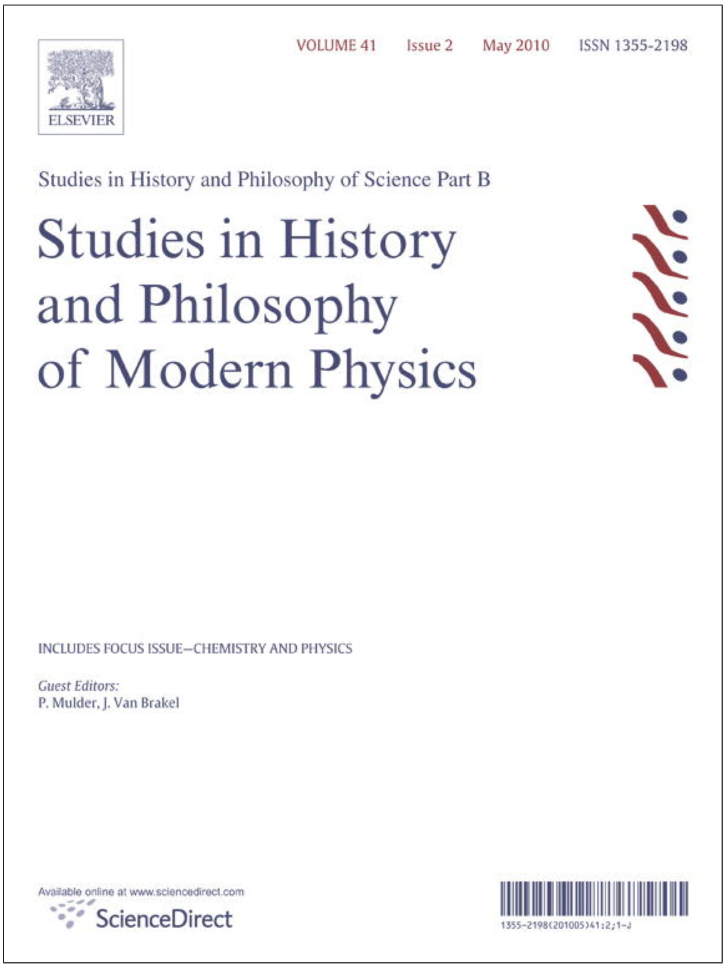
VOLUME 41 Issue 2 May 2010 ISSN 1355-2198
ELSEVIER
Studies in History and Philosophy of Science Part B
Studies in History and Philosophy of Modern Physics
INCLUDES FOCUS ISSUE—CHEMISTRY AND PHYSICS
Guest Editors:
P. Mulder, J. Van Brakel
Available online at www.sciencedirect.com
ScienceDirect

Studies in History and Philosophy of Modern Physics 41 (2010) 93–103

Contents lists available at ScienceDirect

Studies in History and Philosophy of Modern Physics

journal homepage: www.elsevier.com/locate/shpsb

The modal-Hamiltonian interpretation and the Galilean covariance of quantum mechanics ☆

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ARTICLE INFO

Article history:
Received 26 April 2009
Received in revised form
23 December 2009
Accepted 23 February 2010

Keywords:
Quantum mechanics
Modal interpretations
Hamiltonian
Covariance
Invariance
Galilean group

ABSTRACT

The aim of this paper is to analyze the modal-Hamiltonian interpretation of quantum mechanics in the light of the Galilean group. In particular, it is shown that the rule of definite-value assignment proposed by that interpretation has the same properties of Galilean covariance and invariance as the Schrödinger equation. Moreover, it is argued that, when the Schrödinger equation is invariant, the rule can be reformulated in an explicitly invariant form in terms of the Casimir operators of the Galilean group. Finally, the possibility of extrapolating the rule to quantum field theory is considered.

When citing this paper, please use the full journal title *Studies in History and Philosophy of Modern Physics*

1. Introduction

As stressed by Lévi-Leblond (1974), although it is usual to read that non-relativistic quantum mechanics is covariant, and even invariant, under the Galilean transformations, this issue has been scarcely treated in the standard literature on the theory. For instance, the commutation relations defining the Galilean group are often not even quoted in the textbooks on the matter (an exception is Ballentine, 1998). This fact has several undesirable consequences. On the one hand, the meanings of the concepts of covariance and invariance are not precisely elucidated. On the other hand, one is left with no clear idea about in what sense covariance and/or invariance can be predicated of quantum mechanics.

This situation has its counterpart in the field of the interpretation of quantum mechanics: the relevance of the Galilean group is rarely discussed in the impressive amount of literature on the subject. The general premise underlying the present paper is that the relationship between interpretation and Galilean transformations deserves to be seriously analyzed: the fact that the theory is covariant under the Galilean group does not guarantee the same property for the interpretation since, in general, interpretations add interpretative postulates to the formal structure of the theory.

In a recent paper (Lombardi & Castagnino, 2008) we have presented a new realist interpretation of quantum mechanics belonging to the modal family (see Dieks & Vermaas, 1998), where the Hamiltonian plays a decisive role in the main interpretative postulates. In that work, the Galilean group guided us in the understanding of the physical content of the theory; however, the behavior of the interpretation under the Galilean transformations was not explored. The aim of this paper is to address this issue by analyzing whether and under what conditions the modal-Hamiltonian interpretation satisfies the physical constraints imposed by the Galilean group. To this end, we shall begin with clarifying the concepts of covariance and invariance, and with stating the problem at issue in precise terms. On this basis, and after recalling the main features of the Galilean group, we shall consider in what sense quantum mechanics is covariant and under what conditions it is invariant under the transformations of the group. This conceptual framework will provide us with the tools for studying the result of the application of the Galilean transformations to the modal-Hamiltonian

☆ This work is fully collaborative: the order of the names does not mean priority.

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doi:10.1016/j.shpsb.2010.02.002

interpretative postulates. Finally, we shall consider the perspectives opened by this group approach to interpretative issues, in particular, the possibility of extrapolating the approach to the interpretation of quantum field theory.

2. Covariance and invariance

As we have pointed out, the covariance—and even the invariance (see Ballentine, 1998)—of quantum mechanics under the Galilean transformations is usually assumed as a well-known fact. Nevertheless, only in very few cases this assumption is grounded on a conceptual elucidation of the involved notions: the meanings of the words 'invariance' and 'covariance' are taken for granted. Therefore, it is worth beginning with a clarification of those concepts.

A generic item is said to be *symmetric* under a certain transformation when it is *invariant* under that transformation. However, this does not explain yet what kind of items may be endowed with the property of invariance. As Brading and Castellani (2007) stress, the first step is to distinguish between symmetries of objects and symmetries of laws: although related with each other, both cases should not be confused. In fact, as we shall see, the symmetry of a law does not imply the symmetry of the objects (states and operators) contained in the law. Therefore, the philosophical implications of the symmetries of the law and of the involved objects under a particular group of transformations have to be both considered.

Secondly, it is necessary to say a few words about the concept of covariance. In the literature there is no consensus about what 'covariance' means. Often, the term 'invariant' is only applied to objects and the term 'covariant' is retained for equations or laws. Here we shall not follow this path, because the corresponding concepts can be understood in such a way that the difference between the invariance and the covariance of a law makes sense. In rough terms, we shall say that a law is covariant under a certain transformation when its form is left unchanged by that transformation (see Suppes, 2000; Brading & Castellani, 2007). On this basis, we can now introduce the necessary definitions.

Let us consider a set $\mathcal{X}$ of objects $X_i \in \mathcal{X}$, and a group G of transformations $T_\alpha \in G$, where the $T_\alpha : \mathcal{X} \to \mathcal{X}$ act on the X_i as $X_i \to X'_i$. An *object* $X_j \in \mathcal{X}$ is *invariant* under the transformation T_α if, for that transformation, $X'_j = X_j$; in turn, $X_j \in \mathcal{X}$ is *invariant* under the group G if it is invariant under all the transformations $T_\alpha \in G$. In physical theories, the objects to which the transformations apply are usually states s, observables O and differential operators D, and each transformation acts on them in a particular way. For instance, in the fundamental laws of Hamiltonian mechanics—the Hamilton equations, the state is $s = (\boldsymbol{q}, \boldsymbol{p})$, the relevant observable O is the Hamiltonian H, and the differential operators are $D_1 = d/dt$, $D_2 = \partial/\partial \boldsymbol{p}$ and $D_3 = \partial/\partial \boldsymbol{q}$. The time-reversal transformation, which acts on the variable t as $t \to -t$, reverses all the objects whose definitions in function of t are non-invariant under the transformation:

$$s = (\boldsymbol{q},\boldsymbol{p}) \to s' = (\boldsymbol{q}',\boldsymbol{p}') = (\boldsymbol{q},-\boldsymbol{p}) \quad O = H \to O' = H'$$
$$D_1 = d/dt \to D'_1 = d'/dt = -d/dt$$
$$D_2 = \partial/\partial \boldsymbol{p} \to D'_2 = \partial'/\partial \boldsymbol{p} = -\partial/\partial \boldsymbol{p}$$
$$D_3 = \partial/\partial \boldsymbol{q} \to D'_3 = \partial'/\partial \boldsymbol{q} = \partial/\partial \boldsymbol{q}$$

