Available online at www.sciencedirect.com
SCIENCE DIRECT®
Ocean Engineering 31 (2004) 127–138
OCEAN ENGINEERING
www.elsevier.com/locate/oceaneng

# Analytical and experimental investigation on transverse vibrations of solid, circular and annular plates carrying a concentrated mass at an arbitrary position with marine applications

D.V. Bambill *, S. La Malfa, C.A. Rossit, P.A.A. Laura

*Department of Engineering, Institute of Applied Mechanics, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina*

Received 24 April 2003; received in revised form 28 May 2003; accepted 11 June 2003

**Abstract**

This investigation arose from the practical necessity of placing a centrifugal pump rigidly attached to a thin, circular cover plate of a water tank in a medium size ocean vessel. Due to lack of space, it was necessary to locate the system off—center of the circular configuration. It was considered necessary to calculate the fundamental frequency of the coupled system.

The first part of the present study is concerned with the determination of the fundamental frequency of vibration of a circular plate carrying a concentrated mass at an arbitrary position, using a variational approach.

Numerical results are obtained for the stated problem for several combinations of the intervening geometric and mechanical parameters.

An experimental investigation is also performed in the case of clamped plates.

Based on the results for solid circular plates, the fundamental frequency of annular plates with a free inner edge and a concentrated mass is also obtained.

Circular plates are fundamental structural elements in ocean engineering applications: from off-shore platforms to underwater acoustic transducers. In a great variety of circumstances, they must carry operational systems in an eccentric fashion. Since the dynamic performance is always of interest, one must know at least some of the basic dynamic parameters.


* Corresponding author. Fax: +54-291-4595-157/110.
*E-mail address:* dbambill@criba.edu.ar (D.V. Bambill).

0029-8018/$ - see front matter 
doi:10.1016/S0029-8018(03)00116-1

*Keywords:* Solid circular and annular plates; Free inner edge; Variational approach; Experimental results; Fundamental frequency

## 1. Introduction

A survey of the literature reveals that free transverse vibrations of circular plates carrying a rigidly attached point mass have only been studied, when the mass is placed at the plate center (Roberson, 1951; Leissa, 1969 and Laura et al., 1984).

The exact solution, obtained by Roberson, requires a Bessel function expansion of zero order.

On the other hand, if the mass is eccentric with respect to the plate configuration, the exact solution is considerably more complicated, since it would require combinations of Bessel functions of higher order and trigonometric terms in the azimuthal variable. For the present study, since one is specially interested in the fundamental frequency coefficient, it was decided to use, in the case of a solid plate, combinations of polynomials which identically satisfy the plate boundary conditions and trigonometric terms in the angular variable taking into account the lack of radial symmetry when the mass is off-center. The polynomials contain an undetermined exponential parameter, which allows for further optimization of the frequency coefficient once the classical Rayleigh–Ritz method is implemented.

The methodology was also implemented in the case of a circular annular plate with a free inner edge and carrying a concentrated mass. The same coordinate functions as in that case of solid plates were used. Accordingly, the natural boundary conditions were not satisfied at the inner edge but this is legitimate when applying the Rayleigh–Ritz method.

It is felt that the problem tackled in this paper is of general interest in several modern engineering applications: from mechanical systems design to mounting a transformer or an electronic device on a circular printed circuit board.

However, the present investigation was generated by the need to know the fundamental frequency of a thin circular plate carrying a centrifugal pump placed on top of a tank in the power plant of a medium size ocean vessel. Due to space limitations the pump was attached eccentrically with respect to the circular plate.

A limited amount of experimental results was obtained in the case of a circular solid plate and good engineering agreement with the analytical predictions is shown to exist.

It must be emphasized that circular (solid or annular) plates are quasi-universal structural elements in engineering applications: from transducer elements to bulkheads as well as supporting elements in off-shore platforms. In a great variety of instances they must carry operating electromechanical or electronic systems placed off center. Hence the need of knowledge of their basic dynamic structural parameters such as lower natural frequencies of transverse vibration.

