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Projective modules and Gröbner bases for skew PBW extensions

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Contents

Abstract

Many rings and algebras arising in quantum mechanics, algebraic analysis, and non-commutative algebraic geometry can be interpreted as skew PBW (Poincaré–Birkhoff–Witt) extensions. In the present paper we study two aspects of these non-commutative rings: their finitely generated projective modules from a matrix-constructive approach, and the construction of the Gröbner theory for their left ideals and modules. These two topics have interesting applications in functional linear systems and in non-commutative geometry.

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1. Introduction

Many rings and algebras arising in quantum mechanics, algebraic analysis, and non-commutative algebraic geometry can be interpreted as skew PBW (Poincaré–Birkhoff–Witt) extensions. Indeed, Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and their quantizations), well known classes of Ore algebras (for example, the algebra of shift operators and the algebra for multidimensional discrete linear systems), Artamonov quantum polynomials, diffusion algebras, Manin algebras of quantum matrices, Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2, K))$, among many others, are examples of skew PBW extensions.

This type of non-commutative rings were defined firstly in [19] and represent a generalization of PBW extensions introduced by Bell and Goodearl [4]. Some other authors have classified quantum algebras and other non-commutative rings of polynomial type by similar notions: Levandovskyy [29] defined the G-algebras, Bueso, Gómez-Torrecillas and Verschoren [7] introduced the PBW rings, Panov [39] defined the so-called Q-solvable algebras. In all these cases they assume that either the ring of coefficients is a field or the variables commute with the coefficients. As we will see below, for the skew PBW extensions the ring of coefficients is arbitrary and the variables not necessarily commute.

Ring- and module-theoretical properties of skew PBW extensions have been studied in some recent papers [35], [34], [47]. In the present paper we are interested in two aspects of these non-commutative rings: the study of finitely generated projective modules from a matrix-constructive perspective, and the construction of the Gröbner theory for left ideals and modules. These two topics have interesting applications in functional linear systems (as has been done for Ore algebras in [5], [11]–[13], [17], [40]–[46], [54] and [55]), and in non-commutative algebraic geometry (see [49, Section 1.4] about non-commutative Gröbner bases for some quantum algebras).

2. Skew PBW extensions

In this section we recall the definition of skew PBW extensions and some of their elementary properties, and we present examples of this class of non-commutative rings of polynomial type (see [19] and [35]).

2.1. Definitions and elementary examples. We will see next that the skew PBW extensions are a generalization of PBW extensions defined by Bell and Goodearl [4] in 1988.

Definition 1. Let R and A be rings. We say that A is a *skew PBW extension* of R, also called a *σ-PBW extension*, if:

(i) $R \subseteq A$.
(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis

$$\mathrm{Mon}(A) = \mathrm{Mon}\{x_1, \dots, x_n\} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\},$$

where $\mathbb{N} := \{0, 1, 2, \dots\}$.
(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{2.1}$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2.2}$$

Under these conditions we will write $A = \sigma(R)\langle x_1, \dots, x_n \rangle$.

Remark 2. (i) Since $\mathrm{Mon}(A)$ is an R-basis for A, the elements $c_{i,r}$ and $c_{i,j}$ in the above definition are unique.

(ii) If $r = 0$, then $c_{i,0} = 0$: in fact, $0 = x_i 0 = c_{i,0} x_i + s_i$, with $s_i \in R$, but since $\mathrm{Mon}(A)$ is an R-basis, we have $c_{i,0} = 0 = s_i$.

(iii) In Definition 1(iv), $c_{i,i} = 1$: in fact, $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, hence $1 - c_{i,i} = 0 = s_i$.

(iv) Let $i < j$. By (2.2) there exist $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + R x_1 + \cdots + R x_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$, but since $\mathrm{Mon}(A)$ is an R-basis, we have $1 = c_{j,i} c_{i,j}$, i.e., for every $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse.

(v) Each element $f \in A - \{0\}$ has a unique representation in the form $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R - \{0\}$ and $X_i \in \mathrm{Mon}(A)$, $1 \leq i \leq t$.

The following proposition justifies the notation that we have introduced for the skew PBW extensions.

Proposition 3 ([18]). *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r)$$

for each $r \in R$.

A particular case of skew PBW extension is when all the derivations δ_i are zero. Another interesting case is when all the σ_i are bijective and the constants c_{ij} are invertible. We have the following definition.

Definition 4. Let A be a skew PBW extension.

(a) A is *quasi-commutative* if conditions (iii) and (iv) in Definition 1 are replaced by

(iii′) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i. \tag{2.3}$$

(iv′) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{2.4}$$

(b) A is *bijective* if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Some interesting examples of skew PBW extensions are the following.

EXAMPLE 5. (i) Any PBW extension is a bijective skew PBW extension, since in this case $\sigma_i = i_R$ for each $1 \leq i \leq n$ and $c_{i,j} = 1$ for every $1 \leq i, j \leq n$ (see [4]).

(ii) Any *skew polynomial ring* $R[x;\sigma,\delta]$ *of injective type*, i.e., with σ injective, is a skew PBW extension; in this case we have $R[x;\sigma,\delta] = \sigma(R)\langle x\rangle$. If additionally $\delta = 0$, then $R[x;\sigma]$ is quasi-commutative.

(iii) Let $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ be an *iterated skew polynomial ring of injective type*, i.e., the following conditions hold:

- For $1 \leq i \leq n$, σ_i is injective.
- For $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$.
- For $i < j$, $\sigma_j(x_i) = cx_i + d$, where $c, d \in R$ and c has a left inverse.
- For $i < j$, $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_i$.

Then $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ is a skew PBW extension. Under the above conditions we have

$$R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] = \sigma(R)\langle x_1,\dots,x_n\rangle.$$

In particular, any *Ore extension* $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$ *of injective type*, i.e., such that for $1 \leq i \leq n$, σ_i is injective, is a skew PBW extension. In fact, in Ore extensions, for every $r \in R$ and $1 \leq i \leq n$, $\sigma_i(r), \delta_i(r) \in R$, and for $i < j$, $\sigma_j(x_i) = x_i$ and $\delta_j(x_i) = 0$. An important subclass of Ore extensions of injective type are the *Ore algebras of injective type*, i.e., when $R = K[t_1,\dots,t_m]$, $m \geq 0$. Thus, we have

$$K[t_1,\dots,t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] = \sigma(K[t_1,\dots,t_m])\langle x_1,\dots,x_n\rangle.$$

Some concrete examples of Ore algebras of injective type are the following.

The *algebra of shift operators*: let K be a field and $h \in K$. Then the algebra of shift operators is defined by $S_h := K[t][x_h;\sigma_h,\delta_h]$, where $\sigma_h(p(t)) := p(t-h)$, and $\delta_h := 0$ (observe that S_h can also be considered as a skew polynomial ring of injective type). Thus, S_h is a quasi-commutative bijective skew PBW extension.

The *mixed algebra* D_h: As above, let K be a field and $h \in K$. Then the mixed algebra D_h is defined by $D_h := K[t]\big[x; i_{K[t]}, \frac{d}{dt}\big][x_h;\sigma_h,\delta_h]$, where $\sigma_h(x) := x$. Again, D_h is a quasi-commutative bijective skew PBW extension.

The *algebra for multidimensional discrete linear systems* is defined by the formula $D := K[t_1,\dots,t_n][x_1;\sigma_1,0]\cdots[x_n;\sigma_n,0]$, where K is a field and

$$\sigma_i(p(t_1,\dots,t_n)) := p(t_1,\dots,t_{i-1},t_i+1,t_{i+1},\dots,t_n), \quad \sigma_i(x_i) = x_i, \quad 1 \leq i \leq n.$$

Thus, D is a quasi-commutative bijective skew PBW extension. Observe that none of these examples is a PBW extension.

(iv) *Additive analogue of the Weyl algebra*: Let K be a field. The K-algebra $A_n(q_1,\dots,q_n)$ is generated by $x_1,\dots,x_n,y_1,\dots,y_n$ and subject to the relations

$$\begin{aligned}
&x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, \qquad 1\le i,j\le n,\\
&y_ix_j = x_jy_i, \qquad i\neq j,\\
&y_ix_i = q_ix_iy_i+1, \qquad 1\le i\le n,
\end{aligned}$$

where $q_i\in K-\{0\}$. We observe that $A_n(q_1,\dots,q_n)$ is isomorphic to the iterated skew polynomial ring $K[x_1,\dots,x_n][y_1;\sigma_1,\delta_1]\cdots[y_n;\sigma_n,\delta_n]$ over the commutative polynomial ring $K[x_1,\dots,x_n]$:

$$\begin{aligned}
&\sigma_j(y_i):=y_i, \quad \delta_j(y_i):=0, \qquad 1\le i<j\le n,\\
&\sigma_i(x_j):=x_j, \quad \delta_i(x_j):=0, \qquad i\neq j,\\
&\sigma_i(x_i):=q_ix_i, \quad \delta_i(x_i):=1, \qquad 1\le i\le n.
\end{aligned}$$

Thus, $A_n(q_1,\dots,q_n)$ satisfies the conditions of (iii) and is bijective; we have

$$A_n(q_1,\dots,q_n)=\sigma(K[x_1,\dots,x_n])\langle y_1,\dots,y_n\rangle.$$

(v) *Multiplicative analogue of the Weyl algebra*: Let K be a field. The K-algebra $\mathcal{O}_n(\lambda_{ji})$ is generated by $x_1,\dots,x_n$ and subject to the relations

$$x_jx_i=\lambda_{ji}x_ix_j, \qquad 1\le i<j\le n,$$

where $\lambda_{ji}\in K-\{0\}$. We note that $\mathcal{O}_n(\lambda_{ji})$ is isomorphic to the iterated skew polynomial ring $K[x_1][x_2;\sigma_2]\cdots[x_n;\sigma_n]$:

$$\sigma_j(x_i):=\lambda_{ji}x_i, \qquad 1\le i<j\le n.$$

$\mathcal{O}_n(\lambda_{ji})$ satisfies the conditions of (iii), and hence is an iterated skew polynomial ring of injective type but it is not Ore. Thus,

$$\mathcal{O}_n(\lambda_{ji})=\sigma(K[x_1])\langle x_2,\dots,x_n\rangle.$$

Moreover, $\mathcal{O}_n(\lambda_{ji})$ is quasi-commutative and bijective.

(vi) *q-Heisenberg algebra*: Let K be a field. The K-algebra $H_n(q)$ is generated by $x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n$ and subject to the relations

$$\begin{aligned}
&x_jx_i=x_ix_j, \quad z_jz_i=z_iz_j, \quad y_jy_i=y_iy_j, \qquad 1\le i,j\le n,\\
&z_jy_i=y_iz_j, \quad z_jx_i=x_iz_j, \quad y_jx_i=x_iy_j, \qquad i\neq j,\\
&z_iy_i=qy_iz_i, \quad z_ix_i=q^{-1}x_iz_i+y_i, \quad y_ix_i=qx_iy_i, \qquad 1\le i\le n,
\end{aligned}$$

with $q\in K-\{0\}$. Note that $H_n(q)$ is isomorphic to the iterated skew polynomial ring $K[x_1,\dots,x_n][y_1;\sigma_1]\cdots[y_n;\sigma_n][z_1;\theta_1,\delta_1]\cdots[z_n;\theta_n,\delta_n]$ on the commutative polynomial ring $K[x_1,\dots,x_n]$:

$$\begin{aligned}
&\theta_j(z_i):=z_i, \quad \delta_j(z_i):=0, \quad \sigma_j(y_i):=y_i, \qquad 1\le i<j\le n,\\
&\theta_j(y_i):=y_i, \quad \delta_j(y_i):=0, \quad \theta_j(x_i):=x_i, \quad \delta_j(x_i):=0, \quad \sigma_j(x_i):=x_i, \qquad i\neq j,\\
&\theta_i(y_i):=qy_i,\ \delta_i(y_i):=0,\ \theta_i(x_i):=q^{-1}x_i,\ \delta_i(x_i):=y_i,\ \sigma_i(x_i):=qx_i, \qquad 1\le i\le n,
\end{aligned}$$

Since $\delta_i(x_i)=y_i\notin K[x_1,\dots,x_n]$, $H_n(q)$ is not a skew PBW extension of $K[x_1,\dots,x_n]$, however, with respect to K, $H_n(q)$ satisfies the conditions of (iii), and hence $H_n(q)$ is a bijective skew PBW extension of K:

$$H_n(q) = \sigma(K)\langle x_1,\dots,x_n; y_1,\dots,y_n; z_1,\dots,z_n\rangle.$$

REMARK 6. We remark that the skew PBW extensions are not a subclass of the collection of iterated skew polynomial rings: take for example $\mathcal{U}(\mathcal{G})$ or the diffusion algebra (see [35] and Section 2.3 below). On the other hand, the skew polynomial rings are not included in the class of skew PBW extensions: take $R[x;\sigma,\delta]$ with σ not injective.

2.2. Basic properties. Next we present some basic properties of skew PBW extensions. We start with some notation that we will use frequently.

DEFINITION 7. Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \le i \le n$, as in Proposition 3.

(i) For $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, we write $\sigma^\alpha := \sigma_1^{\alpha_1}\cdots\sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$, then $\alpha+\beta := (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$.
(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.
(iii) Let $0 \ne f \in A$. Then $t(f)$ is the finite set of terms that form f, i.e., if $f = c_1X_1 + \cdots + c_tX_t$ with $X_i \in \mathrm{Mon}(A)$ and $c_i \in R-\{0\}$, then $t(f) := \{c_1X_1,\dots,c_tX_t\}$.
(iv) Let f be as in (iii). Then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

The skew PBW extensions can be characterized in a similar way to what was done in [6] for PBW rings.

THEOREM 8 ([18]). *Let A be a left polynomial ring over R with respect to $\{x_1,\dots,x_n\}$, i.e., conditions* (i) *and* (ii) *in Definition* 1 *are satisfied. Then A is a skew PBW extension of R if and only if:*

(a) *For every $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \ne r \in R$ there exist unique $r_\alpha := \sigma^\alpha(r) \in R-\{0\}$ and $p_{\alpha,r} \in A$ such that*
$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}, \tag{2.5}$$
where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \ne 0$. Moreover, if r is left invertible, then r_α is left invertible.
(b) *For all $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*
$$x^\alpha x^\beta = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{2.6}$$
where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha+\beta|$ if $p_{\alpha,\beta} \ne 0$.

