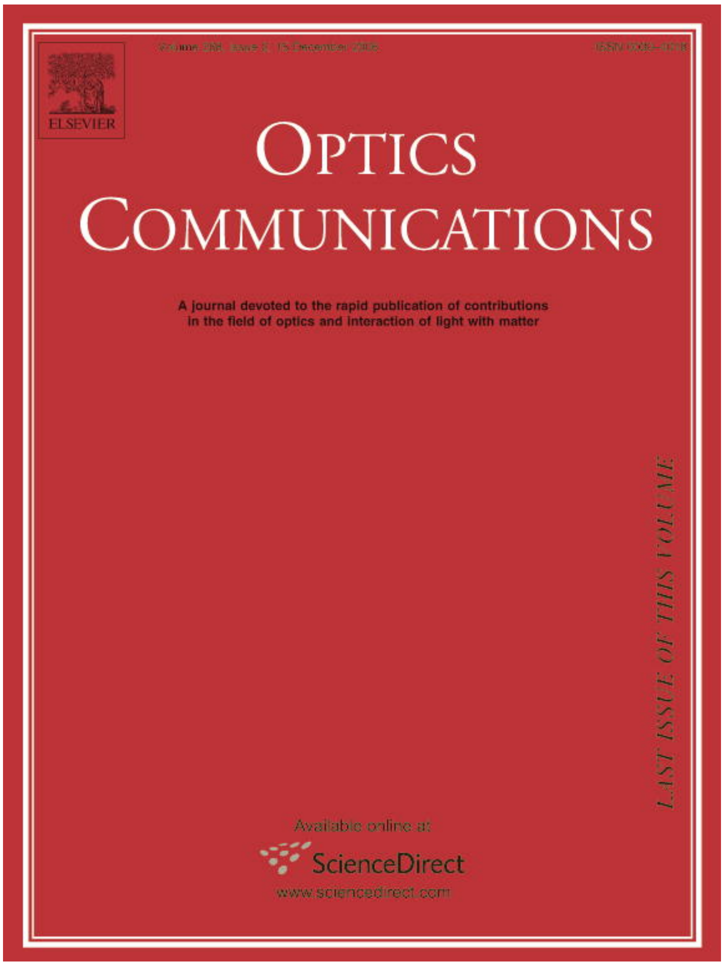
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Optical simulation of the quantum Hadamard operator

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Abstract

A possible way to optically simulate quantum algorithms is by making use of the spatial distribution of light in a laser beam. In this approach, the quantum states are represented by the amplitudes of the electromagnetic field in the beam. Temporal evolution is simulated by using optical elements such as lenses and phase shifters. Different elements are required depending on the operation whose implementation is desired. In this paper, we present an optical module to simulate the Hadamard transformation operating on a single qubit. The system is composed by a set of lenses, a phase plate and a phase grating and it could be used as a part of more complex arrangements. As an example, we make use of our Hadamard optical module as a part of the quantum circuit that solves the Deutsch problem. We show the obtained experimental results and we discuss the limitations of the proposal.

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Keywords: Information processing; Optical data processing

1. Introduction

Due to the current interest in quantum information and computation, the area of optical simulations of quantum information processing has received increasing attention. The main idea of this kind of simulation is that both, the electromagnetic field in classical optics, and the quantum state of a physical system, evolve following the wave equation and satisfy the superposition principle [1]. Therefore, experiments that only involve classical optics could be useful tools to simulate the behavior of quantum computers [2–7]. Optical architectures that can reproduce different quantum algorithms are interesting not only from the academic point of view but also to understand their basic properties. In this sense is important the development of new configurations that can be performed in a laboratory with a low degree of difficulty, a high accuracy and inexpensive equipment.

