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Multipoint Padé approximants as limits of rational functions of best approximation in the complex domain†

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Abstract

In this paper we study the behavior of best L^p-approximations by rational functions to an analytic function on union of disks, when the measure of them tends to zero.

Keywords: best approximation, rational functions, Padé approximant, L^p-norm.

MSC: Primary 41A20, 41A21; Secondary 32A10.

§1. Introduction

Let $X = \{z_j\}_{j=1}^k \subset \mathbb{C}$, $k \in \mathbb{N}$, and let B_j be disjoint pairwise open disks centered at z_j and radius $\beta > 0$. We denote $\mathcal{A}(I)$ the space of analytic functions on $I := \cup_{j=1}^k B_j$, which are continuous on $\overline{I}$. Let $n, m \in \mathbb{N} \cup \{0\}$ and let Π^n be the class of algebraic polynomials with complex coefficients of degree at most n. We consider the set of rational functions

$$\mathcal{R}_m^n = \mathcal{R}_m^n(I) := \left\{ \frac{P}{Q} : P \in \Pi^n, Q \in \Pi^m, Q(z) \neq 0 \text{ for all } z \in I \right\}.$$

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Clearly, we can assume that $\frac{P}{Q} \in \mathcal{R}_m^n$ with $\|Q\|_\infty := \max_{z\in\overline{I}} |Q(z)| = 1$.

If $\|\cdot\|$ is a norm defined on $\mathcal{A}(I)$ and $h \in \mathcal{A}(I)$, for each $0 < \epsilon \le 1$, we write $\|h\|_\epsilon = \|h^\epsilon\|$, where $h^\epsilon(z) = h(\epsilon(z - z_j) + z_j)$, $z \in B_j$. We put

$$||h|| = \left(\sum_{j=1}^{k} \int_{\gamma_j} |h(z)|^p \frac{|dz|}{2\beta k\pi} \right)^{\frac{1}{p}}, \quad 1 < p < \infty,$$

where $\gamma_j : [0, 2\pi] \to \mathbb{C}$ is the path $\gamma_j(t) = z_j + \beta e^{it}$. We observe that if $\gamma_{j,\epsilon} : [0, 2\pi] \to \mathbb{C}$ is the path $\gamma_{j,\epsilon}(t) = z_j + \epsilon\beta e^{it}$, then $||h||_\epsilon^p = \sum_{j=1}^{k} \int_{\gamma_{j,\epsilon}} |h(z)|^p \frac{|dz|}{2\beta k\pi\epsilon}$. We use the notation

$$\|h\|_{B_j} = \left(\int_{\gamma_j} |h(z)|^p |dz| \right)^{\frac{1}{p}}.$$

Let $f \in \mathcal{A}(I)$ and $0 < \epsilon \le 1$. Then $u_\epsilon \in \mathcal{R}_m^n$ is called a *best rational approximation* of f from $\mathcal{R}_m^n$ if

$$\|f - u_\epsilon\|_\epsilon = \inf_{u\in\mathcal{R}_m^n} \|f - u\|_\epsilon. \tag{1.1}$$

It is well known that u_ϵ always exists (see [9, p. 682]).

From now on, we make the assumption that $n + m + 1 = kq + r$, $q \in \mathbb{N} \cup \{0\}$, $0 \le r < k$.

Given $q > 0$ and $u \in \mathcal{R}_m^n$, if $(f - u)^{(s)}(z_j) = 0$, $0 \le s \le q - 1$, $1 \le j \le k$, then u is said to be a *Padé approximant of* f at X. This approximant may not exist, for example, if $X = \{0\}$, $n = m = 1$ and $f(z) = z^2 + 1$ (see [7, p.700]). If it exists and $r = 0$, then it is unique, as it follows immediately from its definition.