In the fundamental law of Newtonian mechanics—Newton's second law—the state is $s = \boldsymbol{x}$, the relevant observables are $O_1 = \boldsymbol{F}$ and $O_2 = m$, and the differential operator is $D = d^2/dt^2$. Under time-reversal they transform as

$$s = \boldsymbol{x} \to s' = \boldsymbol{x}' = \boldsymbol{x} \quad D = d^2/dt^2 \to D' = d'^2/dt^2 = d^2/dt^2$$
$$O_1 = \boldsymbol{F} \to O'_1 = \boldsymbol{F}' \quad O_2 = m \to O'_2 = m'$$

In physics, these objects are combined in equations representing the laws of a theory. In particular, a dynamical law is represented by a differential equation $E(s, O_i, D_j) = 0$ including the state s, certain observables O_i and certain differential operators D_j. When a transformation is applied to all these objects, the law may remain exactly the same, that is, its form may be left invariant by the transformation. This means that the nomological relationship among the transformed objects is the same as that linking the original objects. But it may also be the case that the equation still holds when only the state is transformed, and this implies that the evolution of the state is not affected by the transformation. Precisely, let L be a *law* represented by an equation $E(s, O_i, D_j) = 0$, and let G be a group of transformations $T_\alpha \in G$ acting on the objects involved in the equation as $s \to s'$, $O_i \to O'_i$ and $D_j \to D'_j$, L is *covariant* under the transformation T_α if $E(s', O'_i, D'_j) = 0$, and L is *invariant* under the transformation T_α if $E(s', O_i, D_j) = 0$. Moreover, L is covariant—invariant—under the group G if it is covariant—invariant—under all the transformations $T_\alpha \in G$. A group G of transformations is said to be the *symmetry group* of a theory if the laws of the theory are covariant under the group G; this means that the laws preserve their validity even when the transformations of the group are applied to the involved objects. It is easy to see that the Hamilton equations, $d\boldsymbol{q}/dt = \partial H/\partial \boldsymbol{p}$ and $d\boldsymbol{p}/dt = -\partial H/\partial \boldsymbol{q}$, are covariant under time-reversal when $H' = H$, a condition satisfied when H is time-independent; nevertheless, they are not invariant under time-reversal because $d\boldsymbol{p}'/dt \neq -\partial H/\partial \boldsymbol{q}$. In turn, Newton's second law is covariant under time-reversal when $\boldsymbol{F}' = \boldsymbol{F}$, a condition satisfied when $\boldsymbol{F}$ is time-independent, and it is also invariant since $d^2\boldsymbol{x}'/dt^2 = \boldsymbol{F}/m$.

It is clear that, when a law is covariant under a transformation, and the observables and the differential operators contained in it are invariant under that transformation, the law is also invariant under the transformation: this is the case of Newton's second law under time-reversal. Nevertheless, this is not the only way to obtain the invariance of a law; we shall return to this point in Section 4.2, for the particular case of the Schrödinger equation.

Some authors prefer to speak about the symmetry of a law instead of the covariance of a law. For instance, Earman (2004a) defines symmetry in the language of the model-view of theories as follows. Let $\mathcal{M}$ be the set of the models of a certain mathematical structure, and let $\mathcal{M}_L \subset \mathcal{M}$ be the subset of the models satisfying the law L. A symmetry of the law L is a map $S : \mathcal{M} \to \mathcal{M}$ that preserves $\mathcal{M}_L$, that is, for any $m \in \mathcal{M}_L$, $m' = S(m) \in \mathcal{M}_L$. In our case, where L is represented by a differential equation $E(s, O_i, D_j) = 0$, each model $m \in \mathcal{M}_L$ corresponds to a solution $s = F\ (O_i, s_0)$ of the equation, representing a nomologically possible evolution of the system. Then, the covariance of L under a transformation T—that is, $E(s', O'_i, D'_j) = 0$—implies that $s' = F'(O'_i, s_0)$ is also a solution of the equation and, then, it corresponds to a model $m' \in \mathcal{M}_L$. This means that our definition of covariance and the definition of symmetry in the model-view language are equivalent.

It is interesting to notice that the covariance of a dynamical law—represented by a differential equation — does not imply the invariance of the nomologically possible evolutions—represented by the solutions of the equation—(see Castagnino, Lara, & Lombardi, 2003; Earman, 2004b). In fact, the covariance of L, represented by $E(s, O_i, D_j) = 0$, implies that $s = F\ (O_i, s_0)$ and $s' = F'(O'_i, s_0)$ are both solutions of the equation, but does not imply that $s = s'$; in the model-view language, the symmetry of L does not imply that $m = m'$. Nevertheless, since the invariance of L means that $E(s', O_i, D_j) = 0$, in this case $s = s' = F\ (O_i, s_0)$ or, in the model-view language, $m = m'$.

On the basis of these concepts, now we can explicitly state the conditions of covariance and invariance for the fundamental law of quantum mechanics. Given a group G whose transformations act on states, observables and differential operator as $|\varphi\rangle \to |\varphi'\rangle$, $|O\rangle \to |O'\rangle$ and $d/dt \to d'/dt$, the Schrödinger equation is *covariant* when

$$\frac{d'|\varphi'\rangle}{dt} = -i\hbar H'|\varphi'\rangle \tag{2.1}$$

and it is *invariant* when

$$\frac{d|\varphi'\rangle}{dt} = -i\hbar H|\varphi'\rangle \tag{2.2}$$

3. Stating the problem

Let us recall that a continuous space-time transformation admits two interpretations. Under the active interpretation, the transformation corresponds to a change from a system to another—transformed—system; under the passive interpretation, the transformation consists in a change of the viewpoint—the reference frame—from which the system is described (see Brading & Castellani, 2007). Nevertheless, in both cases the invariance of the fundamental law of a theory under its symmetry group implies that the behavior of the system is not altered by the application of the transformation: in the active interpretation language, the original and the transformed systems are equivalent; in the passive interpretation language, the original and the transformed reference frames are equivalent.

We know that the Galilean group of continuous space-time transformations is the symmetry group of classical and quantum mechanics. According to the passive interpretation language, the invariance of the dynamical laws amounts to the equivalence between inertial reference frames, that is, reference frames time-displaced, space-displaced, space-rotated or uniformly moving with respect to each other: the application of a Galilean transformation does not introduce a modification in the physical situation, but only expresses a change of the perspective from which the system is described. For instance, if a classical particle originally described in the reference frame *RF* is described in a different inertial reference frame *RF*′, the invariance of Newton's second law means that the time-evolution of the particle does not change when described in the new reference frame.

The physical meaning of the action of the Galilean transformations is well-understood in classical mechanics. However, as we have pointed out in the Introduction, this issue is scarcely discussed in the field of quantum mechanics, perhaps under the assumption that the matter is as easy as in the classical case. But we shall see that quantum mechanics is peculiar also regarding to this point. In particular, the properties of the Schrödinger equation under the Galilean group have relevant consequences for interpretation. As stressed by Brown, Suárez, and Bacciagaluppi (1998), any interpretation that selects the set of the definite-valued observables of a quantum system in a given state is committed to considering how that set is transformed under the Galilean group. The study of this question is particularly urging in the case of realist interpretations, which conceive a definite-valued observable as a physical magnitude that objectively acquires an actual value among all its possible values: the actualization of one of the possible values has to be an objective fact. Therefore, when nomological invariance holds, the set of the definite-valued observables of a system should be left invariant by the Galilean transformations: from a realist viewpoint, it would be unacceptable that such a set changed as the mere result of a change in the perspective from which the system is described.

Of course, in order to address this issue in a given interpretative framework, a precise statement of the rule of definite-value ascription is required: for instance, the traditional eigenstate-eigenvalue link (Fine, 1973), the Kochen–Dieks rule based on the biorthogonal decomposition of the state vector (Dieks, 1988; Kochen, 1985), the Vermaas–Dieks rule based on the spectral resolution of the reduced density operator (Clifton, 1995; Vermaas & Dieks, 1995), or the modal-Hamiltonian rule based on the system's Hamiltonian (Lombardi & Castagnino, 2008). But one also needs a clear understanding of the application of the Galilean transformations in quantum mechanics. For this reason, we shall proceed to review this point since, as noticed above, it is addressed with not enough care in the literature.

4. The Galilean group in quantum mechanics

Under the assumption that time can be represented by a variable $t \in \mathbb{R}$ and position can be represented by a variable $\boldsymbol{r} = (x,y,z) \in \mathbb{R}^3$, the Galilean group $\mathcal{G} = \{T_\alpha\}$, with $\alpha = 1$ to 10, is a group of continuous space-time transformations $T_\alpha : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}$ acting as

- Time-displacement: $t \to t' = t + \tau$
- Space-displacement: $\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{\rho}$
- Space-rotation: $\boldsymbol{r} \to \boldsymbol{r}' = R_\theta \boldsymbol{r}$
- Velocity-boost: $\boldsymbol{r} \to \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}t$

where $\tau \in \mathbb{R}$ is a real number representing a time interval, $\boldsymbol{\rho} = (\rho_x,\rho_y,\rho_z) \in \mathbb{R}^3$ is a triple of real numbers representing a space interval, $R_\theta \in \mathcal{M}^{3\times 3}$ is a 3×3 matrix representing a space rotation an angle θ, and $\boldsymbol{u} = (u_x,u_y,u_z) \in \mathbb{R}^3$ is a triple of real numbers representing a constant velocity.

