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INVARIANCE OF A SHIFT-INVARIANT SPACE

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ABSTRACT. A shift-invariant space is a space of functions that is invariant under integer translations. Such spaces are often used as models for spaces of signals and images in mathematical and engineering applications. This paper characterizes those shift-invariant subspaces S that are also invariant under additional (non-integer) translations. For the case of finitely generated spaces, these spaces are characterized in terms of the generators of the space. As a consequence, it is shown that principal shift-invariant spaces with a compactly supported generator cannot be invariant under any non-integer translations.

1. Introduction

A *shift-invariant space* (SIS) is a space of functions that is invariant under integer translations.

They have applications throughout mathematics and engineering, as such spaces are often used as models for spaces of signals and images, see [Grö01], [HW96], [Mal98].

One example of a shift-invariant space is the Paley–Wiener space of functions that are bandlimited to $[-1/2, 1/2]$:

$$PW(\mathbb{R}) = \big\{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}]\big\}.$$

This SIS has the property that it is not only invariant under integer translations, but it is in fact invariant under every real translation. A space with this property is said to be *translation-invariant*. A classical theorem of Fourier analysis (often attributed to Wiener, see for example [Hel64]), completely characterizes the closed translation-invariant subspaces of $L^2(\mathbb{R})$ as being of the form

$$\{f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}\,) \subseteq A\}$$

where $A \subseteq \mathbb{R}$ is measurable.

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In many applications, it is desirable to have a shift-invariant space that possesses extra invariances [CS03], [Web00]. In this paper we characterize those shift-invariant subspaces S that are not only invariant under integer translations, but are also invariant under some particular set of translations $M \subseteq \mathbb{R}$. We show that there are only two possibilities:

- either S is translation-invariant, or
- there exists an $n \in \mathbb{N}$ such that S is invariant under translations by multiples of $1/n$, but not invariant under translations by $1/m$ with $m > n$.

We give several characterizations of those shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant. A trivial way to create such a space is to fix a function $g \in L^2(\mathbb{R})$ and set

$$S \,=\, \overline{\text{span}}\bigl\{g(x - \tfrac{k}{n}) : k \in \mathbb{Z}\bigr\},$$

the closed span of the $\frac{1}{n}\mathbb{Z}$ translates of g. However, we are interested in the more subtle question of recognizing when a given SIS is $\frac{1}{n}\mathbb{Z}$-invariant. For example, in many applications one is presented with a SIS of the form

$$S \,=\, \overline{\text{span}}\bigl\{g(x - k) : k \in \mathbb{Z}\bigr\},$$

and it is not obvious whether such a space possesses any invariants other than translation by integers. We completely determine the invariances of such a space in terms of properties of g and, more generally, characterize any SIS that is $\frac{1}{n}\mathbb{Z}$-invariant.

One interesting corollary of our characterization is that the shift-invariant space generated by a compactly supported function is not invariant under any translations other than $\mathbb{Z}$. Thus, the shift-invariant spaces associated with compactly supported multiresolution analyses and wavelets are already "maximally invariant."

2. Notation and Definitions

We normalize the Fourier transform of $f \in L^1(\mathbb{R})$ as

$$\widehat{f}(\omega) \,=\, \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \omega x} dx.$$

The Fourier transform extends to a unitary operator on $L^2(\mathbb{R})$. Given $\mathcal{F} \subseteq L^2(\mathbb{R})$, we set $\widehat{\mathcal{F}} = \{\widehat{f} : f \in \mathcal{F}\}$.

The translation operator T_a is $T_a f(x) = f(x - a)$. Note that $(T_a f)^\wedge(\omega) = e^{-2\pi i a \omega}\widehat{f}(\omega)$.

A function f is *$b\mathbb{Z}$-periodic* if $T_{bk} f = f$ for all $k \in \mathbb{Z}$. A set $A \subseteq \mathbb{R}$ is *$b\mathbb{Z}$-periodic* if its characteristic function is $b\mathbb{Z}$-periodic.

A *shift-invariant space* (SIS) is a closed subspace S of $L^2(\mathbb{R})$ that is invariant under integer translations. We say that S is $b\mathbb{Z}$-invariant if it is invariant under translation by bk for all $k \in \mathbb{Z}$.

Given $\mathcal{F} \subseteq L^2(\mathbb{R})$, we define

$$\mathcal{T}_{\mathbb{Z}}(\mathcal{F}) \,=\, \{T_j f : f \in \mathcal{F},\, j \in \mathbb{Z}\}.$$

The SIS generated by $\mathcal{F}$ is

$$\mathfrak{S}(\mathcal{F}) = \overline{\operatorname{span}}\big(\mathcal{T}_{\mathbb{Z}}(\mathcal{F})\big) = \overline{\operatorname{span}}\{T_j f : f \in \mathcal{F},\, j \in \mathbb{Z}\}.$$

We call $\mathcal{F}$ a *set of generators* for $\mathfrak{S}(\mathcal{F})$. When $\mathcal{F} = \{f\}$ consists of a single function, we simply write $\mathfrak{S}(f)$.

The *length* of a SIS S is the minimum cardinality of the sets $\mathcal{F}$ such that $S = \mathfrak{S}(\mathcal{F})$. A SIS of length one is called a *principal* SIS. A SIS of finite length is a *finitely generated* SIS.

We will write $W = U \dot{\oplus} V$ to denote the *orthogonal* direct sum of closed subspaces of $L^2(\mathbb{R})$, i.e., the subspaces U, V must be closed and orthogonal, and W is their direct sum.