We define

$$\mathcal{V}_{n,m}^q(f, X) := \{u \in \mathcal{R}_m^n : u \text{ is a Padé approximant of } f \text{ at } X\}.$$

If $q = 0$, no constraint over the rational function is assumed and $\mathcal{V}_{n,m}^q(f, X) = \mathcal{R}_m^n$.
Suppose $\mathcal{V}_{n,m}^q(f, X)$ is not an empty set. We say that $u_0 \in \mathcal{V}_{n,m}^q(f, X)$ is a *best Padé approximant of* f *at* X if

$$\sum_{j=1}^{k} \left| (f - u_0)^{(q)}(z_j) \right|^p \le \sum_{j=1}^{k} \left| (f - u)^{(q)}(z_j) \right|^p, \quad u \in \mathcal{V}_{n,m}^q(f, X).$$

In 1934, J. L. Walsh proved [10] that the Taylor polynomial of degree n for an analytic function f can be obtained by taking the limit as $\epsilon \to 0$ of the best (Tchebychev) approximant from Π^n to f on the disk $|z| \le \epsilon$. Later, in [11] he generalized this result to Padé approximants of analytic functions. In [12], it was shown that the Padé approximant to any function $f \in \mathcal{C}^{n+m+1}[0,\epsilon]$ under suitable conditions is obtained by taking the best rational approximant (with real coefficients) on the interval $[0,\epsilon]$ and then making $\epsilon \to 0$. The same year, this work was generalized to any function in $\mathcal{C}^{n+m+1}[0,\epsilon]$ [4]. In [7], the authors extended the last work to L^p-approximation on k disjoint intervals, $0 < p \le \infty$, in the case where $n+m+1$ is divisible by k. Finally, similar results in Orlicz spaces can be seen in [3] and [6].

In Section 2, we show that there exists at least a best Padé approximant of f at X. In Section 3, we prove that as $\epsilon \to 0$, any net of the best rational approximations u_ϵ approaches a best Padé approximant of f at X on any closed set of I.

§2. Existence of best multipoint Padé approximants

Henceforth, for simplicity we assume $\beta = 1$. Now, we establish an existence theorem of best multipoint Padé approximants.

Theorem 2.1. *Let $f \in \mathcal{A}(I)$. If $\mathcal{V}^q_{n,m}(f,X) \neq \emptyset$, then there exists at least a best Padé approximant of f at X.*

Proof. Let $\left\{\frac{P_l}{Q_l}\right\}_{l\in\mathbb{N}} \subset \mathcal{V}^q_{n,m}(f,X)$ be a sequence satisfying

$$\lim_{l\to\infty} \sum_{j=1}^{k} \left| \left(f - \frac{P_l}{Q_l}\right)^{(q)} (z_j) \right|^p = \inf_{\frac{P}{Q} \in \mathcal{V}^q_{n,m}(f,X)} \sum_{j=1}^{k} \left| \left(f - \frac{P}{Q}\right)^{(q)} (z_j) \right|^p =: E. \tag{2.1}$$

If $q > 0$, then

$$\left(f - \frac{P_l}{Q_l}\right)^{(i)} (z_j) = 0, \quad 0 \le i \le q-1, \quad 1 \le j \le k. \tag{2.2}$$

According to (2.1), there is a constant $M > 0$ such that

$$\left| \left(f - \frac{P_l}{Q_l}\right)^{(i)} (z_j) \right| \le M\delta_{i,q}, \quad 0 \le i \le q, \quad 1 \le j \le k, \quad l \in \mathbb{N}, \tag{2.3}$$

where δ is the Kronecker's delta function. From the Leibniz rule for the ith derivative of a product of two factors, $(fQ_l - P_l)^{(i)}(z_j) = 0$, $0 \leq i \leq q-1$, $1 \leq j \leq k$, and $\left|\left(f - \frac{P_l}{Q_l}\right)^{(q)}(z_j)\right| = \left|(fQ_l - P_l)^{(q)}(z_j).\frac{1}{Q_l(z_j)}\right|$. So, (2.3) and the normalization of Q_l imply

$$|(fQ_l - P_l)^{(i)}(z_j)| \leq M\delta_{i,q}, \quad 0 \leq i \leq q, \quad 1 \leq j \leq k, \quad l \in \mathbb{N}. \tag{2.4}$$

We observe that if $q = 0$, (2.4) is also true, by (2.1).
Let $\frac{S}{T} \in \mathcal{V}^q_{n,m}(f, X)$ and $M_1 = \max\limits_{1\leq j\leq k} \left|\left(f - \frac{S}{T}\right)^{(q)}(z_j)\right|$. Using the Leibniz rule again, we get $\left|\left(\left(\frac{S}{T} - f\right) Q_l\right)^{(i)}(z_j)\right| \leq M_1\delta_{(i,q)}$, $0 \leq i \leq q$, $1 \leq j \leq k$, $l \in \mathbb{N}$. Therefore, from (2.4)

$$\left|\left(\frac{SQ_l - TP_l}{T}\right)^{(i)}(z_j)\right| = \left|\left(\frac{S}{T}Q_l - P_l\right)^{(i)}(z_j)\right| \leq (M_1 + M)\delta_{(i,q)},$$