Since the Galilean group $\mathcal{G}$ is a Lie group, the Galilean transformations T_α can be represented by unitary operators U_α over the Hilbert space, with the exponential parametrization $U_\alpha = e^{iK_\alpha s_\alpha}$, where s_α is a continuous parameter and K_α is a Hermitian operator independent of s_α, called *generator* of the transformation T_α. Then, $\mathcal{G}$ is defined by ten group generators K_α: one time-displacement K_τ, three space-displacements K_{ρ_i}, three space-rotations K_{θ_i}, and three velocity-boosts K_{u_i}, with $i = x, y, z$. The generators of $\mathcal{G}$ form the Galilean algebra, that is, the Lie algebra of the Galilean generators. The combined action of all the transformations is given by

$$U_s = \prod_{\alpha=1}^{10} e^{iK_\alpha s_\alpha} \tag{4.1}$$

In the case of quantum mechanics, the symmetry group is the group corresponding to the central extension of the Galilean algebra, obtained as a semi-direct product between the Galilean algebra and the algebra generated by a central charge, which in this case denotes the mass operator $M = mI$, where I is the identity operator and m is the mass (see Bose, 1995; Weinberg, 1995).[1] In order to simplify the presentation, from now on we shall use the expression 'Galilean group' to refer to the corresponding central extension, and we shall take $\hbar = 1$ as usual.

As a Lie group, the Galilean group is defined by the commutation relations between its generators:

$$\begin{array}{llll} \text{(a)} & [K_{\rho_i},K_{\rho_j}] = 0 & \text{(f)} & [K_{u_i},K_{\rho_j}] = i\delta_{ij}M \\ \text{(b)} & [K_{u_i},K_{u_j}] = 0 & \text{(g)} & [K_{\rho_i},K_\tau] = 0 \\ \text{(c)} & [K_{\theta_i},K_{\theta_j}] = i\varepsilon_{ijk}K_{\theta_j} & \text{(h)} & [K_{\theta_i},K_\tau] = 0 \\ \text{(d)} & [K_{\theta_i},K_{\rho_j}] = i\varepsilon_{ijk}K_{\rho_k} & \text{(i)} & [K_{u_i},K_\tau] = iK_{\rho_i} \\ \text{(e)} & [K_{\theta_i},K_{u_j}] = i\varepsilon_{ijk}K_{u_k} & & \end{array} \tag{4.2}$$

[1] The mass operator as a central charge is a consequence of the projective representation of the Galilean group. Precisely, the action of two transformations is equal to the product of the actions of the two transformations up to a phase; for instance, $U_{\rho_j u_j} = \exp(iK_{\rho_i}) \cdot \exp(iK_{u_j}) \cdot \exp(if(\rho_i,u_j)M)$. Then, a unitary representation of the Galilean group requires the introduction of the central charge $M = mI$ (see Weinberg, 1995).

where ε_{ijk} is the Levi-Civita tensor, such that $i\neq k$, $j\neq k$, $\varepsilon_{ijk}=\varepsilon_{jki}=\varepsilon_{kij}=1$, $\varepsilon_{ikj}=\varepsilon_{jik}=\varepsilon_{kji}=-1$, and $\varepsilon_{ijk}=0$ if $i=j$. In quantum mechanics, when the system is free from external fields, the generators K_α represent the basic magnitudes of the theory: the energy $H=K_\tau$, the three momentum components $P_i=\hbar K_{\rho_i}$, the three angular momentum components $J_i=\hbar K_{\theta_i}$, and the three boost components $G_i=\hbar K_{u_i}$. Then, by taking $\hbar=1$, the commutation relations result

$$\begin{array}{llll} \text{(a)} & [P_i,P_j]=0 & \text{(f)} & [G_i,P_j]=i\delta_{ij}M \\ \text{(b)} & [G_i,G_j]=0 & \text{(g)} & [P_i,H]=0 \\ \text{(c)} & [J_i,J_j]=i\varepsilon_{ijk}J_k & \text{(h)} & [J_i,H]=0 \\ \text{(d)} & [J_i,P_j]=i\varepsilon_{ijk}P_k & \text{(i)} & [G_i,H]=iP_i \\ \text{(e)} & [J_i,G_j]=i\varepsilon_{ijk}G_k & & \end{array} \quad (4.3)$$

The rest of the physical magnitudes can be defined in terms of these basic ones: for instance, the three position components are $Q_i=G_i/m$, the three orbital angular momentum components are $L_i=\varepsilon_{ijk}Q_jP_k$, and the three spin components are $S_i=J_i-L_i$. The—central extension of the—Galilean group has three Casimir operators which, as such, commute with all the generators of the group: they are the mass operator M, the operator S^2, and the internal energy operator $W=H-P^2/2\,\mathrm{m}$. The eigenvalues of the Casimir operators label the irreducible representations of the group; so, in each irreducible representation, the Casimir operators are multiples of the identity: $M=mI$, $S^2=s(s+1)I$, where s is the eigenvalue of the spin S, and $W=wI$, where w is the scalar internal energy.

In the Hilbert formulation of quantum mechanics, corresponding to each Galilean transformation T_α there must be a transformation of states and of observables such that

$$|\varphi\rangle\rightarrow|\varphi'\rangle=U_{s_\alpha}|\varphi\rangle=e^{iK_\alpha s_\alpha}|\varphi\rangle \quad (4.4)$$

$$O\rightarrow O'=U_{s_\alpha}OU_{s_\alpha}^{-1}=e^{iK_\alpha s_\alpha}Oe^{-iK_\alpha s_\alpha} \quad (4.5)$$

Moreover, since

$$O'=e^{iK_\alpha s_\alpha}Oe^{-iK_\alpha s_\alpha}=O\Leftrightarrow[O,K_\alpha]=0 \quad (4.6)$$

in this context the invariance of an observable O under a Galilean transformation T_α amounts to the commutation between O and the corresponding generator K_α.

Now we have all the theoretical elements necessary to consider the Galilean covariance and invariance of quantum mechanics by analyzing how the Galilean transformations affect the Schrödinger equation,

$$\frac{d|\varphi\rangle}{dt}=-iH|\varphi\rangle \quad (4.7)$$

Let us premultiply both members of the equation by $U=e^{iKs}$; by using the property $UU^{-1}=I$ and, then, adding $(dU/dt)|\varphi\rangle$ to both members, we obtain

$$U\frac{d(|\varphi\rangle)}{dt}+\frac{dU}{dt}|\varphi\rangle=-iUHU^{-1}U\frac{d|\varphi\rangle}{dt}+\frac{dU}{dt}|\varphi\rangle \quad (4.8)$$

Therefore,

$$\frac{d(U|\varphi\rangle)}{dt}=-i\left[UHU^{-1}+i\frac{dU}{dt}U^{-1}\right]U|\varphi\rangle \quad (4.9)$$

If we recall the action of the Galilean transformations on states and observables (see Eqs. (4.4) and (4.5)), we can write

$$\frac{d|\varphi'\rangle}{dt}=-i\left[H'+i\frac{dU}{dt}U^{-1}\right]|\varphi'\rangle \quad (4.10)$$

On the basis of this equation, we shall analyze the invariance and the covariance of the Schrödinger equation separately.

4.1. The invariance of the Schrödinger equation

As we have seen, when there are no external fields acting on the system, the Galilean group is defined by the commutation relations (4.3). However, there is a difference between the boost generators G_i and the remaining generators.

In a closed, constant-energy system free from external fields, H is time-independent and the P_i and the J_i are constants of motion (see Eqs. (4.3g,h)). Then, for time-displacements, space-displacements and space-rotations, $dU/dt=de^{iKs}/dt=0$, where K and s stand for H and τ, P_i and ρ_i, and J_i and θ_i, respectively. As a consequence, Eq. (4.10) yields

$$\frac{d|\varphi'\rangle}{dt}=-iH'|\varphi'\rangle \quad (4.11)$$

Moreover, for those transformations, $H'=H$ because (see Eq. (4.6)):

- Time-displacements : $H'=e^{iH\tau}He^{-iH\tau}=H$ since $[H,H]=0$
- Space-displacements : $H'=e^{iP_i\rho_i}He^{-iP_i\rho_i}=H$ since $[P_i,H]=0$ (relation (4.3g))
- Space-rotaions : $H'=e^{iJ_i\theta_i}He^{-iJ_i\theta_i}=H$ since $[J_i,H]=0$ (relation (4.3h)) (4.12)

By applying these results to Eq. (4.11), we prove the invariance of the Schrödinger equation under time-displacements, space-displacements and space-rotations when there are no external fields acting on the system:

$$\frac{d|\varphi'\rangle}{dt}=-iH|\varphi'\rangle \quad (4.13)$$

The case of boost-transformations is different from the previous cases, because the Hamiltonian is not boost-invariant even when the system is free from external fields (for the same claim in classical mechanics, see Butterfield, 2007, p. 6). In fact, under a boost-transformation corresponding to a velocity u_x, H changes as

$$H'=e^{iG_xu_x}He^{-iG_xu_x}\neq H \quad \text{since} \quad [G_x,H]=iP_x\neq 0\ \text{(relation (4.3i))} \quad (4.14)$$

and the generator G_x is

$$G_x=mQ_x=m(Q_{x0}+V_xt)=mQ_{x0}+P_xt \quad (4.15)$$

Since G_x is not time-independent, $dU/dt=\mathrm{d}e^{iG_xu_x}/dt\neq 0$, and Eq. (4.10) yields

$$\frac{d|\varphi'\rangle}{dt}=-i\left[H'+i\frac{de^{iG_xu_x}}{dt}e^{-iG_xu_x}\right]|\varphi'\rangle \quad (4.16)$$

