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Hypergeometric foundations of Fokker–Planck like equations

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ABSTRACT

We discover a deep connection between the Fokker–Planck equation and the hypergeometric differential equation. The same applies to a nonlinear generalization of such equation.

1. Introduction

In this paper we uncover the fact that the celebrated Fokker–Planck (FP) equation [1]

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial x}[K(x)F] + \frac{Q}{2}\frac{\partial^2 F}{\partial x^2} \tag{1.1}$$

exhibits a deep connection with a hypergeometric differential equation. In equation (1.1), F is the distribution function, $K(x)$ is the drift coefficient and Q is the diffusion coefficient (a positive quantity) [1]. The second term on the r.h.s. describes the effects of the fluctuating forces (diffusion term). Without it, (1.1) would describe deterministic motion (overdamped motion of a particle under the force $K(x)$). For the time being, we restrict ourselves to the case $K =$ constant. A similar hypergeometric derivation applies to **a nonlinear generalization** of equation (1.1), in the spirit of the one discussed by Plastino and Plastino [2].

Note that Eq. (1.1) is not just the Fokker–Planck equation, but also (up to appropriate scaling of F) encompasses all advection–diffusion equations (sometimes called convection–diffusion or advection–dispersion equations), with $K(x) =$ drift velocity and $Q/2 =$ diffusion coefficient (see, for instance, [3] and references therein). The discussion given below is therefore more general in its application.

This papers continues a line of research initiated by uncovering manifestations of hypergeometric equations in quantum equations [4].

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2. Deep connection between hypergeometric and Fokker–Planck equations

The ordinary hypergeometric function $F_1^2(a, b; c; z)$ is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Many second-order linear ODEs can be transformed into this equation. Generalized hypergeometric functions include the confluent hypergeometric function (also called Kummer's function) as a special case, which in turn has many particular special functions as special instances, such as elementary functions, Bessel functions, and the classical orthogonal polynomials. In particular, Kummer's function reads [4]

$$\phi(a, b, z) = \sum_{n=0}^{\infty} \frac{a_n}{b_n}\frac{z^n}{n!};\ a,\ b \in \mathcal{R}, \tag{2.1}$$

with a_n, b_n the Pochhammer symbols:

$$a_0 = 1, a_n = a\,(a+1)\,(a+2)\ldots(a+n-1);\ \text{same for } b. \tag{2.2}$$

The confluent hypergeometric (or Kummer's) function satisfies the second-order differential equation [4]:

$$z\phi''(a, b, z) + (b - z)\phi'(a, b, z) - a\phi(a, b, z) = 0, \tag{2.3}$$

and has accordingly two linearly independent solutions. One of them will be connected to the Fokker–Planck equation.

Eq. (2.3), for $a = b$, adopts the appearance

$$z\phi''(a,a,z) + (a-z)\phi'(a,a,z) - a\phi(a,a,z) = 0, \tag{2.4}$$

where primes indicate differentiation with respect to z. Accordingly [see (2.1)],

$$\phi(a,a,z) = e^z. \tag{2.5}$$

Now, if we write z in the fashion

$$z = -(\lambda t + x/\lambda), \tag{2.6}$$

we have for the function ϕ

$$\phi\left[a, a, -\left(\lambda t + \frac{x}{\lambda}\right)\right] = e^{-(\lambda t + \frac{x}{\lambda})}, \tag{2.7}$$

where we express the new quantity λ in terms of an equation involving the two ones K and Q entering Eq. (1.1)

$$\lambda^3 + K\lambda + \frac{Q}{2} = 0. \tag{2.8}$$

This equation for λ exhibits three solutions, one of them real and the other two complex. Since F in (1.1) is a normalized density function, the complex solutions are of no use to us.

Given that ϕ is such that

$$\phi'' = \lambda^2 \frac{\partial^2\phi}{\partial x^2} \quad ; \quad \phi' = -\frac{1}{\lambda}\frac{\partial\phi}{\partial t} \equiv \phi, \tag{2.9}$$

Eq. (2.4) can be recast as

$$z\lambda^2 \frac{\partial^2\phi}{\partial x^2} + a\phi' + \frac{z}{\lambda}\frac{\partial\phi}{\partial t} - a\phi = 0. \tag{2.10}$$

Since $\phi' = \phi$, (2.10) gets simplified to

$$\lambda^3 \frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial t} = 0. \tag{2.11}$$

According to (2.8), Eq. (2.11) becomes

$$-(K\lambda + \frac{Q}{2})\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial t} = 0. \tag{2.12}$$

In addition, since ϕ verifies

$$\lambda \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial\phi}{\partial x}, \tag{2.13}$$

we are led to the following expression for (2.12)

$$K\frac{\partial\phi}{\partial x} - \frac{Q}{2}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial t} = 0, \tag{2.14}$$

which is tantamount to

$$\frac{\partial\phi}{\partial t} + \frac{\partial(K\phi)}{\partial x} - \frac{Q}{2}\frac{\partial^2\phi}{\partial x^2} = 0, \tag{2.15}$$

i.e., Fokker–Planck's equation for K independent of x. Note that, by definition, (2.7) is a solution of (2.15).