REMARK 9. (i) A left inverse of $c_{\alpha,\beta}$ will be denoted by $c'_{\alpha,\beta}$. We observe that if $\alpha = 0$ or $\beta = 0$, then $c_{\alpha,\beta} = 1$ and hence $c'_{\alpha,\beta} = 1$.

(ii) Let $\theta,\gamma,\beta \in \mathbb{N}^n$ and $c \in R$. Then we have the following identities:
$$\sigma^\theta(c_{\gamma,\beta})c_{\theta,\gamma+\beta} = c_{\theta,\gamma}c_{\theta+\gamma,\beta}, \qquad \sigma^\theta(\sigma^\gamma(c))c_{\theta,\gamma} = c_{\theta,\gamma}\sigma^{\theta+\gamma}(c).$$
In fact, since $x^\theta(x^\gamma x^\beta) = (x^\theta x^\gamma)x^\beta$, we have
$$x^\theta(c_{\gamma,\beta}x^{\gamma+\beta} + p_{\gamma,\beta}) = (c_{\theta,\gamma}x^{\theta+\gamma} + p_{\theta,\gamma})x^\beta,$$
$$\sigma^\theta(c_{\gamma,\beta})c_{\theta,\gamma+\beta}x^{\theta+\gamma+\beta} + p = c_{\theta,\gamma}c_{\theta+\gamma,\beta}x^{\theta+\gamma+\beta} + q,$$

with $p = 0$ or $\deg(p) < |\theta+\gamma+\beta|$, and $q = 0$ or $\deg(q) < |\theta+\gamma+\beta|$. From this we get the first identity above. For the second, $x^\theta(x^\gamma c) = (x^\theta x^\gamma)c$, and hence

$$x^\theta(\sigma^\gamma(c)x^\gamma + p_{\gamma,c}) = (c_{\theta,\gamma}x^{\theta+\gamma} + p_{\theta,\gamma})c, \quad \sigma^\theta(\sigma^\gamma(c))c_{\theta,\gamma}x^{\theta+\gamma} + p = c_{\theta,\gamma}\sigma^{\theta+\gamma}(c)x^{\theta+\gamma} + q,$$

with $p = 0$ or $\deg(p) < |\theta+\gamma|$, and $q = 0$ or $\deg(q) < |\theta+\gamma|$.

(iii) If A is quasi-commutative, then from the proof of Theorem 8 (see [18]) we conclude that $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and all $\alpha, \beta \in \mathbb{N}^n$. On the other hand, note that the evaluation function at 0, i.e., $A \to R$, $f \in A \mapsto f(0) \in R$, is a surjective ring homomorphism with kernel $\langle x_1, \dots, x_n\rangle$, the two-sided ideal generated by $x_1, \dots, x_n$. Thus, $A/\langle x_1, \dots, x_n\rangle \cong R$.

(iv) If A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

(v) In $\mathrm{Mon}(A)$ we define

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\ \text{or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\ \text{or} \\ x^\alpha \neq x^\beta,\ |\alpha| = |\beta| \text{ but there is } i \text{ with } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1},\ \alpha_i > \beta_i. \end{cases}$$

It is clear that this is a total order on $\mathrm{Mon}(A)$, called the *deglex* order. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Each element $f \in A - \{0\}$ can be represented in a unique way as $f = c_1x^{\alpha_1} + \cdots + c_tx^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leader monomial* of f and we write $\mathrm{lm}(f) := x^{\alpha_1}$; furthermore, c_1 is the *leader coefficient* of f, written $\mathrm{lc}(f) := c_1$; and $c_1x^{\alpha_1}$ is the *leader term* of f, denoted by $\mathrm{lt}(f) := c_1x^{\alpha_1}$. If $f = 0$, we define $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$, and we set $X \succ 0$ for any $X \in \mathrm{Mon}(A)$ (see also Section 6.1). We observe that

$$x^\alpha \succ x^\beta \Rightarrow \mathrm{lm}(x^\gamma x^\alpha x^\lambda) \succ \mathrm{lm}(x^\gamma x^\beta x^\lambda) \quad \text{for all } x^\gamma, x^\lambda \in \mathrm{Mon}(A).$$

Natural and useful results that we will use later are the following properties.

Proposition 10 ([35]). *Let A be a bijective skew PBW extension of a ring R. Then*

(i) *A is a right R-free module with basis* $\mathrm{Mon}(A)$.
(ii) *If R is a domain, then A is a domain.*

Proposition 11 ([35]). *Let A be a skew PBW extension of R. Then there exists a quasi-commutative skew PBW extension A^σ of R in n variables $z_1, \dots, z_n$ defined by*

$$z_ir = c_{i,r}z_i, \quad z_jz_i = c_{i,j}z_iz_j, \quad 1 \leq i, j \leq n,$$

where $c_{i,r}, c_{i,j}$ are the same constants that define A. If A is bijective, then A^σ is also bijective.

Theorem 12 ([35]). *Let A be an arbitrary skew PBW extension of a ring R. Then A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R & \text{if } m = 0, \\ \{f \in A \mid \deg(f) \leq m\} & \text{if } m \geq 1, \end{cases} \tag{2.7}$$

and the corresponding graded ring Gr(A) *is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then* Gr(A) *is a quasi-commutative bijective skew PBW extension of R.*

The next theorem characterizes the quasi-commutative skew PBW extensions.

THEOREM 13 ([35]). *Let A be a quasi-commutative skew PBW extension of a ring R. Then*

(i) *A is isomorphic to an iterated skew polynomial ring of endomorphism type.*
(ii) *If A is bijective, then each endomorphism is bijective.*

THEOREM 14 (Hilbert basis theorem). *Let A be a bijective skew PBW extension of R. If R is a left Noetherian ring, then so is A.*

Proof. We repeat the proof given in [35]. According to Theorem 12, Gr(A) is a quasi-commutative skew PBW extension, and by the hypothesis, Gr(A) is also bijective. By Theorem 13, Gr(A) is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \le i \le n$. This implies that Gr(A) is a left Noetherian ring, and hence A is left Noetherian (see [38, Theorem 1.6.9]). ■

Many other properties of skew PBW extensions have been studied recently; for example, Ore's and Goldie's theorems were proved in [34], prime ideals were investigated in [33], the groups K_i, $i \ge 0$, of algebraic K-theory were computed in [35], etc. We want to conclude this section with two results that estimate the global and Krull dimensions of bijective skew PBW extensions. We denote by lgld(S) the left global dimension of a ring S and by lKdim(S) its left Krull dimension (see [50] and [38]).

THEOREM 15 ([35]). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R. Then*

$$\mathrm{lgld}(R) \le \mathrm{lgld}(A) \le \mathrm{lgld}(R) + n \quad \text{if } \mathrm{lgld}(R) < \infty.$$

If A is quasi-commutative, then

$$\mathrm{lgld}(A) = \mathrm{lgld}(R) + n.$$

In particular, if R is semisimple, then $\mathrm{lgld}(A) = n$.

THEOREM 16 ([35]). *Let A be a bijective skew PBW extension of a left Noetherian ring R. Then*

$$\mathrm{lKdim}(R) \le \mathrm{lKdim}(A) \le \mathrm{lKdim}(R) + n.$$

If A is quasi-commutative, then

$$\mathrm{lKdim}(A) = \mathrm{lKdim}(R) + n.$$

In particular, if $R = K$ is a field, then $\mathrm{lKdim}(A) = n$.

REMARK 17. The right versions of the above three theorems are also true.

2.3. More examples. Many other important and interesting examples of bijective skew PBW extensions were presented and discussed in [35] and [48]. In this section we recall other key examples; some of them will be used to illustrate the algorithms that will be presented later in this paper.

EXAMPLE 18. According to [24], a *diffusion algebra* $\mathcal{D}$ over a field K is generated by $\{D_i, x_i \mid 1 \leq i \leq n\}$ over K with relations

$$x_i x_j = x_j x_i, \quad x_i D_j = D_j x_i, \quad 1 \leq i, j \leq n.$$
$$c_{ij} D_i D_j - c_{ji} D_j D_i = x_j D_i - x_i D_j, \quad i < j, \quad c_{ij}, c_{ji} \in K^*.$$

Thus, $\mathcal{D} \cong \sigma(K[x_1, \dots, x_n])\langle D_1, \dots, D_n\rangle$ is a bijective non-quasi-commutative skew PBW extension of $K[x_1, \dots, x_n]$. Observe that $\mathcal{D}$ is not a PBW extension or an iterated skew polynomial ring of bijective type (see Example 5).

EXAMPLE 19. Viktor Levandovskyy [29] has defined G-algebras and constructed the theory of Gröbner bases for them (see Section 6 for the Gröbner theory of bijective skew PBW extensions). Let K be a field. A K-algebra A is called a *G-algebra* if $K \subset Z(A)$ (the center of A) and A is generated by a finite set $\{x_1, \dots, x_n\}$ of elements that satisfy the following conditions:

(a) The collection of standard monomials of A is a K-basis of A.
(b) $x_j x_i = c_{ij} x_i x_j + d_{ij}$ for $1 \leq i < j \leq n$, with $c_{ij} \in K - \{0\}$ and $d_{ij} \in A$.
(c) There exists a total order $<_A$ on $\mathrm{Mon}(A)$ such that for $i < j$, $\mathrm{lm}(d_{ij}) <_A x_i x_j$.

According to this definition, G-algebras look more general than skew PBW extensions since d_{ij} is not necessarily linear; however, in G-algebras the coefficients of polynomials are in a field and they commute with the variables $x_1, \dots, x_n$. Note that the class of G-algebras does not include the class of skew PBW extensions over fields. For example, consider the K-algebra $\mathcal{A}$ generated by x, y, z subject to the relations

$$yx - q_2 xy = x, \quad zx - q_1 xz = z, \quad zy = yz, \quad q_1, q_2 \in K.$$

Thus, $\mathcal{A}$ is not a G-algebra in the sense of [29]. Note that if $q_1, q_2 \neq 0$, then $\mathcal{A} \cong \sigma(K)\langle x, y, z\rangle$ is a bijective non-quasi-commutative skew PBW extension of K.

EXAMPLE 20. *Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2, K))$.* Edward Witten introduced and studied a 7-parameter deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, K))$ over the field K, depending on a 7-tuple $\underline{\xi} = (\xi_1, \dots, \xi_7)$ of parameters from K and subject to the relations

$$xz - \xi_1 zx = \xi_2 x, \quad zy - \xi_3 yz = \xi_4 y, \quad yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z.$$

The resulting algebra is denoted by $W(\underline{\xi})$ and it is assumed that $\xi_1\xi_3\xi_5 \neq 0$ (see [29]). Note that if $\xi_2\xi_4\xi_6 \neq 0$, then $W(\underline{\xi}) \cong \sigma(\sigma(K[x])\langle z\rangle)\langle y\rangle$ is a bijective non-quasi-commutative skew PBW extension of $\sigma(K[x])\langle z\rangle$, and in turn, $\sigma(K[x])\langle z\rangle$ is a bijective non-quasi-commutative skew PBW extension of $K[x]$. In [29] it is proved that $W(\underline{\xi})$ is a G-algebra only when $\xi_1 = \xi_3$ and $\xi_2 = \xi_4$. Thus, in general, $W(\underline{\xi})$ is a skew PBW extension but is not a G-algebra.

EXAMPLE 21. In [6] (see also [7]) Bueso, Gómez-Torrecillas and Lobillo defined a type of rings and algebras called *left PBW rings*. Many of the rings and algebras considered in [35] (see also [48]) can also be interpreted as left PBW rings. Next we present an example of a skew PBW extension that is not a left PBW ring. Let K be a field; for any $0 \neq q \in K$, let $\mathcal{R}$ be an algebra generated by the variables a, b, c, d subject to the relations

$$\begin{array}{llll} ba = qab, & db = qbd, & ca = qac, & dc = qcd, \\ bc = \mu cb, & ad - da = (q^{-1} - q)bc, & & \end{array}$$

for some $\mu \in K$. Then $\mathcal{R}$ is not a left PBW ring unless $\mu = 1$ (see [7]). Thus, for $\mu \neq 1$, $\mathcal{R} \cong \sigma(K[b])\langle a, c, d\rangle$ is a bijective non-quasi-commutative skew PBW extension of $K[b]$ that is not a left PBW ring.

3. Finitely generated projective modules

One of the main purposes of the present work is to study finitely generated projective modules over skew PBW extensions. Recall that if S is a ring and P is a module over S, then P is said to be *projective* if there exists an S-module P' and a free S-module F such that $P \oplus P' \cong F$; in particular, P is a *finitely generated projective module* if there exists $r \geq 0$ such that $P \oplus P' \cong S^r$. Note that any free module is projective (the null module $0 = S^0$ is free by definition). Given a ring S, one of the classical questions in homological algebra is to determine if any finitely generated projective S-module is free. It is well known that this is the case when S is a principal ideal domain, or when S is local (see a matrix-constructive proof of this fact below, Proposition 27), or when $S = R[x_1, \ldots, x_n]$ with R a principal ideal domain (Quillen–Suslin theorem, see [27]). For skew PBW extensions, in general, the answer to this question is negative, as the next trivial example shows [27]: if K is a division ring, then $S := K[x, y]$ has a module P such that $P \oplus S \cong S^2$, but P is not free. Thus, instead we can ask if for skew PBW extensions *Serre's theorem* is true, i.e., if any finitely generated projective module P is *stably free*, meaning that there exist $r, s \geq 0$ such that $P \oplus S^s \cong S^r$ (see Definition 36). We will say that a ring S is *PSF* if any finitely generated projective S-module is stably free (Definition 54).

3.1. Serre's theorem. Next we will prove Serre's theorem for bijective skew PBW extensions (see also [35]). Some preliminaries are needed.