In order to know the value of the bracket in the right hand side side of Eq. (4.16), we have to compute both terms in the bracket. By using the Hadamard lemma applied to the Baker–Campbell–Hausdorff formula, $e^BAe^{-B}=A+[B,A]+(1/2!)[B,[B,A]]+(1/3!)[B,[B,[B,A]]]+\cdots$, and by applying the commutation relations (4.3i) and (4.3f), H' results (see Appendix A)

$$H'=e^{iG_xu_x}He^{-iG_xu_x}=H-u_xP_x+\frac{1}{2}Mu_x^2=H+T_B \quad (4.17)$$

where T_B is the boost contribution to the energy. In turn, by means of the lemma and the commutation relation (4.3f), P_x' results (see Appendix A)

$$P_x'=e^{iG_xu_x}P_xe^{-iG_xu_x}=P_x-Mu_x\Rightarrow P'=P+P_B \quad (4.18)$$

where $P_B=(-Mu_x,0,0)$ is the boost contribution to the momentum. Let us recall that, when there are no external fields, the internal energy W is a Casimir operator; therefore, the

Hamiltonian can be written as

$$H=\frac{P^2}{2m}+W \tag{4.19}$$

By means of Eqs. (4.18) and (4.19), it is easy to show that the transformed Hamiltonian can be expressed as

$$H'=\frac{(P+P_B)^2}{2m}+W \tag{4.20}$$

On the other hand, we have to compute the time-derivative $de^{iG_xu_x}/dt$ of Eq. (4.16). By using the identity $e^{A+B}=e^Ae^Be^{-[A,B]/2}$ which holds when $[A,[A,B]]=[B,[A,B]]=0$, and by applying the commutation relation $[P_i,F(Q_j)]=-i\partial F/\partial Q_j$ valid on the Galilean algebra, it can be proved that (see Appendix B)

$$\frac{de^{iG_xu_x}}{dt}=-i\left(u_xP_x-\frac{1}{2}Mu_x^2\right)e^{iG_xu_x} \tag{4.21}$$

When the results (4.17) and (4.21) are introduced into Eq. (4.16), the terms added to H in H' cancel with those coming from the term containing the time-derivative; so, we prove the invariance of the Schrödinger equation also for boost-transformations:

$$\frac{d|\varphi'\rangle}{dt}=-iH|\varphi'\rangle \tag{4.22}$$

Let us summarize the results obtained up to this point. When there are no external fields acting on the system, the Hamiltonian is invariant under time-displacements, space-displacements and space-rotations, but not under boost-transformations. In spite of this fact, the Schrödinger equation is completely invariant under the Galilean group, and this conceptually means that the state vector $|\varphi\rangle$ does not "see" the effect of the transformations: the evolutions of $|\varphi\rangle$ and $|\varphi'\rangle$ are identical. In other words, the time-behavior of the system is independent of the reference frame used for the description.

When the system is affected by external fields, the harmony of the previous results gets lost. In fact, the fields modify the evolution of the system: for instance, if the system is under the action of a non-isotropic potential, we cannot longer expect that its behavior does not change when it is rotated in space. But, in non-relativistic quantum mechanics, fields are not quantized: they are not quantum systems and, as a consequence, their action cannot be conceived as an interaction between systems. Then, the effect of the fields on the system has to be accounted for by its Hamiltonian: the potentials have to modify the form of the Hamiltonian because it is the only observable involved in the time-evolution law. As a consequence, in the presence of fields the Hamiltonian is no longer the generator of time-displacements: it only retains its role as the generator of the dynamical evolution (see Ballentine, 1998; Laue, 1996). This means that we have to give up the commutation relations involving the Hamiltonian, Eqs. (4.3g–i): now these relations hold with the generator of time-displacements d/dt (see Eqs. (4.2g–i)), but not with the Hamiltonian:

$$[P_i,H]\neq 0 \quad [J_i,H]\neq 0 \quad [G_i,H]\neq iP_x \tag{4.23}$$

Therefore, we cannot guarantee the time-independence of the P_i and the J_i, and the result (4.17) cannot be obtained since based on the commutation relation (4.3i), $[G_i,H]=iP_x$. As a consequence, in general, the Schrödinger equation loses its Galilean invariance in the presence of external fields.

4.2. The covariance of the Schrödinger equation

In order to understand the covariance of the Schrödinger equation, let us rewrite Eq. (4.10) as

$$\frac{d|\varphi'\rangle}{dt}-\frac{dU}{dt}U^{-1}|\varphi'\rangle=-iH'|\varphi'\rangle \tag{4.24}$$

It is quite clear that covariance obtains when the differential operator transforms as

$$\frac{d}{dt}\rightarrow\frac{d'}{dt}=\frac{d}{dt}-\frac{dU}{dt}U^{-1} \tag{4.25}$$

This means that the transformed differential operator d'/dt is a covariant time-derivative D/Dt, which makes the Schrödinger equation to be Galilean-covariant in the following sense:

$$\frac{d'|\varphi'\rangle}{dt}=\frac{D|\varphi'\rangle}{Dt}=-iH'|\varphi'\rangle \tag{4.26}$$

As we have seen in the previous subsection, with no external fields applied on the system, H, the P_i and the J_i are time-independent and, as a consequence, $dU/dt=0$. Therefore, from eq. (4.25) we see that the time-derivative is invariant under time-displacements, space-displacements and space-rotations: $d/dt\rightarrow d/dt$. But for boost-transformations this is not the case: the covariance of the Schrödinger equation implies the transformation of the differential operator as $d/dt\rightarrow D/Dt$. This means that covariance under boosts amounts to a sort of "non-homogeneity" of time that requires the covariant adjustment of the time-derivative. This conclusion should not be surprising since, when the system is described in a reference frame RF' at uniform motion with respect to the original frame RF, the boost-transformed state depends on a generator that is a linear function of time (see Eq. (4.15)); then, if the Schrödinger equation is to be valid in RF' where the state is $|\varphi'\rangle$, the transformed time-derivative has to be adjusted to compensate this time-depending transformation of the state.

The case of boost-transformations illustrates a claim advanced in Section 2: although a law is invariant under a transformation when it is covariant and all the involved objects are invariant, this is not the only way to obtain nomological invariance. When the system is free from external fields, the Schrödinger equation is invariant under boost-transformations, in spite of the fact that the Hamiltonian and the differential operator d/dt are not boost-invariant objects.

When there are external fields applied to the system, Eq. (4.26) is still valid. But since now the Hamiltonian includes the action of the fields, the transformed Hamiltonian $H'=UHU^{-1}$ has to be computed in each case. The conditions to be satisfied by the external potentials in order to preserve the covariance of the Schrödinger equation can be deduced by knowing the precise dependence of the Hamiltonian on those potentials (see Brown & Holland, 1999; Colussi & Wickramasekara, 2008). However, this is not the point here; for our purpose it is sufficient to have seen how the operators are Galilean-transformed and in what sense the Schrödinger equation is covariant under the Galilean group.

4.3. Galilean-transformed observables

Some authors adopt a different strategy to address the matter of the Galilean covariance of the Schrödinger equation. By assuming the Galilean invariance of the differential operator d/dt, they preserve the Galilean covariance of the Schrödinger equation by redefining the action of the boost-transformation on certain dynamical magnitudes, in particular, on the Hamiltonian. Precisely, a boost-transformation given by $U=e^{iG_xu_x}$ does not act on H as $H'=UHU^{-1}$, but as (see Brown, Suárez, & Bacciagaluppi, 1998, Eq. (16); Brown & Holland, 1999, Eq. (33))

$$H\rightarrow\widetilde{H}=UHU^{-1}+i\frac{dU}{dt}U^{-1} \tag{4.27}$$

whereas the states and the differential operator transform as

$$|\varphi\rangle\rightarrow|\widetilde{\varphi}\rangle=U|\varphi\rangle \quad d/dt\rightarrow\widetilde{d}/dt=d/dt \tag{4.28}$$

Since these transformations were deliberately designed for preserving the Galilean covariance of the Schrödinger equation, by introducing Eqs. (4.27) and (4.28) into Eq. (4.9), such a covariance is immediately obtained

$$\frac{\widetilde{d}|\widetilde{\varphi}\rangle}{dt} = -i\tilde{H}|\tilde{\varphi}\rangle \tag{4.29}$$

If we take $\tilde{H}$ as the boost-transformed H, when there are no external fields acting on the system the Hamiltonian turns out to be invariant also under boost-transformations, since $\tilde{H}=H$ (introduce Eqs. (4.17) and (4.21) into Eq. (4.27)). Therefore, the invariance of the Schrödinger equation under boosts follows from its covariance, given by Eq. (4.29), and the boost-invariance of the objects involved in the equation, $\widetilde{d}/dt = d/dt$ and $\tilde{H}=H$. In this case, the boost-transformation of H is still unitary—it is the identity—and the choice between H' and $\tilde{H}$ as the boost-transformed Hamiltonian seems to be a matter of convention. However, the preference of $\tilde{H}$ over H' leads to undesirable consequences, both from a mathematical and from a physical point of view.