## 2. Approximate analytical solution

The conventional Rayleigh–Ritz formulation for vibrating circular plates with a concentrated mass at arbitrary position (Fig. 1) involves the following energy functional:

$$\begin{aligned} J(W) = \frac{D}{2}\int_0^{2\pi}\int_0^{a}\Bigg\{ & \left(\frac{\partial^2 W(\bar r,\theta)}{\partial \bar r^2} + \frac{1}{\bar r}\frac{\partial W(\bar r,\theta)}{\partial \bar r} + \frac{1}{\bar r^2}\frac{\partial^2 W(\bar r,\theta)}{\partial \theta^2}\right)^2 \\ & -2(1-\mu)\left[\frac{\partial^2 W(\bar r,\theta)}{\partial \bar r^2}\left(\frac{1}{\bar r}\frac{\partial W(\bar r,\theta)}{\partial \bar r} + \frac{1}{\bar r^2}\frac{\partial W^2(\bar r,\theta)}{\partial \theta^2}\right)\right] \\ & +2(1-\mu)\left[\frac{\partial}{\partial \bar r}\left(\frac{1}{\bar r}\frac{\partial W(\bar r,\theta)}{\partial \theta}\right)\right]^2\Bigg\}\bar r \,\mathrm{d}\bar r\,\mathrm{d}\theta - \frac{1}{2}\omega^2\rho h\int_0^{2\pi}\int_0^{a} W(\bar r,\theta)\bar r\,\mathrm{d}\bar r\,\mathrm{d}\theta \\ & -\frac{1}{2}M\omega^2[W(\bar r_1,\theta_1)]^2 + \frac{1}{2}\left[\int_0^{2\pi} M(\bar r,\theta)\frac{\partial W(\bar r,\theta)}{\partial \bar r}\bar r\,\mathrm{d}\theta\right]_{\bar r = a} \end{aligned} \quad (1)$$

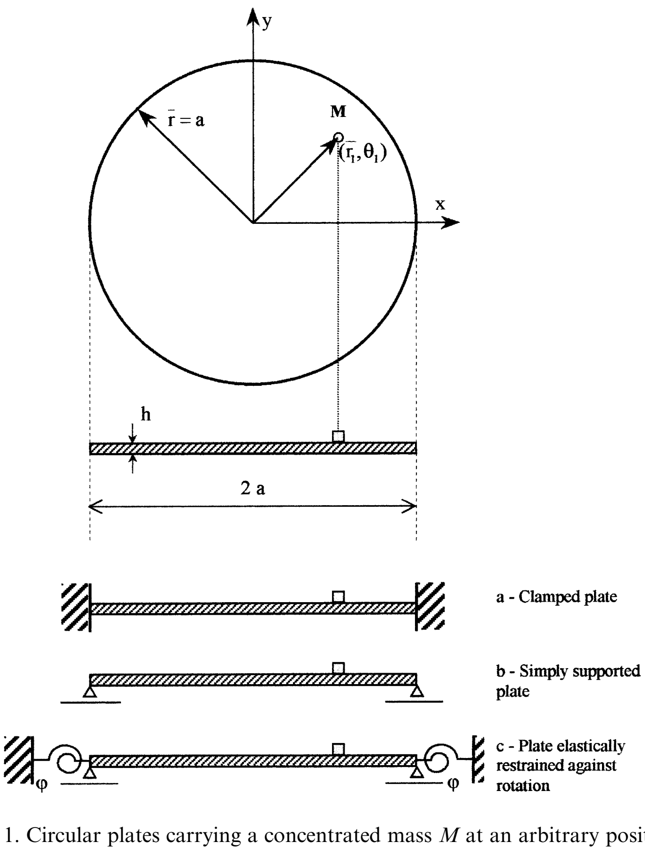


Fig. 1. Circular plates carrying a concentrated mass $M$ at an arbitrary position.

where $W(\bar{r},\theta)$ is the amplitude of the plate's displacement and $M(\bar{r},\theta)$ is the amplitude of the radial bending moment (Bambill, 1994). As the edge rotation is opposed by spiral springs having distributed flexibility $\varphi$ (unit length/moment) it can be expressed:

$$M(a,\theta) = \frac{1}{\varphi}\frac{\partial W(\bar{r},\theta)}{\partial \bar{r}}\bigg|_{\bar{r}=a}$$

Adimensional parameters and coefficients are used to simplify the formulation into a non-dimensional form:

$$r = \frac{\bar{r}}{a} \tag{2a}$$

$$r_1 = \frac{\bar{r}_1}{a} \tag{2b}$$

$$\Omega = \omega\sqrt{\frac{\rho h}{D}}a^2 \tag{2c}$$

$$D = \frac{Eh^3}{12(1-\mu^2)} \tag{2d}$$

$\Omega$ is the vibration dimensionless frequency coefficient, $\rho$ is the density of the plate material, $E$ and $\mu$ are Young's modulus and Poisson's ratio, respectively.