The Lebesgue measure of a set $E \subseteq \mathbb{R}$ is denoted by $|E|$.

The cardinality of a finite set F is denoted by $\#F$.

3. Order of Invariance

Let S be a SIS. If θ is a real number, we will say that S is *invariant under translations by* θ or that S is θ*-invariant* if

$$f \in S \quad \Longrightarrow \quad T_\theta f \in S.$$

We have the following Proposition.

Proposition 3.1. *Let S be a SIS and define,*

$$M = \{\theta \in \mathbb{R} : S \text{ is } \theta\text{-invariant}\}.$$

Then M is a closed additive subgroup of $\mathbb{R}$ containing $\mathbb{Z}$.

Proof. Note that $\mathbb{Z} \subseteq M$ since S is shift-invariant. To see that M is closed, let $\{\theta_j\}$ be a sequence in M such that $\theta_j \to \theta$. Then, given any $f \in S$, we have,

$$\|T_{\theta_j} f - T_\theta f\|_2^2 = \int_{-\infty}^{\infty} |f(x-\theta_j) - f(x-\theta)|^2 \, dx \to 0 \quad \text{as } j \to \infty.$$

So, since S is closed and $T_{\theta_j} \in S$, $T_\theta f$ must be in S and therefore $\theta \in M$.

Let us prove now that M is indeed an additive subgroup of $\mathbb{R}$. Clearly, M is closed under addition. Furthermore, if $n, m \in \mathbb{Z}$ with $n > 0$ and $\theta \in M$ then $n\theta + m \in M$.

We need to see that $-\theta$ is in M for each $\theta \in M$. For this, let us first consider the case that θ is rational. So we can assume that $\theta = p/q$ with $q > 0$.

Then, if $p/q \in M$ we have

$$-\frac{p}{q} = (q-1)\frac{p}{q} - p \in M.$$

Now, if $\theta \in M$ is irrational, then $D \equiv \{n\theta + m : n, m \in \mathbb{Z}, n > 0\} \subseteq M$. Since D is dense in $\mathbb{R}$ and M is closed, then $M = \mathbb{R}$ and so $-\theta$ is in M. □

Since the only closed additive subgroups of $\mathbb{R}$ containing $\mathbb{Z}$ are $\frac{1}{n}\mathbb{Z}$ for some positive integer n or the entire group $\mathbb{R}$, we have the following.

Proposition 3.2. *Let S be a SIS. Then either S is translation-invariant, or there exists a maximum positive integer n such that S is $\frac{1}{n}\mathbb{Z}$-invariant.*

Proposition 3.2 suggests the following definition.

Definition 3.3. Given a shift-invariant space S, we say that S has *invariance order* n if n is the maximum positive integer such that S is $\frac{1}{n}\mathbb{Z}$-invariant. If this maximum does not exist, we say that S has *invariance order* ∞; in this case S is translation-invariant.

Remark 3.4. Note that the invariance order of any SIS is at least 1, since S is $\mathbb{Z}$-invariant. Also, if S has invariance order n, then S is *not* invariant under translation by any real number y in the range $0 < y < 1/n$. Furthermore, if $y \geq 1/n$, S can *only* be invariant under translation by y if y is a multiple of $1/d$ where d divides n. In particular, if the order of invariance of S is a prime number p, there exist no other integers $m > 1$ such that S is invariant under translations by $1/m$.

4. Characterization of $\frac{1}{n}$-Invariance

In this part we will characterize those shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant.

4.1. Notation.

We will use the following notation throughout the remainder of this paper.

Given a fixed positive integer n, we partition the real line into n sets, each of which is $n\mathbb{Z}$-periodic, as follows. For $k = 0, \dots, n-1$ define,

$$B_k = \bigcup_{j \in \mathbb{Z}} ([k, k+1) + nj).$$

Note that B_k implicitly depends on the choice of n.

Given a SIS $S \subseteq L^2(\mathbb{R})$, we associate the following subspaces:

$$U_k = \{f \in L^2(\mathbb{R}) : \widehat{f} = \widehat{g}\,\chi_{B_k} \text{ for some } g \in S\}, \qquad k = 0, \dots, n-1. \tag{1}$$

The spaces U_k are mutually orthogonal since the sets B_k are disjoint (up to sets of measure zero).

If $f \in S$ and $0 \leq k \leq n-1$, then we let f^k denote the function defined by

$$\widehat{f^k} = \widehat{f}\,\chi_{B_k}.$$

Letting P_k denote the orthogonal projection onto $\{f : supp(\hat{f}) \subseteq B_k\}$, we have that

$$U_k = P_k(S) \text{ and } f^k = P_k f.$$

Note that integer translations commute with the projections P_k: if $j \in \mathbb{Z}$ and $k = 0, \dots, n-1$, then

$$T_j P_k = P_k T_j.$$

4.2. **Preliminary results.** We will need the following result from [dBVR94a].

Proposition 4.1 ([dBVR94a])**.** *Let $f \in L^2(\mathbb{R})$ be given. If $g \in \mathfrak{S}(f)$, then there exists a $\mathbb{Z}$-periodic function m such that $\widehat{g} = m\widehat{f}$.*

Conversely, if m is a $\mathbb{Z}$-periodic function such that $m\widehat{f} \in L^2(\mathbb{R})$, then the function g defined by $\widehat{g} = m\widehat{f}$ belongs to $\mathfrak{S}(f)$.