3. Nonlinear Fokker–Planck equation [5]

Anomalous diffusion is exhibited in a variety of physical systems and is therefore the subject of much interest. It can be observed, for example, in general systems such as plasma flow, porous media, and surface growth, as well as in more specific situations such as cetyltrimethylammonium bromide micelles dissolved in salted water and NMR relaxometry of liquids in porous glasses [5]. The main characteristic of anomalous diffusion is the fact that the mean squared displacement is not proportional to time t but rather to some power of it. If the scaling is faster than t, then the pertinent system is superdiffusive while, if it is slower than t, it is subdiffusive. A nonlinear Fokker–Planck diffusion equation has been proposed for those systems with correlated anomalous diffusion, beginning with [2] and followed afterward by, for instance, [6–9]. For an excellent overview, see [5].

For the ordinary hypergeometric function $F_1^2(a,b;c;z)$ we have [10], using now three Pochhammer symbols,

$$F_1^2(a,b;c;z) \equiv F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{a_{(n)} b_{(n)}}{c_{(n)}} \frac{z^n}{n!}; \; (|z| < 1), \tag{3.1}$$

where the series terminates if either a or b is a non-zero integer. A particularly important special case is

$$F(-m, b, b, -z) = (1+z)^m. \tag{3.2}$$

Eq. (3.1) verifies [10]

$$z(1-z)F''(\alpha,\beta;\gamma;z) + [\gamma - (\alpha+\beta+1)z]F'(\alpha,\beta;\gamma;z) - \alpha\beta F(\alpha,\beta;\gamma;z) = 0. \tag{3.3}$$

This second-order equation has two independent solutions, and we will give a physical meaning to just one of these solutions.

If $\beta = \gamma$, then F satisfies [11]

$$F(-\alpha,\gamma;\gamma;-z) = (1+z)^\alpha. \tag{3.4}$$

Focus attention now upon the function

$$f(x,t) = \left[1 + (q-1)\left(\lambda t + \frac{x}{\lambda}\right)\right]^{\frac{1}{1-q}}, \tag{3.5}$$

where λ obeys (for K and Q both constants)

$$\lambda^3 + K\lambda + \frac{Q}{2} = 0. \tag{3.6}$$

We start now a rather lengthy discussion in order to derive Eqs. (3.16) and (3.19) below. Recourse to (3.4) allows one to write

$$F\left[\frac{1}{q-1}, \gamma; \gamma; (1-q)\left(\lambda t + \frac{x}{\lambda}\right)\right] = \left[1 + (q-1)\left(\lambda t + \frac{x}{\lambda}\right)\right]^{\frac{1}{1-q}}, \tag{3.7}$$

and then

$$z = (1-q)\left(\lambda t + \frac{x}{\lambda}\right). \tag{3.8}$$

For $\beta = \gamma$, F [cf. (3.3)] adopts the appearance

$$z(1-z)F''(\alpha,\gamma;\gamma;z) + [\gamma - (\alpha+\gamma+1)z]F'(\alpha,\gamma;\gamma;z) - \alpha\beta F(\alpha,\gamma;\gamma;z) = 0. \tag{3.9}$$

Since F verifies

$$F'' = \frac{\lambda^2}{(1-q)^2}\frac{\partial^2 F}{\partial x^2} \quad ; \quad F' = \frac{1}{\lambda(1-q)}\frac{\partial F}{\partial t}, \tag{3.10}$$

then (3.9) becomes

$$z(1-z)\frac{\lambda^2}{(1-q)^2}\frac{\partial^2 F}{\partial x^2} + \frac{qz}{\lambda(1-q)^2}\frac{\partial F}{\partial t} + \gamma(1-z)F' - \frac{\gamma}{q-1}F = 0, \tag{3.11}$$

and, adequately simplifying,

$$(1-z)\lambda^3\frac{\partial^2 F}{\partial x^2}+q\frac{\partial F}{\partial t}+\frac{\lambda\gamma}{z}(1-q)^2\left[(1-z)F'-\frac{1}{q-1}F\right]=0. \tag{3.12}$$

Again, since F fulfills

$$(1-z)F'-\frac{1}{q-1}F=0, \tag{3.13}$$

Eq. (3.12) becomes

$$(1-z)\lambda^3\frac{\partial^2 F}{\partial x^2}+q\frac{\partial F}{\partial t}=0, \tag{3.14}$$

or, equivalently,

$$\lambda^3 F^{(1-q)}\frac{\partial^2 F}{\partial x^2}+q\frac{\partial F}{\partial t}=0, \tag{3.15}$$

since $F^{(1-q)}(z)=1-z$. Thus, we are in a position to cast (3.15) as

$$\lambda^3\frac{\partial^2 F}{\partial x^2}+\frac{\partial F^q}{\partial t}=0. \tag{3.16}$$

Utilizing (3.6) we can recast things as

$$-\left(\lambda K+\frac{Q}{2}\right)\frac{\partial^2 F}{\partial x^2}+\frac{\partial F^q}{\partial t}=0. \tag{3.17}$$