$0 \leq i \leq q$, $1 \leq j \leq k$, $l \in \mathbb{N}$. As $\|P\| := \max\limits_{0\leq i\leq q} \max\limits_{1\leq j\leq k} |\left(\frac{P}{T}\right)^{(i)}(z_j)|$ is a norm on $\Pi^{k(q+1)-1}$, the equivalence of the norms in $\Pi^{k(q+1)-1}$ implies that $\{SQ_l - TP_l\}$ is uniformly bounded on $\overline{I}$, and consequently $\{TP_l\}$ is uniformly bounded on $\overline{I}$. Since $\|PT\|_\infty$, with $P \in \Pi^n$, is a norm on Π^n, we get that $\{P_l\}$ is uniformly bounded on $\overline{I}$. So, there are two subsequences of $\{P_l\}$ and $\{Q_l\}$, which are denoted in the same way, $P_0 \in \Pi^n$ and $Q_0 \in \Pi^m$ such that $P_l \to P_0$ and $Q_l \to Q_0$ uniformly on $\overline{I}$, as $l \to \infty$. By the Hurwitz Theorem [2, p. 152], $Q_0(z) \neq 0$, $z \in I$. Then $\frac{P_0}{Q_0} \in \mathcal{R}^n_m$ and $\frac{P_l}{Q_l}$ converges to $\frac{P_0}{Q_0}$ uniformly on any closed set of I. From (2.1) and the analytical convergence theorem, $\sum\limits_{j=1}^{k} \left|\left(f - \frac{P_0}{Q_0}\right)^{(q)}(z_j)\right|^p = E$. Since (2.2) implies

$$\left(f - \frac{P_0}{Q_0}\right)^{(i)}(z_j) = 0, \quad 0 \leq i \leq q-1, \quad 1 \leq j \leq k,$$

we conclude that $\frac{P_0}{Q_0}$ is a best Padé approximant of f at X. ■

§3. Convergence of best rational approximations

In [8], the author proved the following characterization result for best approximants based on the one-sided Gateaux derivative, when the approximant set is a linear subspace of a complex Banach space.

Theorem A. *Let $(E, \|\cdot\|)$ be a complex Banach space, S a linear subspace of E and $f \in E \setminus \overline{S}$. Then $s \in S$ is the best approximant of f from S if and only if $\inf_{\phi\in[0,2\pi)} \gamma_\phi(f - s, g) \geq 0$, for all $g \in S$, where $\gamma_\phi(h, g) = \lim_{t\to 0^+} \frac{1}{t}(\|h + te^{i\phi}g\| - \|h\|)$ is the ϕ-Gateaux derivative of $\|\cdot\|$ at h in g in the direction ϕ.*

Let $P_q(z) = z^q$. We denote by $M_{p,q} \in \Pi^{q-1}$ the best approximant of P_q from Π^{q-1} with respect to the norm

$$\|h\|_p = \left(\int_\gamma |h(z)|^p |dz|\right)^{\frac{1}{p}},$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$ is the path $\gamma(t) = e^{it}$ (here, $\Pi^{-1} = \{0\}$).

A straightforward computation shows that $\gamma_\phi(h, g) := \frac{1}{\|h\|_p^{p-1}} \int_\gamma Re\left(|h(z)|^{p-2}h(z)e^{-i\phi}\overline{g(z)}\right)|dz|$, $h \neq 0$. Since

$$\int_\gamma Re\left(|z^q|^{p-2}\, z^q\, e^{-i\phi}\overline{z^s}\right)|dz| = 0, \quad 0 \leq s \leq q-1, \quad \phi \in [0, 2\pi),$$

then $\gamma_\phi(P_q, Q) = 0$ for all $Q \in \Pi^{q-1}$ and $\phi \in [0, 2\pi)$. So, Theorem A implies $M_{p,q} \equiv 0$. We put $\mathcal{K}_p := \|P_q - M_{p,q}\|_p = \|P_q\|_p = (2\pi)^{1/p}$.