We will also need a version of the preceding result for spaces that are $\frac{1}{n}\mathbb{Z}$-invariant instead of shift-invariant. This follows easily by rescaling.

Corollary 4.2. *Let $f \in L^2(\mathbb{R})$ and $n \in \mathbb{N}$ be given, and set*

$$\mathfrak{S}(f, \tfrac{1}{n}\mathbb{Z}) \;=\; \overline{\operatorname{span}}\{T_{j/n} f : j \in \mathbb{Z}\}.$$

If $g \in \mathfrak{S}(f, \frac{1}{n}\mathbb{Z})$, then there exists a $n\mathbb{Z}$-periodic function m such that $\widehat{g} = m\widehat{f}$.

Conversely, if m is an $n\mathbb{Z}$-periodic function such that $m\widehat{f} \in L^2(\mathbb{R})$, then the function g defined by $\widehat{g} = m\widehat{f}$ belongs to $\mathfrak{S}(f, \frac{1}{n}\mathbb{Z})$.

4.3. **Characterization of $\frac{1}{n}$-invariance in terms of subspaces.** The periodicity of the B_k sets yields the following lemma.

Lemma 4.3. *Let S be a SIS. Assume that the subspace $U_k \subseteq S$. Then for each $k = 0, \dots, n-1$, U_k is a SIS that is also $\frac{1}{n}\mathbb{Z}$-invariant.*

Proof. Fix $0 \le k \le n-1$, and choose any $f \in U_k$. There exists a $g \in S$ such that $\widehat{f} = \widehat{g}\,\chi_{B_k}$. Since S is shift-invariant and $g \in S$, we have that $e^{-2\pi i s\omega}\widehat{g}(\omega)$ is in $\widehat{S}$, for all $s \in \mathbb{Z}$. Hence

$$e^{-2\pi i s\omega}\,\widehat{f}(\omega) \;=\; e^{-2\pi i s\omega}\,\widehat{g}(\omega)\,\chi_{B_k}(\omega) \;\in\; \widehat{U}_k.$$

Therefore $T_s f \in U_k$, so U_k is invariant under integer translates.

Suppose now that $f_j \in U_k$ and $f_j \to f$ in $L^2(\mathbb{R})$. Since $U_k \subseteq S$ and S is closed, f must be in S. Further,

$$\|\widehat{f}_j - \widehat{f}\,\|_2^2 \;=\; \|(\widehat{f}_j - \widehat{f}\,)\,\chi_{B_k}\|_2^2 + \|(\widehat{f}_j - \widehat{f}\,)\,\chi_{B_k^{\mathrm{C}}}\|_2^2 \;=\; \|\widehat{f}_j - \widehat{f}\,\chi_{B_k}\|_2^2 + \|\widehat{f}\,\chi_{B_k^{\mathrm{C}}}\|_2^2.$$

Since the left-hand side converges to zero, we must have that $\widehat{f}\,\chi_{B_k^{\mathrm{C}}} = 0$ a.e., and that $\widehat{f}_j \to \widehat{f}\,\chi_{B_k}$ in $L^2(\mathbb{R})$. Since we also have $\widehat{f}_j \to \widehat{f}$, we conclude that

$$\widehat{f} \;=\; \widehat{f}\,\chi_{B_k} \text{ a.e.}$$

Consequently $f \in U_k$, so U_k is closed.

Finally, to see that U_k is $\frac{1}{n}\mathbb{Z}$-invariant, define

$$h(\omega) \;=\; e^{-\frac{2\pi i\omega}{n}} \sum_{j=-k}^{n-1-k} e^{\frac{2\pi i j}{n}}\,\chi_{B_{k+j}}(\omega).$$

Note that $|h(\omega)| = 1$ and that h is $\mathbb{Z}$-periodic. Furthermore, if $\omega \in B_k$ and $-k \le j \le n-1-k$, then $\chi_{B_{k+j}}(\omega)$ can be nonzero only when $j = 0$. Hence:

$$\omega \in B_k \quad\Longrightarrow\quad h(\omega) = e^{-\frac{2\pi i\omega}{n}}.$$

If $f \in U_k$ then, since $\operatorname{supp}(\hat{f}) \subseteq B_k$, we have

$$e^{-\frac{2\pi i\omega}{n}}\,\widehat{f}(\omega) \;=\; h(\omega)\,\widehat{f}(\omega).$$

However, since U_k is $\mathbb{Z}$-invariant, we have by Proposition 4.1 that $h\widehat{f} \in \widehat{U_k}$. Therefore $e^{-\frac{2\pi i\omega}{n}}\,\widehat{f}(\omega) \in \widehat{U_k}$, which implies that $T_{1/n}f \in U_k$. □

This leads to the following characterization.

Theorem 4.4. *If $S \subseteq L^2(\mathbb{R})$ is a SIS, then the following are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$U_k \subseteq S$ for $k = 0, \dots, n-1$.*

(c) *If $f \in S$, then $f^k = P_k f \in S$ for each $k = 0, \dots, n-1$.*

Moreover, in case these hold we have that S is the orthogonal direct sum

$$S \;=\; U_0 \,\dot{\oplus} \dots \dot{\oplus}\, U_{n-1},$$

with each U_k being a (possibly trivial) $\frac{1}{n}\mathbb{Z}$-invariant SIS.

