{a=1}^N \prod_{r=1}^{\tilde{s}_+} |z_a - z_1 + \varepsilon x_r|^{2(j_a + m_a)} (\bar{z}_a - \bar{z}_1 + \bar{\varepsilon} \bar{x}_r)^{-p_a} \\ & \times \prod_{l=1}^{\tilde{s}_-} |z_a - z_1 + \varepsilon y_l|^{2(j_a - m_a)} (\bar{z}_a - \bar{z}_1 + \bar{\varepsilon} \bar{y}_l)^{p_a} \prod_{r=1}^{\tilde{s}_+} \prod_{t=\tilde{s}_+ + 1}^{s_+} |z_1 - \varepsilon x_r - u_t|^2 \\ & \times \prod_{l=1}^{\tilde{s}_-} \prod_{t=\tilde{s}_- + 1}^{s_-} |z_1 - \varepsilon y_l - v_t|^2 \prod_{l=\tilde{s}_- + 1}^{s_-} \prod_{r=1}^{\tilde{s}_+} |z_1 - v_l - \varepsilon x_r|^{2-2k} \\ & \times \prod_{l=1}^{\tilde{s}_-} \prod_{r=\tilde{s}_+ + 1}^{s_+} |z_1 - \varepsilon y_l - u_r|^{2-2k} \end{aligned} \tag{88}$$

and where the sums over \tilde{s}_\pm and the multiplicity factor $C(s_\pm, \tilde{s}_\pm) = \frac{\Gamma(s_\pm + 1)}{\Gamma(\tilde{s}_\pm + 1)\Gamma(s_\pm - \tilde{s}_\pm + 1)}$ stand by considering all the different ways of selecting \tilde{s}_\pm among s_\pm screening operators $\Phi_\pm(z)$ to be taken as going to z_1 in the factorization limit. As it can be seen directly from the definition of the variables x and y in (84)–(86), the factorization limit is realized by taking the limit $\varepsilon \rightarrow 0$. In order to explicitly identify the sequence of poles that appears in this limit it will be useful to

expand the above expression in terms of powers of ε , namely

$$|W(\varepsilon; z_a, u_r, v_l)|^2 \equiv W(\varepsilon; z_a, u_r, v_l)\bar{W}(\bar{\varepsilon}; \bar{z}_a, \bar{u}_r, \bar{v}_l) = \sum_{n, \bar{n} \in \mathbb{N}^2} |W_{n, \bar{n}}(z_a, u_r, v_l)|^2 \cdot \varepsilon^n \bar{\varepsilon}^{\bar{n}}, \tag{89}$$

where

$$|W_{n, \bar{n}}(z_a, u_r, v_l)|^2 = \frac{\partial_\varepsilon^n \partial_{\bar{\varepsilon}}^{\bar{n}} |W(\varepsilon; z_a, u_r, v_l)|^2_{(\varepsilon=\bar{\varepsilon}=0)}}{\Gamma(n+1)\Gamma(\bar{n}+1)}. \tag{90}$$

Then, we can collect the whole dependence of $(\varepsilon, \bar{\varepsilon})$ and therefore write down the following expression for the power expansion in the neighborhood of $\varepsilon \simeq 0$ and write

$$\int d^2\varepsilon \sum_{n, \bar{n} \in \mathbb{N}^2} |W_{n, \bar{n}}(z_a, u_r, v_l)|^2 \varepsilon^{\eta(j_a, m_a) + n} \bar{\varepsilon}^{\eta(j_a, -\bar{m}_a) + \bar{n}},$$

where we have denoted

$$\begin{aligned} \eta(j_a, m_a) &= (\tilde{s}_+ + \tilde{s}_-)(j_1 + j_2) + \frac{1}{2}(\tilde{s}_+ + \tilde{s}_-)(\tilde{s}_+ + \tilde{s}_- + 1) \\ &\quad - k\tilde{s}_+\tilde{s}_- - 2\frac{j_1 j_2}{k-2} + 2\frac{m_1 m_2}{k} + (m_1 + m_2)(\tilde{s}_+ - \tilde{s}_-) + q_1^i q_{2i}. \end{aligned}$$

