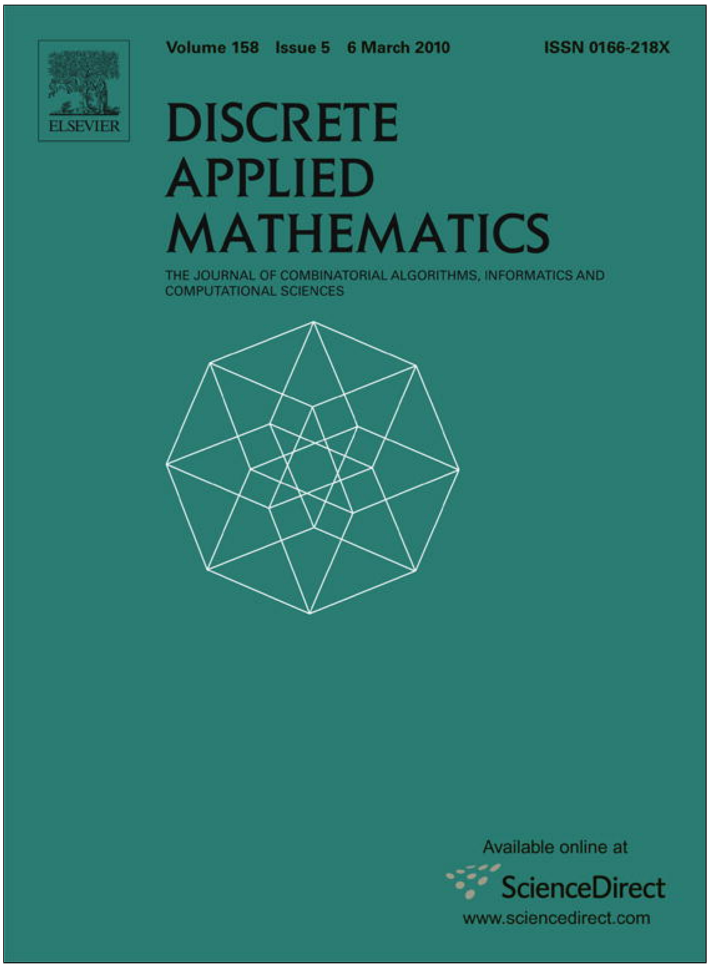
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Additive edge labelings

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A B S T R A C T

Let $G = (V, E)$ be a graph and d a positive integer. We study the following problem: for which labelings $f_E : E \to \mathbb{Z}_d$ is there a labeling $f_V : V \to \mathbb{Z}_d$ such that $f_E(i, j) = f_V(i) + f_V(j) \pmod{d}$, for every edge $(i, j) \in E$? We also explore the connections of the equivalent multiplicative version to toric ideals. We derive a polynomial algorithm to answer these questions and to obtain all possible solutions.

1. Introduction

Graph labeling is a broad subject encompassing a myriad of variants. In general, it involves assigning a value to each vertex or each edge of a graph, subject to some restrictions. For an extensive list of references on the subject, see the dynamic survey [2].

A classic example of graph labeling is graph coloring. Other examples are harmonious labelings [3] and felicitous labelings [7]. In the present work, we study a problem similar to these, but dropping the one-to-one conditions and allowing modular arithmetic over an arbitrary positive integer d. The particular case $d = 2$ is applied in [1] to the study of monotone dynamical systems.

Let $G = (V, E)$ be a graph and let $\mathbb{Z}_d$ denote as usual the set of integers modulo d. A function $f_E : E \to \mathbb{Z}_d$ is called an e-labeling of G and a function $f_V : V \to \mathbb{Z}_d$ is called a v-labeling. The pair (G, f_E) denotes the graph G with its edges labeled with f_E, and we say it is an *e-labeled graph*.

In this paper, we answer completely the question of when a given labeling of the edges of G with integers modulo d admits a labeling of the nodes of G such that the label of each edge is the sum, modulo d, of the labels of its vertices. More formally, we study the following problem.

Problem 1.1. Let (G, f_E) be an e-labeled graph. When is there a v-labeling f_V of G such that

$$f_E((v, v')) = f_V(v) + f_V(v') \pmod{d} \tag{1.1}$$

for every edge $(v, v') \in E$?

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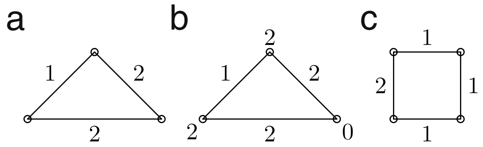

Fig. 1. (a) An e-labeling of a graph over $\mathbb{Z}_3$ and (b) a valid v-labeling of it. (c) A non-additive e-labeling of a graph, again over $\mathbb{Z}_3$.

Definition 1.2. We say that a v-labeling f_V satisfying (1.1) is a *valid v-labeling* of (G, f_E). If such an f_V exists, we say that f_E is an *additive e-labeling* of G. We also say that (G, f_E) is an *additively e-labeled graph*.

Note that we are not imposing the restriction that adjacent vertices have different labels.