The approximate functional relation for the deflection of the plate is assumed to be:

$$W(r,\theta) \cong W_a(r,\theta) = \sum_{j=0}^{N} A_j f_j(r) g_j(\theta) \tag{3}$$

where:

$$\begin{aligned}
f_0(r) &= 1 + \alpha_0 r^\gamma + \beta_0 r^2 \\
f_1(r) &= \left(1 + \alpha_1 r^\gamma + \beta_1 r^2\right) r^2 \\
f_2(r) &= \left(1 + \alpha_2 r^\gamma + \beta_2 r^2\right) r^3 \\
&\vdots \\
f_j(r) &= (1 + \alpha_j r^\gamma + \beta_j r^2) r^{j+1} \\
&\vdots \\
f_N(r) &= \left(1 + \alpha_N r^\gamma + \beta_N r^2\right) r^{N+1}
\end{aligned}$$

and

$$g_j(\theta) = \cos j\theta \qquad \text{for} \qquad j = 0, 1, 2, \ldots N$$

The $\alpha_j$'s and $\beta_j$'s coefficients are determined by applying the boundary conditions in terms of the deflection and its derivatives at $r = \bar{r}/a = 1$.

The governing boundary conditions are:

$$W(r,\theta)|_{r=1} = 0 \tag{4a}$$

$$\left.\frac{\partial W(r,\theta)}{\partial r}\right|_{r=1} = -\phi D\left[\left.\frac{\partial^2 W(r,\theta)}{\partial r^2}\right|_{r=1} + \mu\left(\frac{1}{r}\frac{\partial W(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W(r,\theta)}{\partial \theta^2}\right)_{r=1}\right] \tag{4b}$$

where $\phi = \varphi D/a$ is the adimensional flexibility coefficient of the boundary springs.

The corresponding coefficients of the coordinate functions are accordingly functions of $\mu$, $\phi$ and $\gamma$, for $j = 0$

$$\alpha_0 = \frac{1}{a^\gamma}\frac{2[1+(1+\mu)\phi]}{(\gamma-2)[1+(1+\mu+\gamma)\phi]}$$

$$\beta_0 = \frac{1}{a^2}\frac{\gamma[1+(\gamma+\mu-1)\phi]}{(\gamma-2)[1+(1+\mu+\gamma)\phi]}$$

and for $j = 1, 2, \ldots, N$

$$\alpha_j = \frac{1}{a^\gamma}\frac{2[1+(3+\mu+2j)\phi]}{(\gamma-2)[1+(3+\mu+\gamma+2j)\phi]}$$

$$\beta_j = \frac{1}{a^2}\frac{\gamma[1+(\gamma+\mu+1+2j)\phi]}{(\gamma-2)[1+(3+\mu+\gamma+2j)\phi]}$$

For the case of a simply supported plate $\phi = \varphi D/a \to \infty$

Hence, for $j = 0$

$$\alpha_0 = \frac{1}{a^\gamma}\frac{2(1+\mu)}{(\gamma-2)(1+\mu+\gamma)}$$

$$\beta_0 = -\frac{1}{a^2}\frac{\gamma(\gamma+\mu-1)}{(\gamma-2)(1+\mu+\gamma)}$$

and for $j = 1, 2, \ldots, N$

$$\alpha_j = -\frac{1}{a^\gamma}\frac{2(3+\mu+2j)}{(\gamma-2)(3+\mu+\gamma+2j)}$$

$$\beta_j = -\frac{1}{a^2}\frac{\gamma(\gamma+\mu+1+2j)}{(\gamma-2)(3+\mu+\gamma+2j)}$$

On the other hand for a clamped plate one has $\phi = \varphi D/a \to 0$.

Accordingly:

$$\alpha_0 = \alpha_1 = \cdots = \alpha_N = \frac{1}{a^\gamma}\frac{2}{(\gamma-2)}$$

$$\beta_0 = \beta_1 = \cdots = \beta_N = -\frac{1}{a^2}\frac{\gamma}{(\gamma-2)}$$

Replacing Eq. (3) into the energy functional (1) and minimizing it with respect to the $A_i$ parameters.

$$\frac{\partial J(W)}{\partial A_i} = 0, \qquad i = 0, 1, 2, \ldots, N \tag{5}$$

a trascendental equation in $\Omega(\gamma)$ is found using the non-triviality condition.

The optimization parameter $\gamma$ allows for the determination of an improved eigenvalue $\Omega_1$ (the lowest root of the trascendental equation),

$$\frac{\partial \Omega_1(\gamma)}{\partial \gamma} = 0 \tag{6}$$

The analytical approach was also used to calculate the frequency coefficients of annular plates with a free inner edge (Fig. 2), at the second stage of the analytical investigation.