Hence, the pole condition is $\eta(j_a, m_a) + n = \eta(j_a, -\bar{m}_a) + \bar{n} = -1$ and can be written as follows

$$\begin{aligned} &\frac{(j_1 + j_2 - (\frac{k}{2} - 1)(\tilde{s}_+ + \tilde{s}_-))(j_1 + j_2 - (\frac{k}{2} - 1)(\tilde{s}_+ + \tilde{s}_-) + 1)}{k-2} \\ &+ \frac{(q_1^i + q_2^i)(q_{1i} + q_{2i})}{2} + \frac{(m_1 + m_2 + \frac{k}{2}(\tilde{s}_+ - \tilde{s}_-))^2}{k} - 1 = 0, \end{aligned}$$

and analogously for $(m_a, \tilde{s}_\pm) \leftrightarrow (\bar{m}_a, \tilde{s}_\mp)$. Besides, if the physical constraint $h_{(j_a, m_a, q_a)} = \bar{h}_{(j_a, \bar{m}_a, q_a)} = 1$ is assumed to hold, the condition above can be written as a Virasoro constraint as well

$$h_{(j, m, q)} = -\frac{j(j+1)}{k-2} + \frac{m^2}{k} + \frac{q^i q_i}{2} = 1 \tag{91}$$

and analogously for $\bar{h}_{(j, \bar{m}, q)}$, where the quantum numbers $j, p = m + \bar{m}$ and $\omega = (m - \bar{m})/k$ are given by

$$j = j_1 + j_2 - \frac{k-2}{2}(\tilde{s}_+ + \tilde{s}_-), \tag{92}$$

$$p = p_1 + p_2, \quad q^i = q_1^i + q_2^i, \tag{93}$$

$$\omega = \omega_1 + \omega_2 + \tilde{s}_+ - \tilde{s}_- \tag{94}$$

with $0 \leq \tilde{s}_\pm \leq s_\pm$. Consequently, the following identifications hold

$$\sum_{a=3}^N j_a + j + 1 = \frac{k-2}{2}(s_+ + s_- - \tilde{s}_+ - \tilde{s}_-), \tag{95}$$

$$\sum_{a=3}^N p_a + p = 0, \quad \sum_{a=3}^N q_a^i + q^i = 0, \tag{96}$$

$$\sum_{a=3}^N \omega_a + \omega = s_- - \tilde{s}_- + \tilde{s}_+ - s_+ \tag{97}$$

with $i \in \{0, 2, 3, 4, \dots, d\}$.

Hence, in this way, we have obtained the quantum numbers of the intermediate channels in the tree-level scattering processes in sine-Liouville string theory. Again, it was achieved by writing the pole condition arising in the factorization limit as a Virasoro constraint for these physical states.

An interesting comment on the functional form of the correlators in the coincidence limit is related to the factors involved in the different terms of the power expansion in ε . The characteristic property of the realization described here is the explicit form in which the different contributions of internal states can be recognized. Notice that in the limit $\varepsilon \rightarrow 0$, the following factors arise in the expression (87)

$$\prod_{t=\tilde{s}_++1}^{s_+} ((z_1 - u_t)^{j_1+j_2+m_1+m_2+\tilde{s}_+-\tilde{s}_--k\tilde{s}_-} (\bar{z}_1 - \bar{u}_t)^{j_1+j_2-\bar{m}_1-\bar{m}_2+\tilde{s}_+-\tilde{s}_--k\tilde{s}_-})$$

$$\times \prod_{t=\tilde{s}_-+1}^{s_-} ((z_1 - v_t)^{j_1+j_2-m_1-m_2+\tilde{s}_+-\tilde{s}_--k\tilde{s}_+} (\bar{z}_1 - \bar{v}_t)^{j_1+j_2+\bar{m}_1+\bar{m}_2+\tilde{s}_+-\tilde{s}_--k\tilde{s}_+})$$

and, in an analogous way,

$$\prod_{a=3}^N |z_1 - z_a|^{-4\frac{ja(j_1+j_2)}{k-2} + 2j_a(\tilde{s}_++\tilde{s}_-) + 2q_{ai}(q_1^i+q_2^i)} (z_1 - z_a)^{\frac{2}{k}m_a(m_1+m_2)+m_a(\tilde{s}_+-\tilde{s}_-)}$$

$$\times (\bar{z}_1 - \bar{z}_a)^{\frac{2}{k}\bar{m}_a(\bar{m}_1+\bar{m}_2)-\bar{m}_a(\tilde{s}_+-\tilde{s}_-)}.$$