Proof. (a) ⇒ (b). Assume that S is $\frac{1}{n}\mathbb{Z}$-invariant and fix $0 \le k \le n-1$ and $f \in U_k$. By definition of U_k, we have that $\widehat{f} = \widehat{g}\,\chi_{B_k}$ for some $g \in S$. Since χ_{B_k} is $n\mathbb{Z}$-periodic and bounded, Corollary 4.2 implies that $f \in \mathfrak{S}(g, \frac{1}{n}\mathbb{Z}) \subseteq S$.

(b) ⇒ (a). Suppose that $U_k \subseteq S$ for each $k = 0, \dots, n-1$.

Note that Lemma 4.3 implies that U_k is $\frac{1}{n}\mathbb{Z}$-invariant, and we also have that the U_k are mutually orthogonal since the sets B_k are disjoint.

Suppose that $f \in S$. Then $f = f^0 + \cdots + f^{n-1}$ where $\widehat{f^k} = \widehat{f}\,\chi_{B_k}$. This implies that $f \in U_0 \,\dot{\oplus} \dots \dot{\oplus}\, U_{n-1}$, and consequently S is the orthogonal direct sum

$$S \;=\; U_0 \,\dot{\oplus} \dots \dot{\oplus}\, U_{n-1}.$$

As each U_k is $\frac{1}{n}\mathbb{Z}$-invariant, it follows that S is $\frac{1}{n}\mathbb{Z}$-invariant as well.

(b) ⇔ (c). This is a restatement of the definition of U_k. □

Corollary 4.5. *Let S be a SIS. If there exists a $k \in \{0, \dots, n-1\}$ such that* $\operatorname{supp}(\widehat{f}\,) \subseteq B_k$ *for all $f \in S$, then S is $\frac{1}{n}\mathbb{Z}$-invariant.*

Remark 4.6. It is interesting to note that the subspaces U_k satisfy:

$$U_k = P_k(S) = \{f \in L^2(\mathbb{R}) : supp(\hat{f}) \subseteq B_k\} \cap S.$$

That is, the projections and the restrictions of S yield valid tests for $\frac{1}{n}\mathbb{Z}$-invariance.

4.4. **Characterization of $\frac{1}{n}$-invariance in terms of generators.** We will show now that the conditions for $\frac{1}{n}\mathbb{Z}$-invariance can be formulated in terms of properties of a set of generators of the SIS.

Theorem 4.7. *Let $\mathcal{F}$ be a set of generators for a SIS S, i.e., $S = \mathfrak{S}(\mathcal{F})$. Then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$P_k\mathcal{F} = \{f^k : f \in \mathcal{F}\} \subseteq S$ for $k = 0, \dots, n-1$.*

Proof. (a) $\Rightarrow$ (b). This is a consequence of Theorem 4.4.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. Then, by hypothesis, $V_k = \mathfrak{S}(P_k\mathcal{F}) \subseteq S$, and, by Corollary 4.5, V_k is $\frac{1}{n}\mathbb{Z}$-invariant. Furthermore, $V_j \perp V_k$ when $j \neq k$. If $f \in \mathcal{F}$, then $f = f^0 + \cdots + f^{n-1} \in V_1 \dot\oplus \cdots \dot\oplus V_{n-1}$. Consequently, $S = \mathfrak{S}(\mathcal{F}) = V_1 \dot\oplus \cdots \dot\oplus V_{n-1}$. As each V_k is $\frac{1}{n}\mathbb{Z}$-invariant, it follows that S is as well. □

It is known that it is always possible to choose a (possibly infinite) set of generators of a SIS in such a way that the integer translates of the generators actually forms a frame for the SIS (see Theorem 4.16). This is particularly important in applications, and we examine this situation next.

Recall that a countable collection of vectors $\{v_\alpha : \alpha \in \Lambda\}$ forms a *frame* for a Hilbert space H if there exist constants A, B (called *frame bounds*) such that

$$\forall\, w \in H, \quad A\,\|w\|^2 \;\le\; \sum_{\alpha\in\Lambda} |\langle w, v_\alpha\rangle|^2 \;\le\; B\|w\|^2. \tag{2}$$

If we can take $A = B = 1$, then the frame is called a *Parseval frame.*

The next result shows that if the integer translates of the generators of a SIS form a frame, then the set of integer translations of the "cutoffs" of the generators remains a frame.

Theorem 4.8. *Assume that S is a SIS that is $\frac{1}{n}\mathbb{Z}$-invariant, and that $\mathcal{F} \subseteq S$ is such that $\mathcal{T}_\mathbb{Z}(\mathcal{F})$ is a frame for S with frame bounds A, B. Then*

$$\mathcal{T}_\mathbb{Z}(P_k\mathcal{F}) \;=\; \big\{T_j f^k : f \in \mathcal{F},\, j \in \mathbb{Z}\big\}$$

is a frame for $U_k = \mathfrak{S}(P_k\mathcal{F})$ with frame bounds A, B. Further,

$$\mathcal{T}_\mathbb{Z}\bigg(\bigcup_{k=0}^{n-1} P_k\mathcal{F}\bigg) \;=\; \big\{T_j f^k : f \in \mathcal{F},\, j \in \mathbb{Z},\, k = 0, \dots, n-1\big\}$$

is a frame for S with frame bounds A, B.