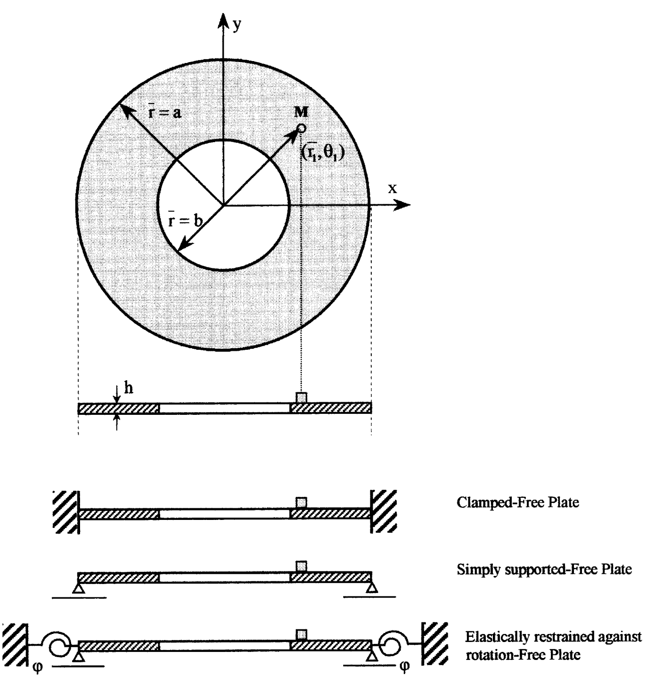


Fig. 2. Annular plates carrying a concentrated mass $M$ at an arbitrary position.

For the annular plate, the functional is:

$$J(W) = \frac{D}{2}\int_0^{2\pi}\int_b^a \left\{ \left( \frac{\partial^2 W(\bar{r},\theta)}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial W(\bar{r},\theta)}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2 W(\bar{r},\theta)}{\partial \theta^2} \right)\bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\theta \right.$$
$$-2(1-\mu)\left[ \frac{\partial^2 W(\bar{r},\theta)}{\partial \bar{r}^2} \left( \frac{1}{\bar{r}}\frac{\partial W(\bar{r},\theta)}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial W^2(\bar{r},\theta)}{\partial \theta^2} \right) \right]$$
$$\left. +2(1-\mu)\left[ \frac{\partial}{\partial \bar{r}} \left( \frac{1}{\bar{r}}\frac{\partial W(\bar{r},\theta)}{\partial \theta} \right) \right]^2 \right\} \bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\theta - \frac{1}{2}\omega^2 \rho h \int_0^{2\pi}\int_0^a W(\bar{r},\theta)\bar{r}\, \mathrm{d}\bar{r}\, \mathrm{d}\theta$$
$$-\frac{1}{2}M\omega^2[W(\bar{r}_1,\theta_1)]^2 \tag{7}$$

Obviously, in the case of elastically restrained edges, one must add to the governing functional, the mechanical energy stored in the restraints.

The formulation is reduced into a non-dimensional form by using the adimensional coordinate functions and coefficients (2a–d). The parameter $r_\mathrm{i} = b/a$ results now in order to define the geometric configuration.

Expression (3) is used for the deflection's amplitude of the annular plate too:

$$W(r,\theta) \cong W_a(r,\theta) = \sum_{j=0}^{N} A_j f_j(r) g_j(\theta)$$

The coordinate functions $f_j(r)$ and $g_j(\theta)$ satisfy only the conditions at the outer boundary. The mass is at the $(\bar{r}_1, \theta_1)$ position.

## 3. Numerical results and experimental investigation

Table 1 presents the fundamental vibrating coefficients $\Omega_1$ for a rigidly clamped plate carrying a concentrated mass at its center. The mass adimensional parameter $M/M_\mathrm{p}$ is referred to the mass of the circular solid plate, $(M_\mathrm{p} = \pi r^2 h\rho)$.

The first column of coefficients has been taken from Leissa classical treatise (Leissa, 1969), the second column is from Laura et al. (1984) and the third column depicts the results obtained in the present investigation.

Table 1
Frequency coefficients $\Omega_1 = \omega_1\sqrt{\rho h/D}a^2$ for a rigidly clamped circular plate carrying a concentrated mass at its center

| $M/M_\mathrm{p}$ | Leissa (1969) | Laura et al. (1984) | Present study |
|---|---|---|---|
| 0 | 10.214 | 10.22 | 10.226 |
| 0.05 | 9.012 | 9.01 | 9.012 |
| 0.10 | 8.111 | 8.11 | 8.111 |
| 0.20 | 7.000 | 6.87 | 6.872 |
| 0.50 | 5.000 | 5.02 | 5.023 |
| 1.00 | 3.750 | 3.75 | 3.759 |