In fact, these are precisely the contributions required to express the original N -point functions in terms of the product of two different correlators on the sphere, since these factors allow to recover the $n = \bar{n} = 0$ contribution, which is simply interpreted as the product of two correlators involving only tachyonic (non-excited) states. Moreover, once the higher derivative terms arising in the power expansion (90) are taken into account, then the derivatives ∂_ε acting on the factors in (88) generate contributions which reproduce the contractions arising in the operator product expansions of the form $\partial_{z_1} \langle \rho(z_1) \rho(z_a) \rangle \simeq (z_a - z_1)^{-1}$, e.g.,

$$\partial_\varepsilon \prod_{a=2}^N \prod_{r=1}^{\tilde{s}_+} (z_a - z_1 + \varepsilon x_r)^{j_a+m_a}_{(\varepsilon=0)} = \prod_{a=1}^N \prod_{r=1}^{\tilde{s}_+} (z_a - z_1)^{j_a+m_a} \sum_{l=1}^{\tilde{s}_+} x_l \sum_{b=2}^N (j_b + m_b) (z_b - z_1)^{-1}$$

where, after evaluating $\varepsilon = 0$, each step of the sum over the index b must be put in correspondence with each term arising in the operator product expansion between an operator $\partial\rho(z_1)$ and an exponential field $e^{j\sqrt{\frac{2}{k-2}}\rho(z_a)}$ (and consequently, this is analogous for the contributions of the Y^μ fields). Then, we recover the higher order contributions interpreted as different terms in the sum over excited intermediate states.

Let us make a brief remarks about the symmetry under the interchange $(m, \bar{m}, s_\pm) \leftrightarrow (\bar{m}, m, s_\mp)$. This invariance of the whole expression reflects the fact that the dissipation of the winding number in the two-dimensional theory is characterized directly by the relative amount s_+ and s_- corresponding to both screening operators $\Phi_+(z)$ and $\Phi_-(z)$ respectively; this is due to

the chiral nature of the interaction term (76). This is because of the symmetry existing under $\omega \leftrightarrow -\omega$.

3.6. On winding number conservation

Before concluding, a word on the prescription for computing correlators in sine-Liouville. Actually, let us consider in detail the structure of the correlation functions representing processes that violate the winding number conservation in this CFT. This computation, originally presented in Ref. [44], is similar to previous analysis done for the case of the WZNW model, although some substantial differences do exist between calculations in both CFTs. Perhaps, the main difference to be emphasized regards the role played by the screening operators: while in sine-Liouville theory the violation of the winding number is produced by the presence of the sine-Liouville interaction term itself (which is manifestly represented in the free field realization by the insertion of a different amounts of both screening operators s_+ and s_-), the violation of the winding number in the $SL(2, R)_k$ WZNW model is more confusing and involves the inclusion of conjugate representations of the identity operator (which is often called the “spectral flow operator”, see [6,7,27]). An interesting (curious) feature of the non-conservation of the winding in these CFTs is the existence of a bound for such violation. In particular, in Ref. [7] Maldacena and Ooguri explained how to understand the upper bound for the violation of the total winding number in $SL(2, R)_k$ WZNW correlators as being related to the $\widehat{sl}(2)_k$ symmetry of the theory (see Appendix D of Ref. [7]). Then, the question arises as to whether could be possible to conclude that, in analogous way, the upper bound for the violation of winding number in sine-Liouville theory turns out to be related to such symmetry as well. If this is indeed the case (and it seems to be due to the conjectured FZZ duality), then it should be feasible to verify that such a strange feature (i.e., the fact that the non-vanishing correlators satisfy $|\sum_{a=1}^N \omega_a| \leq N - 2$) turns out to be related to the fact that the FZZ sine-Liouville model satisfies a very particular relation between the radius R of the compact direction $Y^1(z)$ and the background charge of the field $\rho(z)$; namely $R = \sqrt{\frac{k}{2}}$, $Q_\rho = \frac{1}{\sqrt{k-2}}$. Presumably, there would be no reason for the upper bound of the violation of winding number to exist in the “generic” sine-Liouville field theory besides the FZZ dual radius $R = \sqrt{\frac{k}{2}}$.