Various works have been performed in this line. In a pioneer paper, Cerf et al. proposed optical configurations to simulate quantum circuits [2] and presented an all optical simulation of the Grover's search algorithm [3]. In these works, single photons were used for the representation of quantum bits (qubits) and the implementation of universal quantum gates was made by using simple optical components like beam splitters and phase shifters in the context of quantum optical experiments. Around the same time, Spreeuw [4] suggested for the first time, the analogy between entangled states of qubits and certain configurations of classical electromagnetic light waves. The information was codified in the polarization states of a single classical light beam (polarization classical bits) or in the amplitudes and phases of 2^n different spatially separated parallel beams (position classical bits). Entangled states were emulated by combining these two kinds of cbits representations. By using spatially distributed cbits, Bhattacharya et al. [6] simulated

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several iterations of the Grover's search algorithm. Recently, Puentes et al. [7] implemented a simulation of the Deutsch–Jozsa's algorithm and one step of the Grover's algorithm by using programmable liquid crystal displays.

In general, the described simulations involve unitary operations that can be implemented by means of linear optical elements. For instance, Spreeuw has proposed the use of a beam splitter to simulate the Hadamard operator. By adding to the beam splitter a phase variation, any unitary operation can be simulated. Although the representation of unitary operations with beam splitters is conceptually simple and easy to be understood, its implementation presents some experimental drawbacks. In fact, the experimental implementation of this proposal requires the additional use of mirrors to redirect the light arriving from each cbit. The stability requirements of this architecture are similar to those needed for interferometric systems. In addition, a real beam splitter will cause two images that interfere each other corrupting with fringes the signal of interest.

On the contrary, our proposal is based on an imaging architecture, which renders the system more robust to noise and instabilities. We show here a compact optical system that can simulate the quantum Hadamard operation working on a single bit. The Hadamard transformation is a unitary operation frequently used in quantum computing. Many quantum algorithms make use of this transformation in some intermediate stage, so our experimental setup could be inserted as a module in the whole process. As example we show the performance of our Hadamard module when it is inserted in a quantum circuit that solves experimentally the Deutsch problem [1].

In Section 2, some fundamental concepts are presented. The notion of qubit and its optical representation are discussed and we also describe the Hadamard operator acting on 1 qubit. In Section 3, the optical simulation of the Hadamard operator is detailed and it is shown the experimental implementation. In Section 4, it is presented the Deutsch problem and the corresponding experimental results. Finally, in Section 5, we discuss the limitations of the proposed system and we give some conclusions.

2. Background

As a classical computer operates over classical bits (cbits) a quantum computer operates on quantum bits (qubits). The computational basis for the case of 1 qubit is composed by two vectors usually denoted as $|0\rangle$ and $|1\rangle$. The more general state of a qubit is a linear combination of the form $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex coefficients that satisfy the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. Thus, the quantum state can be described by a vector $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ in a bidimensional complex space. In the case of n-qubits, a 2^n dimensional vector is needed to characterize the quantum state.

The optical simulation of quantum algorithms can be done by representing each of the 2^n coefficients by means of the electromagnetic amplitudes of 2^n arbitrarily chosen spatial portions of a laser beam [5,6,8].

Before describing the simulation of the Hadamard transformation, let us briefly introduce how the states can be emulated as a spatial distribution of light [7]. Let us suppose that the input plane is limited to a square region, we assign the upper region to one state of the basis and the bottom region to the other one. We have selected a zone contained in the upper region to represent the state $|0\rangle$. The state $|1\rangle$ is represented by a zone included in the bottom region of the square. These examples are shown in Fig. 1a. In Fig. 1b, it is shown an arbitrary state of a single qubit where the coefficients α and β in the superposition are arbitrary complex numbers. The representation of such state requires a medium where amplitudes and phases could be represented as we have previously discussed [7]. Following the same idea a state of n qubits can be represented by assigning 2^n places in the square. Obviously, the larger the base the higher should be the resolution of the physical medium where the trasmitance will be represented. In Fig. 1c, it is shown a possible organization of the input plane for representing a 2 qubit state. In this paper, we propose an architecture that can be used to perform the Hadamard transformation over the first bit of a state with an arbitrary dimension. First, we show results of the transformation applied to a state corresponding to a single qubit. In Section 4, results for the same operation applied to a 2 qubit state are shown.