Proposition 3.1. *Let $f \in \mathcal{A}(I)$ and $\frac{S}{T} \in \mathcal{V}^q_{n,m}(f, X)$. Then*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^q} \left\|\left(f - \frac{S}{T}\right)^\epsilon\right\|_{B_j} = \frac{1}{q!}\left|\left(f - \frac{S}{T}\right)^{(q)}(z_j)\right| \mathcal{K}_p, \quad 1 \leq j \leq k.$$

Proof. If $q = 0$ the result is obvious. Now assume $q > 0$. As $f - \frac{S}{T}$ is an analytic function on I and $\left(f - \frac{S}{T}\right)^{(s)}(z_j) = 0$, $0 \leq s \leq q-1$, $1 \leq j \leq k$, expanding $f - \frac{S}{T}$ by its Taylor polynomial at z_j, $1 \leq j \leq k$, up to order $q - 1$, we have

$$\frac{1}{\epsilon^q}\left(f - \frac{S}{T}\right)^\epsilon(z) = \frac{1}{\epsilon^q}\left(f - \frac{S}{T}\right)(\epsilon(z - z_j) + z_j) = \frac{(z - z_j)^q}{2\pi i}\int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S}{T}\right)(w)}{(w - (\epsilon(z - z_j) + z_j))(w - z_j)^q}dw, \tag{3.1}$$

for $z \in \overline{B_j}$, $0 < \epsilon < \lambda < 1$. Since for each $z \in \overline{B_j}$,

$$\left|\int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S}{T}\right)(w)}{(w - (\epsilon(z - z_j) + z_j))(w - z_j)^q}dw - \int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S}{T}\right)(w)}{(w - z_j)^{q+1}}dw\right|$$
$$= \left|\int_{\gamma_{j,\lambda}} \frac{\epsilon(z - z_j)\left(f - \frac{S}{T}\right)(w)}{((w - z_j) - \epsilon(z - z_j))(w - z_j)^{q+1}}dw\right| \leq \frac{\epsilon}{\lambda^{q+1}(\lambda - \epsilon)}\int_{\gamma_{j,\lambda}} \left|\left(f - \frac{S}{T}\right)(w)\right| |dw|,$$

from (3.1) and the Cauchy differentiation formula we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}\left(f-\frac{S}{T}\right)^{\epsilon}(z)=\frac{(z-z_j)^q}{2\pi i}\int_{\gamma_{j,\lambda}}\frac{\left(f-\frac{S}{T}\right)(w)}{(w-z_j)^{q+1}}dw=\frac{1}{q!}\left(f-\frac{S}{T}\right)^{(q)}(z_j)(z-z_j)^q,$$

uniformly in $z\in\overline{B_j}$. Therefore,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}\left\|\left(f-\frac{S}{T}\right)^{\epsilon}\right\|_{B_j}=\frac{1}{q!}\left|\left(f-\frac{S}{T}\right)^{(q)}(z_j)\right|\left\|(z-z_j)^q\right\|_{B_j}.$$

Now, substituting $z-z_j$ by w into the above equality,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^q}\left\|\left(f-\frac{S}{T}\right)^{\epsilon}\right\|_{B_j}=\frac{1}{q!}\left|\left(f-\frac{S}{T}\right)^{(q)}(z_j)\right|\left\|w^q-M_{p,q}(w)\right\|_p=\frac{1}{q!}\left|\left(f-\frac{S}{T}\right)^{(q)}(z_j)\right|\mathcal{K}_p.$$

This finishes the proof. ∎

Remark 3.2. Let $f\in\mathcal{A}(I)$ and $\frac{S}{T}\in\mathcal{V}^q_{n,m}(f,X)$. Then $\left\|f-\frac{S}{T}\right\|_{\epsilon}=O(\epsilon^q)$ as $\epsilon\to 0$. In fact, we see that

$$\left|\frac{1}{\epsilon^q}\left(f-\frac{S}{T}\right)^{\epsilon}(z)\right|\le\frac{1}{2\pi\lambda^q(\lambda-\epsilon)}\int_{\gamma_{j,\lambda}}\left|\left(f-\frac{S}{T}\right)(w)\right||dw|,\quad z\in\overline{B_j},\quad 0<\epsilon<\lambda<1,$$

which is clear from (3.1).