Once we know that an e-labeling f_E of a graph G is additive, we can investigate how many valid v-labelings it admits. We denote this number by $\kappa(G, f_E)$. For example, the graph of Fig. 1(a), with the edge labels in $\mathbb{Z}_3$, is additive and admits a unique valid v-labeling over $\mathbb{Z}_3$, shown in Fig. 1(b), whereas the e-labeling of the graph of Fig. 1(c) is not additive.

We characterize completely the existence of valid v-labelings in Theorem 2.8 and we compute $\kappa(G, f_E)$ in Theorem 2.9. Moreover, we present a polynomial algorithm to decide the existence of valid v-labelings of an e-labeled graph (G, f_E) in Theorem 4.5. We also show that we can enumerate all such valid v-labelings in polynomial time. We derive these complexity results in Section 4 from the cost of computing the Smith Normal Form (SNF) [6] of the incidence matrix of the graph [4] and Theorem 4.4.

In Section 5, we comment on the equivalent multiplicative version Problem 5.1 of Problem 1.1, linking graphs and toric ideals. In particular, we obtain in Theorem 5.2 a modular version of classic results on the implicitization of toric parametrizations.

2. Characterization of additive e-labelings

In this section, we show that, not surprisingly, if a given e-labeling is additive, this imposes restrictions on the cycles in G. Throughout this work, the term cycle will not necessarily mean *simple* cycle. Theorem 2.8 shows that these restrictions are in fact sufficient.

If $C = (V, E)$ is a cycle of length k in G, we number its nodes "consecutively" $v_1, \dots, v_k$ and its edges $e_1, \dots, e_k$, where $e_i = (v_i, v_{i+1})$ for all $i < k$, and $e_k = (v_k, v_1)$.

Definition 2.1. We say that an e-labeled graph (G, f_E) has the *even cycle property* if every cycle of even length in G, with edges $e_1, \dots, e_{2k}$, satisfies

$$\sum_{l \text{ odd}} f_E(e_l) = \sum_{l \text{ even}} f_E(e_l) \pmod{d}.$$

Definition 2.2. Let d be an even positive integer. We say that an e-labeled graph (G, f_E) has the *odd cycle property* if every cycle of odd length in G, with edges $e_1, \dots, e_{2k+1}$, satisfies

$$\frac{d}{2}\sum_{l=1}^{2k+1} f_E(e_l) = 0 \pmod{d}. \tag{2.1}$$

Equivalently, the odd cycle property holds if the sum $\sum_{l=1}^{2k+1} f_E(e_l)$ is an even number for all odd cycles in G. Note that if $d = 2$, then both properties mean that the number of 1's in the edges of a cycle of any length is even. This case was studied in [1] in a multiplicative setting as in Section 4.

Definition 2.3. Let (G, f_E) be an e-labeled graph. We say that (G, f_E) is *compatible* if one of the following conditions holds.

- d is odd and (G, f_E) has the even cycle property.
- d is even and (G, f_E) has both the even and the odd cycle properties.

Remark 2.4. The preceding definitions take into account *all* the cycles of a graph, not just its simple cycles. The example in Fig. 2 shows two simple cycles joined at a vertex. Assume $d \geq 3$. Then, each cycle, considered as a graph, satisfies the odd cycle property. Furthermore, each cycle trivially satisfies the even cycle property. However, the whole graph does not satisfy the even cycle property, because the non-simple cycle obtained by traversing both triangles in succession does not satisfy it.

We show in Theorem 4.4 that the number of conditions to be checked to ensure that (G, f_E) is compatible is in fact "small".

Lemma 2.5. *Assume d is even. Let (G, f_E) be a connected e-labeled graph satisfying the even cycle property. Let C be any odd cycle in G. Then, (G, f_E) satisfies the odd cycle property if and only if (2.1) holds for C.*

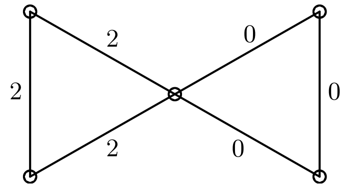

Fig. 2. Two simple cycles joined at a vertex.

Proof. Suppose that (G, f_E) satisfies the even cycle property and that (2.1) holds for C. Let C' be an odd cycle in G. Let $v \in C$ and $v' \in C'$. Since G is connected, there is a path P from v to v'. Let $e_1, \ldots, e_{2k+1}, e'_1, \ldots, e'_{2s+1}$ and $e^P_1, \ldots, e^P_r$ be the edges of C, C' and P, such that v is a vertex of e_1 and of e^P_1 and that v' is a vertex of e'_1 and e^P_r. The even cycle property of (G, f_E) applied to the even cycle made up of C, P from v to v', C' and then P from v' to v, implies that

$$\sum_{i=1}^{2k+1}(-1)^{i+1} f_E(e_i) + \sum_{i=1}^{r}(-1)^{i} f_E(e^P_i) + \sum_{i=1}^{2s+1}(-1)^{r+i} f_E(e'_i) + \sum_{i=1}^{r}(-1)^{i} f_E(e^P_i) = 0 \pmod{d}.$$

If we multiply both sides by $d/2$, and since $d/2 = -d/2 \pmod{d}$, we obtain $\frac{d}{2}\left(\sum_{i=1}^{2k+1} f_E(e_i) + \sum_{i=1}^{2s+1} f_E(e'_i)\right) = 0 \pmod{d}$. Using the odd cycle property of (C, f_E), we obtain $\frac{d}{2}\sum_{i=1}^{2s+1} f_E(e'_i) = 0 \pmod{d}$, which means that (2.1) holds for C' too. □

The following two lemmas show that compatibility is a necessary condition for additivity. We leave their straightforward proofs to the reader.