First, this strategy has an unpalatable *ad hoc* flavor. In fact, the Galilean covariance of the Schrödinger equation, rather than a result, turns out to be an *a priori* truth: no matter the particular form the equation has, the transformations are specifically defined for preserving its covariance. But the price to pay for covariance so obtained is to admit that a given transformation acts in different ways on different observables. Precisely, the transformation $H \to \tilde{H}$ given by Eq. (4.27), although unitary when there are no external fields, becomes non-unitary in the presence of those fields. This means that some observables, in particular the Hamiltonian, transform with what Brown, Suárez, and Bacciagaluppi (1998, p. 297) call a "*sui generis*" non-unitary transformation. But non-unitarity breaks the basic features of the Galilean group. On the one hand, non-unitary transformations cannot be combined in a single operation as that given by Eq. (4.1), which expresses the sequence of the ten elementary transformations. On the other hand, non-unitary transformations do not preserve the commutation relations among transformed observables. Precisely, given two observables A and B such that $[A, B]=C$, it is easy to see that the application of a transformation represented by a unitary operator U yields

$$[A,B]=C \Rightarrow [A',B'] = [UAU^{-1},UBU^{-1}] = UCU^{-1} = C' \tag{4.30}$$

This property is what preserves the commutation relations that define the Galilean group also for the transformed observables; for instance, $[P'_i,P'_j]=0$ or $[J'_i,G'_j]=i\varepsilon_{ijk}G'_k$. But if we use the "*sui generis*" transformation (4.27), this property gets lost. In particular, if $A \to \tilde{A}$ and $B \to \tilde{B}$, with some algebra we obtain

$$[\tilde{A},\tilde{B}] = UCU^{-1} + i\left[U(B-A)\frac{dU^{-1}}{dt} + \frac{dU}{dt}(B-A)U^{-1}\right] \tag{4.31}$$

The right hand side of this equation can be identified neither with $\tilde{C}$ (C transformed with Eq. (4.27)) nor with C' (C unitarily transformed). Therefore, the adoption of non-unitary transformations seems to be a too high price to pay for preserving the Galilean covariance of the Schrödinger equation, to the extent that this strategy leaves us with no clear idea about what the Galilean group means when deprived from its basic features.

From the physical viewpoint, a transformation as that of Eq. (4.27) also leads to undesirable consequences. According to that equation, when there are no external fields the boost-transformed Hamiltonian is $\tilde{H}=H$. This means that, if H is the Hamiltonian of the system when described in the reference frame RF, then $\tilde{H}=H$ is the Hamiltonian of the same system when described in the uniformly moving reference frame RF'. But this conclusion disagrees with the physical fact that the total energy of a system changes in an additive kinetic value when we change our descriptive perspective from RF to RF'. And this change in the total energy has its empirical manifestation as a Doppler shift in the energy spectrum (see Cohen-Tannoudji, Diu, & Laloë, 1977), which cannot be accounted for by the boost-transformation (4.27).

Summing up, if we want to preserve the formal structure of the Galilean group and the physical meaning of its transformations, we are not free to decide the form of the transformations. The argument of this subsection shows that any claim about the Galilean covariance of the Schrödinger equation must be based on the adequate transformations of the observables, in particular, of the Hamiltonian. This conclusion not only contributes to the understanding of the role of the Galilean group in quantum mechanics, but also has relevant implications for interpretation, as we shall see in the following section.

5. The modal-Hamiltonian interpretation in the light of the Galilean group

The modal-Hamiltonian interpretation of quantum mechanics (Lombardi & Castagnino, 2008) belongs to the modal family: it is a realist, non-collapse interpretation according to which the quantum state describes the possible properties of a system but not its actual properties. Here we shall only recall the interpretative postulates relevant to our discussion.

The first step is to identify the systems that populate the quantum ontology. By adopting an algebraic perspective, a quantum system is defined in the following terms:

Systems postulate (SP): A quantum system $\mathcal{S}$ is represented by a pair $(\mathcal{O},H)$ such that (i) $\mathcal{O}$ is a space of self-adjoint operators on a Hilbert space $\mathcal{H}$, representing the observables of the system, (ii) $H \in \mathcal{O}$ is the time-independent Hamiltonian of the system $\mathcal{S}$, and (iii) if $\rho_0 \in \mathcal{O}'$ (where $\mathcal{O}'$ is the dual space of $\mathcal{O}$) is the initial state of $\mathcal{S}$, it evolves according to the Schrödinger equation in its von Neumann version.

Of course, any quantum system can be partitioned in many ways; however, not any partition will lead to parts which are, in turn, quantum systems (see Harshman & Wickramasekara, 2007). On this basis, a composite system is defined as

Composite systems postulate (CSP): A quantum system represented by $\mathcal{S}:(\mathcal{O},H)$, with initial state $\rho_0 \in \mathcal{O}'$, is *composite* when it can be partitioned into two quantum systems $\mathcal{S}^1:(\mathcal{O}^1,H^1)$ and $\mathcal{S}^2:(\mathcal{O}^2,H^2)$ such that (i) $\mathcal{O}=\mathcal{O}^1 \otimes \mathcal{O}^2$ and (ii) $H = H^1 \otimes I^2 + I^1 \otimes H^2$ (where I^1 and I^2 are the identity operators in the corresponding tensor product spaces). In this case, the initial states of $\mathcal{S}^1$ and $\mathcal{S}^2$ are obtained as the partial traces $\rho_0^1 = Tr_{(2)}\rho_0$ and $\rho_0^2 = Tr_{(1)}\rho_0$; we say that $\mathcal{S}^1$ and $\mathcal{S}^2$ are *subsystems* of the composite system, $\mathcal{S}=\mathcal{S}^1 \cup \mathcal{S}^2$. If the system is not composite, it is *elemental.*

Since the contextuality of quantum mechanics prevents us from consistently assigning actual values to all the observables of a quantum system in a given state (see Kochen & Specker, 1967), the second step is to identify the set of the definite-valued observables of the system by means of a rule of definite-value assignment which, in this case, selects the observables that acquire an actual value:

Actualization rule (AR): Given an elemental quantum system represented by $\mathcal{S}:(\mathcal{O},H)$, the actual-valued observables of $\mathcal{S}$ are H and all the observables commuting with H and having, at least, the same symmetries as H.

For the conceptual advantages of the modal-Hamiltonian interpretation, we refer the reader to the original work (Lombardi & Castagnino, 2008). This brief sketch is sufficient for our present purpose of analyzing how the interpretation behaves under the action of the Galilean transformations.

5.1. Transformation of systems

According to the modal-Hamiltonian postulate SP, a quantum system $\mathcal{S}$ is represented by a pair $(\mathcal{O},H)$. In turn, any Galilean transformation $T_\alpha \in G$ has to apply to $\mathcal{S}$ as $\mathcal{S}:(\mathcal{O},H) \to \mathcal{S}':(\mathcal{O}',H')$. However, as we have seen in Section 2, a group G of transformations $T_\alpha \in G$ is an automorphism $T_\alpha : \mathcal{X} \to \mathcal{X}$; then, the Galilean group $\mathcal{G}$ applies to the observables of the system in such a way that

$$\forall T_\alpha \in \mathcal{G}, \quad \text{if } O \in \mathcal{O} \text{ and } O \to O', \text{ then } O' \in \mathcal{O} \tag{5.1}$$

In other words, the space of observables of a quantum system is closed under the transformations of the Galilean group,

$$\forall T_\alpha \in \mathcal{G}, \quad \mathcal{O} \to \mathcal{O}' \tag{5.2}$$

This feature is physically reasonable, since one does not expect that the mere application of a Galilean transformation on the system $\mathcal{S}$ modifies its identity by modifying its space of observables $\mathcal{O}$ (see Georgi, 1982). Therefore, the result of the application of the Galilean transformations to a quantum system will only depend on the way in which the Hamiltonian is transformed:

$$\forall T_\alpha \in \mathcal{G}, \quad \mathcal{S}:(\mathcal{O},H) \to \mathcal{S}':(\mathcal{O},H') \tag{5.3}$$

where H transforms *unitarily* as $H' = U_\alpha H U_\alpha^{-1}$, $U_\alpha = e^{iK_\alpha s_\alpha}$, and K_α is the generator of the transformation T_α.

As we have seen, in the presence of external fields the Schrödinger equation is covariant but not invariant under the Galilean group. This physically means that the system changes its behavior when merely displaced in space or time, rotated in space or uniformly moved: the covariance of the law is preserved by the transformation of the objects involved in it, in particular, of the Hamiltonian. According to the modal-Hamiltonian postulate SP, the original and the transformed systems are not the same system because the Hamiltonian has changed. Although this might sound surprising, it is completely consistent with the interpretative framework of the modal-Hamiltonian interpretation. In fact, a system that preserves its identity when Galilean-transformed in the presence of fields is a system having a principle of identity that reidentifies it under the change of all its properties. Such a principle has to be given by a sort of substance, conceived as a characterless substratum to which properties are "stuck": the permanence of that substratum is what endows the system with its identity under property change. But in the modal-Hamiltonian interpretation, a quantum system is not a substance acting as a bearer of properties, but a bundle of properties represented by the observables of the space $\mathcal{O}$ (see Lombardi & Castagnino, 2008, Section 8). Since there is no substratum that preserves the identity of the system when the properties change, the identity of the system changes with the change of the properties.