PROPOSITION 22 ([38, Proposition 7.7.4]). *Let S be a filtered ring. If* $\mathrm{Gr}(S)$ *is left regular, then so is S.*

PROPOSITION 23 ([38, Theorem 7.7.5]). *If R is a left regular and left Noetherian ring and σ is an automorphism, then $R[x;\sigma]$ is left regular.*

PROPOSITION 24 ([38, Theorem 12.3.2]). *If B is a filtered ring with filtration $\{B_p\}_{p\geq 0}$ such that* $\mathrm{Gr}(B)$ *is left Noetherian, left regular, and flat as a right B_0-module, then B is PSF when B_0 is PSF.*

THEOREM 25. *Let A be a bijective skew PBW extension of a ring R. If R is a left regular and left Noetherian ring, then A is left regular.*

Proof. Theorems 12 and 13 say that $\mathrm{Gr}(A)$ is isomorphic to an iterated skew polynomial ring of automorphism type with coefficients in R; hence the result follows from Propositions 23 and 22. ∎

Theorem 26 (Serre's theorem). *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R such that R is left Noetherian, left regular and PSF. Then A is PSF.*

Proof. By Theorem 12, A is filtered, $A_0 = R$, and $\mathrm{Gr}(A)$ is a quasi-commutative bijective skew PBW extension of R; Theorem 14 says that $\mathrm{Gr}(A)$ is left Noetherian, and Theorem 25 implies that $\mathrm{Gr}(A)$ is left regular. Moreover, $\mathrm{Gr}(A)$ is flat as a right R-module (see Proposition 10); then assuming that R is PSF we deduce from Proposition 24 that A is PSF. ■

From Serre's theorem we conclude that the study of finitely generated projective modules over bijective skew PBW extensions is reduced to the investigation of stably free modules (of course under certain conditions on the ring R of coefficients). In a more general framework, and as preparatory material for the next sections, we are interested in studying when stably free modules over non-commutative rings are free. A well known result in this direction is Stafford's theorem that we will prove later. Many characterizations of stably free modules will also be presented. There are different techniques to investigate stably free modules; one of the purposes of the present work is to combine homological and matrix-constructive methods.

3.2. $\mathcal{RC}$ and $\mathcal{IBN}$ rings. In this section we gather some notation and well known elementary properties of linear algebra for left modules over non-commutative rings. All rings are non-commutative and modules will be left modules; S will represent an arbitrary non-commutative ring; S^r is the left S-module of columns of size $r \times 1$; if $S^s \xrightarrow{f} S^r$ is an S-homomorphism, then there is a matrix associated to f in the canonical bases of S^r and S^s, denoted $F := m(f)$, and arranged by columns, i.e., $F \in M_{r\times s}(S)$. In fact, if f is given by

$$S^s \xrightarrow{f} S^r, \qquad \mathbf{e}_j \mapsto \mathbf{f}_j,$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ is the canonical basis of S^s, then f can be represented by a matrix, i.e., if $\mathbf{f}_j := [f_{1j} \ \cdots \ f_{rj}]^T$, then the matrix of f in the canonical bases of S^s and S^r is

$$F := [\mathbf{f}_1 \ \cdots \ \mathbf{f}_s] = \begin{bmatrix} f_{11} & \cdots & f_{1s} \\ \vdots & & \vdots \\ f_{r1} & \cdots & f_{rs} \end{bmatrix} \in M_{r\times s}(S).$$

Note that $\mathrm{Im}(f)$ is the column module of F, i.e., the left S-module generated by the columns of F, denoted by $\langle F\rangle$:

$$\mathrm{Im}(f) = \langle f(\mathbf{e}_1), \dots, f(\mathbf{e}_s)\rangle = \langle \mathbf{f}_1, \dots, \mathbf{f}_s\rangle = \langle F\rangle.$$

Moreover, observe that if $\mathbf{a} := (a_1, \dots, a_s)^T \in S^s$, then

$$f(\mathbf{a}) = (\mathbf{a}^T F^T)^T. \tag{3.1}$$

In fact,

$$f(\mathbf{a}) = a_1 f(\mathbf{e}_1) + \cdots + a_s f(\mathbf{e}_s) = a_1\mathbf{f}_1 + \cdots + a_s\mathbf{f}_s$$

$$= a_1 \begin{bmatrix} f_{11} \\ \vdots \\ f_{r1} \end{bmatrix} + \cdots + a_s \begin{bmatrix} f_{1s} \\ \vdots \\ f_{rs} \end{bmatrix} = \begin{bmatrix} a_1 f_{11} + \cdots + a_s f_{1s} \\ \vdots \\ a_1 f_{r1} + \cdots + a_s f_{rs} \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_1 & \cdots & a_s \end{bmatrix} \begin{bmatrix} f_{11} & \cdots & f_{r1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{rs} \end{bmatrix} \right)^T = (\mathbf{a}^T F^T)^T.$$

Observe that the function $m : \mathrm{Hom}_S(S^s, S^r) \to M_{r\times s}(S)$ is bijective; moreover, if $S^r \xrightarrow{g} S^p$ is a homomorphism, then the matrix of gf in the canonical bases is $m(gf) = (F^T G^T)^T$. Thus, $f : S^r \to S^r$ is an isomorphism if and only if $F^T \in \mathrm{GL}_r(S)$. Finally, let $C \in M_r(S)$; the columns of C form a basis of S^r if and only if $C^T \in \mathrm{GL}_r(S)$.

We also recall that

$$\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}) := \{\mathbf{a} := (a_1, \ldots, a_s)^T \in S^s \mid a_1\mathbf{f}_1 + \cdots + a_s\mathbf{f}_s = \mathbf{0}\}.$$

Note that

$$\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}) = \ker(f), \tag{3.2}$$

but $\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}) \neq \ker(F)$ since we have

$$\mathbf{a} \in \mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}) \;\Leftrightarrow\; \mathbf{a}^T F^T = \mathbf{0}. \tag{3.3}$$

A matrix characterization of finitely generated (f.g.) projective modules can be formulated in the following way.

Proposition 27. *Let S be an arbitrary ring and M an S-module. Then M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F^T is idempotent and $M = \langle F \rangle$.*

Proof. ($\Rightarrow$): If $M = 0$, then $F = 0$. Let $M \neq 0$. There exist $s \geq 1$ and M' such that $S^s = M \oplus M'$; let $f : S^s \to S^s$ be the projection on M and F the matrix of f in the canonical basis of S^s. Then $f^2 = f$ and $(F^T F^T)^T = F$, so $F^T F^T = F^T$; note that $M = \mathrm{Im}(f) = \langle F \rangle$.

($\Leftarrow$): Let $f : S^s \to S^s$ be the homomorphism defined by F (see (3.1)). From $F^T F^T = F^T$ we get $f^2 = f$; moreover, since $M = \langle F \rangle$, we have $\mathrm{Im}(f) = M$ and hence M is a direct summand of S^s, i.e., M is f.g. projective (observe that the complement M' of M is $\ker(f)$ and f is the projection on M). ∎

Remark 28. (i) When S is commutative, or when we consider right modules instead of left modules, (3.1) says that $f(\mathbf{a}) = F\mathbf{a}$. Moreover, in such cases $\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_s\}) = \ker(F)$ and the matrix of a composite homomorphism gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^r \to S^r$ is an isomorphism if and only if $F \in \mathrm{GL}_r(S)$; moreover, $C \in \mathrm{GL}_r(S)$ if and only if its columns form a basis of S^r. In addition, Proposition 27 says that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F \rangle$.

(ii) If the matrices of homomorphisms of left modules are arranged by rows instead of by columns, i.e., if $S^{1\times s}$ is the left free module of row vectors of length s and the matrix of the homomorphism $S^{1\times s} \xrightarrow{f} S^{1\times r}$ is defined by

$$F' = \begin{bmatrix} f'_{11} & \cdots & f'_{1r} \\ \vdots & & \vdots \\ f'_{s1} & \cdots & f'_{sr} \end{bmatrix} := \begin{bmatrix} f_{11} & \cdots & f_{r1} \\ \vdots & & \vdots \\ f_{1s} & \cdots & f_{rs} \end{bmatrix} \in M_{s\times r}(S),$$

then

$$f(a_1, \ldots, a_s) = (a_1, \ldots, a_s)F', \tag{3.4}$$

i.e., $f(\mathbf{a}^T) = \mathbf{a}^T F^T$. Thus, the values given by (3.4) and (3.1) agree since $F' = F^T$. Moreover, the composite homomorphism gf means that g acts first, followed by f, and hence the matrix of gf is given by $m(gf) = m(g)m(f)$. Note that $f : S^{1\times r} \to S^{1\times r}$ is an isomorphism if and only if $m(f) \in \mathrm{GL}_r(S)$; moreover, $C \in \mathrm{GL}_r(S)$ if and only if its rows form a basis of $S^{1\times r}$. This left-row notation is used in [14]. Observe that with this notation, the proof of Proposition 27 says that M is a f.g. projective S-module if and only if there exists a square matrix F over S such that F is idempotent and $M = \langle F\rangle$, but in this case $\langle F\rangle$ represents the module generated by the rows of F. Note that Proposition 27 could be formulated in this way: in fact, the set of idempotents matrices of $M_s(S)$ coincides with the set $\{F^T \mid F \in M_s(S),\ F^T \text{ idempotent}\}$.

Definition 29 ([27]). Let S be a ring.

(i) S satisfies the *rank condition* ($\mathcal{RC}$) if for any integers $r, s \geq 1$, given an epimorphism $S^r \xrightarrow{f} S^s$, we have $r \geq s$.

(ii) S is an $\mathcal{IBN}$ ring (*Invariant Basis Number*) if for any integers $r, s \geq 1$, $S^r \cong S^s$ if and only if $r = s$.

Proposition 30 ([18]). *Let S be a ring.*

(i) *S is $\mathcal{RC}$ if and only if, for any matrix $F \in M_{s\times r}(S)$,*

if F has a right inverse then $r \geq s$.

(ii) *S is $\mathcal{RC}$ if and only if, for any matrix $F \in M_{s\times r}(S)$,*

if F has a left inverse then $s \geq r$.

Proposition 31. *$\mathcal{RC} \Rightarrow \mathcal{IBN}$.*

Proof. Let $S^r \xrightarrow{f} S^s$ be an isomorphism. Then f is an epimorphism, and hence $r \geq s$; considering f^{-1} we get $s \geq r$. ∎

Example 32. Most rings are $\mathcal{RC}$, and hence $\mathcal{IBN}$.

(i) Any field K is $\mathcal{RC}$: Let $K^r \xrightarrow{f} K^s$ be an epimorphism. Then $\dim(K^r) = r = \dim(\ker(f)) + s$, so $r \geq s$.

(ii) Let S and T be rings and let $S \xrightarrow{f} T$ be a ring homomorphism. If T is an $\mathcal{RC}$ ring then so is S. In fact, T is a right S-module, $t \cdot s := tf(s)$. Suppose that $S^r \xrightarrow{f} S^s$ is an epimorphism. Then $T \otimes_S S^r \xrightarrow{i_T \otimes f} T \otimes_S S^s$ is also an epimorphism of left T-modules, i.e., we have an epimorphism $T^r \to T^s$, so $r \geq s$ (a similar result and proof is valid for the $\mathcal{IBN}$ property).

(iii) We can apply the property proved in (ii) in many situations. For example, any commutative ring S is $\mathcal{RC}$: Let J be a maximal ideal of S. Then the canonical homomorphism $S \to S/J$ shows that S is $\mathcal{RC}$ since S/J is a field.

(iv) Any ring S with finite uniform dimension (Goldie dimension, see [38] and [22]) is $\mathcal{RC}$: In fact, suppose that $S^r \xrightarrow{f} S^s$ is an epimorphism. Then $S^r \cong S^s \oplus M$ and hence $r\,\mathrm{udim}(S) = s\,\mathrm{udim}(S) + \mathrm{udim}(M)$, so $r \geq s$.

(v) Since any left Noetherian ring S has finite uniform dimension, S is $\mathcal{RC}$. In particular, any left Artinian ring is $\mathcal{RC}$.

Since the objects studied in the present monograph are the skew PBW extensions, it is natural to investigate the $\mathcal{IBN}$ and $\mathcal{RC}$ properties for these rings.

PROPOSITION 33. *Let B be a filtered ring. If $\mathrm{Gr}(B)$ is $\mathcal{RC}$ ($\mathcal{IBN}$), then B is $\mathcal{RC}$ ($\mathcal{IBN}$).*

Proof. Let $\{B_p\}_{p\geq 0}$ be the filtration of B and $f : B^r \to B^s$ an epimorphism. For $M := B^r$ we consider the standard positive filtration given by

$$F_0(M) := B_0 \cdot e_1 + \cdots + B_0 \cdot e_r, \quad F_p(M) := B_pF_0(M), \quad p \geq 1,$$

where $\{e_i\}_{i=1}^r$ is the canonical basis of B^r. Let $e_i' := f(e_i)$. Then B^s is generated by $\{e_i'\}_{i=1}^r$ and $N := B^s$ has standard positive filtration given by

$$F_0(N) := B_0 \cdot e_1' + \cdots + B_0 \cdot e_r', \quad F_p(N) := B_pF_0(N), \quad p \geq 1.$$

Note that f is filtered and strict: In fact, $f(F_p(M)) = B_pf(F_0(M)) = B_p(B_0 \cdot f(e_1) + \cdots + B_0 \cdot f(e_r)) = B_p(B_0 \cdot e_1' + \cdots + B_0 \cdot e_r') = B_pF_0(N) = F_p(N)$. This implies that $\mathrm{Gr}(M) \xrightarrow{\mathrm{Gr}(f)} \mathrm{Gr}(N)$ is surjective. If we prove that $\mathrm{Gr}(M)$ and $\mathrm{Gr}(N)$ are free over $\mathrm{Gr}(B)$ with bases of r and s elements, respectively, then from the hypothesis we conclude that $r \geq s$ and hence B is $\mathcal{RC}$.