Table 2
Frequency coefficients $\Omega_1 = \omega_1\sqrt{\rho h/D}a^2$ for a simply supported circular plate carrying a concentrated mass at its center ($\mu = 0.30$)

| $M/M_p$ | Leissa (1969) | Laura et al. (1984) | Present study |
|---|---|---|---|
| 0 | 4.935 | 4.93 | 4.936 |
| 0.05 | – | – | 4.547 |
| 0.10 | – | 4.23 | 4.231 |
| 0.20 | 3.767 | 3.75 | 3.750 |
| 0.50 | 2.945 | 2.92 | 2.913 |
| 1.00 | 2.291 | 2.25 | 2.255 |

Table 2 depicts natural frequency coefficients $\Omega_1$ of a simply supported circular plate (with $\mu = 0.30$) with a concentrated mass at its center. In both tables, it is to be observed that the agreement between results obtained by other researchers and present ones is good.

It is important to remark that the presence of the function $g_j(\theta)$ inside Eq. (3), which takes into account the angular dependence, disturbs the result when the punctual mass is at the center. As it was expected, the function $W(r, \theta)$ is not a very convenient approximation in this case, because the $g_j(\theta)$ term disturbs the symmetry.

Tables 3–5 depict the values of the fundamental frequency coefficients for circular plates with different boundary conditions. The mass relation $M/M_p$ varies from 0.05 to twice the mass of the solid plate, while the position of the mass changes from a position near to the center ($r_1 = 0.05$) to the boundary: $r_1 = 1$.

Fig. 3 shows how the frequency coefficient $\Omega_1$ decreases as the mass ratio increases from $M/M_p = 0$ to $M/M_p = 2$ for a circular plate clamped at its boundary as it was to be expected.

It is to be observed that the variation of $\Omega_1$ is more noticeable when the mass M is near the center of the plate ($M/M_p = 0.10$; $\Omega_1 = 9.07$ and $M/M_p = 2$; $\Omega_1 =$

Table 3
Frequency fundamental coefficients $\Omega_1 = \omega_1\sqrt{\rho h/D}a^2$ for a clamped circular plate with a concentrated mass $M$ at $(r_1, \theta_1)$ position

| Mass position $(r_1, \theta_1)$ | $M/M_p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a$ | 0.05 | 0.10 | 0.20 | 0.30 | 0.50 | 1.00 | 1.50 | 2.00 |
| 0 | 9.01 | 8.11 | 6.87 | 6.05 | 5.02 | 3.76 | 3.13 | 2.74 |
| 0.10 | 9.07 | 8.21 | 7.01 | 6.22 | 5.19 | 3.91 | 3.27 | 2.87 |
| 0.20 | 9.22 | 8.45 | 7.34 | 6.58 | 5.56 | 4.25 | 3.57 | 3.14 |
| 0.30 | 9.41 | 8.76 | 7.77 | 7.05 | 6.05 | 4.70 | 3.97 | 3.50 |
| 0.40 | 9.63 | 9.12 | 8.27 | 7.61 | 6.64 | 5.24 | 4.46 | 3.95 |
| 0.50 | 9.84 | 9.48 | 8.82 | 8.25 | 7.33 | 5.88 | 5.03 | 4.47 |
| 0.60 | 10.02 | 9.82 | 9.39 | 8.96 | 8.17 | 6.70 | 5.77 | 5.13 |
| 0.70 | 10.14 | 10.06 | 9.87 | 9.66 | 9.18 | 7.92 | 6.92 | 6.19 |
| 0.80 | 10.20 | 10.18 | 10.14 | 10.10 | 9.99 | 9.61 | 9.04 | 8.39 |
| 0.90 | 10.22 | 10.22 | 10.22 | 10.21 | 10.21 | 10.19 | 10.18 | 10.16 |

Table 4
Frequency fundamental coefficients $\Omega_1 = \omega_1 \sqrt{\rho h/D} a^2$ for a simply supported circular plate with a concentrated mass $M$ at ($r_1$, $\theta_1$) position ($\mu = 0.30$)