When there are no external fields, on the contrary, the issue of the Galilean transformation of a system requires to be considered with care. In this case, the Schrödinger equation is invariant under the Galilean group, and this means that the application of a Galilean transformation does not introduce a modification in the physical situation, but only expresses a change in the perspective from which the system is described. As a consequence, we can expect that, in this case, the system does not change its identity as the result of being Galilean-transformed: the system should be a Galilean-invariant object. In the context of the modal-Hamiltonian interpretation, the invariance of the system under time-displacements, space-displacements and space-rotations follows directly from the invariance of the Hamiltonian under those transformations (see Eqs. (4.12)):

$$\mathcal{S}:(\mathcal{O},H) \to \mathcal{S}':(\mathcal{O},H') = \mathcal{S}:(\mathcal{O},H) \tag{5.4}$$

But the situation is, again, completely different for boost-transformations: although the Schrödinger equation is invariant, the Hamiltonian is not invariant under boosts (see Eq. (4.14)). We shall analyze this case in detail.

Let us consider a quantum system not affected by external fields, represented by $\mathcal{S}:(\mathcal{O},H)$. In a generic reference frame *RF*, the Hamiltonian reads $H = P^2/2m + W = K + W$ (see Eq. (4.19)), where the kinetic energy $K = P^2/2m$ only depends on the total momentum relative to *RF*, and the internal energy W does not depend on the position and the momentum relative to *RF*, but only depends on differences of positions and, eventually, on their derivatives. Therefore, it can be guaranteed that $[K, W] = 0$ and, as a consequence, H can be expressed as

$$H = K + W = H_K \otimes I_W + I_K \otimes H_W \tag{5.5}$$

where H_K is the kinetic Hamiltonian acting on the Hilbert space $\mathcal{H}_K$, H_W is the internal energy Hamiltonian acting on the Hilbert space $\mathcal{H}_W$, and I_K and I_W are the identity operators of the respective tensor-product spaces (for examples in well-known models, see Ardenghi, Castagnino, & Lombardi, 2009). According to the modal-Hamiltonian postulate CSP, Eq. (5.5) implies that the system $\mathcal{S}$ is a composite system $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$, whose elemental subsystems are:

- A system represented by $\mathcal{S}_W : (\mathcal{O}_W, H_W)$, where $\mathcal{O}_W$ is the space of observables acting on $\mathcal{H}_W$, and $H_W \in \mathcal{O}_W$ represents the internal energy.
- A system represented by $\mathcal{S}_K : (\mathcal{O}_K, H_K)$, where $\mathcal{O}_K$ is the space of observables acting on $\mathcal{H}_K$, and $H_K \in \mathcal{O}_K$ represents the kinetic energy.

If we now apply a boost-transformation of velocity u_x to the system $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$, the unitarily transformed Hamiltonian is $H' = H + T_B$ (see Eq. (4.17)) and, then, it can expressed as

$$H' = H + T_B = \frac{P^2}{2m} + W + T_B = K' + W \tag{5.6}$$

where K' is the transformed kinetic energy (see Eqs. (4.18) and (4.20)):

$$K = \frac{P^2}{2m} \Rightarrow K' = K + T_B = \frac{P^2}{2m} + T_B = \frac{(P + P_B)^2}{2m} \tag{5.7}$$

For the same reasons as before, $[K', W] = 0$ and, as a consequence, H' can be written as

$$H' = K' + W = H'_K \otimes I_W + I_K \otimes H_W \tag{5.8}$$

where $H'_K = H_K + H_B$ is the transformed kinetic Hamiltonian acting on $\mathcal{H}_K$. Therefore, the boost-transformed system is again a composite system $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}'_K$, whose elemental subsystems are the original $\mathcal{S}_W$ and the system $\mathcal{S}'_K : (\mathcal{O}_K, H'_K)$ now defined by a kinetic energy H'_K that adds the kinetic energy H_B of the boost to the original kinetic energy H_K.

This argument shows that, when there are no external fields, a boost-transformation acts on a system represented by $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$ as

$$\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K \to \mathcal{S}' = \mathcal{S}_W \cup \mathcal{S}'_K \tag{5.9}$$

When, in particular, $\mathcal{S}$ is described in the reference frame at rest with respect to its center of mass, $P = 0$; then, $\mathcal{S}$ is an elemental system with Hamiltonian $H = W$, on which a boost acts as

$$\mathcal{S} = \mathcal{S}_W \to \mathcal{S}' = \mathcal{S}_W \cup \mathcal{S}'_K \tag{5.10}$$

where the subsystem $\mathcal{S}'_K$ is now defined only by the kinetic energy of the boost. Therefore, the subsystem $\mathcal{S}_W$, carrying the internal energy of the system, is boost-invariant, in agreement with the fact that the internal energy W is a Casimir operator of the

Galilean group. The application of a boost-transformation only affects the subsystem $\mathcal{S}_K$ by adding the kinetic energy of the boost to its Hamiltonian:

$$\mathcal{S}_W \to \mathcal{S}'_W = \mathcal{S}_W \quad H'_W = H_W \tag{5.11}$$

$$\mathcal{S}_K \to \mathcal{S}'_K \quad H'_K = H_K + H_B \tag{5.12}$$

This result leads us to ask ourselves about the ontological status of both subsystems.

On the one hand, when there are no external fields, the action of a boost-transformation has a well-defined manifestation in the energy spectrum of the composite system $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$: the boost produces a Doppler shift on the energy of $\mathcal{S}$. But we also know that energy is defined up to a constant value: the relevant information about the energy spectrum of a system is contained in its internal energy, and the kinetic energy only introduces a shift of that spectrum. Therefore, the boost-invariant subsystem $\mathcal{S}_W$ carries the physically meaningful structure of the energy spectrum, and $\mathcal{S}_K$ represents an energy shift which, although observable, is physically non relevant and merely relative to the reference frame used for the description. On the other hand, even the composite or elemental character of the system $\mathcal{S}$ depends on the particular reference frame selected. In fact, in the reference frame *RF* at rest with respect to the center of mass, $\mathcal{S} = \mathcal{S}_W$ is an elemental system; when, in turn, we decide to describe the system in a reference frame *RF′* uniformly moving with respect to *RF*, the system turns out to be composite, $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$.

Both considerations point to the same direction: the *objective* content of the description is given by the internal energy. In other words, the objective description of a system is $\mathcal{S}_w$, that is, the description in the reference frame at rest with respect to the center of mass, where $H = W : \mathcal{S}_W$ is completely invariant under the Galilean group. On the contrary, $\mathcal{S}_K$, which carries the kinetic energy, is a sort of "pseudo-system", whose identity is modified by a mere change of the descriptive perspective, and may even "appear" and "disappear" as a consequence of such a change. In order to express this idea with precision, we shall use the terminology introduced in our previous work (Lombardi & Castagnino, 2008), which allowed us to distinguish between the physical language and its ontological reference: the symbol "[●]" denotes the ontological item referred to by the word "●" of the physical language.[2] On this basis, we can say that $\mathcal{S} = \mathcal{S}_W \cup \mathcal{S}_K$ and $\mathcal{S}_W$ refer to the same *ontological system*:

$$[\mathcal{S}] = [\mathcal{S}_W \cup \mathcal{S}_K] = [\mathcal{S}_W] \tag{5.13}$$

where the symbol '=' strictly denotes *logical identity* (that is, if $a = b$, then a and b are two names for the same item). Therefore, when ontological systems are free from external fields, they are invariant under all the transformations of the Galilean group, in particular, under boosts,

$$[\mathcal{S}] = [\mathcal{S}_W \cup \mathcal{S}_K] = [\mathcal{S}_W] \to [\mathcal{S}'] = [\mathcal{S}_W \cup \mathcal{S}'_K] = [\mathcal{S}_W] = [\mathcal{S}] \tag{5.14}$$

The intuition about a strong link between invariance and objectivity is rooted in a natural idea: what is objective should not depend on the particular perspective used for the description; or, in group-theoretical terms, what is objective according to a theory is what is invariant under the symmetry group of the theory. This idea is not new. It was widely discussed in the context of special and general relativity with respect to the ontological status of space and time: "Henceforth space for itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality" (Minkowski, 1923, p. 75). The claim that objectivity means invariance is also a central thesis of Weyl's book *Symmetry* (1952). In recent times, the idea has strongly reappeared in several works. For instance, in her deep analysis of quantum field theory, Auyang (1995) makes her general concept of "object" to be founded on its invariance under transformations among all representations. In turn, the assumption that invariance is the root of objectivity is the central theme of Nozick's book *Invariances: The structure of the objective world* (2001).[3] Our conclusion about the objective description of a quantum system is in complete agreement with the general idea behind those works: when the Galilean group leaves invariant the Schrödinger equation, the objective description of the system is also invariant and, as a consequence, the ontological system is left unaffected by the Galilean transformations.