Since every e_i belongs to $F_0(M)$ and since $F_p(M) = \sum_{i=1}^r \bigoplus B_p \cdot e_i$, M is filtered-free with filtered-basis $\{e_i\}_{i=1}^r$, so $\mathrm{Gr}(M)$ is graded-free with graded-basis $\{\overline{e_i}\}_{i=1}^r$, $\overline{e_i} := e_i + F_{-1}(M) = e_i$ (recall that by definition of positive filtration, $F_{-1}(M) := 0$). For $\mathrm{Gr}(N)$ note that N is also filtered-free with respect to the filtration $\{F_p(N)\}_{p\geq 0}$ given above: Indeed, we will show next that the canonical basis $\{f_j\}_{j=1}^s$ of N is a filtered basis. If $f_j = x_{j1} \cdot e_1' + \cdots + x_{jr} \cdot e_r'$, with $x_{ji} \in B_{p_{ij}}$, let $p := \max\{p_{ij}\}$, $1 \leq i \leq r$, $1 \leq j \leq s$. Then $f_j \in F_p(N)$, moreover, for every q, $B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s \subseteq B_{q-p}F_p(N) \subseteq F_q(N)$ (recall that for $k < 0$, $B_k = 0$). In turn, let $x \in F_q(N)$. Then $x = b_1 \cdot f_1 + \cdots + b_s \cdot f_s$ and in $\mathrm{Gr}(N)$ we have $\overline{x} \in \mathrm{Gr}(N)_q$, $\overline{x} = \overline{b_1} \cdot \overline{f_1} + \cdots + \overline{b_s} \cdot \overline{f_s}$. If $b_j \in B_{u_j}$, let $u := \max\{u_j\}$. Then $\overline{b_j} \cdot \overline{f_j} \in \mathrm{Gr}(N)_{u+p}$, so $q = u + p$, i.e., $u = q - p$ and hence $x \in B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s$. Thus, we have proved that $B_{q-p} \cdot f_1 \oplus \cdots \oplus B_{q-p} \cdot f_s = F_q(N)$ for every q, and consequently $\{f_j\}_{j=1}^s$ is a filtered basis of N. From this we conclude that $\mathrm{Gr}(N)$ is graded-free with graded-basis $\{\overline{f_j}\}_{j=1}^s$, $\overline{f_j} := f_j + F_{p-1}(N)$.

We can repeat this proof for the $\mathcal{IBN}$ property but assuming that f is an isomorphism. ∎

COROLLARY 34. *Let A be a skew PBW extension of a ring R. Then A is $\mathcal{RC}$ [$\mathcal{IBN}$] if and only if R is $\mathcal{RC}$ [$\mathcal{IBN}$].*

Proof. We only consider the proof for $\mathcal{RC}$, the case $\mathcal{IBN}$ is completely analogous.

($\Rightarrow$): Since $R \hookrightarrow A$, Example 32 shows that if A is $\mathcal{RC}$, then R is $\mathcal{RC}$.

($\Leftarrow$): We consider first the skew polynomial ring $R[x;\sigma]$ of endomorphism type. Then $R[x;\sigma] \to R$ given by $p(x) \mapsto p(0)$ is a ring homomorphism, so $R[x;\sigma]$ is $\mathcal{RC}$ since R is $\mathcal{RC}$. By Theorem 13, $\mathrm{Gr}(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$, so $\mathrm{Gr}(A)$ is $\mathcal{RC}$. It remains to apply Proposition 33. ∎

Remark 35. (i) The condition $\mathcal{IBN}$ for rings is independent of the side we are considering the modules. In fact, if we define left $\mathcal{IBN}$ rings and right $\mathcal{IBN}$ rings, depending on left or right free S-modules, then S is left $\mathcal{IBN}$ if and only if S is right $\mathcal{IBN}$ (see [32]). The same is true for the $\mathcal{RC}$ property.

(ii) *From now on we will assume that all rings considered are* $\mathcal{RC}$.

3.3. Characterizations of stably free modules

Definition 36. Let M be an S-module and $t \geq 0$ an integer. Then M is *stably free* of rank $t \geq 0$ if there exists an integer $s \geq 0$ such that $S^{s+t} \cong S^s \oplus M$.

The rank of M is denoted by $\mathrm{rank}(M)$. Note that any stably free module M is finitely generated and projective. Moreover, as we will show in the next proposition, $\mathrm{rank}(M)$ is well defined, i.e., $\mathrm{rank}(M)$ is unique for M.

Proposition 37. *Let $t, t', s, s' \geq 0$ be integers such that $S^{s+t} \cong S^s \oplus M$ and $S^{s'+t'} \cong S^{s'} \oplus M$. Then $t' = t$.*

Proof. We have $S^{s'} \oplus S^{s+t} \cong S^{s'} \oplus S^s \oplus M$ and $S^s \oplus S^{s'+t'} \cong S^s \oplus S^{s'} \oplus M$; then since S is an $\mathcal{IBN}$ ring, we have $s' + s + t = s + s' + t'$, and hence $t' = t$. ∎

Corollary 38. *M is stably free of rank $t \geq 0$ if and only if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$, with $r \geq s$ and $t = r - s$.*

Proof. If M is stably free of rank t, then $S^{s+t} \cong S^s \oplus M$ for some integers $s, t \geq 0$. Taking $r := s + t$ we get the result. Conversely, if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$ with $r \geq s$, then $S^{s+r-s} \cong S^s \oplus M$, i.e., M is stably free of rank $r - s$. ∎

Proposition 39. *Let M be an S-module and let $r, s \geq 0$ be integers such that $S^r \cong S^s \oplus M$. Then $r \geq s$.*

Proof. The canonical projection $S^r \to S^s$ is an epimorphism, but since we are assuming that S is $\mathcal{RC}$, we have $r \geq s$. ∎

Corollary 40. *M is stably free if and only if there exist integers $r, s \geq 0$ such that $S^r \cong S^s \oplus M$.*

Proof. This is a direct consequence of Corollary 38 and Proposition 39. ∎

Next we present other characterizations of stably free modules over non-commutative rings.

Theorem 41. *Let M be an S-module. Then the following conditions are equivalent:*

(i) *M is stably free.*

(ii) *M is projective and has a finite free resolution*

$$0 \to S^{t_k} \xrightarrow{f_k} S^{t_{k-1}} \xrightarrow{f_{k-1}} \cdots \xrightarrow{f_1} S^{t_0} \xrightarrow{f_0} M \to 0.$$

In this case

$$\operatorname{rank}(M) = \sum_{i=0}^{k} (-1)^i t_i. \tag{3.5}$$

(iii) *M is isomorphic to the kernel of an epimorphism of free modules:* $M \cong \ker(\pi)$, $\pi : S^r \to S^s$.

(iv) *M is projective and has a finite presentation* $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, *where* $\ker(f_0)$ *is stably free.*

(v) *M has a finite presentation* $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$, *where* f_1 *has a left inverse.*

Proof. See [28, Chapter 21], [36], and [38, Chapter 11]. ■

Definition 42. A finite presentation

$$S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0 \tag{3.6}$$

of an S-module M is *minimal* if f_1 has a left inverse.

Corollary 43. *Let M be an S-module. Then M is stably free if and only if M has a minimal presentation.*

Proof. This is Theorem 41(i)⇔(v). ■

Unimodular matrices are closely related to stably free modules.

Definition 44. Let F be a matrix over S of size $r \times s$.

(i) Let $r \geq s$. Then F is *unimodular* if F has a left inverse.
(ii) Let $s \geq r$. Then F is *unimodular* if F has a right inverse.

The set of unimodular column matrices of size $r \times 1$ is denoted by $\mathrm{Um}_c(r, S)$, and $\mathrm{Um}_r(s, S)$ is the set of unimodular row matrices of size $1 \times s$.

Remark 45. Note that a column matrix is unimodular if and only if the left ideal generated by its entries coincides with S, and a row matrix is unimodular if and only if the right ideal generated by its entries is S.

We can add some other characterizations of stably free modules (cf. [45, Lemma 16]).

Corollary 46 ([18]). *Let M be an S-module. Then the following conditions are equivalent:*

(i) *M is stably free.*
(ii) *M is projective and has a finite system of generators* $\mathbf{f}_1, \ldots, \mathbf{f}_r$ *such that* $\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_r\})$ *is the module generated by the columns of a matrix* F_1 *of size* $r \times s$ *such that* F_1^T *has a right inverse.*
(iii) *M is projective and has a finite system of generators* $\mathbf{f}_1, \ldots, \mathbf{f}_r$ *such that* $\mathrm{Syz}(\{\mathbf{f}_1, \ldots, \mathbf{f}_r\})$ *is the module generated by the columns of a matrix* F_1 *of size* $r \times s$ *such that* F_1^T *is unimodular.*

Another interesting result about stably free modules over arbitrary $\mathcal{RC}$ rings is presented next. For this, we recall that if M is a finitely presented left S-module with a presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ and F_1 is the matrix of f_1 in the canonical bases, then the right S-module M^T defined by $M^T := S^s/\mathrm{Im}(f_1^T)$, where $f_1^T : S^r \to S^s$ is the homomorphism of right free S-modules induced by the matrix F_1^T, is called the *transposed module* of M. Thus, M^T is given by the presentation $S^r \xrightarrow{f_1^T} S^s \to M^T \to 0$.

THEOREM 47 ([9]). *Let M be an S-module with an exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. Then $M^T \cong \mathrm{Ext}^1_S(M, S)$ and the following conditions are equivalent:*

(i) *M is stably free.*
(ii) *M is projective.*
(iii) *$M^T = 0$.*
(iv) *F_1^T has a right inverse.*
(v) *f_1 has a left inverse.*

3.4. Stafford's theorem: a constructive proof. A well known result due Stafford says that any left ideal of the Weyl algebras $D := A_n(K)$ or $B_n(K)$, with $\mathrm{char}(K) = 0$, is generated by two elements (see [51] and [45]). From Stafford's theorem it follows that any stably free left module M over D with $\mathrm{rank}(M) \geq 2$ is free. [45] gives a constructive proof of this result that we want to study for arbitrary $\mathcal{RC}$ rings. Actually, we will consider the generalization given in [45]; this result says that any stably free left S-module M with $\mathrm{rank}(M) \geq \mathrm{sr}(S)$ is free, where $\mathrm{sr}(S)$ denotes the stable rank of the ring S. Our proof have been adapted from [45], however we do not need the involution of ring S used in [45] since we are using left notation for modules and column representation for homomorphism. This could justify our special left-column notation. In order to apply the results to bijective skew PBW extensions we will estimate the stable rank of such extensions.

DEFINITION 48. Let S be a ring and $\mathbf{v} := [v_1 \ \cdots \ v_r]^T \in \mathrm{Um}_c(r, S)$ a unimodular column vector. Then $\mathbf{v}$ is called *stable* (reducible) if there exist $a_1, \ldots, a_{r-1} \in S$ such that $\mathbf{v}' := [v_1 + a_1 v_r \ \ldots \ v_{r-1} + a_{r-1} v_r]^T$ is unimodular. We say that the *left stable rank* of S is $d \geq 1$, denoted $\mathrm{sr}(S) = d$, if d is the least positive integer such that every unimodular column vector of length $d+1$ is stable. We say that $\mathrm{sr}(S) = \infty$ if for every $d \geq 1$ there exists a non-stable unimodular column vector of length $d + 1$.

REMARK 49. The right stable rank of S is defined in a similar way, and the two ranks coincide. We now list some well known properties of the stable rank (see [2], [3], [8], [38], [45], [51], [52], [53], or [21]).

PROPOSITION 50 ([45]). *Let S be a ring and $\mathbf{v} := [v_1 \ \cdots \ v_r]^T$ a unimodular stable column vector over S. Then there exists $U \in E_r(S)$ such that $U\mathbf{v} = \mathbf{e}_1$.*

Next we present a lemma that enables one to check when a stably free module is free.

LEMMA 51. *Let S be a ring and M a stably free S-module given by a minimal presentation $S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$. Let $g_1 : S^r \to S^s$ be such that $g_1 f_1 = i_{S^s}$. Then the following conditions are equivalent:*

(i) *M is free of dimension $r-s$.*

(ii) *There exists a matrix $U \in \mathrm{GL}_r(S)$ such that $UG_1^T = \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right]$, where G_1 is the matrix of g_1 in the canonical bases. In that case, the last $r-s$ columns of U^T form a basis for M. Moreover, the first s columns of U^T form the matrix F_1 of f_1 in the canonical bases.*

(iii) *There exists a matrix $V \in \mathrm{GL}_r(S)$ such that G_1^T coincides with the first s columns of V, i.e., G_1^T can be completed to an invertible matrix V of $\mathrm{GL}_r(S)$.*

Proof. By the hypothesis, the exact sequence $0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0$ splits, so F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$, i.e., $F_1^T G_1^T = I_s$. Moreover, there exists $g_0 : M \to S^r$ such that $f_0 g_0 = i_M$. From this we also get the split sequence $0 \to M \xrightarrow{g_0} S^r \xrightarrow{g_1} S^s \to 0$. Note that $M \cong \ker(g_1)$.

(i)⇒(ii): We have $S^r = \ker(g_1) \oplus \mathrm{Im}(f_1)$; by the hypothesis, $\ker(g_1)$ is free. If $s = r$ then $\ker(g_1) = 0$ and hence f_1 is an isomorphism, so $f_1 g_1 = i_{S^s}$, i.e., $G_1^T F_1^T = I_s$. Thus, we can take $U := F_1^T$.

Let $r > s$. If $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ is the canonical basis of S^s, then $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ is a basis of $\mathrm{Im}(f_1)$ with $\mathbf{u}_i := f_1(\mathbf{e}_i)$, $1 \le i \le s$; let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis of $\ker(g_1)$ with $p = r-s$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{u}_1, \dots, \mathbf{u}_s\}$ is a basis of S^r. We define $S^r \xrightarrow{h} S^r$ by $h(\mathbf{e}_i) := \mathbf{u}_i$ for $1 \le i \le s$, and $h(\mathbf{e}_{s+j}) = \mathbf{v}_j$ for $1 \le j \le p$. Clearly h is bijective; moreover, $g_1 h(\mathbf{e}_i) = g_1(\mathbf{u}_i) = g_1 f_1(\mathbf{e}_i) = \mathbf{e}_i$ and $g_1 h(\mathbf{e}_{s+j}) = g_1(\mathbf{v}_j) = \mathbf{0}$, i.e., $H^T G_1^T = \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right]$. Let $U := H^T$. Observe that the last p columns of U^T form a basis of $\ker(g_1) \cong M$ and the first s columns of U^T form F_1.