Now we will briefly discuss how operates the Hadamard transformation. Let us consider the computational basis of 1 qubit, in this basis the Hadamard transformation H can be represented by the 2×2 matrix:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{1}$$

Note that the effect of applying H to each element of a basis of 1 qubit is to obtain a superposition of both states as follows:

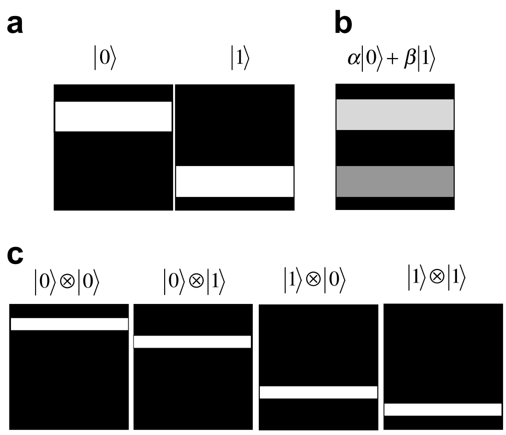

Fig. 1. Example of the input transmittance of the system: in (a) the two states of a basis; in (b) an arbitrary state of a single qubit; in (c) the four states basis of 2 qubits space.

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad (2)$$

where we have made explicit the matrix and the bracket notations.

3. Hadamard operator

In order to simulate this operator, we propose the experimental set-up shown in Fig. 2. A laser is expanded and filtered by the spatial filter SF, and collimated by lens L_0. The collimated beam impinges onto the plane Π where the quantum state is represented. To illustrate the Hadamard transformation we represent three examples: the state $|0\rangle$, the state $|1\rangle$ and an equally weighted combination of both states $|0\rangle + |1\rangle$. To this end, a binary screen is used to simulate each state. The dimensions of the zones with transmittance 1 (white rectangles in Fig. 1) were 0.18 cm × 2 cm and they were separated 0.95 cm. The next element is a phase plate that introduces a retardance of π (with an error of approximately 1%) in the bottom half. The phase plate was constructed by deposition of a transparent film over a plane glass plate. Lens L_1 (focal length 30 cm) allows to obtain the Fourier transform of the input plane Π. In the Fourier plane a phase grating is placed. The phase grating is constructed in such a way that the three principal orders (−1, 0 and 1) have identical intensities. The frequency of the grating is selected to produce diffracted orders whose separation in the final plane is equal to the distance between the 2 qubit images.

In our case, we synthesized an holographic bleached grating with period of 75 l/mm. A third lens L_2 (focal length 30 cm) is used to obtain the inverse Fourier transform which is projected onto a screen and registered by a CCD. It should be noted that this architecture with some modifications, allows to obtain the optical simulation of any U(2) operation. In fact, this can be done by using a grating with different efficiency for the three principal orders (1, 0 and −1) and eventually two arbitrary phase plates: one in front of, and one behind the Fourier plane [1].

In Fig. 3, we show a scheme of how the Hadamard transformation in 1 qubit is obtained by means of the set composed by the grating and the phase plate. Two elements of the basis and their corresponding outputs are shown in the left side and in the right side, respectively.

When the state $|0\rangle$ is used as input, the result in the output plane are the orders 0 and +1. This corresponds to an image containing the states $|0\rangle$ and $|1\rangle$ with equal weights. Instead, if the entrance is the state $|1\rangle$, orders 0 and −1 will be registered. The −1 order (which is dephased in π with respect to the orders 0 and +1) plus the phase of π introduced by the plate reproduces in the output plane the state $|0\rangle$; the 0 order with the addition of the phase π reproduces the state $-|1\rangle$. The orders that do not appear in the square image (in dashed line in the figure) are not registered by the camera.