Lemma 2.6. *If (G, f_E) is an additive e-labeled graph, then (G, f_E) has the even cycle property.*

Lemma 2.7. *If d is even, and (G, f_E) is an additive e-labeled graph, then G has the odd cycle property.*

In fact, the compatibility conditions are sufficient for additivity.

Theorem 2.8. *An e-labeled graph (G, f_E) is additive if and only if it is compatible.*

Let $G_1, \ldots, G_r$ be the connected components of G, and let f_E be an e-labeling of G. Then,

$$\kappa(G, f_E) = \prod_i \kappa(G_i, f_E).$$

In particular, an e-labeled graph is additive if and only if each connected component is additive. Therefore, we restrict ourselves to connected graphs.

Theorem 2.9. *Let (G, f_E) be a connected additive e-labeled graph.*
If (G, f_E) has no odd simple cycles, $\kappa(G, f_E) = d$.
If (G, f_E) has at least one odd simple cycle, then

- *if d is odd, then $\kappa(G, f_E) = 1$*
- *if d is even, then $\kappa(G, f_E) = 2$*

We prove Theorems 2.8 and 2.9 in Section 3.

3. Proof of Theorems 2.8 and 2.9

Lemmas 2.6 and 2.7 show that compatibility is a necessary condition for additivity. We now turn our attention to sufficient conditions and to the number of valid v-labelings that an additive e-labeled graph admits, through a series of preparatory lemmas.

Lemma 3.1. *Let (G, f_E) be a connected additive e-labeled graph, and suppose that f_V and f'_V are valid v-labelings of (G, f_E). If there is $v \in V$ such that $f_V(v) = f'_V(v)$, then $f_V = f'_V$.*

The proof follows easily by induction on the distance between v and v'. As a consequence, once we fix the label for one vertex in a connected additive e-labeled graph, the rest of the vertex labels are fixed. Furthermore, Lemma 3.1 implies that $\kappa(G, f_E) \leq d$.

Definition 3.2. Given a simple cycle C and three vertices v, v' and v'' in C, we define $C[v, v', v'']$ as the simple path in C from v to v' that contains v''. Conversely, $C[v, v', \overline{v''}]$ is the simple path from v to v' in C that *does not* contain v'' (see Fig. 3.)

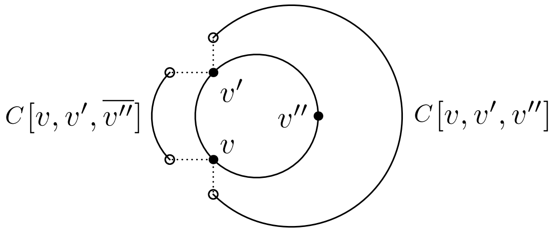

Fig. 3. Two simple paths from v to v' in C.

Definition 3.3. Let P be a path with vertices $v_1, \dots, v_k$ and edges $e_1 = (v_1, v_2), \dots, e_{k-1} = (v_{k-1}, v_k)$, and consider an e-labeling f_E of P. Let $\varphi_P : \mathbb{Z}_d \to \mathbb{Z}_d$ be defined by

$$\varphi_P(c) = (-1)^{k-1} c + \sum_{l=1}^{k-1} (-1)^{k-1-l} f_E(e_l) \pmod{d}. \tag{3.1}$$

In other words, $\varphi_P(c)$ is the label that v_k would have if we assigned label c to v_1 and propagated it through P.

Remark 3.4. Let (C, f_E) be an additive e-labeled simple cycle. Let v, v', v'' be in C and set $C_1 = C[v, v', v'']$ and $C_2 = C[v, v', \overline{v''}]$. For any valid v-labeling f_V of (C, f_E), we have that $\varphi_{C_1}(f_V(v)) = \varphi_{C_2}(f_V(v)) = f_V(v')$.

We now prove Theorems 2.8 and 2.9 for simple cycles of odd length.