(ii)⇒(i): Let $U_{(k)}$ be the kth row of U. Then

$$UG_1^T = [U_{(1)} \ \cdots \ U_{(s)} \ \cdots \ U_{(r)}]^T G_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix},$$

so $U_{(i)} G_1^T = \mathbf{e}_i^T$, $1 \le i \le s$, $U_{(s+j)} G_1^T = \mathbf{0}$, $1 \le j \le p$, with $p := r - s$. This means that $(U_{(s+j)})^T \in \ker(g_1)$ and hence $\langle (U_{(s+j)})^T \mid 1 \le j \le p \rangle \subseteq \ker(g_1)$. On the other hand, let $\mathbf{c} \in \ker(g_1) \subseteq S^r$; then $\mathbf{c}^T G_1^T = \mathbf{0}$ and $\mathbf{c}^T U^{-1} U G_1^T = \mathbf{0}$, thus $\mathbf{c}^T U^{-1} \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right] = \mathbf{0}$ and hence $(\mathbf{c}^T U^{-1})^T \in \ker(l)$, where $l : S^r \to S^s$ is the homomorphism with matrix $\begin{bmatrix} I_s & 0 \end{bmatrix}$. Let $\mathbf{d} = [d_1, \dots, d_r]^T \in \ker(l)$. Then $[d_1, \dots, d_r] \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right] = \mathbf{0}$ and from this we conclude that $d_1 = \cdots = d_s = 0$, i.e., $\ker(l) = \langle \mathbf{e}_{s+1}, \dots, \mathbf{e}_{s+p} \rangle$. From $(\mathbf{c}^T U^{-1})^T \in \ker(l)$ we get $(\mathbf{c}^T U^{-1})^T = a_1 \cdot \mathbf{e}_{s+1} + \cdots + a_p \cdot \mathbf{e}_{s+p}$, so $\mathbf{c}^T U^{-1} = (a_1 \cdot \mathbf{e}_{s+1} + \cdots + a_p \cdot \mathbf{e}_{s+p})^T$, i.e., $\mathbf{c}^T = (a_1 \cdot \mathbf{e}_{s+1} + \cdots + a_p \cdot \mathbf{e}_{s+p})^T U$, and from this we get $\mathbf{c} \in \langle (U_{(s+j)})^T \mid 1 \le j \le p \rangle$. This proves that $\ker(g_1) = \langle (U_{(s+j)})^T \mid 1 \le j \le p \rangle$; but since U is invertible, $\ker(g_1)$ is free of dimension p. We have also proved that the last p columns of U^T form a basis for $\ker(g_1) \cong M$.

(ii)⇔(iii): $UG_1^T = \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right]$ if and only if $G_1^T = U^{-1} \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right]$. But the first s columns of $U^{-1} \left[\begin{smallmatrix} I_s \\ 0 \end{smallmatrix}\right]$ coincide with the first s columns of U^{-1}; taking $V := U^{-1}$ we get the result. ∎

THEOREM 52. *Let S be a ring. Then any stably free S-module M with* $\mathrm{rank}(M) \ge \mathrm{sr}(S)$ *is free of dimension* $\mathrm{rank}(M)$.

Proof. Since M is stably free, it has a minimal presentation, and hence it is given by an exact sequence

$$0 \to S^s \xrightarrow{f_1} S^r \xrightarrow{f_0} M \to 0;$$

moreover, note that $\operatorname{rank}(M) = r - s$. Since this sequence splits, F_1^T admits a right inverse G_1^T, where F_1 is the matrix of f_1 in the canonical bases and G_1 is the matrix of $g_1 : S^r \to S^s$, with $g_1 f_1 = i_{S^s}$. The idea of the proof is to find a matrix $U \in \mathrm{GL}_r(S)$ such that $UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}$ and then apply Lemma 51.

We have $F_1^T G_1^T = I_s$, and from this we find that the first column $\mathbf{g}_1$ of G_1^T is unimodular. But since $r > r - s \geq \operatorname{sr}(S)$, it follows that $\mathbf{g}_1$ is stable, and by Proposition 50, there exists $U_1 \in E_r(S)$ such that $U_1\mathbf{g}_1 = \mathbf{e}_1$. If $s = 1$, we finish since $G_1^T = \mathbf{g}_1$.

Let $s \geq 2$. We have

$$U_1 G_1^T = \begin{bmatrix} 1 & * \\ 0 & F_2 \end{bmatrix}, \qquad F_2 \in M_{(r-1)\times(s-1)}(S).$$

Note that $U_1 G_1^T$ has a left inverse (for instance $F_1^T U_1^{-1}$), and the form of this left inverse is

$$L = \begin{bmatrix} 1 & * \\ 0 & L_2 \end{bmatrix}, \qquad L_2 \in M_{(s-1)\times(r-1)}(S),$$

and hence $L_2 F_2 = I_{s-1}$. The first column of F_2 is unimodular, and since $r - 1 > r - s \geq \operatorname{sr}(S)$ we again apply Proposition 50 to obtain a matrix $U_2' \in E_{r-1}(S)$ such that

$$U_2' F_2 = \begin{bmatrix} 1 & * \\ 0 & F_3 \end{bmatrix}, \qquad F_3 \in M_{(r-2)\times(s-2)}(S).$$

Let

$$U_2 := \begin{bmatrix} 1 & 0 \\ 0 & U_2' \end{bmatrix} \in E_r(S).$$

Then

$$U_2 U_1 G_1^T = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & F_3 \end{bmatrix}.$$

By induction on s and multiplying on the left by elementary matrices we get $U \in E_r(S)$ such that

$$UG_1^T = \begin{bmatrix} I_s \\ 0 \end{bmatrix}. \ \blacksquare$$

Corollary 53 (Stafford). *Let $D := A_n(K)$ or $B_n(K)$, with* $\operatorname{char}(K) = 0$. *Then any stably free left D-module M satisfying* $\operatorname{rank}(M) \geq 2$ *is free.*

Proof. The result follows from Theorem 52 since $\operatorname{sr}(D) = 2$. $\blacksquare$

4. Hermite rings

Rings for which all stably free modules are free have attracted special attention in homological algebra. In this section we will consider a matrix-constructive interpretation of such rings. The material presented here can be considered as preparatory for the next section where we study the Hermite condition for skew PBW extensions. Recall that all rings considered are $\mathcal{RC}$ (see Remark 35).

4.1. Matrix descriptions of Hermite rings

DEFINITION 54. Let S be a ring.

(i) S is a *PF ring* if every f.g. projective S-module is free.
(ii) S is a *PSF ring* if every f.g. projective S-module is stably free.
(iii) S is an *Hermite ring*, a property denoted by H, if any stably free S-module is free.

The right versions of the above rings (i.e., for right modules) are defined in a similar way and denoted by PF_r, PSF_r and H_r, respectively. We say that S is a $\mathcal{PF}$ ring if S is PF and PF_r simultaneously; similarly we define the properties $\mathcal{PSF}$ and $\mathcal{H}$. However, we will prove below that these properties are left-right symmetric, i.e., they can be denoted simply by $\mathcal{PF}$, $\mathcal{PSF}$ and $\mathcal{H}$.

From Definition 54 we get

$$H \cap PSF = PF. \tag{4.1}$$

The following theorem gives a matrix description of H rings (see [14] and compare with [31] for the particular case of commutative rings; in [8] a different and independent proof of this theorem for right modules is presented).

THEOREM 55 ([18]). *Let S be a ring. Then the following conditions are equivalent:*

(i) *S is H.*
(ii) *For every $r \geq 1$, any unimodular row matrix $\mathbf{u}$ over S of size $1 \times r$ can be completed to an invertible matrix of $\mathrm{GL}_r(S)$ by adding $r-1$ new rows.*
(iii) *For every $r \geq 1$, if $\mathbf{u}$ is a unimodular row matrix of size $1 \times r$, then there exists a matrix $U \in \mathrm{GL}_r(S)$ such that $\mathbf{u}U = (1, 0, \ldots, 0)$.*
(iv) *For every $r \geq 1$, given a unimodular matrix F of size $s \times r$, $r \geq s$, there exists $U \in \mathrm{GL}_r(S)$ such that*

$$FU = [I_s \quad | \quad 0].$$

REMARK 56. In a similar way to Remark 28, if we consider right modules and the right S-module structure on the module S^r of columns vectors, the conditions of the previous theorem can be reformulated properly (see [18]).

4.2. Matrix characterization of PF rings. In [14] some matrix characterizations of projective-free rings are given. In this subsection we present another matrix interpretation of this important class of rings. The main result here (Corollary 60) extends [31, Theorem 6.2.2]. This result has also been proved independently in [8, Proposition 11.4.9]. A matrix proof of a Kaplansky theorem about finitely generated projective modules over local rings is also included.

THEOREM 57 ([18]). *Let S be an Hermite ring and M a f.g. projective module given by the column module of a matrix $F \in M_s(S)$, with F^T idempotent. Then M is free with*

$\dim(M) = r$ *if and only if there exists* $U \in M_s(S)$ *such that* $U^T \in \mathrm{GL}_s(S)$ *and*

$$(U^T)^{-1} F^T U^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T. \tag{4.2}$$

In that case, a basis of M *is given by the last* r *rows of* $(U^T)^{-1}$.

From the previous theorem we get the following matrix description of PF rings.

COROLLARY 58 ([18]). *Let* S *be a ring. Then* S *is PF if and only if for each* $s \geq 1$, *given a matrix* $F \in M_s(S)$ *with* F^T *idempotent, there exists a matrix* $U \in M_s(S)$ *such that* $U^T \in \mathrm{GL}_s(S)$ *and*

$$(U^T)^{-1} F^T U^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T, \tag{4.3}$$

where $r = \dim(\langle F \rangle)$, $0 \leq r \leq s$.

REMARK 59. (i) If we consider right modules instead of left modules, then the previous corollary can be reformulated in the following way: S is PF_r if and only if for each $s \geq 1$, given an idempotent $F \in M_s(S)$, there exists $U \in \mathrm{GL}_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{4.4}$$

where $r = \dim(\langle F \rangle)$, $0 \leq r \leq s$, and $\langle F \rangle$ represents the right S-module generated by the columns of F. The proof is as in the commutative case (see [31]).

(ii) Considering again left modules and arranging the matrices of homomorphisms by rows and composing homomorphisms from the left to the right (see Remark 28), we get the characterization (4.4) for the PF property. However, observe that in this case $\langle F \rangle$ represents the left S-module generated by the rows of F. Note that Corollary 58 could have been formulated this way: In fact,

$$\begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}$$

and we can rewrite (4.3) as (4.4) changing F^T to F (see Remark 28) and $(U^T)^{-1}$ to U.

(iii) If S is a commutative ring, of course $PF = PF_r = \mathcal{PF}$. Moreover, we will prove in Corollary 61 that the projective-free property is left-right symmetric for general rings.

COROLLARY 60. S *is PF if and only if for each* $s \geq 1$, *given an idempotent matrix* $F \in M_s(S)$, *there exists a matrix* $U \in \mathrm{GL}_s(S)$ *such that*

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix}, \tag{4.5}$$

where $r = \dim(\langle F \rangle)$, $0 \leq r \leq s$, *and* $\langle F \rangle$ *represents the left* S*-module generated by the rows of* F.

Proof. This is the content of Remark 59(ii). ■

COROLLARY 61. *Let* S *be a ring. Then* S *is PF if and only if* S *is* PF_r, *i.e.,* $PF = PF_r = \mathcal{PF}$.

Proof. Let $F \in M_s(S)$ be an idempotent matrix. If S is PF, then there exists $P \in \mathrm{GL}_s(S)$ such that

$$UFU^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I_r \end{bmatrix},$$

where r is the dimension of the left S-module generated by the rows of F. Observe that UFU^{-1} is also idempotent, and the matrices $X := UF$ and $Y := U^{-1}$ satisfy $UFU^{-1} = XY$ and $F = YX$. Then from [14, Proposition 0.3.1] we conclude that the left S-module generated by the rows of UFU^{-1} coincides with the left S-module generated by the rows of F, and also the right S-module generated by the columns of UFU^{-1} coincides with the right S-module generated by the columns of F. This implies that the S-module generated by the rows of F coincides with the right S-module generated by the columns of F. This means that S is PF_r. The symmetry of the problem completes the proof. ■

Another interesting matrix characterization of $\mathcal{PF}$ rings is given in [14, Proposition 0.4.7]: a ring S is $\mathcal{PF}$ if and only if given an idempotent matrix $F \in M_s(S)$ there exist matrices $X \in M_{s\times r}(S)$, $Y \in M_{r\times s}(S)$ such that $F = XY$ and $YX = I_r$. A similar matrix interpretation can be given for *PSF* rings by using [14, Proposition 0.3.1] and Corollary 40:

PROPOSITION 62. *Let S be a ring. Then*

(i) *S is PSF if and only if given an idempotent matrix $F \in M_r(S)$ there exist $s \geq 0$ and matrices $X \in M_{(r+s)\times r}(S)$, $Y \in M_{r\times (r+s)}(S)$ such that*

$$\begin{bmatrix} F & 0 \\ 0 & I_s \end{bmatrix} = XY \quad \text{and} \quad YX = I_r.$$

(ii) $PSF = PSF_r = \mathcal{PSF}$.

For the H property we have a similar characterization that proves the symmetry of this condition.

PROPOSITION 63 ([18]). *Let S be a ring. Then*

(i) *S is H if and only if given an idempotent matrix $F \in M_r(S)$ with $\left[\begin{smallmatrix} F & 0 \\ 0 & 1 \end{smallmatrix}\right] = XY$ and $YX = I_r$ for some $X \in M_{(r+1)\times r}(S)$ and $Y \in M_{r\times (r+1)}(S)$, there exist $X' \in M_{r\times (r-1)}(S)$ and $Y' \in M_{(r-1)\times r}(S)$ such that $F = X'Y'$ and $Y'X' = I_{r-1}$.*

(ii) $H = H_r = \mathcal{H}$.

We conclude this subsection by giving a matrix-constructive proof of a well known theorem of Kaplansky.

PROPOSITION 64. *Any local ring S is $\mathcal{PF}$.*

Proof. Let M a projective left S-module. By Remark 28(ii), there exists an idempotent matrix $F = [f_{ij}] \in M_s(S)$ such that the module generated by the rows of F coincides with M. According to Corollary 60, we need to show that there exists $U \in \mathrm{GL}_s(S)$ such that the relation (4.5) holds. The proof is by induction on s.

$s = 1$: In this case $F = [f_{ij}] = [f]$; since S is local, its idempotents are trivial, so $f = 1$ or $f = 0$ and hence M is free.