Finally, we show the results obtained with this set-up. In Fig. 4, the first column represents the quantum states to be

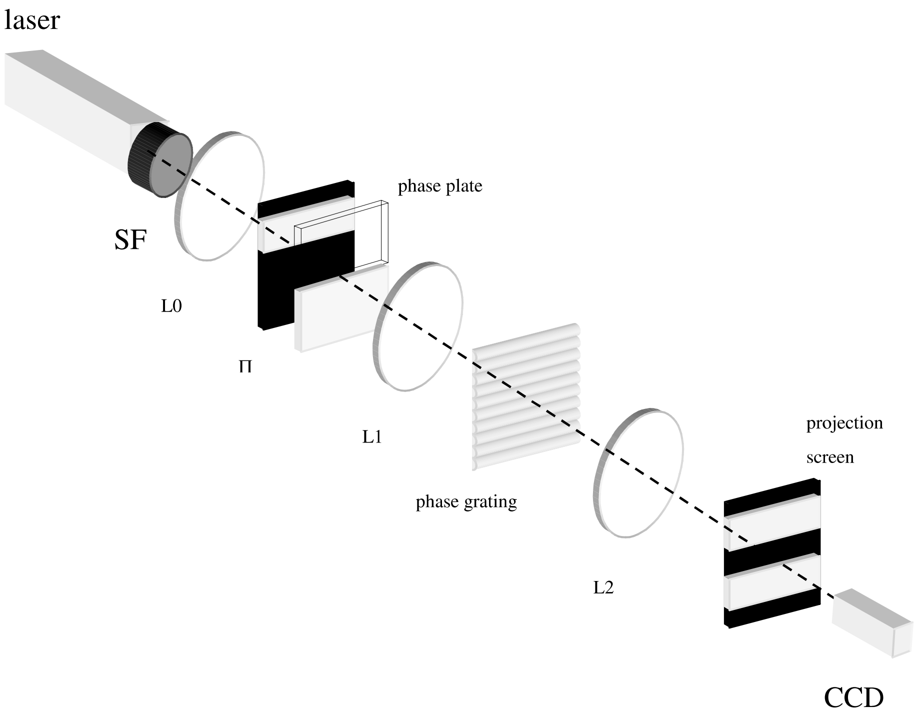

Fig. 2. Experimental set-up to obtain the Hadamard transformation.

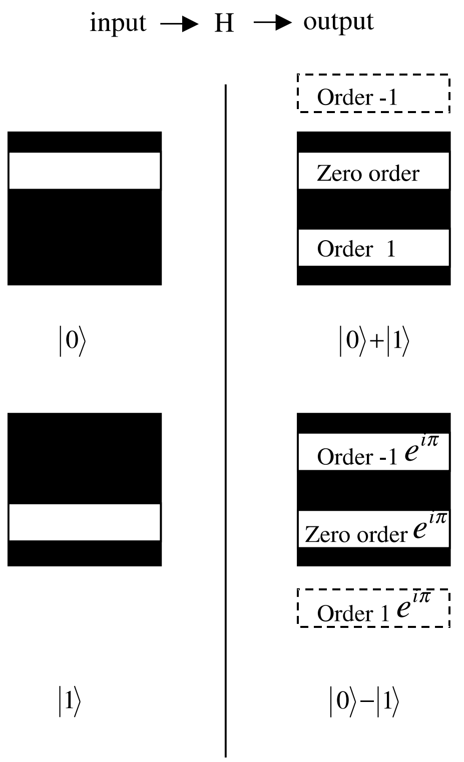

Fig. 3. Schematic demonstration of the Hadamard transformation.