Proof. By hypothesis,

$$\forall\, g \in S, \quad A\,\|g\|_2^2 \le \sum_{j\in\mathbb{Z}} \sum_{f\in\mathcal{F}} |\langle g, T_j f\rangle|^2 \;\le\; B\,\|g\|_2^2. \tag{3}$$

Suppose that $g \in U_k$. Then since P_k commutes with integer translations, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, T_j P_k f\rangle|^2 &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, P_k T_j f\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle P_k g, T_j f\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle g, T_j f\rangle|^2.\end{aligned}$$

Combining this with (3), we see that $\mathcal{T}_{\mathbb{Z}}(P_k\mathcal{F})$ is a frame for U_k with frame bounds A, B.

Suppose now that $g \in S$. Then since S is the orthogonal direct sum of the U_k, we have that

$$\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}}\sum_{k=0}^{n-1} |\langle g, T_j P_k f\rangle|^2 = \sum_{k=0}^{n-1}\sum_{j\in\mathbb{Z}}\sum_{f\in\mathcal{F}} |\langle P_k g, T_j f\rangle|^2 \le B\sum_{k=0}^{n-1} \|P_k g\|_2^2 = B\,\|g\|_2^2.$$

The estimate from below is similar, so we see that $\mathcal{T}_{\mathbb{Z}}\bigl(\bigcup_{k=0}^{n-1} P_k\mathcal{F}\bigr)$ is a frame for S, with frame bounds A, B. □

4.5. **Characterization of $\frac{1}{n}$-invariance in terms of fibers.** A useful tool in the theory of shift-invariant spaces is based on early work of Helson [Hel64]. An $L^2(\mathbb{R})$ function is decomposed into "fibers." This produces a characterization of SIS in terms of closed subspaces of $\ell^2(\mathbb{Z})$ (the fiber spaces). For a detailed description of this approach, see [Bow00] and the references therein.

Definition 4.9. Given $f \in L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber* $\widehat{f}_\omega$ of f at ω is the sequence

$$\widehat{f}_\omega = \bigl\{\widehat{f}(\omega + k)\bigr\}_{k\in\mathbb{Z}}.$$

If f is in $L^2(\mathbb{R})$, then the fiber $\widehat{f}_\omega$ belongs to $\ell^2(\mathbb{Z})$ for almost every $\omega \in [0,1)$.

Definition 4.10. Given a subspace V of $L^2(\mathbb{R})$ and $\omega \in [0,1)$, the *fiber space* of V at ω is

$$\mathcal{J}_V(\omega) = \overline{\bigl\{\widehat{f}_\omega : f \in V \text{ and } \widehat{f}_\omega \in \ell^2(\mathbb{Z})\bigr\}},$$

where the closure is taken in the norm of $\ell^2(\mathbb{Z})$.

The map assigning to each ω the fiber space $\mathcal{J}_V(\omega)$ is known in the literature as the *range function* of V.

For a proof that, for almost every ω, $\mathcal{J}_V(\omega)$ is a well-defined closed subspace of $\ell_2(\mathbb{Z})$ and that shift-invariant spaces can be characterized through range functions, see [Bow00], [Hel64].

We will need the following two results.

Proposition 4.11 ([Hel64])**.** *If S is a SIS, then*

$$S = \bigl\{f \in L^2(\mathbb{R}) : \widehat{f}_\omega \in \mathcal{J}_S(\omega) \text{ for a.e. } \omega\bigr\}.$$

Proposition 4.12. *Let S_1 and S_2 be SISs. If $S = S_1 \dot{\oplus} S_2$, then*

$$\mathcal{J}_S(\omega) \;=\; \mathcal{J}_{S_1}(\omega) \dot{\oplus} \mathcal{J}_{S_2}(\omega), \quad a.e.\ \omega.$$

The converse of Proposition 4.12 is also true, but will not be needed.

Combining Theorem 4.4 with Proposition 4.11 yields the following characterization of $\frac{1}{n}$-invariance in terms of the fiber spaces.

Theorem 4.13. *Let S be a SIS. Then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *$\mathcal{J}_{U_k}(\omega) \subseteq \mathcal{J}_S(\omega)$ for almost every ω and each $k = 0, \dots, n-1$.*

For the finitely generated case we can obtain a slightly simpler characterization of $\frac{1}{n}\mathbb{Z}$-invariance.

Theorem 4.14. *If S is a finitely generated SIS, then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *For almost every $\omega \in [0,1)$,*

$$\dim\bigl(\mathcal{J}_S(\omega)\bigr) \;=\; \sum_{k=0}^{n-1} \dim\bigl(\mathcal{J}_{U_k}(\omega)\bigr).$$

Proof. (a) $\Rightarrow$ (b). If S is $\frac{1}{n}\mathbb{Z}$-invariant then $S = \dot{\bigoplus}_{k=0}^{n-1} U_k$. This is an orthogonal direct sum, so Proposition 4.12 implies that $\mathcal{J}_S(\omega) = \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ for a.e. ω, with this sum also orthogonal. The equality of dimensions in statement (b) therefore holds.

(b) $\Rightarrow$ (a). Suppose that statement (b) holds. It is clear that the inclusion $\mathcal{J}_S(\omega) \subseteq \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ holds for a.e. ω. Since the spaces U_k are orthogonal, Proposition 4.12 implies that, for a.e. ω, the spaces $\mathcal{J}_{U_k}(\omega)$ are also orthogonal. Counting dimensions and applying statement (b), we conclude that $\mathcal{J}_S(\omega) = \dot{\bigoplus}_{k=0}^{n-1} \mathcal{J}_{U_k}(\omega)$ for a.e. ω.