Lemma 3.5. *If (C, f_E) is a compatible e-labeled simple cycle of odd length then it is additive. If d is odd, $\kappa(C, f_E) = 1$. If d is even, $\kappa(C, f_E) = 2$.*

Proof. Let $v_1, \dots, v_{2k+1}$ be the nodes of the cycle. Suppose that we have a valid v-labeling f_V. We want to see which are the possible values of $f_V(v_1)$. We need

$$\varphi_{C[v_1, v_{2k+1}, v_2]}(f_V(v_1)) = \varphi_{C[v_1, v_{2k+1}, \overline{v_2}]}(f_V(v_1)) \pmod{d}.$$

Replacing each side by its definition, we obtain

$$(-1)^{2k} f_V(v_1) + \sum_{l=1}^{2k} (-1)^{2k-l} f_E(e_l) = f_E(e_{2k+1}) - f_V(v_1) \pmod{d},$$

and since $2k$ is even, this expression is equivalent to

$$2 f_V(v_1) = \sum_{l=1}^{2k+1} (-1)^{l+1} f_E(e_l) \pmod{d}. \tag{3.2}$$

If d is odd, then 2 is invertible modulo d and Eq. (3.2) has a unique solution. That implies that there is at most one possible value for $f_V(v_1)$. Since propagating this value yields a valid v-labeling, there is a unique valid v-labeling of (C, f_E).

If d is even, then we use the odd cycle condition. Recall that this implies that the sum of the labels of the edges in the cycle is an even number. Since changing the sign of some summands does not alter the parity of a sum, the right-hand side of (3.2), $\ell := \sum_{l=1}^{2k+1} (-1)^{l+1} f_E(e_l)$, is also even. Eq. (3.2) is then of the form $2X = 2b \pmod{2c}$. This equation has exactly two solutions: $X = b$ and $X = b + c$. This means that $f_V(v_1)$ is either $\ell/2$ or $(\ell + d)/2$. Since propagating these two values for $f_V(v_1)$ yields valid v-labelings, the proof is complete. $\square$

The proof of Lemma 3.5 allows us to deduce the following

Corollary 3.6. *Let (C, f_E) be an additive e-labeled simple cycle of odd length, with d even. If f_V and f'_V are its two different valid v-labelings, then $f_V(v) = f'_V(v) + d/2 \pmod{d}$ for all $v \in V$.*

Let (G, f_E) be an e-labeled graph. In the following proofs, given a subgraph C of G, we will denote by (C, f_E) the graph C labeled with the restriction of f_E to the edges of C.

Lemma 3.7. *Let (G, f_E) be a compatible e-labeled connected graph. Let C and C' be two cycles of odd length in G. Let $e_1, \dots, e_r$ be the edges of C and $e'_1, \dots, e'_s$ be the edges of C'. Assume that C and C' share at least one vertex v_1, such that both e_1 and e'_1 are incident to v_1. Then,*

$$\sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) = \sum_{l=1}^{s} (-1)^{s-l} f_E(e'_l) \pmod{d}.$$

Lemma 3.7 follows from the even cycle condition applied to the cycle $e_1, \dots, e_r, e'_1, \dots, e'_s$.

Proof of Theorems 2.8 and 2.9. Let (G, f_E) be a compatible e-labeled graph. Without loss of generality, we can assume that it is connected. We prove the theorems by constructing a valid v-labeling of it.

If G has odd simple cycles, call one of them C. Choose a valid v-labeling f of (C, f_E). Pick a vertex v in C and set $\ell = f(v)$. If G has no odd cycles, choose any vertex v in G and label it with any ℓ in $\mathbb{Z}_d$.

We build a valid v-labeling f_V of (G, f_E) by propagating the label of v to the rest of the graph. For that, set $f_V(v) = \ell$. For any vertex $v' \in V$, choose a path P from v to v' and set $f_V(v') = \varphi_P(\ell)$, where φ_P is as in (3.1). We have to prove that f_V is well-defined and that it is a valid v-labeling of (G, f_E).

Given v' and two simple paths P_1 and P_2 from v to v', we have to prove that $\varphi_{P_1}(\ell) = \varphi_{P_2}(\ell)$. Let $e_1, \dots, e_r$ and $e'_1, \dots, e'_s$ be the edges of P_1 and P_2, respectively, and assume that v is an endpoint of e_1 and e'_1. We call C' the cycle formed by the union of P_1 and P_2.

If the sum of the lengths of P_1 and P_2 is even, we can use the even cycle property of (G, f_E) applied to C'. That is,

$$f_E(e_1) - f_E(e_2) + \cdots + (-1)^{r+1} f_E(e_r) + (-1)^{r+2} f_E(e'_s) + \cdots - f_E(e'_1) = 0 \pmod d.$$

This condition is equivalent to the identity

$$\sum_{l=1}^{r} (-1)^l f_E(e_l) = \sum_{l=1}^{s} (-1)^l f_E(e'_l) \pmod d. \tag{3.3}$$

We have $\varphi_{P_1}(\ell) = (-1)^r \ell + \sum_{l=1}^{r} (-1)^{r-l} f_E(e_l) \pmod d$, and $\varphi_{P_2}(\ell) = (-1)^s \ell + \sum_{l=1}^{s} (-1)^{s-l} f_E(e_l) \pmod d$. Since r and s have the same parity, it follows that $\varphi_{P_1}(\ell) = \varphi_{P_2}(\ell)$.