| Mass position ($r_1$, $\theta_1$) | $M/M_p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a$ | 0.05 | 0.10 | 0.20 | 0.30 | 0.50 | 1.00 | 1.50 | 2.00 |
| 0 | 4.54 | 4.23 | 3.75 | 3.39 | 2.93 | 2.25 | 1.90 | 1.67 |
| 0.10 | 4.55 | 4.25 | 3.78 | 3.43 | 2.95 | 2.29 | 1.94 | 1.71 |
| 0.20 | 4.58 | 4.29 | 3.85 | 3.52 | 3.05 | 2.39 | 2.03 | 1.79 |
| 0.30 | 4.62 | 4.37 | 3.96 | 3.64 | 3.18 | 2.53 | 2.16 | 1.91 |
| 0.40 | 4.68 | 4.46 | 4.09 | 3.80 | 3.36 | 2.70 | 2.32 | 2.06 |
| 0.50 | 4.74 | 4.56 | 4.25 | 3.99 | 3.58 | 2.92 | 2.53 | 2.26 |
| 0.60 | 4.79 | 4.66 | 4.42 | 4.21 | 3.84 | 3.21 | 2.80 | 2.51 |
| 0.70 | 4.85 | 4.77 | 4.61 | 4.46 | 4.17 | 3.60 | 3.19 | 2.89 |
| 0.80 | 4.89 | 4.86 | 4.78 | 4.70 | 4.54 | 4.16 | 3.82 | 3.54 |
| 0.90 | 4.92 | 4.91 | 4.89 | 4.87 | 4.84 | 4.73 | 4.62 | 4.51 |

2.869 when the mass is at $r_1 = 0.10$) (clamped plate) than when the mass is placed close to the outer boundary. Fig. 4 corresponds to the case of simply supported circular plates.

Some experiments were performed on solid circular plates to investigate the effect of the mass.

The first set of tests was performed on a clamped stainless steel plate of 0.09 cm of thickness. The diameter of the plate was 39 cm.

The concentrated mass was located at the geometric center ($r_1 = 0$) and at two positions out of the center ($r_1 = 1/3;\ 2/3$).

The mass ratios $M/M_p$ were of 0.442 and 0.530, respectively.

A second experience was made on a thin clamped iron plate ($h = 1.24$ cm) with three different mass ratios $M/M_p = 0.326$, 0.475 and 0.600. The frequency values, expressed in Hertz, are in Table 7. The deviation is very small when the mass is

Table 5
Frequency fundamental coefficients $\Omega_1 = \omega_1 \sqrt{\rho h/D} a^2$ for a circular plate carrying a concentrated mass $M$ at ($r_1$, $\theta_1$) position and with a rotational elastic constrain along the edge ($\phi = \varphi a/D = 0.05$; $\mu = 0.30$)

| Mass position ($r_1$, $\theta_1$) | $M/M_p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a$ | 0.05 | 0.10 | 0.20 | 0.30 | 0.50 | 1.00 | 1.50 | 2.00 |
| 0 | 8.20 | 7.30 | 6.28 | 5.58 | 4.67 | 3.52 | 2.94 | 2.58 |
| 0.10 | 8.04 | 7.35 | 6.35 | 5.66 | 4.76 | 3.61 | 3.02 | 2.65 |
| 0.20 | 8.14 | 7.51 | 6.58 | 5.92 | 5.04 | 3.87 | 3.26 | 2.86 |
| 0.30 | 8.28 | 7.73 | 6.90 | 6.28 | 5.41 | 4.22 | 3.57 | 3.15 |
| 0.40 | 8.43 | 7.99 | 7.26 | 6.69 | 5.86 | 4.63 | 3.95 | 3.50 |
| 0.50 | 8.59 | 8.27 | 7.68 | 7.17 | 6.37 | 5.12 | 4.38 | 3.89 |
| 0.60 | 8.74 | 8.53 | 8.11 | 7.71 | 6.99 | 5.72 | 4.93 | 4.38 |
| 0.70 | 8.85 | 8.75 | 8.53 | 8.29 | 7.77 | 6.58 | 5.72 | 5.11 |
| 0.80 | 8.91 | 8.89 | 8.82 | 8.75 | 8.57 | 7.95 | 7.21 | 6.55 |
| 0.90 | 8.94 | 8.93 | 8.93 | 8.92 | 8.90 | 8.86 | 8.80 | 7.73 |

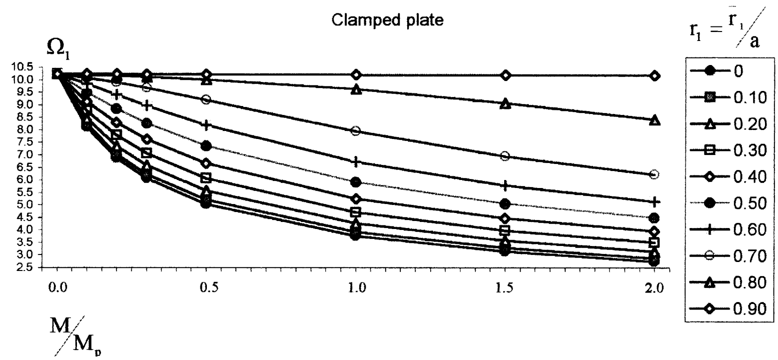


Fig. 3. Frequency fundamental coefficients of clamped plates carrying a concentrated mass.

located at $r_1 = 1/3$ and slightly larger deviation is observed when the mass is located at $r_1 = 2/3$.