Suppose now that $f \in U_k$. Then $\widehat{f}_\omega \in \mathcal{J}_{U_k}(\omega) \subseteq \mathcal{J}_S(\omega)$ for a.e. ω, so Proposition 4.11 implies that $f \in S$. Thus $U_k \subseteq S$ for each k, so it follows from Theorem 4.4 that S is $\frac{1}{n}\mathbb{Z}$-invariant. ☐

Remark 4.15. Given a SIS V, the function $D_V(\omega) \equiv \dim\bigl(\mathcal{J}_V(\omega)\bigr)$ defined for $\omega \in [0,1)$ is known in the literature as the *Dimension function* or *Multiplicity function* of the shift-invariant space V. So, condition (b) of Theorem 4.14 is a statements about dimension functions of the shift invariant spaces involved.

4.6. **The Bownik decomposition and $\frac{1}{n}$-invariance.** In [Bow00], Bownik obtained a decomposition for general shift-invariant spaces, extending the earlier works [dBVR94a] and [dBDR94b], which applied to the finitely generated case. We will apply this decomposition to shift-invariant spaces that are $\frac{1}{n}\mathbb{Z}$-invariant.

Theorem 4.16 (Bownik)**.** *Let $S \subseteq L^2(\mathbb{R})$ be a SIS. Then for each $j \in \mathbb{N}$ we can find a function $\varphi_j \in L^2(\mathbb{R})$ such that $\mathcal{T}_{\mathbb{Z}}(\varphi_j)$ is a Parseval frame for $\mathfrak{S}(\varphi_j)$, and furthermore*

$$S = \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_j).$$

Note that a consequence of this theorem is that every SIS always has a set of generators whose integer translates form a Parseval frame of the SIS.

By applying Theorem 4.16 to each space U_k, we obtain the following result.

Theorem 4.17. *Let S be a SIS that is $\frac{1}{n}\mathbb{Z}$-invariant. Then there exist functions $\varphi_{k,j} \in L^2(\mathbb{R})$ such that*

$$S = \dot{\bigoplus_{k=0}^{n-1}} \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_{k,j}),$$

with the following properties holding.

(a) *$\mathcal{T}_{\mathbb{Z}}(\varphi_{k,j})$ is a Parseval frame for $\mathfrak{S}(\varphi_{k,j})$.*

(b) *$\mathfrak{S}(\varphi_{k,j}) \subseteq U_k$ for each $j \in \mathbb{N}$, and*

$$U_k = \dot{\bigoplus_{j\in\mathbb{N}}} \mathfrak{S}(\varphi_{k,j}).$$

(c) *Each space $\mathfrak{S}(\varphi_{k,j})$ is $\frac{1}{n}\mathbb{Z}$-invariant.*

5. Finitely Generated Shift-Invariant Spaces and $\frac{1}{n}$-Invariance

In this section we will apply some of the general results obtained so far to the particular case of finitely generated shift-invariant spaces. We will use the concept of the Gramian. This is a common tool in the study of finitely generated shift-invariant spaces; see for example [dBDR94b],[RS95],[ACM07].

5.1. Characterization of $\frac{1}{n}$-invariance in terms of the Gramian.

Definition 5.1. Let $\Phi = \{\varphi_1, \dots, \varphi_m\}$ be a collection of finitely many functions in $L^2(\mathbb{R})$. Then the *Gramian* G_Φ of Φ is the $m \times m$ matrix of $\mathbb{Z}$-periodic functions

$$[G_\Phi(\omega)]_{ij} = \Big\langle (\widehat{\varphi}_i)_\omega, (\widehat{\varphi}_j)_\omega \Big\rangle = \sum_{k\in\mathbb{Z}} \widehat{\varphi}_i(\omega+k)\,\overline{\widehat{\varphi}_j(\omega+k)}, \qquad \omega \in \mathbb{R}, \tag{4}$$

where $(\widehat{\varphi}_j)_\omega$ is the fiber of φ_j at ω.

We consider now the SIS $S = \mathfrak{S}(\Phi)$ generated by the set $\Phi = \{\varphi_1, \dots, \varphi_m\}$. It is known [dBDR94b] that if $f \in \mathfrak{S}(\Phi)$, then there exist $\mathbb{Z}$-periodic functions $a_1, \dots, a_m$ such that

$$\widehat{f}(\omega) = \sum_{j=1}^{m} a_j(\omega)\,\widehat{\varphi}_j(\omega), \qquad \text{a.e. } \omega.$$

This implies that the fiber spaces $\mathcal{J}_S(\omega)$ are generated by the fibers of the generators of S at ω (see also [Bow00]). That is, for almost every ω we have that

$$\mathcal{J}_S(\omega) \ = \ \operatorname{span}\big\{(\widehat{\varphi}_j)_\omega : j = 1, \dots, m\big\}.$$

Therefore

$$\dim\big(\mathcal{J}_S(\omega)\big) \ = \ \operatorname{rank}[G_\Phi(\omega)]$$

for almost every ω.

In the same way, since the SIS U_k is generated by $\Phi^k = P_k\Phi = \{\varphi_1^k, \dots, \varphi_m^k\}$, where $\varphi_j^k = P_k\varphi_j$, we have for almost every ω that the fiber spaces $\mathcal{J}_{U_k}(\omega)$ satisfy

$$\mathcal{J}_{U_k}(\omega) \ = \ \operatorname{span}\Big\{\Big(\widehat{\varphi_j^k}\Big)_\omega : j = 1, \dots, m\Big\}.$$

Let us denote by G_{Φ^k} the Gramian matrix associated with the generators of U_k. Then, as above we have that $\dim(\mathcal{J}_{U_k}(\omega)) = \operatorname{rank}[G_{\Phi^k}(\omega)]$ for almost every ω and $k = 0, \dots, n-1$. Now Theorem 4.14 can be restated in the following way.