If r is odd and s is even, the cycle C' has odd length. The equality $\varphi_{P_1}(\ell) = \varphi_{P_2}(\ell)$ is equivalent to

$$2\ell = \sum_{l=1}^{r} (-1)^{l+1} f_E(e_l) + \sum_{l=1}^{s} (-1)^{l+1} f_E(e_l) \pmod d. \tag{3.4}$$

The right-hand side of (3.4) is the alternating sum of the labels of the edges of the odd cycle C', starting at v. By Lemma 3.7, this sum is equal, modulo d, to the alternating sum of the labels of the edges of C, starting at v. By Lemma 3.5, this sum is equivalent to 2ℓ, which is what we needed to prove.

We now know that f_V is a well-defined labeling. Similar arguments show that it is also a valid v-labeling of (G, f_E). □

Remark 3.8. Given a compatible e-labeled graph (G, f_E), it is not difficult to see that if we add any edge e to G, there is an extension of f_E that assigns a label to e such that the resulting e-labeled graph is compatible. So any partial compatible labeling of a graph G can be extended to an additive e-labeling.

4. An efficient additivity test

Theorems 2.8 and 2.9 give a theoretical characterization of additive e-labeled graphs. These results are not practical per se, since they involve verifying certain conditions on all the cycles of a graph. In this section, we develop a polynomial algorithm to test for additivity.

We tackle this problem by studying the incidence matrix A_G of $G = (V, E)$. The matrix A_G has size $n \times m$, where $n = |V|$ and $m = |E|$, and its entries are defined by

$$[A_G]_{i,j} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident with the edge } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

We use the Smith Normal Form (SNF) S of A_G together with the left and right multipliers U, V. Here, $U \in \mathbb{Z}^{n\times n}$, $V \in \mathbb{Z}^{m\times m}$, $S \in \mathbb{Z}^{n\times m}$ have the following properties:

- U and V are unimodular (so that U^{-1} and V^{-1} are integer matrices),
- S is a diagonal matrix, with $s_{i,i} | s_{i+1,i+1}$ for all i, and
- $A_G = USV$.

Let $\mathbf{0}$ be the $n \times m - n$ matrix of 0's. Then the SNF S of A_G is [4]

$$\begin{bmatrix} D & \mathbf{0} \end{bmatrix}, \tag{4.1}$$

where D is an $n \times n$ diagonal matrix, with $D_{i,i} = 1$ for $i \le n - 1$, and $D_{n,n} = \alpha$. Here, $\alpha = 0$ if G is bipartite (i.e. has no odd cycles) and 2 otherwise.

Definition 4.1. Let $G = (V, E)$ be a graph and let C be any cycle of G. We associate a vector $\omega_C \in \mathbb{Z}^{|E|}$ with C. We index the coordinates of ω_C using the edges of G. Label the consecutive edges of C $e_1, e_2, \dots, e_{k-1}, e_k$, with e_1 any edge of the cycle. If C is an even cycle, we adjoin $(-1)^{i+1}$ to e_i:

$$e_1, -e_2, \dots, (-1)^{i+1} e_i, \dots, e_{k-1}, -e_k. \tag{4.2}$$

If C is an odd cycle and d is even, we adjoin $d/2$ to each edge:

$$\frac{d}{2}e_1, \dots, \frac{d}{2}e_i, \dots, \frac{d}{2}e_k. \tag{4.3}$$

Since C need not be a simple cycle, some edges may appear more than once in (4.2) and (4.3). Let $e'_1, \dots, e'_r$ be the *distinct* edges of C. For each distinct edge e'_i, we define $\omega_{e'_i}$ to be the sum of the coefficients of each appearance of e'_i in (4.2) or (4.3). For example, if an edge e'_i appears twice, both times accompanied by a 1, then the corresponding $\omega_{e'_i}$ is 2. If one of the appearances has a 1 and the other one a (-1), then $\omega_{e'_i}$ is 0.

Given a cycle C, we define $\omega_C \in \mathbb{Z}^E$ as

$$(\omega_C)_{(u,v)} = \begin{cases} \omega_{(u,v)} & \text{if } (u,v) \text{ is in } C, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that in (4.2), the choice of e_1 may swap the 1's and the -1's. This only changes ω_C into $-\omega_C$.

Remark 4.2. Let C be a cycle of G. If the length of C is even, then the sum of the coordinates of ω_C is 0. If the length of C is odd, then the sum of the coordinates of ω_C is $d/2 \pmod d$.

Let $\pi_d : \mathbb{Z}^{|E|} \to {\mathbb{Z}_d}^{|E|}$ denote the projection $\pi_d(x)_{(u,v)} = (x_{(u,v)}) \pmod d$, and $\mathcal{C}$ the set of even cycles in G. The integer kernel of A_G is computed in [9], and is shown to be the submodule spanned by $\{\omega_C, C \in \mathcal{C}\}$:

$$\ker_{\mathbb{Z}}(A_G) = \langle \omega_C, C \in \mathcal{C} \rangle. \tag{4.4}$$

We prove a modular version of this result. Given $M \in \mathbb{Z}^{a\times b}$, we define $\ker_{\mathbb{Z}_d}(M) = \{\mathbf{x} \in {\mathbb{Z}_d}^b, M\mathbf{x} = 0 \pmod d\}$.