$s = 2$: In view of the fact that S is local, two possibilities may arise:

• f_{11} *is invertible.* Then one can find $G \in \mathrm{GL}_2(S)$ such that $GFG^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right]$ for some $f \in S$. For this it is enough to take

$$G = \begin{bmatrix} 1 & f_{11}^{-1}f_{12} \\ -f_{21}f_{11}^{-1} & 1 \end{bmatrix};$$

to show that this matrix is invertible with inverse

$$G^{-1} = \begin{bmatrix} f_{11} & -f_{12} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12} + 1 \end{bmatrix}$$

we can use the relations that exist between the entries of F. Let us check for example that $GG^{-1} = I_2$:

$$f_{11} + f_{11}^{-1}f_{12}f_{21} = 1 \quad \text{because } f_{11}^2 + f_{12}f_{21} = f_{11} \text{ and } f_{11} \text{ is invertible;}$$

$$-f_{12} - f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{12} + f_{11}^{-1}f_{12}$$
$$= -f_{12} + (1 - f_{11}^{-1}f_{12}f_{21})f_{11}^{-1}f_{12} = -f_{12} + f_{11}f_{11}^{-1}f_{12} = 0;$$

$$-f_{21}f_{11}^{-1}f_{11} + f_{21} = 0;$$

$$f_{21}f_{11}^{-1}f_{12} - f_{21}f_{11}^{-1}f_{12} + 1 = 1.$$

Similar calculations show that $G^{-1}G = I_2$. Since F is idempotent, f is also idempotent; applying the case $s = 1$ we get the result.

• $1 - f_{11}$ *is invertible.* In the same way as above, we can find $H \in \mathrm{GL}_2(S)$ such that $HFH^{-1} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & g \end{smallmatrix}\right]$. It is enough to take

$$H = \begin{bmatrix} 1 & -(1 - f_{11})^{-1}f_{12} \\ f_{21} & -f_{21}(1 - f_{11})^{-1}f_{12} + 1 \end{bmatrix}.$$

Note that

$$H^{-1} = \begin{bmatrix} 1 - f_{11} & (1 - f_{11})^{-1}f_{12} \\ -f_{21} & 1 \end{bmatrix}.$$

Indeed $HH^{-1} = I_2$:

$$1 - f_{11} + (1 - f_{11})^{-1}f_{12}f_{21} = 1 - f_{11} + f_{11} = 1 \quad \text{because } f_{12}f_{21} = (1 - f_{11})f_{11};$$

$$(1 - f_{11})^{-1}f_{12} - (1 - f_{11})^{-1}f_{12} = 0;$$

$$f_{21}(1 - f_{11}) + f_{21}(1 - f_{11})^{-1}f_{12}f_{21} - f_{21} = f_{21}(1 - f_{11}) + f_{21}f_{11} - f_{21} = 0;$$

$$f_{21}(1 - f_{11})^{-1}f_{12} - f_{21}(1 - f_{11})^{-1}f_{12} + 1 = 1.$$

An analogous calculation shows that $H^{-1}H = I_2$. Note that g is an idempotent of S, hence $g = 0$ or $g = 1$ and the statement follows.

Now suppose that the result holds for $s-1$; two possibilities for f_{11} can occur:

- f_{11} *is invertible.* Taking

$$G = \begin{bmatrix} 1 & f_{11}^{-1}f_{12} & f_{11}^{-1}f_{13} & \dots & f_{11}^{-1}f_{1s} \\ -f_{21}f_{11}^{-1} & 1 & 0 & \dots & 0 \\ -f_{31}f_{11}^{-1} & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \\ -f_{s1}f_{11}^{-1} & 0 & 0 & \dots & 1 \end{bmatrix}$$

we see that $G \in \mathrm{GL}_s(S)$ and its inverse is

$$G^{-1} = \begin{bmatrix} f_{11} & -f_{12} & -f_{13} & \dots & -f_{1s} \\ f_{21} & -f_{21}f_{11}^{-1}f_{12}+1 & -f_{21}f_{11}^{-1}f_{13} & \dots & -f_{21}f_{11}^{-1}f_{1s} \\ f_{31} & -f_{31}f_{11}^{-1}f_{12} & -f_{31}f_{11}^{-1}f_{13}+1 & \dots & -f_{31}f_{11}^{-1}f_{1s} \\ \vdots & & & \vdots & \\ f_{s1} & -f_{s1}f_{11}^{-1}f_{12} & -f_{s1}f_{11}^{-1}f_{13} & \dots & -f_{s1}f_{11}^{-1}f_{1s}+1 \end{bmatrix}.$$

Indeed, $GG^{-1} = I_s$:

$f_{11} + f_{11}^{-1}f_{12}f_{21} + \cdots + f_{11}^{-1}f_{1s}f_{s1} = 1$ because $f_{11}^2 + f_{12}f_{21} + \cdots + f_{1s}f_{s1} = f_{11}$;

$$-f_{12} - f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{12} + f_{11}^{-1}f_{12} - f_{11}^{-1}f_{13}f_{31}f_{11}^{-1}f_{12} - \cdots - f_{11}^{-1}f_{1s}f_{s1}f_{11}^{-1}f_{12}$$
$$= -f_{12} + (1 - f_{11}^{-1}\sum_{i=2}^{s} f_{1i}f_{i1})f_{11}^{-1}f_{12} = -f_{12} + f_{11}f_{11}^{-1}f_{12} = 0;$$

$\vdots$

$$-f_{1s} - f_{11}^{-1}f_{12}f_{21}f_{11}^{-1}f_{1s} - f_{11}^{-1}f_{13}f_{31}f_{11}^{-1}f_{1s} - \cdots - f_{11}^{-1}f_{1s}f_{s1}f_{11}^{-1}f_{1s} + f_{11}^{-1}f_{1s}$$
$$= -f_{1s} + (1 - f_{11}^{-1}\sum_{i=2}^{s} f_{1i}f_{i1})f_{11}^{-1}f_{1s} = -f_{1s} + f_{11}f_{11}^{-1}f_{1s} = 0;$$

$-f_{21}f_{11}^{-1}f_{11} + f_{21} = 0;$ $\quad f_{21}f_{11}^{-1}f_{12} - f_{21}f_{11}^{-1}f_{12} + 1 = 1;$

$f_{21}f_{11}^{-1}f_{1i} - f_{21}f_{11}^{-1}f_{1i} = 0$ for every $3 \leq i \leq s$;

$\vdots$

$-f_{s1}f_{11}^{-1}f_{11} + f_{s1} = 0;$

$f_{s1}f_{11}^{-1}f_{1i} - f_{s1}f_{11}^{-1}f_{1i} = 0$ for every $2 \leq i \leq s-1$;

$f_{s1}f_{11}^{-1}f_{1s} - f_{s1}f_{11}^{-1}f_{1s} + 1 = 1.$

Similarly, $G^{-1}G = I_s$. Moreover,

$$GFG^{-1} = \begin{bmatrix} 1 & 0_{1,s-1} \\ 0_{s-1,1} & F_1 \end{bmatrix},$$

where $F_1 \in M_{s-1}(S)$ is idempotent. Now we can apply the induction hypothesis.

- $1 - f_{11}$ *is invertible.* Taking

$$H = \begin{bmatrix} 1 & -(1-f_{11})^{-1}f_{12} & -(1-f_{11})^{-1}f_{13} & \dots & -(1-f_{11})^{-1}f_{1s} \\ f_{21} & -f_{21}(1-f_{11})^{-1}f_{12}+1 & -f_{21}(1-f_{11})^{-1}f_{13} & \dots & -f_{21}(1-f_{11})^{-1}f_{1s} \\ f_{31} & -f_{31}(1-f_{11})^{-1}f_{12} & -f_{31}(1-f_{11})^{-1}f_{13}+1 & \dots & -f_{31}(1-f_{11})^{-1}f_{1s} \\ \vdots & & & \vdots & \\ f_{s1} & -f_{s1}(1-f_{11})^{-1}f_{12} & -f_{s1}(1-f_{11})^{-1}f_{13} & \dots & -f_{s1}(1-f_{11})^{-1}f_{1s}+1 \end{bmatrix}$$

we see that $H \in \mathrm{GL}_s(S)$ with inverse given by

$$H^{-1} = \begin{bmatrix} 1-f_{11} & (1-f_{11})^{-1}f_{12} & (1-f_{11})^{-1}f_{13} & \dots & (1-f_{11})^{-1}f_{1s} \\ -f_{21} & 1 & 0 & \dots & 0 \\ -f_{31} & 0 & 1 & \dots & 0 \\ \vdots & & & \vdots & \\ -f_{s1} & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Indeed, $HH^{-1} = I_s$:

$$1 - f_{11} + (1-f_{11})^{-1}\sum_{i=2}^{s} f_{1i}f_{i1} = 1 - f_{11} + f_{11} = 1 \quad \text{because} \sum_{i=2}^{s} f_{1i}f_{i1} = (1-f_{11})f_{11};$$

$$f_{21}(1-f_{11}) + f_{21}\sum_{i=1}^{s}(1-f_{11})^{-1}f_{1i}f_{i1} - f_{21} = -f_{21}f_{11} + f_{21}f_{11} = 0;$$

$$f_{21}(1-f_{11})^{-1}f_{12} - f_{21}(1-f_{11})^{-1}f_{12} + 1 = 1;$$

$$f_{21}(1-f_{11})^{-1}f_{1i} - f_{21}(1-f_{11})^{-1}f_{1i} = 0 \quad \text{for } 3 \le i \le s;$$

$$\vdots$$

$$f_{s1}(1-f_{11}) + f_{s1}\sum_{i=1}^{s}(1-f_{11})^{-1}f_{1i}f_{i1} - f_{s1} = -f_{s1}f_{11} + f_{21}f_{11} = 0;$$

$$f_{s1}(1-f_{11})^{-1}f_{1i} - f_{s1}(1-f_{11})^{-1}f_{1i} = 0 \quad \text{for } 3 \le i \le s-1;$$

$$f_{s1}(1-f_{11})^{-1}f_{1s} - f_{s1}(1-f_{11})^{-1}f_{1s} + 1 = 1.$$

Similarly, $H^{-1}H = I_s$. Furthermore,

$$HFH^{-1} = \begin{bmatrix} 0 & 0_{1,s-1} \\ 0_{s-1,1} & F_2 \end{bmatrix}$$

with $F_2 \in M_{s-1}(S)$ idempotent. One more time we apply the induction hypothesis. ■

5. d-Hermite rings and skew PBW extensions

Under suitable conditions on the ring R of coefficients, most of the bijective skew PBW extensions are $\mathcal{PSF}$ (see Theorem 26). A different situation occurs for the $\mathcal{H}$ property. In fact, as observed before, if K is a division ring, then $S := K[x,y]$ has a module M such that $M \oplus S \cong S^2$, but M is not free, i.e., S is not $\mathcal{H}$. Another example occurs in Weyl algebras: Let K be a field with $\mathrm{char}(K) = 0$. The Weyl algebra $A_1(K) = K[t]\big[x; \frac{d}{dt}\big]$ is not $\mathcal{H}$ since there exist stably free modules of rank 1 over $A_n(K)$ that are not free ([14, Corollary 1.5.3]; see also [38, Example 11.1.4]). In this section we will study a condition weaker than the $\mathcal{H}$ property for skew PBW extensions: the d-Hermite condition. Recall that we always assume that all rings are $\mathcal{RC}$.

5.1. d-Hermite rings. The following proposition suggests the definition of d-Hermite rings.

Proposition 65. *Let S be a ring. For any integer $d \geq 0$, the following statements are equivalent:*

(i) *Any stably free module of rank $\geq d$ is free.*
(ii) *Any unimodular row matrix over S of length $\geq d+1$ can be completed to an invertible matrix over S.*
(iii) *For every $r \geq d+1$, if $\mathbf{u}$ is a unimodular row matrix of size $1 \times r$, then there exists a matrix $U \in \mathrm{GL}_r(S)$ such that $\mathbf{u}U = (1, 0, \ldots, 0)$, i.e., $\mathrm{GL}_r(S)$ acts transitively on $\mathrm{Um}_r(r, S)$.*
(iv) *For every $r \geq d+1$, given a unimodular matrix F of size $s \times r$, $r \geq s$, there exists $U \in \mathrm{GL}_r(S)$ such that*

$$FU = \begin{bmatrix} I_s & | & 0 \end{bmatrix}.$$

Proof. We can repeat the proof of [18, Theorem 2] taking $r \geq d+1$. ■

Definition 66. Let S be a ring and $d \geq 0$ an integer. Then S is *d-Hermite*, a property denoted by *d-$\mathcal{H}$*, if S satisfies any of the conditions in Proposition 65.

The next result extends Proposition 63.

Proposition 67. *The d-Hermite condition is left-right symmetric.*

Corollary 68. *Let S be a ring. Then S is $\mathrm{sr}(S)$-$\mathcal{H}$.*

Proof. This follows from Definition 66 and Theorem 52. ■

Corollary 69. *Let S be a ring. If $\mathrm{sr}(S) = 1$, then S is $\mathcal{H}$.*

Proof. According to Corollary 68, S is 1-$\mathcal{H}$; however, it is well known that rings with stable rank 1 are cancellable (see [16]), so by [18, Proposition 12], S is $\mathcal{H}$. ■

Remark 70. (i) Observe that 0-Hermite rings coincide with $\mathcal{H}$ rings, and for commutative rings, 1-Hermite also coincides with $\mathcal{H}$ (see [27, Theorem I.4.11]). If K is a field with $\mathrm{char}(K) = 0$, by Corollary 53, $A_1(K)$ is 2-$\mathcal{H}$, but as we observed before, $A_1(K)$ is not 1-$\mathcal{H}$. In general, $\mathcal{H} \subsetneq 1\text{-}\mathcal{H} \subsetneq 2\text{-}\mathcal{H} \subsetneq \cdots$ (see [14]).

(ii) Note that $\mathcal{H} = 1\text{-}\mathcal{H} \cap \mathcal{WF}$ (a ring S is $\mathcal{WF}$, *weakly finite*, if for all $n \geq 0$, $P \oplus S^n \cong S^n$ if and only if $P = 0$).

(iii) Any left Artinian ring S is $\mathcal{H}$ since $\mathrm{sr}(S) = 1$. In particular, semisimple and semilocal rings are $\mathcal{H}$.

(iv) Rings with big stable rank can be Hermite, for example $\mathrm{sr}(\mathbb{R}[x_1, \ldots, x_n]) = n+1$ (see [38, Theorem 11.5.9]), but by the Quillen–Suslin theorem, $\mathbb{R}[x_1, \ldots, x_n]$ is $\mathcal{H}$.

5.2. Stable rank. Corollaries 53 and 68 motivate the task of computing the stable rank of bijective skew PBW extensions. For this purpose we need to recall the famous stable range theorem. This theorem relates the stable rank and the Krull dimension of a ring. The original version of this classical result is due to Bass (1968, [3]) and states that if S is a commutative Noetherian ring and $\mathrm{Kdim}(S) = d$ then $\mathrm{sr}(S) \leq d+1$. Heitmann extended the theorem to arbitrary commutative rings (1984, [23]). In 2004 Lombardi et al. ([15, Theorem 2.4]; see also [37]) proved the theorem again for arbitrary commutative rings using the Zariski lattice of a ring and the boundary ideal of an element. This proof is elementary and constructive. In 1981 Stafford [52] proved a non-commutative version of the theorem for left Noetherian rings.