5.2. Transformation of the actualization rule

The actualization rule AR is not an object but a postulate, a sort of "interpretative law" that adds content to the theory. Then, the questions about its covariance and its invariance make sense. If we call $DVO(\mathcal{S})$ the set of the definite-valued observables of the elemental quantum system represented by $\mathcal{S} : (\mathcal{O}, H)$, according to AR, $DVO(\mathcal{S}) = \{H, A_i\}$, where the A_i are the observables commuting with H and having, at least, the same symmetries as H. Although AR is not a differential equation, it is easy to apply the concepts of covariance and invariance to it:

- AR is Galilean-covariant if, $\forall T_\alpha \in G$, the set $DVO(\mathcal{S})$ transforms as

$$DVO(\mathcal{S}) = \{H, A_i\} \to DVO(\mathcal{S}') = \{H', A'_i\} \tag{5.15}$$

- AR is Galilean-invariant if, $\forall T_\alpha \in G$, the set $DVO(\mathcal{S})$ transforms as

$$DVO(\mathcal{S}) \to DVO(\mathcal{S}') = DVO(\mathcal{S}) \tag{5.16}$$

The covariance of AR follows directly from the *unitarity* of the Galilean transformations. In fact, each Galilean transformation can be viewed as a definite rotation of the eigenvectors of all the observables $O \in \mathcal{O}$ in the Hilbert space $\mathcal{H}$. This rotation preserves all the commutation relations under the transformations; so, if $[H, A_i] = 0$, then $[H, A'_i] = 0$ (see Eq. (4.30)). But, as a mere rotation, such a transformation also preserves all the remaining relations between observables; as a consequence, if A_i has, at least, the same symmetries as H, then A'_i has, at least, the same symmetries as H'. Therefore, if AR selects H and the A_i as the definite-valued observables of $\mathcal{S}$, when we transform these observables we obtain H' and A'_i, which are precisely the observables selected as definite-valued by AR applied to $\mathcal{S}'$.

On the other hand, it is quite clear that, when the system is free from external fields, AR in invariant under time-displacements, space-displacements and space-rotations, since the system is invariant under those transformations (see Eq. (5.4)):

$$\mathcal{S} = \mathcal{S}' \Rightarrow DVO(\mathcal{S}') = DVO(S) \tag{5.17}$$

The difficulty comes, again, from the boost-transformation, which does not leave the Hamiltonian invariant. Since $DVO(\mathcal{S})$ depends on H, it changes with the change of H, and this seems to be a serious problem for the modal-Hamiltonian interpretation: as a realist interpretation, the *DVO* selected by its rule of definite-value assignment should not be modified by a mere change of

[2] In that paper (Lombardi & Castagnino, 2008), we have said that an observable O ontologically represents a *type-property* $\lceil O \rceil$, and its eigenvalues o_i ontologically represent its *case-properties* $\lceil O : o_i \rceil$. Moreover, given a composite system $S = S^1 \cup S^2$, the observables A^1 of S^1 and $A = A^1 \otimes I^2$ of S represent the same type-property $\lceil A \rceil = \lceil A_1 \rceil$ with the same case-properties $[A : a_i^1] = [A^1 : a_i^1]$.

[3] The issue of symmetries and invariances has become a increasingly debated topic in the philosophy of physics during the last years. See also the works of Brading & Castellani (2003, 2007) and Earman (2002, 2004a, 2004b).

reference frame, like that represented by a boost on a system free from external fields. Nevertheless, the problem vanishes when we recall that AR applies to *elemental* quantum systems.

As we have seen in the previous subsection, when there are no external fields, the system represented by $\mathcal{S}:(\mathcal{O},H)$ is, in general, a composite system $\mathcal{S}=\mathcal{S}_W\cup\mathcal{S}_K$, where the subsystem represented by $\mathcal{S}_W:(\mathcal{O}_W,H_W)$ carries the internal energy and the subsystem represented by $\mathcal{S}_K:(\mathcal{O}_K,H_K)$ carries the kinetic energy. Therefore, AR applies to the elemental subsystems independently:

- Since $\mathcal{S}_W$ is boost-invariant, $DVO(\mathcal{S}_W)=DVO(\mathcal{S}'_W)$: the set of the definite-valued observables of $\mathcal{S}'_W$ remains unaltered by the transformation.
- Under a boost, $\mathcal{S}_K:(\mathcal{O}_K,H_K)$ transforms as $\mathcal{S}'_K:(\mathcal{O}_K,H'_K)$, where $H'_K=H_K+H_B$ (see Eq. (5.12)); then $DVO(\mathcal{S}_K)\neq DVO(\mathcal{S}'_K)$.

However, as we have argued, the system $\mathcal{S}_K$ has no ontological reference to the extent that it is a mere artifact of the descriptive perspective. In ontological terms, $[\mathcal{S}']=[\mathcal{S}_W]=[\mathcal{S}]$ (see Eq. (5.14)). As a consequence, when applied to the objective description of the system, given by $\mathcal{S}_W$, AR turns out to be also boost-invariant.

5.3. An invariant version of the actualization rule

In this subsection we shall focus only on the situation where there are not external fields acting on the system: it is in this situation that the Schrödinger equation is invariant under the Galilean group and, as a consequence, we can expect to have a reformulation of AR in an explicitly Galilean-invariant form.

The idea presented at the end of the previous subsection can be expressed with precision in ontological terms by means of the ontological language introduced in our previous work (Lombardi & Castagnino, 2008). With that terminology, AR can be reformulated as (see Note 2):

Actualization rule′ (AR′): Given an ontological system $[\mathcal{S}]$ free from external fields and represented by $\mathcal{S}:(\mathcal{O},H)=\mathcal{S}_W\cup\mathcal{S}_K$, where the subsystems $[\mathcal{S}_W]$ and $[\mathcal{S}_K]$ are represented by $\mathcal{S}_W:(\mathcal{O}_W,H_W)$ and $\mathcal{S}_K:(\mathcal{O}_K,H_K)$, the set $\mathcal{A}([\mathcal{S}])$ of the actual type-properties of $[\mathcal{S}]$ (the type-properties of $[\mathcal{S}]$ that acquire actual values) is $\mathcal{A}([\mathcal{S}])=\mathcal{A}([\mathcal{S}_W])=\{[W],[A_i]\}$, where $[W]$ is the type-property represented by $W=H_W\otimes I_K$, and the type-properties $[A_i]$ are represented by the observables A_i commuting with W and having, at least, the same symmetries as W.

Under this form, the actualization rule is explicitly Galilean-invariant since $[\mathcal{S}]$ and $[\mathcal{S}_W]$ are only different names for the same ontological system, and $[\mathcal{S}_W]$ is Galilean-invariant[4]:

$$[\mathcal{S}]=[\mathcal{S}_W]\Rightarrow\mathcal{A}([\mathcal{S}])=\mathcal{A}([\mathcal{S}_W])=\mathcal{A}([\mathcal{S}'_W])=\mathcal{A}([\mathcal{S}']) \quad (5.18)$$

This shows that, when we go beyond the physical language and think in ontological terms, the seeming non-invariance of the actualization rule under boosts shows itself as a result of the descriptive language. In spite of the change of the physical representation of the system under boosts, with no external fields acting on the system the set of its actual type-properties is invariant under all the Galilean transformations due precisely to the invariance of the ontological system. In other words, the identity and the behavior of the real quantum system behind the description are not modified by a mere change of the descriptive perspective, in agreement with the physical meaning of the Galilean group.

Although AR′ is conceptually meaningful from an ontological viewpoint, we may consider the possibility of reformulating it in such a way that it results Galilean-invariant also in the physical language. The natural way to reach this goal is to appeal to the Casimir operators of the Galilean group: if the actualization rule has to select a Galilean-invariant set of definite-valued observables, such a set must depend on those Casimir operators, which are invariant under all the transformations of the Galilean group. Precisely,

Actualization rule″ (AR″): Given a quantum system free from external fields and represented by $\mathcal{S}:(\mathcal{O},H)$, its definite-valued observables are the observables C_i represented by the Casimir operators of the Galilean group in the corresponding irreducible representation, and all the observables commuting with the C_i and having, at least, the same symmetries as the C_i.

Since the Casimir operators of the Galilean group are M, S^2 and W, this reformulation of the rule is in agreement with the original AR when applied to a system free from external fields:

- The definite-valuedness of M and S^2, postulated by AR″, follows from AR: these observables commute with H and do not break its symmetries because, in non-relativistic quantum mechanics, both are multiples of the identity in any irreducible representation. The fact that M and S^2 always acquire definite values is completely natural from a physical viewpoint, since mass and spin are properties supposed to be always possessed by any quantum system and measurable in any physical situation.
- The definite-valuedness of W might seem to be in conflict with AR because W is not the Hamiltonian: whereas W is Galilean-invariant, H changes under the action of a boost. However, as we have seen, this is not a real obstacle when the elemental subsystems to which AR applies are considered from an ontological viewpoint.

In addition to supplying an explicitly invariant version of the rule of definite-value assignment in the physical language, AR″ leads us to a final reflection. As we have seen, the identity and the behavior of any ontological quantum system free from external fields are invariant under the Galilean group. On the other hand, from a realist viewpoint, the fact that certain observables acquire an actual definite value is an objective fact in the behavior of the system; therefore, the set of definite-valued observables selected by a realist interpretation must be also Galilean-invariant. But the Galilean-invariant observables are always functions of the Casimir operators of the Galilean group. As a consequence, one is led to the conclusion that any realist interpretation that intends to preserve the objectivity of actualization may not stand very far from the modal-Hamiltonian interpretation.

6. Conclusions and perspectives

In spite of the impressive literature on the interpretation of quantum mechanics, the constraints imposed by the group properties of the theory on interpretation have not been sufficiently studied. The aim of this paper has been to address this issue in the context of the modal-Hamiltonian interpretation, with its particular definition of quantum system and its specific rule of definite-value assignment. The arguments developed in the paper led us to the following conclusions:

- The modal-Hamiltonian actualization rule mirrors the Galilean-covariance/invariance of the Schrödinger equation:
  - In general, the rule is as covariant as the Schrödinger equation.