Proposition 4.3. *Let G be a connected graph, and let A_G be its incidence matrix. Then,*

(i) *If d is odd or if G has no odd cycles, then* $\ker_{\mathbb{Z}_d}(A_G) = \pi_d(\ker_{\mathbb{Z}}(A_G))$;
(ii) *If d is even and there is an odd cycle C' in G, then*

$$\ker_{\mathbb{Z}_d}(A_G) = \pi_d(\ker_{\mathbb{Z}}(A_G)) \oplus \langle \pi_d(\omega_{C'}) \rangle.$$

Proof. Let S be the SNF of A_G, and U,V such that $A_G = USV$, as described in (4.1). Equivalently, $U^{-1}A_G = SV$. Therefore, $\ker_{\mathbb{Z}}(A_G) = \ker_{\mathbb{Z}}(SV)$ and $\ker_{\mathbb{Z}_d}(A_G) = \ker_{\mathbb{Z}_d}(SV)$, implying that

$$\ker_{\mathbb{Z}}(A_G) = V^{-1}\ker_{\mathbb{Z}}(S) \quad \text{and} \quad \ker_{\mathbb{Z}_d}(A_G) = \pi_d(V^{-1}\ker_{\mathbb{Z}_d}(S)). \tag{4.5}$$

Let $\mathbf{x} = (x_1, \dots, x_m) \in \ker_{\mathbb{Z}_d}(S)$. This means that

$$S\mathbf{x}^t = (x_1, \dots, \alpha x_n) = 0 \pmod d. \tag{4.6}$$

If G has no odd cycles, $\alpha = 0$ and so Eq. (4.6) holds if and only if $x_i = 0 \pmod d$ for $i \in \{1, \dots, n-1\}$. Then, $\ker_{\mathbb{Z}_d}(S) = \langle z_n, \dots, z_m \rangle$ and $\ker_{\mathbb{Z}}(S) = \langle z_n, \dots, z_m \rangle$, where $\{z_1, \dots, z_m\}$ denotes the canonical basis either in $\mathbb{Z}^m$ or in ${\mathbb{Z}_d}^m$. Therefore, we have $\ker_{\mathbb{Z}_d}(S) = \pi_d(\ker_{\mathbb{Z}}(S))$, whence $\ker_{\mathbb{Z}_d}(A_G) = \pi_d(\ker_{\mathbb{Z}}(A_G))$.

If G has an odd cycle, $\alpha = 2$ and so when d is odd, Eq. (4.6) holds if and only if $x_i = 0 \pmod d$ for $i \in \{1, \dots, n\}$. Again $\ker_{\mathbb{Z}_d}(S) = \pi_d(\ker_{\mathbb{Z}}(S))$, implying $\ker_{\mathbb{Z}_d}(A_G) = \pi_d(\ker_{\mathbb{Z}}(A_G))$.

Assume $\alpha = 2$ and d even. From Eq. (4.6), we now deduce that

$$\ker_{\mathbb{Z}_d}(S) = \langle z_{n+1}, \dots, z_m \rangle \oplus \left\langle \frac{d}{2} z_n \right\rangle \quad \text{and} \quad \ker_{\mathbb{Z}}(S) = \langle z_{n+1}, \dots, z_m \rangle. \tag{4.7}$$

Combining Eqs. (4.5) and (4.7), we have

$$\ker_{\mathbb{Z}_d}(A_G) = \langle \pi_d(V^{-1}z_{n+1}), \dots, \pi_d(V^{-1}z_m) \rangle \oplus \left\langle \pi_d\left(V^{-1}\frac{d}{2}z_n\right)\right\rangle. \tag{4.8}$$

$$\ker_{\mathbb{Z}}(A_G) = \langle V^{-1}z_{n+1}, \dots, V^{-1}z_m \rangle. \tag{4.9}$$

Let C' be an odd cycle of G. Then, $\pi_d(\omega_{C'}) \in \ker_{\mathbb{Z}_d}(A_G)$. To see why, recall that entry e_j of $\omega_{C'}$ is $d/2$ times the number of occurrences of the edge e_j in C'. For every vertex v_i of the cycle, the number of edges that enter and leave it must be the same. That means that the v_i-th entry of $A_G\omega_{C'}$ is an even number times $d/2$ (if vertex v_i is in the cycle) or 0. In both cases, $A_G\omega_{C'} = 0 \pmod d$. Now, since $\pi_d(\omega_{C'}) \in \ker_{\mathbb{Z}_d}(A_G)$, we must have

$$\pi_d(\omega_{C'}) = \sum_{l=n+1}^{m} \gamma_l \pi_d(V^{-1}z_l) + \varepsilon \pi_d\left(V^{-1}\frac{d}{2}z_n\right), \tag{4.10}$$

where $\varepsilon, \gamma_l \in \mathbb{Z}_d$ and ε is 0 or 1, since $\langle \frac{d}{2} z_n \rangle = \{0, \frac{d}{2} z_n\}$. The first summand consists of multiples of the projections of even cycles (see (4.4)). This means that if we take the sum of the coordinates of both sides of Eq. (4.10), we get $\varepsilon = 1$ (see Remark 4.2.) Set $\gamma = \sum_{l=n+1}^{m} \gamma_l \pi_d(V^{-1} z_l)$. Then

$$\pi_d\left(V^{-1}\frac{d}{2}z_n\right) = \gamma - \pi_d(\omega_{C'}). \tag{4.11}$$