Proposition 3.3. *Let $f\in\mathcal{A}(I)$ and $\left\{\frac{S_\epsilon}{T_\epsilon}\right\}$ be a net of best rational approximants of f from $\mathcal{R}^n_m$ with respect to $\|\cdot\|_\epsilon$. Suppose $\mathcal{V}^q_{n,m}(f,X)\neq\emptyset$. Then $\{S_\epsilon\}$ and $\{T_\epsilon\}$ are uniformly bounded on compact sets as $\epsilon\to 0$. Moreover, if $q>0$ and $\{S_{\epsilon_l}\}$, $\{T_{\epsilon_l}\}$ are convergent subsequences to S_* and T_*, respectively, then*

$$\left(f-\frac{S_*}{T_*}\right)^{(i)}(z_j)=0,\quad 1\le j\le k,\quad 0\le i\le q-1. \tag{3.2}$$

Proof. Since $\|T_\epsilon\|_\infty=1$, $0<\epsilon\le 1$, the net $\{T_\epsilon\}$ is uniformly bounded on compact sets.

Let $\frac{S}{T} \in \mathcal{V}^q_{n,m}(f,X)$. As $T_\epsilon \neq 0$, then $0 < m_j(\epsilon) := \max\limits_{z\in\overline{B_j}} |T^\epsilon_\epsilon(z)| \leq 1, 1 \leq j \leq k$. So,

$$\frac{\|(S_\epsilon T - T_\epsilon S)^\epsilon\|_{B_j}}{\epsilon^q m_j(\epsilon)} \leq \frac{1}{\epsilon^q}\frac{\|(S_\epsilon T - T_\epsilon S)^\epsilon\|_{B_j}}{m_j(\epsilon)\max\limits_{z\in\overline{B_j}}|T^\epsilon(z)|} \leq \frac{1}{\epsilon^q}\left\|\left(\frac{S_\epsilon}{T_\epsilon} - \frac{S}{T}\right)^\epsilon\right\|_{B_j} \leq \frac{(2k\pi)^{1/p}}{\epsilon^q}\left\|\frac{S_\epsilon}{T_\epsilon} - \frac{S}{T}\right\|_\epsilon$$
$$\leq \frac{2(2k\pi)^{1/p}}{\epsilon^q}\left\|f - \frac{S}{T}\right\|_\epsilon, \quad 1 \leq j \leq k.$$

From Remark 3.2 we get

$$\|(S_\epsilon T - T_\epsilon S)^\epsilon\|_{B_j} = O(\epsilon^q m_j(\epsilon)) \quad \text{as} \quad \epsilon \to 0, \quad 1 \leq j \leq k.$$

Since $(S_\epsilon T - T_\epsilon S)^\epsilon \in \Pi^{k(q+1)}$ on B_j, by Bernstein's inequality [1, Corollary 5.1.6] and the equivalence of the norms in $\Pi^{k(q+1)}$ we have

$$\left|(S_\epsilon T - T_\epsilon S)^{(i)}(z_j)\right| = O(\epsilon^{q-i}) \quad \text{as} \quad \epsilon \to 0, \quad 1 \leq j \leq k, \quad 0 \leq i \leq q. \tag{3.3}$$

But $S_\epsilon T - T_\epsilon S \in \Pi^{k(q+1)}$, then there exist $M > 0$ and $\epsilon_1 > 0$ such that

$$\|S_\epsilon T - T_\epsilon S\|_\infty \leq M, \quad 0 < \epsilon \leq \epsilon_1.$$

Finally, as

$$\|T_\epsilon S\|_\infty \leq \|S\|_\infty, \quad 0 < \epsilon \leq \epsilon_1,$$

and $\|PT\|_\infty$, $P \in \Pi^n$, is a norm on Π^n, from the equivalence of the norms in Π^n we conclude that $\{S_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$.
Now, suppose that $q > 0$ and $\{S_{\epsilon_l}\}$, $\{T_{\epsilon_l}\}$ are convergent subsequences to S_* and T_*, respectively, and let $1 \leq j \leq k$ and $0 \leq i \leq q-1$. From (3.3),

$$(S_* T - T_* S)^{(i)}(z_j) = 0,$$

and according to the Hurwitz Theorem, $T_*(z) \neq 0$, $z \in I$. Therefore, using the Leibniz rule, $\left(\frac{S}{T} - \frac{S_*}{T_*}\right)^{(i)}(z_j) = 0$. Since $\left(f - \frac{S}{T}\right)^{(i)}(z_j) = 0$, we get (3.2). ■

Remark 3.4. If $\mathcal{V}^q_{n,m}(f,X) \neq \emptyset$ and $r = 0$, then $\mathcal{V}^q_{n,m}(f,X) = \left\{\frac{S}{T}\right\}$. So, from the above proof it follows that $\frac{S_*}{T_*} = \frac{S}{T}$.