[4] Let us recall the relationship between type-properties of $[\mathcal{S}]$ and $[\mathcal{S}_W]$ in the modal-Hamiltonian interpretation (see IP4 of Lombardi & Castagnino, 2008; see also Note 2): since $\mathcal{S}=\mathcal{S}_W\cup\mathcal{S}_K$, the observable H_W of $[\mathcal{S}_W]$ and the observable $W=H_W\otimes I_K$ of $[\mathcal{S}]$ represent the same type property, $[H_W]=[W]$, with the same case-properties, $[H_W:w_\alpha]=[W:w_\alpha]$.

   - When the Schrödinger equation is invariant—no external fields acting on the system—, the rule is also invariant when expressed in ontological terms.
- In the Galilean-invariant situation:
   - The ontological system identified by the modal-Hamiltonian interpretation is Galilean-invariant, in agreement with the idea that the Galilean transformations express a mere change of the perspective selected for describing the system.
   - The modal-Hamiltonian actualization rule can be reformulated under an explicitly Galilean-invariant form in terms of the Casimir operators of the Galilean group, leading to results that agree with usual assumptions in the practice of physics.

The last conclusion opens up a promising new research path. In non-relativistic quantum mechanics, the external fields acting on a system are not quantized, and this fact is what breaks down the harmony of the free case: the Schrödinger equation loses its Galilean invariance, and the Hamiltonian is no longer the generator of time-displacements in the Galilean group. In quantum field theory (QFT), on the contrary, fields are quantum items and not "external" fields affecting the behavior of the quantum system. As a consequence, the generators of the Poincaré group do not need to be reinterpreted in the presence of "external" factors, and the the dynamical laws are always Poincaré-invariant. These features of QFT lead us to consider whether the actualization rule, expressed in terms of the Casimir operators of the Galilean group in non-relativistic quantum mechanics, can be transferred to QFT by changing accordingly the symmetry group: the definite-valued observables of a system in QFT would be those represented by the Casimir operators of the Poincaré group, and the observables commuting with them and having, at least, the same symmetries. Since M and S^2 are the only Casimir operators of the Poincaré group, they would always be definite-valued observables. This conclusion would stand in agreement with a usual physical assumption in QFT: elemental particles always have definite values of mass and spin, and those values are precisely what define the different kinds of elemental particles of the theory. Of course, these brief remarks do not amount to a full interpretation of QFT, but they point towards a research program whose viability deserves to be examined in the future.

Acknowledgements

This paper was partially supported by grants of the National Agency of Scientific and Technological Research (ANCYT), the National Council of Scientific and Technical Research (CONICET), the Buenos Aires University (UBA), the Argentine Society of Philosophical Analysis (SADAF) and the University of the Latin American Educational Center (UCEL).

Appendix A

Under a boost-transformation of velocity u_x, the Hamiltonian changes as

$$H' = e^{iG_x u_x} H e^{-iG_x u_x} \tag{A.1}$$

So, H' can be computed by means of the Hadamard's lemma applied to the Baker–Campbell–Hausdorff formula (see Dynkin, 1947; Greiner & Reinhardt, 1995),

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B,[B,[B,A]]] + \cdots \tag{A.2}$$

Then, H' can be expressed as

$$H' = H + [iG_x u_x, H] + \frac{1}{2!}[iG_x u_x,[iG_x u_x,H]] + \frac{1}{3!}[iG_x u_x,[iG_x u_x,[iG_x u_x,H]]] + \cdots \tag{A.3}$$

By the commutation relation (4.3i), we know that

$$[iG_x u_x, H] = iu_x[G_x,H] = iu_x iP_x = -u_x P_x \tag{A.4}$$

Then,

$$[iG_x u_x,[iG_x u_x,H]] = [iG_x u_x, -u_x P_x] = -iu_x^2[G_x,P_x] \tag{A.5}$$

By the commutation relation (4.3f), we obtain

$$-iu_x^2[G_x,P_x] = -iu_x^2 iM = u_x^2 M \tag{A.6}$$

Then, by means of Eqs. (A.5) and (A.6), the fourth term of the r.h.s. of Eq. (A.3) results

$$[iG_x u_x,[iG_x u_x,[iG_x u_x,H]]] = [iG_x u_x, u_x^2 M] = 0 \tag{A.7}$$

Therefore, all the terms following the fourth term of the r.h.s. of Eq. (A.3) are also zero. So, by introducing Eqs. (A.4)–(A.7) into Eq. (A.3), we obtain

$$H' = H - u_x P_x + \frac{1}{2} M u_x^2 = H + T_B \tag{A.8}$$

where T_B is the boost contribution to the energy.

Under a boost-transformation with velocity u_x, the momentum in direction x changes as

$$P'_x = e^{iG_x u_x} P_x e^{-iG_x u_x} \tag{A.9}$$

Again, on the basis of the Hadamard's lemma (see Eq. (A.2)), P'_x can be expressed as

$$P'_x = P_x + [iG_x u_x, P_x] + \frac{1}{2!}[iG_x u_x,[iG_x u_x,P_x]] + \frac{1}{3!}[iG_x u_x,[iG_x u_x,[iG_x u_x,P_x]]] + \cdots \tag{A.10}$$

By the commutation relation (4.3f), we know that

$$[iG_x u_x, P_x] = iu_x[G_x,P_x] = iu_x iM = -Mu_x \tag{A.11}$$

Then,

$$[iG_x u_x,[iG_x u_x,P_x]] = [iG_x u_x, -Mu_x] = 0 \tag{A.12}$$

Therefore, all the terms following the third term of Eq. (A.10) are also zero. By introducing Eqs. (A.11) and (A.12) into Eq. (A.10), we obtain

$$P'_x = P_x - Mu_x \tag{A.13}$$

On the basis of Eq. (A.8), we can write

$$H' = H + T_B = \frac{P^2}{2m} - u_x P_x + \frac{1}{2} M u_x^2 = W + \frac{1}{2m}(P_x^2 - 2Mu_x P_x + M^2 u_x^2 + P_y^2 + P_z^2) \tag{A.14}$$

Therefore,

$$H' = W + \frac{(P_x - Mu_x)^2 + P_y^2 + P_z^2}{2m} \tag{A.15}$$

If we call the boost-momentum $P_B = (-Mu_x, 0, 0)$, the transformed Hamiltonian results

$$H' = W + \frac{(P - P_B)^2}{2m} \tag{A.16}$$

It is interesting to note that, when we have to give up the commutation relations (4.3g–i), involving H due to the action of external fields, Eq. (A.8) and, as a consequence, Eq. (A.16) are no longer valid since they rely on the relation (4.3i), but Eq. (A.13) still holds because its derivation does not require the abandoned relations.

Appendix B

Under a boost-transformation of velocity u_x, the boost-generator G_x reads

$$G_x = mQ_x = m(Q_{x0} + V_x t) = mQ_{x0} + P_x t \tag{B.1}$$

By knowing that

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \tag{B.2}$$

which holds when $[A,[A,B]]=[B,[A,B]]=0$, the exponential $e^{iG_x u_x}$ results

$$e^{iG_x u_x} = e^{imQ_{x0}u_x} e^{-iP_x u_x t} e^{-(i/2)mu_x^2 t} \tag{B.3}$$

Then, the time-derivative $de^{iG_x u_x}/dt$ has to be computed as

$$\frac{de^{iG_x u_x}}{dt} = e^{imQ_{x0}u_x}(-i)P_x u_x e^{-iP_x u_x t} e^{-(i/2)mu_x^2 t} + e^{imQ_{x0}u_x} e^{-iP_x u_x t}(-i/2)mu_x^2 e^{-(i/2)mu_x^2 t} \tag{B.4}$$

The second term of the r.h.s. of Eq. (B.4) is simply $(-i/2)e^{iG_x u_x}$. But the exponential $e^{iG_x u_x}$ cannot be directly reconstructed in the first term because $[e^{imQ_{x0}u_x}, P_x] \neq 0$. This commutator can be computed by knowing that, in the Galilean algebra (see Cohen-Tannoudji, Diu, & Lalöe, 1977, Eq. (48), p. 172),

$$[P_i, F(Q_i)] = -i\frac{\partial F}{\partial Q_i} \tag{B.5}$$

Therefore,

$$[e^{imQ_{x0}u_x}, P_x] = -[P_x, e^{imQ_{x0}u_x}] = i\frac{\partial e^{imQ_{x0}u_x}}{\partial Q_{x0}} = -mu_x e^{imQ_{x0}u_x} \tag{B.6}$$

This means that

$$e^{imQ_{x0}u_x} P_x = P_x e^{imQ_{x0}u_x} - mu_x e^{imQ_{x0}u_x} = (P_x - mu_x) e^{imQ_{x0}u_x} \tag{B.7}$$

By introducing Eq. (B.7) into Eq. (B.4) we obtain

$$\frac{de^{iG_x u_x}}{dt} = -i\left[(P - mu_x)u_x + \frac{1}{2}mu_x^2\right] e^{iG_x u_x} \tag{B.8}$$

and, finally, the time-derivative $de^{iG_x u_x}/dt$ results

$$\frac{de^{iG_x u_x}}{dt} = -i\left[Pu_x - \frac{1}{2}mu_x^2\right] e^{iG_x u_x} \tag{B.9}$$

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