Theorem 5.2. *If $S = \mathfrak{S}(\Phi)$ is the SIS generated by $\Phi = \{\varphi_1, \dots, \varphi_m\}$, then the following statements are equivalent.*

(a) *S is $\frac{1}{n}\mathbb{Z}$-invariant.*

(b) *For almost every $\omega \in [0, 1)$ we have*

$$\operatorname{rank}[G_\Phi(\omega)] \ = \ \sum_{k=0}^{n-1} \operatorname{rank}[G_{\Phi^k}(\omega)].$$

5.2. **Implications for frequency support.** As a consequence of Theorem 5.2 we deduce an interesting result about the supports of the Fourier transforms of the generators of a SIS.

Theorem 5.3. *Let $S = \mathfrak{S}(\Phi)$ be the SIS generated by $\Phi = \{\varphi_1, \dots, \varphi_m\}$, and define*

$$E_j \ = \ \big\{\omega \in [0, 1) : \operatorname{rank}[G_\Phi(\omega)] = j\big\}, \quad j = 0, \dots, m.$$

If S is $\frac{1}{n}\mathbb{Z}$-invariant, then for each interval $I \subseteq \mathbb{R}$ of length n, we have that for each $h = 1, \dots, m$

$$\big|\big\{\omega \in I : \widehat{\varphi_h}(\omega) = 0\big\}\big| \ \geq \ \sum_{j=0}^{n-1} (n - j)\, |E_j|.$$

In particular if $n > m$ we have,

$$\big|\big\{\omega \in I : \widehat{\varphi_h}(\omega) = 0\big\}\big| \ \geq \ n - m.$$

Proof. The measurability of the sets E_j follows from the results of Helson [Hel64], e.g., see [BK06] for an argument of this type.

It is enough to prove the theorem for the interval $I = [0, n)$.

We note that the set

$$K_1 \ = \ \{\omega \in [0, 1) : (\widehat{\varphi_h})_\omega \in \ell^2(\mathbb{Z}) \text{ for } h = 1, \dots, m\}$$

has full measure. Therefore

$$K_n \;=\; \bigcup_{k=0}^{n-1} (K_1 + k)$$

is a subset of $[0, n)$ of measure n.

Fix any particular $j \in \{0, \dots, m\}$. By Theorem 5.2,

$$\operatorname{rank}[G_\Phi(\omega)] \;=\; \sum_{k=0}^{n-1} \operatorname{rank}[G_{\Phi^k}(\omega)], \qquad \text{a.e. } \omega. \tag{5}$$

Therefore, for $j < n$, if $\omega \in E_j$ then at least $n - j$ terms on the right-hand side of equation (5) must vanish. That is, given such an ω, there exists a subset of $\{0, \dots, n-1\}$ with at least $n - j$ elements such that for each k in this subset

$$\operatorname{rank}[G_{\Phi^k}(\omega)] = 0.$$

In particular, for each $h \in \{1, \dots, m\}$ we have that, for $j < n$ and $\omega \in E_j$,

$$\#\bigl\{k \in \{0, \dots, n-1\} : \widehat{\varphi_h}(\omega + k) = 0\bigr\} \ge n - j. \tag{6}$$

On the other hand, using that the sets E_j are disjoint, we have

$$\begin{aligned}
&\bigl\{\omega \in K_n : \widehat{\varphi_h}(\omega) = 0\bigr\} \\
&\quad = \bigl\{\omega + k : \omega \in K_1,\ 0 \le k \le n-1 \text{ and } \widehat{\varphi_h}(\omega + k) = 0\bigr\} \\
&\quad = \bigcup_{j=0}^{m} \bigl\{\omega + k : \omega \in K_1 \cap E_j,\ 0 \le k \le n-1 \text{ and } \widehat{\varphi_h}(\omega + k) = 0\bigr\}.
\end{aligned}$$

Consequently, from (6) and the last equation it follows that,

$$\begin{aligned}
&\bigl|\bigl\{\omega \in K_n : \widehat{\varphi_h}(\omega) = 0\bigr\}\bigr| \\
&\quad = \sum_{j=0}^{m} \bigl|\bigl\{\omega + k : \omega \in K_1 \cap E_j, 0 \le k \le n-1 \text{ and } \widehat{\varphi_h}(\omega + k) = 0\bigr\}\bigr| \\
&\quad = \sum_{j=0}^{m} \int_{E_j} \#\bigl\{k \in \{0, \dots, n-1\} : \widehat{\varphi_h}(\omega + k) = 0\bigr\}\, d\omega \\
&\quad \ge \sum_{j=0}^{n-1} (n - j)|E_j|.
\end{aligned}$$

Furthermore, if $n > m$,

$$\sum_{j=0}^{m} (n - j)\,|E_j| \;\ge\; \sum_{j=0}^{m} (n - m)\,|E_j| \;=\; (n - m).$$

The measurability of the function $\omega \longmapsto \#\big\{k \in \{0, \dots, n-1\} : \widehat{\varphi_h}(\omega + k) = 0\big\}$, follows from the fact that

$$\big\{w \in E_j : \#\big\{k \in \{0, \dots, n-1\} : \widehat{\varphi_h}(\omega + k) = 0\big\} \geq s\big\}$$
$$= \bigcup_{0 \leq k_1 < \cdots < k_s < n} \bigcap_{i=1}^{s} \big\{\omega \in E_j : \widehat{\varphi_h}(\omega + k_i) = 0\big\}.$$

□

Note that if S is a principal SIS, say $S = \mathfrak{S}(\varphi)$, then the Gramian is scalar-valued; since

$$G_\varphi(\omega) \,=\, \big\langle \widehat{\varphi}_\omega,\, \widehat{\varphi}_\omega \big\rangle \,=\, \sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\omega + k)|^2.$$

Applying Theorem 5.3 to this case, we obtain the following corollary.