Now, take any $\mathbf{x} \in \ker_{\mathbb{Z}_d}(A_G)$, $\mathbf{x} = \sum_{l=n+1}^{m} \beta_l \pi_d(V^{-1} z_l) + \beta \pi_d(V^{-1}\frac{d}{2} z_n)$. Plugging in Eq. (4.11) and setting $\tilde{\beta}_l = \beta_l + \gamma_l$, we have

$$\mathbf{x} = \sum_{l=n+1}^{m} \tilde{\beta}_l \pi_d(V^{-1} z_l) + (-\beta)\pi_d(\omega_{C'}),$$

which shows that $\ker_{\mathbb{Z}_d}(A_G) = \pi_d(\ker_{\mathbb{Z}}(A_G)) \oplus \langle \pi_d(\omega_{C'}) \rangle$. □

Let (G, f_E) be an e-labeled graph and $\omega \in \mathbb{Z}^E$. We set

$$\langle \omega, f_E \rangle := \sum_{(u,v)\in E} \omega_{(u,v)} f_E((u, v)).$$

The results we have discussed allow us to obtain the following

Theorem 4.4. *Let (G, f_E) be an e-labeled connected graph. Let A_G be the incidence matrix of G. The following statements are equivalent.*

(i) *(G, f_E) is a compatible e-labeled graph.*
(ii) *$\langle \pi_d(\omega_C), f_E \rangle = 0 \pmod d$, for every cycle C of G.*
(iii) *$\langle \omega, f_E \rangle = 0 \pmod d$, for all $\omega \in \ker_{\mathbb{Z}_d}(A_G)$.*
(iv) *If d is odd or G has no odd cycles, $\langle \omega, f_E \rangle = 0 \pmod d$, for all ω belonging to the projection of a finite set of generators of $\ker_{\mathbb{Z}}(A_G)$. If d is even and has an odd cycle, $\langle \omega, f_E \rangle = 0 \pmod d$, for all ω belonging to a finite set of generators of $\ker_{\mathbb{Z}}(A_G)$ and for ω_C, for some odd cycle C.*
(v) *(G, f_E) is an additive e-labeled graph.*

Proof. Clause (i) is equivalent to clause (v) by Theorem 2.8. Clause (ii) is a restatement of clause (i) using a different notation. Clauses (ii) and (iii) are equivalent by Proposition 4.3. Clauses (iii) and (iv) also follow from that proposition: the finite sets described in clause (iv) were shown to be generators of $\ker_{\mathbb{Z}_d}(A_G)$. □

The equivalence of clauses (iv) and (v) in Theorem 4.4 provides the following complexity result.

Theorem 4.5. *Let (G, f_E) be an e-labeled connected graph. The additivity of (G, f_E) can be tested in time polynomial in the size of the graph. Furthermore, we can obtain all its valid v-labelings in polynomial time too.*

Proof. We compute the Smith Normal Form (SNF) S of A_G described in the proof of Proposition 4.3, together with the left and right multipliers U and V. This computation can be carried out using the polynomial algorithm presented in [6], modified to work with rectangular matrices in the way the authors of that paper suggest.

We saw in Proposition 4.3 that we can obtain generators of $\ker_{\mathbb{Z}_d}(A_G)$ from the columns of V^{-1}. If G has no odd cycles (i.e. $\alpha = 0$), we use the last $m - n + 1$ columns. If $\alpha = 2$ and d is odd, we use the last $m - n$ columns. If $\alpha = 2$ and d is even, we use the last $m - n$ columns and $d/2$ times its n-th column. To check the additivity of (G, f_E), we just need to verify that these generators satisfy the conditions stated in clause (iv) of Theorem 4.4.

Once we know that (G, f_E) is additive, we can efficiently obtain all its valid v-labelings. We must first know whether G has an odd cycle or not. This can be read directly off the SNF S of A_G: G has an odd cycle if and only if the diagonal of S contains a 2. Having no odd cycles is classically known to be equivalent to G being bipartite (cf. for instance [5], p. 18) and can be checked in time $O(n + m)$. We can also obtain an odd cycle in G as a byproduct of this check.

If G has no odd cycles, we can assign any of the d possible labels to an arbitrary vertex, and then propagate the label to the rest of the graph using breadth-first search (BFS). If G does have odd cycles, choose one of them and call it C. Choose a vertex v_1 in C. Formula (3.2) shows which label (or labels, if d is even) we can assign to v_1 in order to obtain valid v-labelings of (G, f_E). We then propagate the label of v_1 to the rest of the graph using BFS. □

Remark 4.6. Given a graph G, consider the *cycle space* of G [5]. It is the $\mathbb{Z}_2$-vector space generated by the fundamental cycles of G. That is, the cycles obtained when adding an edge of G to a spanning tree.