Proposition 71 (Stable range theorem, [52]). *Let S be a left Noetherian ring with* $\mathrm{lKdim}(S) = d$. *Then* $\mathrm{sr}(S) \leq d + 1$.

From this we get the following modest result.

Proposition 72. *Let R be a left Noetherian ring with finite left Krull dimension and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ a bijective skew PBW extension of R. Then*

$$1 \leq \mathrm{sr}(A) \leq \mathrm{lKdim}(R) + n + 1,$$

and A is d-$\mathcal{H}$, with $d := \mathrm{lKdim}(R) + n + 1$.

Proof. The inequalities follow from Proposition 71 and [35, Theorem 4.2]. The second statement follows from Corollary 68. ■

Example 73. The results in [35] for the Krull dimension of many interesting examples of bijective skew PBW extensions can be combined with Proposition 72 in order to get an upper bound for the stable rank. With this we can also estimate the d-Hermite condition. Table 1 gives such estimations.

Remark 74. The values presented in Table 1 can be improved for some particular classes of skew PBW extensions. For example, it is well known that $\mathrm{sr}(A_n(K)) = 2$ if $\mathrm{char}(K) = 0$ (see Remark 49). A challenging problem is to give exact values for the stable rank of all examples of bijective PBW extensions presented in [35].

5.3. Kronecker's theorem. Closely related to the stable range theorem is Kronecker's theorem saying that if S is a commutative ring with $\mathrm{Kdim}(S) < d$, then every finitely generated ideal I of S has the same radical as an ideal generated by d elements. In this subsection we want to investigate this theorem for non-commutative rings using the Zariski lattice and the boundary ideal, but generalizing these tools and their properties to non-commutative rings. The main result will be applied to skew PBW extensions.

Definition 75. Let S be a ring and $\mathrm{Spec}(S)$ the set of all prime ideals of S. The *Zariski lattice* of S is defined by

$$\mathrm{Zar}(S) := \{D(X) \mid X \subseteq S\} \quad \text{with} \quad D(X) := \bigcap_{X \subseteq P \in \mathrm{Spec}(S)} P.$$

$\mathrm{Zar}(S)$ is ordered by inclusion. The description of the Zariski lattice is presented in the next proposition. $\langle X\}, \langle X\rangle, \{X\rangle$ will represent the left, two-sided, and right ideal of S generated by X, respectively. $\vee$ denotes the sup and $\wedge$ the inf.

Proposition 76 ([20]). *Let S be a ring, I, I_1, I_2, I_3 two-sided ideals of S, $X \subseteq S$, and $x_1, \dots, x_n, x, y \in S$. Then*

(i) $D(X) = D(\langle X\}) = D(\langle X\rangle) = D(\{X\rangle)$.
(ii) $D(I) = \mathrm{rad}(S)$ *if and only if* $I \subseteq \mathrm{rad}(S)$. *In particular,* $D(0) = \mathrm{rad}(S)$.
(iii) $D(I) = S$ *if and only if* $I = S$.
(iv) $I \subseteq D(I)$ *and* $D(D(I)) = D(I)$. *Moreover, if* $I_1 \subseteq I_2$, *then* $D(I_1) \subseteq D(I_2)$.

Table 1. Stable rank for some examples of bijective skew PBW extensions

Ring	Upper bound
Habitual polynomial ring $R[x_1,\dots,x_n]$	$\dim(R)+n+1$
Ore extension of bijective type $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$	$\dim(R)+n+1$
Weyl algebra $A_n(K)$	$2n+1$
Extended Weyl algebra $B_n(K)$	$n+1$
Universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, K a commutative ring	$\dim(K)+n+1$
Tensor product $R\otimes_K\mathcal{U}(\mathcal{G})$	$\dim(R)+n+1$
Crossed product $R*\mathcal{U}(\mathcal{G})$	$\dim(R)+n+1$
Algebra of q-differential operators $D_{q,h}[x,y]$	3
Algebra of shift operators S_h	3
Mixed algebra D_h	4
Discrete linear system $K[t_1,\dots,t_n][x_1;\sigma_1]\cdots[x_n;\sigma_n]$	$2n+1$
Linear partial shift operators $K[t_1,\dots,t_n][E_1,\dots,E_n]$	$2n+1$
Linear partial shift operators $K(t_1,\dots,t_n)[E_1,\dots,E_n]$	$n+1$
Linear partial differential operators $K[t_1,\dots,t_n][\partial_1,\dots,\partial_n]$	$2n+1$
Linear partial differential operators $K(t_1,\dots,t_n)[\partial_1,\dots,\partial_n]$	$n+1$
Linear partial difference operators $K[t_1,\dots,t_n][\varDelta_1,\dots,\varDelta_n]$	$2n+1$
Linear partial difference operators $K(t_1,\dots,t_n)[\varDelta_1,\dots,\varDelta_n]$	$n+1$
Linear partial q-dilation operators $K[t_1,\dots,t_n][H_1^{(q)},\dots,H_m^{(q)}]$	$n+m+1$
Linear partial q-dilation operators $K(t_1,\dots,t_n)[H_1^{(q)},\dots,H_m^{(q)}]$	$m+1$
Linear partial q-differential operators $K[t_1,\dots,t_n][D_1^{(q)},\dots,D_m^{(q)}]$	$n+m+1$
Linear partial q-differential operators $K(t_1,\dots,t_n)[D_1^{(q)},\dots,D_m^{(q)}]$	$m+1$
Diffusion algebra	$2n+1$
Additive analogue of the Weyl algebra $A_n(q_1,\dots,q_n)$	$2n+1$
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$	$n+1$
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3,K))$	4
3-dimensional skew polynomial algebra	4
Dispin algebra $\mathcal{U}(\mathrm{osp}(1,2))$	4
Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2,K))$	4
Complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$	11
Algebra $\mathbf{U}$	$3n+1$
Manin algebra $\mathcal{O}_q(M_2(K))$	5
Coordinate algebra of the quantum group $\mathrm{SL}_q(2)$	5
q-Heisenberg algebra $\mathbf{H}_n(q)$	$3n+1$
Quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2,K))$	4
Hayashi algebra $W_q(J)$	$3n+1$
Differential operators on a quantum space $S_{\mathbf{q}}$, $D_{\mathbf{q}}(S_{\mathbf{q}})$	$2n+1$
Witten's deformation of $\mathcal{U}(\mathfrak{sl}(2,K))$	4
Quantum Weyl algebra of Maltsiniotis $A_n^{\mathbf{q},\lambda}$, K a commutative ring	$\dim(K)+2n+1$
Quantum Weyl algebra $A_n(q,p_{i,j})$	$2n+1$
Multiparameter Weyl algebra $A_n^{Q,\Gamma}(K)$	$2n+1$
Quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(K^{2n}))$	$2n+1$
Quadratic algebra in three variables	4

(v) *Let* $\{I_j\}_{j\in\mathcal{J}}$ *be a family of two-sided ideals of* S. *Then* $D(\sum_{j\in\mathcal{J}}I_j)=\bigvee_{j\in\mathcal{J}}D(I_j)$. *In particular,* $D(x_1,\dots,x_n)=D(x_1)\vee\cdots\vee D(x_n)$.

(vi) $D(I_1I_2)=D(I_1)\wedge D(I_2)$. *In particular,* $D(\langle x\rangle\langle y\rangle)=D(x)\wedge D(y)$.

(vii) $D(x+y)\subseteq D(x,y)$.

(viii) *If* $\langle x\rangle\langle y\rangle \subseteq D(0)$*, then* $D(x,y) = D(x+y)$.

(ix) *If* $x \in D(I)$*, then* $D(I) = D(I,x)$.

(x) *If* $\overline{S} := S/I$*, then* $D(\overline{J}) = \overline{D(J)}$ *for any two-sided ideal* J *of* S *containing* I.

(xi) $u \in D(I)$ *if and only if* $\overline{u} \in \operatorname{rad}(S/I)$*. In that case, if* $u \in D(I)$*, there exists* $k \geq 1$ *such that* $u^k \in I$.

(xii) $\operatorname{Zar}(S)$ *is distributive:*

$$D(I_1) \wedge [D(I_2) \vee D(I_3)] = [D(I_1) \wedge D(I_2)] \vee [D(I_1) \wedge D(I_3)],$$
$$D(I_1) \vee [D(I_2) \wedge D(I_3)] = [D(I_1) \vee D(I_2)] \wedge [D(_1) \vee D(I_3)].$$

Definition 77. Let S be a ring and $v \in S$. The *boundary ideal* of v is defined by $I_v := \langle v\rangle + (D(0) : \langle v\rangle)$, where $(D(0) : \langle v\rangle) := \{x \in S \mid \langle v\rangle x \subseteq D(0)\}$.

Note that $I_v \neq 0$ for every $v \in S$. On the other hand, if v is invertible or if $v = 0$, then $I_v = S$. If S is a domain and $v \neq 0$, then $I_v = \langle v\rangle$.

Definition 78. Let S be a ring such that $\operatorname{lKdim}(S)$ exists. We say that S *satisfies the boundary condition* if for any $d \geq 0$ and $v \in S$,

$$\operatorname{lKdim}(S) \leq d \ \Rightarrow\ \operatorname{lKdim}(S/I_v) \leq d-1.$$

Example 79. (i) Any commutative Noetherian ring satisfies the boundary condition: indeed, for commutative Noetherian rings, the classical Krull dimension and the Krull dimension coincide, so we can apply [37, Theorem 13.2].

(ii) Any prime ring S with left Krull dimension satisfies the boundary condition: in fact, for prime rings, any non-zero two sided ideal is essential, so $\operatorname{lKdim}(S/I_v) < \operatorname{lKdim}(S)$ (see [38, Proposition 6.3.10]).

(iii) Any domain with left Krull dimension satisfies the boundary condition: indeed, any domain is a prime ring.

Theorem 80 (Kronecker; see [20]). *Let* S *be a domain such that* $\operatorname{lKdim}(S)$ *exists. If* $\operatorname{lKdim}(S) < d$ *and* $u_1, \dots, u_d, u \in S$*, then there exist* $x_1, \dots, x_d \in S$ *such that*

$$D(u_1, \dots, u_d, u) = D(u_1 + x_1u, \dots, u_d + x_du).$$

Corollary 81. *Let* S *be a domain such that* $\operatorname{lKdim}(S)$ *exists. If* $\operatorname{lKdim}(S) < d$ *and* $u_1, \dots, u_{d+1} \in S$ *are such that* $\langle u_1, \dots, u_{d+1}\rangle = S$*, then there exist elements* $x_1, \dots, x_d \in S$ *such that* $\langle u_1 + x_1u_{d+1}, \dots, u_d + x_du_{d+1}\rangle = S$.

Proof. The statement follows directly from Proposition 76(iii) and Theorem 80. ∎

Corollary 82. *Let* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *be a bijective skew PBW extension of a left Noetherian domain* R*. If* $\operatorname{lKdim}(R) < d$ *and* $u_1, \dots, u_{d+n}, u \in A$*, then there exist* $y_1, \dots, y_{d+n} \in A$ *such that*

$$D(u_1, \dots, u_{d+n}, u) = D(u_1 + y_1u, \dots, u_{d+n} + y_{d+n}u).$$

Proof. This follows directly from Proposition 10, Theorem 14, [35, Theorem 4.2], and Theorem 80. ∎

6. Gröbner bases for skew PBW extensions

In order to make constructive the theory of projective modules, stably free modules and Hermite rings studied in the previous sections, we will now study the theory of Gröbner bases of left ideals and modules for bijective skew PBW extensions. This theory was initially investigated in [19], [25] and [26] for the particular case of quasi-commutative bijective skew PBW extensions. We will extend the theory to arbitrary bijective skew PBW extensions, in particular, Buchberger's algorithm will be established for the general bijective case. *Note that our theory applies to all examples listed in Table* 1.

We start by recalling the basic facts of Gröbner theory for arbitrary skew PBW extensions; we will use the notation of Definition 7.

6.1. Monomial orders in skew PBW extensions. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be an arbitrary skew PBW extension of R, and let $\succeq$ be a total order defined on $\mathrm{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we will write $x^\alpha \succ x^\beta$. By $x^\beta \preceq x^\alpha$ we mean that $x^\alpha \succeq x^\beta$. Let $f \neq 0$ be a polynomial in A. If

$$f = c_1X_1 + \cdots + c_tX_t$$

with $c_i \in R - \{0\}$ and $X_1 \succ \cdots \succ X_t$ the monomials of f, then $\mathrm{lm}(f) := X_1$ is the *leading monomial* of f, $\mathrm{lc}(f) := c_1$ is the *leading coefficient* of f and $\mathrm{lt}(f) := c_1X_1$ is the *leading term* of f. If $f = 0$, we define $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$, and we set $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Thus, we extend $\succeq$ to $\mathrm{Mon}(A) \cup \{0\}$.

Definition 83. Let $\succeq$ be a total order on $\mathrm{Mon}(A)$. We say that $\succeq$ is a *monomial order* on $\mathrm{Mon}(A)$ if the following conditions hold:

(i) For all $x^\beta, x^\alpha, x^\gamma, x^\lambda \in \mathrm{Mon}(A)$,

$$x^\beta \succeq x^\alpha \ \Rightarrow\ \mathrm{lm}(x^\gamma x^\beta x^\lambda) \succeq \mathrm{lm}(x^\gamma x^\alpha x^\lambda).$$

(ii) $x^\alpha \succeq 1$ for every $x^\alpha \in \mathrm{Mon}(A)$.

(iii) $\succeq$ is *degree compatible*, i.e., $|\beta| \geq |\alpha| \Rightarrow x^\beta \succeq x^\alpha$.

Monomial orders are also called *admissible orders*. Condition (iii) is needed in the proof of the following proposition, which in turn will be used in the division algorithm (Theorem 93).

Proposition 84 ([19, Proposition 12]). *Every monomial order on* $\mathrm{Mon}(A)$ *is a well-order. Thus, there are no infinite decreasing chains in* $\mathrm{Mon}(A)$.

From now on we will assume that $\mathrm{Mon}(A)$ is endowed with some monomial order.