In order to verify this idea, let us make some remarks on Fukuda–Hosomichi representation of sine-Liouville correlators: in a very careful analysis of the integral representation, the authors of [44] were able to show that it is feasible to translate the integrals $\prod_{r,l} \int d^2 u_r \int d^2 v_l$ over the whole complex plane into the product of contour integrals. Then, the integral representation (78) turns out to be described by standard techniques developed in the context of rational conformal field theory. Such techniques were used and extended in [44] in order to define a precise prescription to evaluate the correlators by giving the formula for the contour integrals in the particular case of sine-Liouville field theory. The first step in the calculation was to decompose the u_r complex variables (respectively v_l) into two independent real parameters (i.e., the real and imaginary part of u_r) which take values in the whole real line. Secondly, a Wick rotation for the imaginary part of (u_r) was performed in order to introduce a shifting parameter ε which was subsequently used to elude the poles in z_a . Then, the contours are taken in such a way that the poles at $v_r \rightarrow z_a$ are avoided by considering the alternative order with respect to this inserting points. A detailed description for the prescription can be found in the Section 3 of the paper, where the authors refer to the quoted works by Dotsenko and Fateev [45–47]. In the computation for sine-Liouville, Fukuda and Hosomichi proved that, in the case of three-point function, the

winding can be violated up to $|\sum_{a=1}^{N=3} \omega_a| \leq N - 2 = 1$ and, presumably, this is the same for generic N . The key point in obtaining such a constraint is noticing that the integrand that arises in the Coulomb gas-like prescription in sine-Liouville model contains contributions of the form (see the last line in (87))

$$\int d^2v_r d^2v_t |v_r - v_t|^2 \dots \tag{98}$$

for $0 \leq r, t \leq s_-$ and the same for the points u_l with $0 \leq l \leq s_+$, and where the dots “...” stand for “other dependences” on the inserting points v_r and u_l ; these points are those where the screenings of the kind $\Phi_-(u)$ and $\Phi_+(v)$ are respectively inserted. As explained in [44], and raised at the level of “lemma”, “the integral vanishes for certain alignments of contours” due to the fact that the exponent of $|v_r - v_t|$ in (98) is $+2$. Conversely, in the case where such exponent is generic enough (let us say 2ρ , following the notation of [44]), the “integral has a phase ambiguity due to the multi-valuedness of $|v_r - v_t|^{2\rho}$ in the integrand”. Then, it is concluded that those integrals containing two contours of v_r and v_t just next to each other, then the integral vanishes. And this precisely happens for all the contributions of those correlators satisfying $|s_+ - s_-| = |\sum_{a=1}^N \omega_a| > N - 2$. This led Fukuda and Hosomichi to prove that, for the three-point function, there are only three terms that contribute: one with $\sum_{a=1}^3 \omega_a = 1$, a second with $\sum_{a=1}^3 \omega_a = -1$ and the conserving one, $\sum_{a=1}^3 \omega_a = 0$.

Here, we can make two comments regarding this point: first, notice that an identical feature appears in our free field representation of WZNW correlators (58) and, in general, for the N -point function as well; namely: due to the presence of $M \leq N - 2$ additional screening operators $\Phi_{\text{aux}}(w)$ we get contributions like (98) in the integral representation (with w_i instead of v_r) due to the OPE

$$\Phi_{\text{aux}}(w_i)\Phi_{\text{aux}}(w_j) \sim |w_i - w_j|^2 \dots \tag{99}$$

This is exactly what happens in sine-Liouville theory and consequently explains the upper bound for the violation of winding number in our free field realization of (1) as well.