One might be tempted to think that checking the compatibility conditions on these generators suffices to verify the compatibility of a graph with labels in $\mathbb{Z}_d$ for any d, as in the case $d = 2$. However, consider for instance the graph in Fig. 4, in which we marked the spanning tree with edges $\{e_{14}, e_{23}, e_{24}\}$: the sum of the two fundamental triangle cycles C_1 and C_2 (represented by their 0, 1 vectors) equals the square cycle C only when $d = 2$. This situation is depicted informally in Fig. 5. However, if d is odd we do not impose any conditions on C_1 and C_2, and so this cannot insure the even cycle condition we need to check. When $d \neq 2$ is even, we get $\frac{d}{2}$ times the even cycle condition, which again is not sufficient to insure additivity. Consider for instance the labeling $f(e_{12}) = f(e_{24}) = f(e_{34}) = 1, f(e_{14}) = f(e_{23}) = 0$ and $d = 4$. The odd cycle property is verified for C_1 and C_2 but the labeling is not additive.

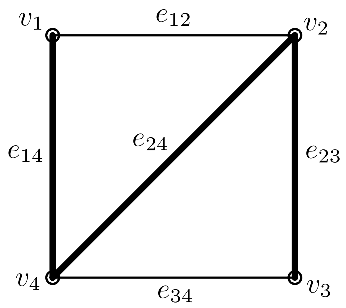

Fig. 4. A spanning tree of a graph.

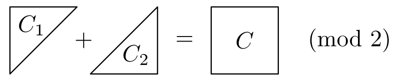

Fig. 5. Adding two odd cycles to obtain an even one.

5. Multiplicative version

In the previous sections, we used labelings that assigned integers modulo d to the edges and vertices of a graph. But actually, everything we wrote is also valid if the labels belong to any finite cyclic group, via the isomorphism with $\mathbb{Z}_d$. In particular, we can use labelings in $\mathbb{G}_d$, the d-th roots of unity. In this case, the isomorphism between $\mathbb{Z}_d$ and $\mathbb{G}_d$ is given by

$$k \mapsto e^{2\pi \mathrm{i} k/d}.$$

This alternate formulation links our problem with the theory of toric ideals. As a general text on this subject, we refer the reader to [8].

Let us state this equivalent version. Let $G = (V, E)$ be a connected graph and d an integer greater than 1. Let $v_1, \ldots, v_n$ be the vertices of G and let $e_1, \ldots, e_m$ be its edges. We work with complex variables x_{v_i} for each $v_i \in V$, and y_{e_i} for each $e_i \in E$. The value of x_{v_i} corresponds to the label of vertex v_i, and the value of y_{e_i} corresponds to the label of edge e_i. We can restate Problem 1.1 in this multiplicative setting.

Problem 5.1. For which $\mathbf{y} \in \mathbb{G}_d{}^m$ are there $\mathbf{x} \in \mathbb{G}_d{}^n$ such that

$$y_{e_i} = x_{u_i} x_{v_i}, \tag{5.1}$$

holds for every edge $e_i = (u_i, v_i) \in E$?

According to a classic result for toric parametrizations, given a vector $\mathbf{y} \in (\mathbb{C}^*)^m$ of complex nonzero numbers, there is an $\mathbf{x} \in (\mathbb{C}^*)^n$ satisfying (5.1) if and only if $y^{\mathbf{u}} = y_1^{u_1} \cdots y_m^{u_m} = 1$, for every $\mathbf{u} = (u_1, \ldots, u_m) \in \ker_{\mathbb{Z}}(A_G)$. Furthermore, under these conditions, the number of such solutions is

$$g = \gcd(\{\text{maximal minors of } A_G\}), \tag{5.2}$$

provided that $g \neq 0$, in which case there are infinitely many solutions. We deduce from (4.1) that $g = 2$ or 0, depending on whether G has an odd cycle or not, respectively. It was this result which prompted us to study the incidence matrix of G in connection with Problem 1.1.

We now state a modular version of the toric result. We impose the additional restriction that

$$x_{v_i}^d = 1, \tag{5.3}$$

for all $v_i \in V$. Together with (5.1), this implies that $y_{e_i} \in \mathbb{G}_d$.

Theorem 5.2. *Let $G = (V, E)$ be a connected graph. Given $\mathbf{y} \in \mathbb{G}_d{}^m$, there exists $\mathbf{x} \in \mathbb{G}_d{}^n$ satisfying (5.1) if and only if*

$$y^{\mathbf{u}} = 1,$$

for every $\mathbf{u} \in \ker_{\mathbb{Z}_d}(A_G)$. If g is 0, there are d solutions to (5.1) and (5.3) simultaneously. If g is 2 and d is even, there are two solutions. Otherwise, there is a unique solution.

The result can be translated from Theorem 2.9. Alternatively, we could prove that given $\mathbf{y} \in \mathbb{G}_d{}^m$, there are as many solutions $\mathbf{x} \in \mathbb{G}_d{}^n$ as stated using the knowledge of g in (5.2), by checking how many of the complex solutions $\mathbf{x} \in (\mathbb{C}^*)^n$ consist of d-th roots of unity.

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