Corollary 5.4. *Let $\varphi \in L^2(\mathbb{R})$ be given. If the SIS $\mathfrak{S}(\varphi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n > 1$, then $\widehat{\varphi}$ must vanish on a set of infinite Lebesgue measure. Furthermore, for each interval $I \subseteq \mathbb{R}$ of length n, we have that*

$$\big|\big\{\omega \in I : \widehat{\varphi}(\omega) = 0\big\}\big| \,\geq\, n\,|E_0| + (n-1)\,|E_1| \,\geq\, n - 1,$$

where $E_0 = \{\omega \in [0, 1) : G_\varphi(\omega) = 0\}$ and $E_1 = \{\omega \in [0, 1) : G_\varphi(\omega) \neq 0\}$

This yields the following fact regarding the order of invariance of a principal SIS generated by a compactly supported function.

Proposition 5.5. *If a nonzero function $\varphi \in L^2(\mathbb{R})$ has compact support, then $\mathfrak{S}(\varphi)$ has invariance order one. That is $\mathfrak{S}(\varphi)$ is not $\frac{1}{n}\mathbb{Z}$-invariant for any $n > 1$.*

Proof. Because of Corollary 5.4, if $\mathfrak{S}(\varphi)$ is $\frac{1}{n}\mathbb{Z}$-invariant with $n > 1$, then $\widehat{\varphi}$ must vanish on a set of positive measure. Since φ has compact support, the Paley–Wiener Theorem implies that $\varphi = 0$ a.e. □

It is not difficult to construct a function $\varphi \in L^2(\mathbb{R})$ such that $\widehat{\varphi}$ is compactly supported in frequency yet the SIS $\mathfrak{S}(\varphi)$ is not translation-invariant. In fact, we have the following consequence of Corollary 5.4.

Corollary 5.6. *If $\varphi \in L^2(\mathbb{R})$ and $\mathfrak{S}(\varphi)$ is translation-invariant, then* $|\mathrm{supp}(\widehat{\varphi})| \leq 1$.

Remark 5.7. As one of the referees pointed out, Proposition 5.5 is known and follows readily from Proposition 4.1.

Likewise, Corollary 5.6 can be obtained from properties of the dimension function of a SIS. For this, observe that since $\mathfrak{S}(\varphi)$ is translation invariant then,

$$D_{\mathfrak{S}(\varphi)} = \sum_{k \in \mathbb{Z}} \chi_{supp(\hat{\varphi})}(\omega + k).$$

Now using that $\mathfrak{S}(\varphi)$ is principal, we have $D_{\mathfrak{S}(\varphi)} \leq 1$. Thus integrating both sides over $[0, 1]$, yields $|\mathrm{supp}(\widehat{\varphi})| \leq 1$. For properties of the dimension function of a SIS, see for example [BM99] or [BR03].

Remark 5.8. Assume now that $\varphi \in L^2(\mathbb{R})$ and $\mathfrak{S}(\varphi)$ is translation-invariant. Using the argument in Remark 5.7 we have that $\sum_{k\in\mathbb{Z}} \chi_{\text{supp}(\hat{\varphi})}(\omega + k) \leq 1$. This implies that $\text{supp}(\hat{\varphi})$ is a subset of a set of representatives of the quotient $\mathbb{R}/\mathbb{Z}$. That is, $\text{supp}(\hat{\varphi})$ is a subset of a tile of $\mathbb{R}$, what is a refinement of Corollary 5.6. Translation-invariance of multiresolution analyses is connected to tilings of $\mathbb{R}$. See for example [Mad92].

5.3. **Application to multiresolution analyses.** For definitions and details on wavelets and multiresolution analyses, see [Dau92] or [Mal89].

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis (MRA) of $L^2(\mathbb{R})$. By definition, $V_0 = \mathfrak{S}(\varphi)$ for some $\varphi \in L^2(\mathbb{R})$, called the *scaling function*, and V_j is the image of V_0 under the unitary operator $D_{2^j} f(x) = 2^{j/2} f(2^j x)$.

The preceding results imply that if φ is compactly supported (as is the case for the Daubechies scaling functions, for example), then the SIS V_0 has invariance order exactly 1. Thus V_0 is invariant only under integer translations. The same remarks apply to the associated wavelet ψ and wavelet SIS $W_0 = \mathfrak{S}(\psi)$ if ψ is compactly supported.

Further, if φ is compactly supported, then at resolution level j, the subspace V_j is invariant exactly under translations $2^j\mathbb{Z}$, and similarly for the wavelet space W_j if ψ is compactly supported. This includes all the spaces associated with the Daubechies scaling functions and wavelets, for example.

6. Higher dimensions

The results of this article are for the line. The higher-dimensional case is much more involved due to the more complex structure of the closed additive subgroups of $\mathbb{R}^d$. This will be the subject of a forthcoming article.

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