The second comment we find interesting is that we can actually give an answer about “how to relate the $\widehat{sl}(2)_k$ symmetry of sine-Liouville at the dual FZZ radius $R \sim \sqrt{k/2}$ and the existence of the upper bound for the violation of the winding”. The relation precisely comes from the fact that the exponent $+2$ in (98) does correspond to the particular value of the radius,³ leading to the very particular OPE

$$\Phi_{\pm}(v_r)\Phi_{\pm}(v_t) \sim |v_r - v_t|^{2\rho_{k,R}} \dots = |v_r - v_t|^2 \dots, \tag{100}$$

which mimics (99) because of $\rho_{k,\sqrt{k/2}} = 1$. The fact of demanding $\Phi_{\pm}(z)$ as being $(1, 1)$ -operators is not enough for affirming that the exponent in (100) is to be $2\rho_{k,R} = 2$; instead, this only holds for the “dual radius” $R = \sqrt{k/2}$ if the theory is requested to have the appropriate central charge (75).

This actually shows that the selection rule $|\sum_{a=1}^N \omega_a| \leq N - 2$ does correspond to a very particular point of the space of parameters of sine-Liouville field theory; i.e., the particular point where it turns out to be dual to the $SL(2, R)_k/U(1)$ WZNW model. We point out again that the same happens in our realization (58).

³ The author thanks J.M. Maldacena for a conversation about this point.

4. Conclusions

The Ribault formula (1), as well as the particular case $M = N - 2$ studied in Refs. [15,16], turns out to be a useful tool to study correlation functions in the non-compact WZNW model [18]. This is because such formula states a very concise relation between these observables and those of the Liouville field theory, which is certainly a better understood CFT. Here, we have gone further by arguing that, besides a mathematical coincidence between correlators of a pair of CFTs, Ribault formula is actually suggesting the equivalence between the worldsheet theory of AdS_3 winding strings and a tachyonic linear dilaton background that contains Liouville theory as a particular factor; see (10). Moreover, since (1) was rigorously proven independently of such a realization, this actually does demonstrate that string theory in AdS_3 can be represented in such a way. The description of worldsheet dynamics of strings in curved space in terms of flat linear dilaton backgrounds establishes an interesting relation between curved and flat exact solutions of non-critical string theory. The discrete light-cone Liouville description of AdS_3 string theory, as well as the quoted FZZ duality between the 2D black hole and the sine-Liouville model, are examples of such kind of description. According to what we discussed here, what Ribault formula is actually stating is an equivalence between the non-linear σ -model on AdS_3 and the CFT on the (tachyonic) linear dilaton background described by (20). And this equivalence holds at the level of the N -point correlation functions. The fact that this equivalence is reminiscent of the FZZ duality and makes contact with the discrete light-cone approach turns out to be one of the main observations of this note. We also showed the connection with the Wakimoto free field representation of $\widehat{sl}(2)_k$ algebra.

By analyzing the prescription for computing correlation functions and the factorization limit of these observables, we showed that the N -point function factorizes in two pieces, each one preserving the total winding number up to $N_{1,2} - 2$ units (with $N_1 + N_2 = N + 2$). By comparing with the Coulomb gas-like realization for the case of sine-Liouville field theory, we argued that such upper bound turns out to be related to the $\widehat{sl}(2)_k$ symmetry of the theory and this fact is clearly manifested in the integral representation of the correlators. The intermediate states in the four-point functions can be obtained by studying the pole structure in the factorization, and both states with winding number $\omega = 0$ and $\omega = \pm 1$ arise in the intermediate channels. This is consistent with previous results.

Then, in this note we proposed a new representation of string theory in Lorentzian AdS_3 space and this explicitly realize the Stoyanovsky–Ribault–Teschner formula (1) in terms of free fields. Besides, this prepares the basis for the next step: the study of the matrix model formulation within a similar context.

Note

In Ref. [17] it is asserted that Prof. V. Fateev has also considered a free field realization of the formula (1) in an unpublished paper. The author of this note would be interested in confirming the agreement between that realization and the one presented here. In any case, the realization analyzed here presents more evidence in favour of the conjecture made in [17] about the validity of the correspondence between WZNW and Liouville correlators for the maximally winding violating case (see Section 3.2.2 of [17]).

Note added

After the first version of this paper appeared in arXives, the author received a copy of the unpublished work [48] from Prof. V. Fateev. There, a free field representation of the theory is presented and several results agree with those of Section 2 here. This manifestly confirms the agreement between both analysis. The author would like to thank Vladimir Fateev for kindly sharing his note.

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