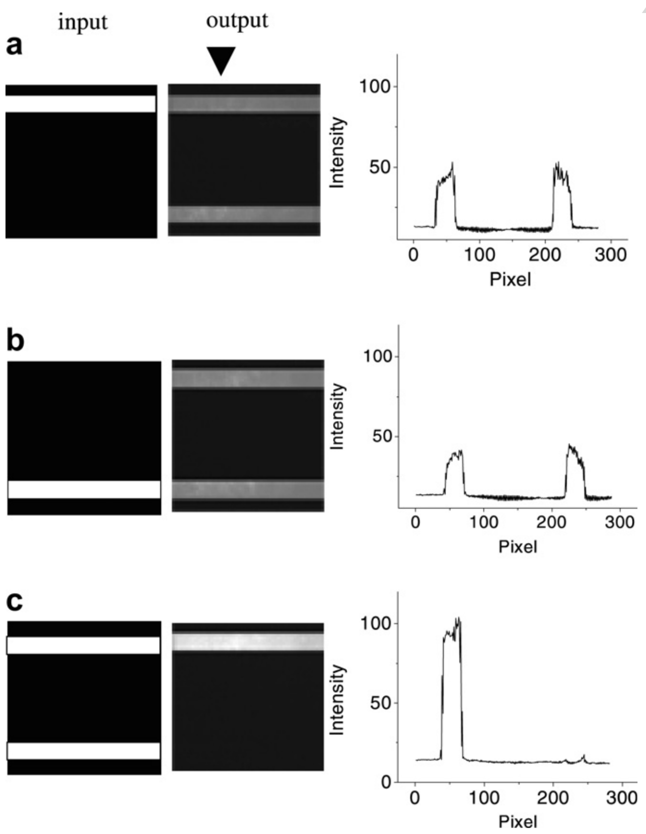

Fig. 4. Experimental results: the first column schematize the input, in the second column the captured images are shown and in the third column line intensity profiles of the images are depicted. (a), (b) and (c) correspond to inputs $|0\rangle$, $|1\rangle$ and $|0\rangle + |1\rangle$, respectively.

transformed, the second column shows the obtained images and finally in the third column intensity line profiles of these images are shown, in arbitrary units. The arrow in the second column indicates where the line profiles were obtained. Fig. 4a corresponds to the state $|0\rangle$ as the input of the system, Fig. 4b corresponds to state $|1\rangle$ as input, and Fig. 4c corresponds to a linear combination of both states $|0\rangle + |1\rangle$. It can be observed that a good reproduction of the expected results is obtained.

Fig. 4a and b show that the final image is composed by two approximately equal weighted states. It should be pointed out that the intensity captured by the CCD corresponds to the square modulus of the states described by Eq. (2). As a consequence the final intensity distributions in both cases must be the same. Although these results are the expected it is not possible to obtain from them an evidence of the difference of signs between the two linear combination expressed by Eq. (2). Nevertheless, this difference could be revealed by transforming a linear combination as that shown in Fig. 4c. From Eq. (2) is obtained $H((|0\rangle + |1\rangle)/\sqrt{2}) = |0\rangle$. The result is an image where only a region corresponding to state $|0\rangle$ is present. Moreover, the intensity profile reveals that the intensity of such state is approximately two times greater than the weight of each state in the previous two cases. We can conclude that the reinforcement of the state $|0\rangle$ and the vanishing of the state $|1\rangle$ are due to an interference effect with the appropriate phases and amplitudes.

4. Example: application to the Deutsch problem

Let us briefly describe the Deutsch algorithm. Let $f(x)$ be a function whose domain and image is the set $\{0,1\}$ i.e., f: $\{0,1\} \to \{0,1\}$. The function is either constant ($f(0) = f(1)$) or balanced ($f(0) \neq f(1)$). The goal of the algorithm is to decide whether $f(x)$ is constant or balanced. In a classical way, we need to evaluate $f(x)$ twice. The quantum algorithm solves this problem by just calling once the circuit that evaluates $f(x)$. This circuit represent a 2 qubits unitary operator U_f that acts as $U_f|x\rangle \otimes |y\rangle = |x\rangle \otimes \ |y \oplus f(x)\rangle$ where the symbol $\oplus$ denotes the binary sum. The algorithm (which has been divided into two stages or modules for simplicity) works as follows: first, we must prepare the initial state as the following combination (omitting the normalization constant) $(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$. Acting with U_f on this initial state we find:

$$U_f\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$
$$=\frac{(-1)^{f(0)}(|0\rangle\otimes|0\rangle-|0\rangle\otimes|1\rangle)+(-1)^{f(1)}(|1\rangle\otimes|0\rangle-|1\rangle\otimes|1\rangle)}{2}$$
$$=\frac{(-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} \quad (3)$$

U_f can be interpreted as generating a selective phase shift to each term of the first qubit state depending on the value of the function f. In a second module, we perform the

Hadamard transform of the state of the first qubit. Then, the probability to detect the state $|x=0\rangle$ in the first qubit is equal to $(((-1)^{f(0)}+(-1)^{f(1)})/2)^2$. This probability is therefore equal to zero if the function is balanced and equal to one if the function is constant. Thus, by detecting the final state, we find out what class does the function in the oracle belong to. We summarize the algorithm in Fig. 5. In this scheme, the upper horizontal line represents the first qubit and the lower horizontal line represents the second qubit. The initial states of these qubits are specified in the left of the scheme. Unitary operators acting on 2 qubit space (like U_f) are representing as a square that includes the two lines meanwhile the 1 qubit operators (like the 1 qubit Hadamard) includes only the line of the qubit that it affects. The input states are on the left of the square operators and the output states are on the right, so the circuit must be read from the left to the right.

In order to test the optically simulated Hadamard operator, we performed an optical simulation of the Deutsch algorithm. As we have mentioned above we divided the process into two modules. The first module consists in the optical simulated U_f operator. In Fig. 6, we show the result of the complete optical simulation. In the first column the four possible functions are described and in the second column, the phase shift induced by U_f is shown. In the upper zone the amplitude is multiplied by the factor $(-1)^{f(0)}$ modifying the phase of the state $|0\rangle$ as required by U_f. Similarly, the lower zone induces a phase shift of $\pi f(1)$ on the state $|1\rangle$. The complete experimental set up is similar to the one sketched in Fig. 2. An Ar laser is used to illuminate the system. The input state and the operator U_f are represented in a single spatial light modulator (SLM) working in phase mode. As it has been demonstrated [8] an arbitrary complex function could be represented in this type of medium. On the top of Fig. 6, we show the input state given by Eq. (3). The spatial light modulator consists in a Sony liquid crystal display TV (LCTV) that combined with two polarizers and two wave plates, acts as a pure phase modulator [8]. The LCTV (model LCX012BL) was extracted from a commercial video-projector and is a VGA resolution panel (640 × 480 pixels) with square pixels of 34 μm size separated by a distance of 41.3 μm. The resulting image was projected onto a screen and registered by a CCD.

In the third column of Fig. 6 the captured image associated with the resulting states after application of the Hadamard module is shown. Finally, in the fourth column an intensity line profile (corresponding to the dashed line drawn on the images of the third column) is depicted. It should be noted that in the case of constant functions, the result is an image where only the region corresponding to the state $|0\rangle$ of the first qubit is significantly present in accordance with the expected result. Also, in the case of balanced functions only the states with projection $|1\rangle$ on the first qubit are non-negligible. These results are in a good agreement with the predicted probability distribution.

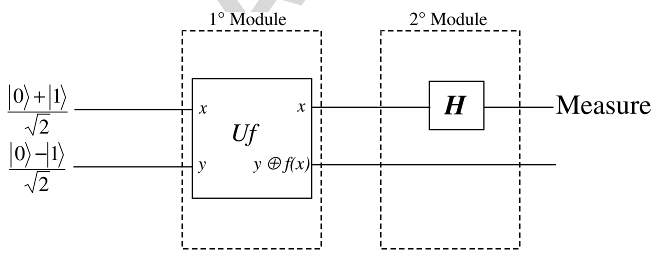

Fig. 5. Quantum circuit that solves the Deutsch problem.