A comparison was also performed with an experimental result available in the literature (Delaplane and Kerlin, 1990) for a thin aluminum clamped circular plate with a concentrated mass $M/M_p = 0.903$ at $r_1 = 0.38$. The corresponding measured frequency is 54 Hz and the value obtained using the formulation developed in the present study for this case is 54.12 Hz. For the lower relation of $M/M_p = 0.041$, the agreement between experimental and analytical results is not good as Delaplane and Kerlin show in their study.

Tables 6 and 7 depict comparisons of analytical and experimental results performed on stainless steel and steel plates, respectively, for several mass ratios and mass positions. In general, a good engineering agreement is observed.

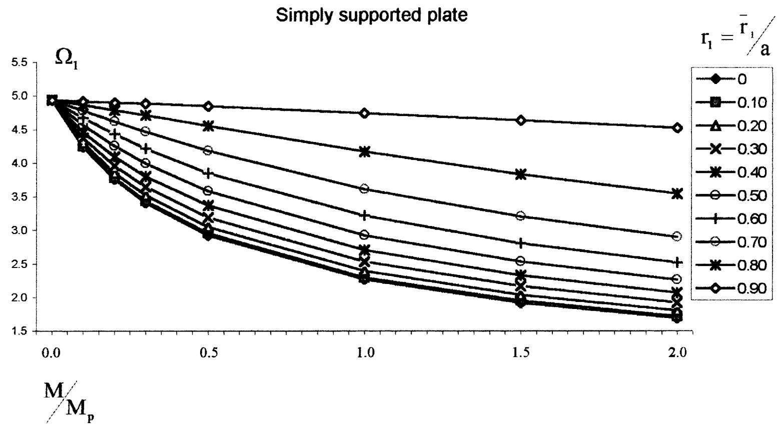


Fig. 4. Frequency fundamental coefficients of simple supported plates carrying a concentrated mass.

Table 6
Frequency coefficients in Hertz for a thin stainless steel circular plate, clamped at its outer boundary and carrying a concentrated mass $M$ at $(r_1, \theta_1)$

| Mass position $(r_1, \theta_1)$ | $M/M_p = 0.442$ | | | $M/M_p = 0.630$ | | |
|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a$ | Experimental results (Hz) | Analytical results (Hz) | Percentage deviation (%) | Experimental results (Hz) | Analytical results (Hz) | Percentage deviation (%) |
| 0 | 29.1 | – | – | 27.5 | – | – |
| 1/3 | 28.8 | 28.1 | 2.6 | 25.9 | 25.2 | 2.3 |
| 2/3 | 32.6 | 35.3 | 7.8 | 28.7 | 30.4 | 5.5 |

Table 7
Frequency coefficients in Hertz for a thin steel circular plate, clamped at its outer boundary and carrying a concentrated mass $M$ at $(r_1, \theta_1)$

| Mass position $(r_1, \theta_1)$ | $M/M_p = 0.326$ | | | $M/M_p = 0.475$ | | | $M/M_p = 0.600$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a$ | Experimental results (Hz) | Analytical results (Hz) | Percentage deviation (%) | Experimental results (Hz) | Analytical results (Hz) | Percentage deivation (%) | Experimental results (Hz) | Analytical results (Hz) | Percentage deviation (%) |
| 0 | 56.2 | – | – | 49.9 | – | – | 44.8 | – | – |
| 1/3 | 58.3 | 58.31 | 0.02 | 55.3 | 52.7 | 5.0 | 50.1 | 48.8 | 2.6 |
| 2/3 | 60.6 | 77.95 | 22.2 | 78.8 | 78.7 | 0.8 | 68.3 | 71.0 | 3.9 |

Table 8 presents fundamental eigenvalues for clamped-free annular plates when the point mass is attached at the inner boundary, while Table 9 deals with, the simply supported-free case, $M_p = \pi\rho h(1 - r_i^2)$.