In the following theorem, we obtain best multipoint Padé approximants as limits of rational functions of best L^p-approximation on complex domain.

Theorem 3.5. *Let $f \in \mathcal{A}(I)$ and $\left\{\frac{S_\epsilon}{T_\epsilon}\right\}$ be a net of best rational approximants of f from $\mathcal{R}_m^n$ with respect to $\|\cdot\|_\epsilon$. Suppose that there exists a unique best Padé approximant of f at X, say $\frac{S}{T}$. Then $\frac{S_\epsilon}{T_\epsilon}$ is convergent to $\frac{S}{T}$ uniformly on any closed subset of I, as $\epsilon \to 0$.*

Proof. It is sufficient to prove that if $\{S_{\epsilon_l}\}$ and $\{T_{\epsilon_l}\}$ are convergent subsequences to S_* and T_*, respectively, then $\frac{S_*}{T_*} = \frac{S}{T}$. Since $\|T_{\epsilon_l}\|_\infty = 1$, for all l, then $\|T_*\|_\infty = 1$ and by the Hurwitz theorem $T_*(z) \neq 0$, $z \in I$. Therefore, $\frac{S_{\epsilon_l}}{T_{\epsilon_l}}$ is convergent to $\frac{S_*}{T_*}$ uniformly on any closed subset of I, as $l \to \infty$. Next, we show that $\frac{S_*}{T_*} = \frac{S}{T}$. If $q = 0$, then

$$\sum_{j=1}^{k} \left| \left(f - \frac{S_*}{T_*}\right)(z_j) \right|^p = \lim_{l\to\infty} \left\| f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}} \right\|_{\epsilon_l}^p \leq \lim_{l\to\infty} \left\| f - \frac{S}{T} \right\|_{\epsilon_l}^p = \sum_{j=1}^{k} \left| \left(f - \frac{S}{T}\right)(z_j) \right|^p.$$

Now assume $q > 0$. Let $1 \leq j \leq k$, $0 \leq i \leq q-1$ and $0 < \lambda < 1$. Since $f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}$ are analytic functions on B_j, expanding $f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}$ by its Taylor polynomial at z_j, $1 \leq j \leq k$, up to order $q-1$, we have

$$\frac{1}{\epsilon_l^q}\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{\epsilon_l}(z) = \sum_{i=0}^{q-1} \frac{1}{i!}\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{(i)}(z_j)\epsilon_l^{i-q}(z-z_j)^i + \frac{R_q\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}, z_j, \lambda, \epsilon_l(z-z_j)+z_j\right)}{\epsilon_l^q}, \quad (3.4)$$

for each $z \in \overline{B_j}$, where $R_q(h, a, \lambda, z) = \dfrac{(z-a)^q}{2\pi i}\displaystyle\int_{\gamma_{a,\lambda}} \frac{h(w)}{(w-z)(w-a)^q}dw$. Since

$$\begin{aligned}
&\left| \int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)(w)}{(w - (\epsilon_l(z-z_j)+z_j))(w-z_j)^q}dw - \int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S_*}{T_*}\right)(w)}{(w-z_j)^{q+1}}dw \right| \\
&= \left| \int_{\gamma_{j,\lambda}} \frac{1}{(w-z_j)^q} \left(\frac{\left(\frac{S_*}{T_*} - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)(w)}{(w-z_j) - \epsilon_l(z-z_j)} + \epsilon_l \frac{(z-z_j)\left(f - \frac{S_*}{T_*}\right)(w)}{((w-z_j)-\epsilon_l(z-z_j))(w-z_j)} \right) dw \right| \\
&\leq \frac{1}{\lambda^q(\lambda-\epsilon_l)} \int_{\gamma_{j,\lambda}} \left| \left(\frac{S_*}{T_*} - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)(w) \right| |dw| + \frac{\epsilon_l}{\lambda^{q+1}(\lambda-\epsilon_l)} \int_{\gamma_{j,\lambda}} \left| \left(f - \frac{S_*}{T_*}\right)(w) \right| |dw|,
\end{aligned}$$