Remembering that F obeys

$$\lambda K\frac{\partial^2 F}{\partial x^2}=-K\frac{\partial F^q}{\partial x}=-\frac{\partial(KF^q)}{\partial x}, \tag{3.18}$$

we obtain from (3.17)

$$\frac{\partial F^q}{\partial t}+\frac{\partial(KF^q)}{\partial x}-\frac{Q}{2}\frac{\partial^2 F}{\partial x^2}=0, \tag{3.19}$$

a nonlinear Fokker–Planck equation. If we set

- $g=F^q$
- $2-q^*=1/q$,

we immediately ascertain that Eq. (3.19), expressed in terms of g and q^*, coincides with the nonlinear FP discussed in [2] with regards to power-law q-entropies

$$\frac{\partial g}{\partial t}+\frac{\partial(Kg)}{\partial x}-\frac{Q}{2}\frac{\partial^2 g^{2-q^*}}{\partial x^2}=0. \tag{3.20}$$

Consider now the stationary case (F independent of t) and assume, following Ref. [2], that K can depend upon x. Then, we have for (3.19)

$$\frac{\partial(K(x)F^q)}{\partial x}-\frac{Q}{2}\frac{\partial^2 F}{\partial x^2}=0, \tag{3.21}$$

whose solution is

$$F(x)=\left[1+\frac{2(q-1)}{Q}V(x)\right]^{\frac{1}{1-q}}, \tag{3.22}$$

where $\frac{dV(x)}{dx}=-K(x)$.

4. Conclusions

We have shown that the Fokker–Planck equation and its nonlinear generalization [see, for instance, [2]] are contained within the structure of hypergeometric linear differential equations, for constant drift K. The FP-extensions to general drifts $K(x)$ have to be postulated like in the ordinary cases.

As seen also in [4], physical linear differential equations (Schrödinger, Klein–Gordon, Fokker–Planck) are contained in the confluent hypergeometric one (with two Pochhammer symbols), while its q-nonlinear counterparts appeal to the general hypergeometric equation (three Pochhammer symbols). Of the three parameters, the first is the so-called Tsallis' nonextensivity parameter q [12]. This gives an answer to the long-standing question for the meaning of q [12].

We have displayed a general solution for the Ornstein–Uhlenbeck equation of constant drift that possibly might be new, although we cannot ascertain it.

We also give an exact solution of the nonlinear FP equation when F does not depend upon the time.

Appendix A. Separation of variables in the Ornstein–Uhlenbeck process $K=-x$

The Ornstein–Uhlenbeck (OU) process is a stochastic process that, loosely, describes the velocity of a massive Brownian particle under the influence of friction, represented by $-x$ [13]. The OU process is stationary, Gaussian, and Markovian, being the only nontrivial evolution that satisfies these three conditions, up to allowing for linear transformations of the space and time variables. We believe that this well-known process of linear drift [1] is worth revisiting for didactic purposes. We start with

$$\frac{\partial F}{\partial t}+\frac{\partial(KF)}{\partial x}-\frac{Q}{2}\frac{\partial^2 F}{\partial x^2}=0, \tag{A.1}$$

$$F(x,t)=G(t)H(x), \tag{A.2}$$

which leads to

$$\frac{1}{G}\frac{\partial G}{\partial t}=\frac{1}{H}\left[\frac{Q}{2}\frac{\partial^2 H}{\partial x^2}-\frac{\partial(KH)}{\partial x}\right]=-\lambda, \tag{A.3}$$

with $\lambda>0$. From here we are immediately led to

$$\frac{\partial G}{\partial t}+\lambda G=0, \tag{A.4}$$

$$\frac{Q}{2}\frac{d^2H}{dx^2}-\frac{d(KH)}{dx}+\lambda H=0. \tag{A.5}$$

For the linear case $K=-x$ we first obtain for G

$$G(t)=e^{-\lambda t}. \tag{A.6}$$

Applying the Fourier transform to (A.5) we find

$$\frac{Q}{2}\alpha^2\hat{H}+\alpha\frac{d\hat{H}}{d\alpha}-\lambda\hat{H}=0, \tag{A.7}$$

where $\hat{H}$ is the Fourier transform of H of variable α. One solves (A.7) and gets

$$\hat{H}(\alpha)=|\alpha|^{\lambda}e^{-\frac{Q\alpha^2}{4}}, \tag{A.8}$$

and from (A.8) we encounter for H

$$H(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|\alpha|^{\lambda}e^{-\frac{Q\alpha^2}{4}}e^{-i\alpha x}\,d\alpha. \tag{A.9}$$

Thus we have for F the general expression

$$F(x,t)=\frac{1}{2\pi}\int\limits_{0}^{\infty}\int\limits_{-\infty}^{\infty}\lambda a(\lambda)e^{-\lambda t}|\alpha|^{\lambda}e^{-\frac{Q\alpha^2}{4}}e^{-i\alpha x}\,d\alpha\,d\lambda, \tag{A.10}$$

where $a(\lambda)$ must verify

$$\int_0^\infty a(\lambda)\, d\lambda = 1. \tag{A.11}$$

Eq. (A.10) may have been obtained before, but we were unable to find such derivation in the vast FP-literature available to us.

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