Table 8
Frequency fundamental coefficients $\Omega_1 = \omega_1\sqrt{\rho h/D}a^2$ for clamped-free annular plates with a concentrated mass $M$ attached at the inner free boundary $(r_i, \theta_1)$

| Mass position $\Omega = \omega_1\sqrt{\rho h/D}a^2$ | $M/M_p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a = r_i$ | 0.05 | 0.10 | 0.20 | 0.30 | 0.50 | 1.00 | 1.50 | 2.00 |
| 0.05 | 9.01 | 8.10 | 6.83 | 6.00 | 4.96 | 3.70 | 3.07 | 2.69 |
| 0.10 | 9.03 | 8.07 | 6.76 | 5.92 | 4.87 | 3.62 | 3.00 | 2.62 |
| 0.20 | 9.21 | 8.19 | 6.85 | 5.99 | 4.94 | 3.68 | 3.06 | 2.67 |
| 0.30 | 10.06 | 8.99 | 7.55 | 6.62 | 5.47 | 4.08 | 3.39 | 2.97 |
| 0.40 | 11.98 | 10.72 | 9.00 | 7.88 | 6.50 | 4.83 | 4.01 | 3.50 |
| 0.50 | 15.60 | 13.88 | 11.50 | 9.97 | 8.11 | 5.94 | 4.89 | 4.25 |
| 0.60 | 22.40 | 19.42 | 14.66 | 12.35 | 9.82 | 7.08 | 5.82 | 5.06 |
| 0.70 | 33.19 | 30.32 | 20.04 | 16.77 | 13.26 | 9.52 | 7.82 | 6.79 |
| 0.80 | 61.01 | 47.27 | 35.21 | 29.28 | 23.69 | 16.47 | 13.50 | 11.71 |

Table 9
Frequency fundamental coefficients $\Omega_1 = \omega_1\sqrt{\rho h/D}a^2$ for simply supported-free annular plates with a concentrated mass $M$ attached at the inner free boundary ($r_i$, $\theta_1$), ($\mu = 0.30$)

| Mass position ($r_i$, $\theta_1$) | $M/M_p$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $r_1 = \bar{r}_1/a = r_i$ | 0.05 | 0.10 | 0.20 | 0.30 | 0.50 | 1.00 | 1.50 | 2.00 |
| 0.05 | 4.54 | 4.22 | 3.74 | 3.38 | 2.89 | 2.23 | 1.88 | 1.65 |
| 0.10 | 4.53 | 4.20 | 3.70 | 3.33 | 2.84 | 2.18 | 1.83 | 1.61 |
| 0.20 | 4.41 | 4.06 | 3.55 | 3.18 | 2.70 | 2.07 | 1.74 | 1.53 |
| 0.30 | 4.31 | 3.97 | 3.48 | 3.13 | 2.66 | 2.04 | 1.72 | 1.51 |
| 0.40 | 4.38 | 4.05 | 3.56 | 3.21 | 2.73 | 2.10 | 1.77 | 1.55 |
| 0.50 | 4.67 | 4.32 | 3.81 | 3.43 | 2.92 | 2.25 | 1.89 | 1.66 |
| 0.60 | 5.51 | 4.95 | 4.30 | 3.87 | 3.29 | 2.52 | 2.12 | 1.86 |
| 0.70 | 6.68 | 6.45 | 6.02 | 5.65 | 5.05 | 4.08 | 3.50 | 3.11 |
| 0.80 | 8.82 | 8.16 | 7.12 | 6.33 | 5.30 | 3.99 | 3.33 | 2.92 |

As a general conclusion one may say that the fact that simple polynomial approximations combined with trigonometric expressions, coupled with a classical variational approach, allow for the solution of a complex elastodynamics problem in a simple manner, is rather remarkable.

## Acknowledgements

The authors are indebted to Mr. Osvaldo Alvarez, Chief Technician (CIC, Buenos Aires Province) for his valuable cooperation in the preparation of the experimental models and to Mrs. M. Susana Grenada in the preparation of the manuscript of the present study.

The present study has been sponsored by CONICET Research and Development Program, by Secretaría General de Ciencia y Tecnologia of Universidad Nacional del Sur and by Rocca Foundation (TECHINT). The cooperation of the Mechanical Systems Analysis Group (FRBB—Universidad Tecnológica Nacional) when performing the experimental phase of the investigation is gratefully acknowledged.

## References

Bambill, D.V., 1994. A variant of the weighted residuals method and its application to the solution of vibrations problems of continuous media (in Spanish). Master's dissertation, Department of Engineering, Universidad Nacional del Sur (Bahía Blanca, Argentina).

Delaplane, N.C., Kerlin, R.L., 1990. Driving-point impedance measurements of a clamped circular plate driven by a no central force. The Journal of the Acoustical Society of America 88 (1), 222–230.

Laura, P.A.A., Laura, P.A., Diez, G., Cortinez, V.H., 1984. A note on vibrating circular plates carrying concentrated masses. Mechanics Research Communications 11 (6), 397–400.

Leissa, A.W., 1969. NASA SP 160, Vibration of Plates.

Roberson, R.E., 1951. Vibrations of a clamped circular plate carrying concentrated mass. Journal of Applied Mechanics 18 (4), 349–352.