for each $z \in \overline{B_j}$, then

$$\lim_{l\to\infty} \frac{1}{\epsilon_l^q} R_q\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}, z_j, \lambda, \epsilon_l(z-z_j)+z_j\right) = \frac{(z-z_j)^q}{2\pi i}\int_{\gamma_{j,\lambda}} \frac{\left(f - \frac{S_*}{T_*}\right)(w)}{(w-z_j)^{q+1}}dw, \tag{3.5}$$

uniformly in $z \in \overline{B_j}$. As $\frac{S_*}{T_*}$ is an analytic function on B_j, from (3.5) and the Cauchy differentiation formula we have

$$\lim_{l\to\infty} \frac{1}{\epsilon_l^q} R_q\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}, z_j, \lambda, \epsilon_l(z-z_j)+z_j\right) = \frac{1}{q!}\left(f - \frac{S_*}{T_*}\right)^{(q)}(z_j)(z-z_j)^q, \tag{3.6}$$

uniformly in $z \in \overline{B_j}$. According to Remark 3.2, $\left\|\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{\epsilon_l}\right\|_{B_j} = O(\epsilon_l^q)$ as $l \to \infty$. So, (3.4) and (3.6) imply

$$\left\|\sum_{i=0}^{q-1} \frac{1}{i!}\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{(i)}(z_j)\epsilon_l^i (z-z_j)^i\right\|_{B_j} = O(\epsilon_l^q) \quad \text{as} \quad l \to \infty.$$

Therefore, by the equivalence of the norms in Π^{q-1}, we get

$$\left|\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{(i)}(z_j)\right| = O(\epsilon_l^{q-i}) \quad \text{as} \quad l \to \infty. \tag{3.7}$$

Now, from (3.4), (3.6) and (3.7) there exist a subsequence of $\{\epsilon_l\}$, which is denoted in the same way, and numbers $a_{ij} \in \mathbb{C}$, $0 \le i \le q-1$, $1 \le j \le k$, such that

$$\lim_{\epsilon_l\to 0} \frac{1}{\epsilon_l^q}\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{\epsilon_l}(z) = \frac{1}{q!}\left(f - \frac{S_*}{T_*}\right)^{(q)}(z_j)\,(z-z_j)^q - \sum_{i=0}^{q-1} a_{ij}(z-z_j)^i, \tag{3.8}$$

uniformly in $z \in \overline{B_j}$. Thus, substituting $z - z_j$ by w into (3.8) gives

$$\begin{aligned}
\lim_{\epsilon_l\to 0} \frac{1}{\epsilon_l^q}\left\|\left(f - \frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right)^{\epsilon_l}\right\|_{B_j} &= \left\|\frac{1}{q!}\left(f - \frac{S_*}{T_*}\right)^{(q)}(z_j)\,(z-z_j)^q - \sum_{i=0}^{q-1} a_{ij}(z-z_j)^i\right\|_{B_j} \\
&\ge \left|\frac{1}{q!}\left(f - \frac{S_*}{T_*}\right)^{(q)}(z_j)\right| \left\|w^q - M_{p,q}(w)\right\|_p \\
&= \frac{1}{q!}\left|\left(f - \frac{S_*}{T_*}\right)^{(q)}(z_j)\right| \mathcal{K}_p,
\end{aligned}$$

for each $1 \leq j \leq k$. Since $\left\{\frac{S_{\epsilon_l}}{T_{\epsilon_l}}\right\}$ is a sequence of best rational approximants of f from $\mathcal{R}_m^n$ with respect to $\|\cdot\|_\epsilon$, by Proposition 3.1 we get

$$\sum_{j=1}^{k}\left|\left(f-\frac{S_*}{T_*}\right)^{(q)}(z_j)\right|^p \leq \sum_{j=1}^{k}\left|\left(f-\frac{S}{T}\right)^{(q)}(z_j)\right|^p .$$

Therefore, $\frac{S_*}{T_*}$ is a best Padé approximant of f at X by (3.2). Finally, by hypothesis $\frac{S_*}{T_*} = \frac{S}{T}$. This finishes the proof. ■

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