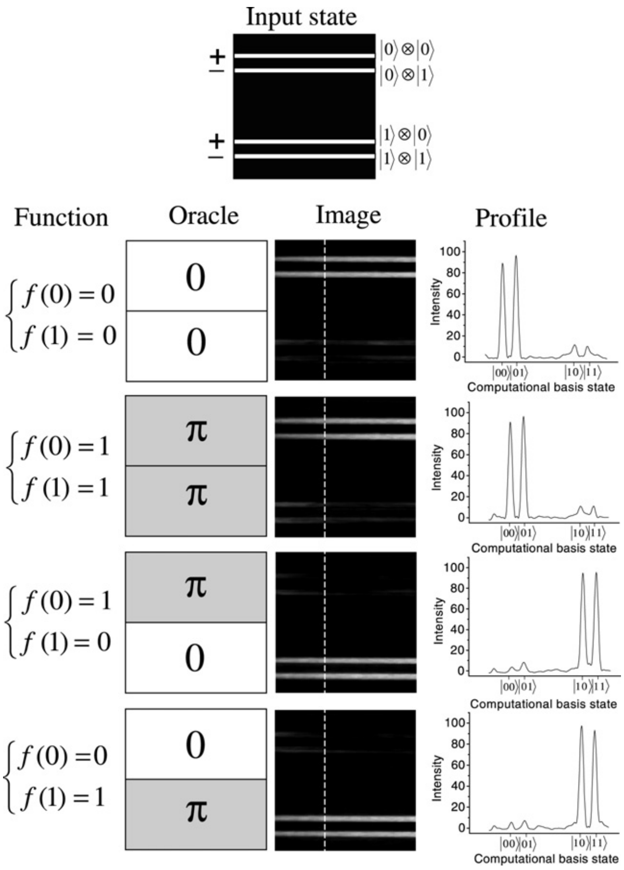

Fig. 6. Experimental results for the Deutsch Problem: in the top the input state is shown. Bellow, in the first column we describe the one bit to one bit function that evaluates the oracle. In the second column the optical representation of the oracle is shown. In the third column are the images of the output state after Hadamard transformation of the first qubit captured by CCD. In the fourth and last column, the line intensity profile of the images are depicted.

5. Discussion and remarks

It can be noted that, as in other optical simulations, the proposed set-up has limitations concerning the number of bits that can be processed. In our case, this limitation is mainly associated to the number of states that can be accommodated in the input plane. On one hand it depends on the dimension of the optical system and, on the other hand, on the resolution of the media where the states will be represented. Other points that must be considered are that the high coherence of the light source introduces speckle noise and the aberrations of the optical elements can introduce undesired phases. These effects can be observed, for instance, in Fig. 4 where the intensity profiles are partially corrupted by noise and irregularities. However, the system can reproduce correctly the predicted results. It should be noted that if the module were used in some stage of an iterative process, a previous estimation of losses must be done in order to insurance the intensity is measurable after the whole process. We estimate that by optimizing the manufacture of the optical elements the losses by reflection and diffraction could be approximately reduced up to 75% in each trip. For instance, by using the standard laser and camera that we have described above we estimate that the final state corresponding to 10 iterations could be detected with a good contrast. In order to increase the number of iterations, a more powerful source and/or a more sensitive camera must be used.

Summarizing, we have shown an arrangement that allows to obtain the Hadamard transformation in 1 bit. The experimental results are promising. The set-up could be included in a more complex optical system in which this transformation is needed in some stage of the process. As example, we have proposed an experiment that makes use of the optically simulated Hadamard operator in the quantum circuit that solves the Deutsch problem. This problem is a simple but non-trivial subject with some relevance in quantum computing. In the future, we plan to use the Hadamard set up described above as a step in an optically simulated quantum random walk.

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