Definition 85. Let $x^\alpha, x^\beta \in \mathrm{Mon}(A)$. We say that x^α *divides* x^β, denoted by $x^\alpha \mid x^\beta$, if there exist $x^\gamma, x^\lambda \in \mathrm{Mon}(A)$ such that $x^\beta = \mathrm{lm}(x^\gamma x^\alpha x^\lambda)$. We will also say that any monomial $x^\alpha \in \mathrm{Mon}(A)$ divides the polynomial zero.

Proposition 86 ([19, Proposition 14]). *Let* $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ *and* $f, g \in A - \{0\}$. *Then:*

(a) $\mathrm{lm}(x^\alpha g) = \mathrm{lm}(x^\alpha \mathrm{lm}(g)) = x^{\alpha+\exp(\mathrm{lm}(g))}$, *i.e.,*

$$\exp(\mathrm{lm}(x^\alpha g)) = \alpha + \exp(\mathrm{lm}(g)).$$

In particular,

$$\operatorname{lm}(\operatorname{lm}(f)\operatorname{lm}(g)) = x^{\exp(\operatorname{lm}(f))+\exp(\operatorname{lm}(g))}, \quad i.e.,$$
$$\exp\big(\operatorname{lm}(\operatorname{lm}(f)\operatorname{lm}(g))\big) = \exp(\operatorname{lm}(f)) + \exp(\operatorname{lm}(g))$$

and

$$\operatorname{lm}(x^\alpha x^\beta) = x^{\alpha+\beta}, \quad i.e., \quad \exp(\operatorname{lm}(x^\alpha x^\beta)) = \alpha+\beta. \tag{6.1}$$

(b) *The following conditions are equivalent:*

(i) $x^\alpha \mid x^\beta$.

(ii) *There exists a unique* $x^\theta \in \operatorname{Mon}(A)$ *such that* $x^\beta = \operatorname{lm}(x^\theta x^\alpha) = x^{\theta+\alpha}$ *and hence* $\beta = \theta + \alpha$.

(iii) *There exists a unique* $x^\theta \in \operatorname{Mon}(A)$ *such that* $x^\beta = \operatorname{lm}(x^\alpha x^\theta) = x^{\alpha+\theta}$ *and hence* $\beta = \alpha + \theta$.

(iv) $\beta_i \geq \alpha_i$ *for* $1 \leq i \leq n$, *with* $\beta := (\beta_1, \ldots, \beta_n)$ *and* $\alpha := (\alpha_1, \ldots, \alpha_n)$.

Remark 87. We note that a least common multiple of monomials of $\operatorname{Mon}(A)$ exists: In fact, let $x^\alpha, x^\beta \in \operatorname{Mon}(A)$. Then $\operatorname{lcm}(x^\alpha, x^\beta) = x^\gamma \in \operatorname{Mon}(A)$, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i := \max\{\alpha_i, \beta_i\}$ for each $1 \leq i \leq n$.

6.2. Reduction in skew PBW extensions. Some natural computational conditions on R will be assumed from now (see [30]).

Definition 88. A ring R is *left Gröbner soluble* (*LGS*) if the following conditions hold:

(i) R is left Noetherian.

(ii) Given $a, r_1, \ldots, r_m \in R$ there exists an algorithm which decides whether a is in the left ideal $Rr_1 + \cdots + Rr_m$, and if so, finds $b_1, \ldots, b_m \in R$ such that $a = b_1 r_1 + \cdots + b_m r_m$.

(iii) Given $r_1, \ldots, r_m \in R$ there exists an algorithm which finds a finite set of generators of the left R-module

$$\operatorname{Syz}_R[r_1 \ \ldots \ r_m] := \{(b_1, \ldots, b_m) \in R^m \mid b_1 r_1 + \cdots + b_m r_m = 0\}.$$

Remark 89. The above three conditions are needed in order to guarantee a Gröbner theory in the rings of coefficients, in particular, to have an effective solution of the membership problem in R (see (ii) in Definition 90 below). *From now on we will assume that* $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ *is a skew PBW extension of* R, *where* R *is an LGS ring and* $\operatorname{Mon}(A)$ *is endowed with some monomial order.*

Definition 90. Let F be a finite set of non-zero elements of A, and let $f, h \in A$. We say that f *reduces to* h *by* F *in one step*, denoted $f \xrightarrow{F} h$, if there exist $f_1, \ldots, f_t \in F$ and $r_1, \ldots, r_t \in R$ such that

(i) $\operatorname{lm}(f_i) \mid \operatorname{lm}(f)$, $1 \leq i \leq t$, i.e., there exists $x^{\alpha_i} \in \operatorname{Mon}(A)$ such that $\operatorname{lm}(f) = \operatorname{lm}(x^{\alpha_i}\operatorname{lm}(f_i))$, or equivalently, $\alpha_i + \exp(\operatorname{lm}(f_i)) = \exp(\operatorname{lm}(f))$.

(ii) $\operatorname{lc}(f) = r_1\sigma^{\alpha_1}(\operatorname{lc}(f_1))c_{\alpha_1,f_1} + \cdots + r_t\sigma^{\alpha_t}(\operatorname{lc}(f_t))c_{\alpha_t,f_t}$, where c_{α_i,f_i} are defined as in Theorem 8, i.e., $c_{\alpha_i,f_i} := c_{\alpha_i,\exp(\operatorname{lm}(f_i))}$.

(iii) $h = f - \sum_{i=1}^t r_i x^{\alpha_i} f_i$.

We say that f *reduces to h by F*, denoted $f \xrightarrow{F}_{+} h$, if there exist $h_1, \dots, h_{t-1} \in A$ such that

$$f \xrightarrow{F} h_1 \xrightarrow{F} h_2 \xrightarrow{F} \cdots \xrightarrow{F} h_{t-1} \xrightarrow{F} h.$$

Furthermore, f is *reduced* (also called *minimal*) with respect to F if $f = 0$ or there is no one step reduction of f by F, i.e., (i) or (ii) fails. Otherwise, we will say that f is *reducible* with respect to F. If $f \xrightarrow{F}_{+} h$ and h is reduced with respect to F, then we say that h is a *remainder* for f with respect to F.

Remark 91. (i) By Theorem 8, the coefficients c_{α_i,f_i} in the previous definition are unique and satisfy

$$x^{\alpha_i} \operatorname{lm}(f_i) = c_{\alpha_i,f_i} x^{\alpha_i+\exp(\operatorname{lm}(f_i))} + p_{\alpha_i,f_i},$$

where $p_{\alpha_i,f_i} = 0$ or $\deg(p_{\alpha_i,f_i}) < |\alpha_i + \exp(\operatorname{lm}(f_i))|$, $1 \le i \le t$.

(ii) $\operatorname{lm}(f) \succ \operatorname{lm}(h)$ and $f - h \in \langle F\}$, where $\langle F\}$ is the left ideal of A generated by F.

(iii) The remainder of f is not unique.

(iv) By definition we will assume that $0 \xrightarrow{F} 0$.

From the reduction relation we get the following interesting properties.

Proposition 92 ([19, Proposition 20]). *Let A be a skew PBW extension such that $c_{\alpha,\beta}$ is invertible for each $\alpha, \beta \in \mathbb{N}^n$. Let $f, h \in A$, $\theta \in \mathbb{N}^n$ and $F = \{f_1, \dots, f_t\}$ be a finite set of non-zero polynomials of A. Then*

(i) *If $f \xrightarrow{F} h$, then there exists $p \in A$ with $p = 0$ or $\operatorname{lm}(x^\theta f) \succ \operatorname{lm}(p)$ such that $x^\theta f + p \xrightarrow{F} x^\theta h$. In particular, if A is quasi-commutative, then $p = 0$.*

(ii) *If $f \xrightarrow{F}_{+} h$ and $p \in A$ is such that $p = 0$ or $\operatorname{lm}(h) \succ \operatorname{lm}(p)$, then $f + p \xrightarrow{F}_{+} h + p$.*

(iii) *If $f \xrightarrow{F}_{+} h$, then there exists $p \in A$ with $p = 0$ or $\operatorname{lm}(x^\theta f) \succ \operatorname{lm}(p)$ such that $x^\theta f + p \xrightarrow{F}_{+} x^\theta h$. If A is quasi-commutative, then $p = 0$.*

(iv) *If $f \xrightarrow{F}_{+} 0$, then there exists $p \in A$ with $p = 0$ or $\operatorname{lm}(x^\theta f) \succ \operatorname{lm}(p)$ such that $x^\theta f + p \xrightarrow{F}_{+} 0$. If A is quasi-commutative, then $p = 0$.*

The next theorem is the theoretical support of the division algorithm for skew PBW extensions.

Theorem 93 ([19, Theorem 21]). *Let $F = \{f_1, \dots, f_t\}$ be a finite set of non-zero polynomials in A and $f \in A$. Then the division algorithm below produces polynomials $q_1, \dots, q_t, h \in A$, with h reduced with respect to F, such that $f \xrightarrow{F}_{+} h$ and*

$$f = q_1 f_1 + \cdots + q_t f_t + h,$$

with

$$\operatorname{lm}(f) = \max\{\operatorname{lm}(\operatorname{lm}(q_1)\operatorname{lm}(f_1)), \dots, \operatorname{lm}(\operatorname{lm}(q_t)\operatorname{lm}(f_t)), \operatorname{lm}(h)\}.$$

Division algorithm in A

INPUT: *$f, f_1, \ldots, f_t \in A$ with $f_j \neq 0$ $(1 \leq j \leq t)$*
OUTPUT: *$q_1, \ldots, q_t$, $h \in A$ with $f = q_1 f_1 + \cdots + q_t f_t + h$,*
h reduced with respect to $\{f_1, \ldots, f_t\}$ and
$\mathrm{lm}(f) = \max\{\mathrm{lm}(\mathrm{lm}(q_1)\,\mathrm{lm}(f_1)), \ldots, \mathrm{lm}(\mathrm{lm}(q_t)\,\mathrm{lm}(f_t)), \mathrm{lm}(h)\}$
INITIALIZATION: $q_1 := 0$, $q_2 := 0, \ldots, q_t := 0$, $h := f$
WHILE *$h \neq 0$ and there exists j such that $\mathrm{lm}(f_j)$ divides $\mathrm{lm}(h)$* **DO**
Calculate $J := \{j \mid \mathrm{lm}(f_j)$ divides $\mathrm{lm}(h)\}$
FOR $j \in J$ **DO**
Calculate $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(\mathrm{lm}(f_j)) = \exp(\mathrm{lm}(h))$
IF *the equation $\mathrm{lc}(h) = \sum_{j\in J} r_j \sigma^{\alpha_j}(\mathrm{lc}(f_j))c_{\alpha_j,f_j}$ is soluble,*
where c_{α_j,f_j} are defined as in Theorem 8 **THEN**
Calculate one solution $(r_j)_{j\in J}$
$h := h - \sum_{j\in J} r_j x^{\alpha_j} f_j$
FOR $j \in J$ **DO**
$q_j := q_j + r_j x^{\alpha_j}$
ELSE
Stop

The following example illustrates the above procedure.

Example 94. We consider the *diffusion algebra* $\mathcal{A}$ in Example 18 with $n = 2$, $K = \mathbb{Q}$, $c_{12} = -2$ and $c_{21} = -1$. In this bijective skew PBW extension, $D_2D_1 = 2D_1D_2 + x_2D_1 - x_1D_2$ and the automorphisms σ_1 and σ_2 are the identity. We consider the deglex order with $D_1 \succ D_2$ and the polynomials $f_1 := x_1x_2D_1D_2$, $f_2 := x_2D_1$, $f_3 = x_1D_2$, $f = x_1x_2^2D_1^2D_2 + x_1^2x_2D_2$ in $\mathcal{A}$. We want to divide f by f_1, f_2 and f_3.

Step 1. We start with $h := f$, $q_1 := 0$, $q_2 := 0$, $q_3 := 0$. Since $\mathrm{lm}(f_j) \mid \mathrm{lm}(f)$ for $j = 1, 2, 3$, we compute $\alpha_j = (\alpha_{j1}, \alpha_{j2}) \in \mathbb{N}^2$ such that $\alpha_j + \exp(\mathrm{lm}(f_j)) = \exp(\mathrm{lm}(h))$ and the corresponding value of $\sigma^{\alpha_j}(\mathrm{lc}(f_j))c_{\alpha_j,\beta_j}$, where $\beta_j = \exp(\mathrm{lm}(f_j))$:

$$
\begin{aligned}
&(\alpha_{11}, \alpha_{12}) + (1,1) = (2,1) \;\Rightarrow\; \alpha_{11} = 1,\, \alpha_{12} = 0,\\
&\sigma^{\alpha_1}(\mathrm{lc}(f_1))c_{\alpha_1,\beta_1} = x_1x_2,\\
&(\alpha_{21}, \alpha_{22}) + (1,1) = (2,1) \;\Rightarrow\; \alpha_{21} = 1,\, \alpha_{22} = 1,\\
&\sigma^{\alpha_1}(\mathrm{lc}(f_2))c_{\alpha_2,\beta_2} = 2x_2,\\
&(\alpha_{31}, \alpha_{32}) + (1,1) = (2,1) \;\Rightarrow\; \alpha_{31} = 2,\, \alpha_{32} = 0,\\
&\sigma^{\alpha_1}(\mathrm{lc}(f_3))c_{\alpha_3,\beta_3} = x_1.
\end{aligned}
$$

Now, we solve the equation

$$\mathrm{lc}(h) = x_1x_2^2 = r_1(x_1x_2) + r^2(2x_2) + r_3(x_1) \;\Rightarrow\; r_1 = 3x_2,\, r_2 = -\tfrac{1}{2}x_1x_2,\, r_3 = -x_2^2,$$

and with the relations defining $\mathcal{A}$, we compute

$$
\begin{aligned}
h &= h - (r_1x^{\alpha_1}f_1 + r_2x^{\alpha_2}f_2 + r_3x^{\alpha_3}f_3)\\
&= h - 3x_1x_2^2D_1^2D_2 + \tfrac{1}{2}x_1x_2^2(2D_1^2D_2 + x_2D_1^2 - x_1D_1D_2)\\
&= \tfrac{1}{2}x_1x_2^3D_1^2 - \tfrac{1}{2}x_1^2x_2^2D_1D_2 + x_1^2x_2D_2.
\end{aligned}
$$

We also compute

$$q_1 := 3x_2D_1, \quad q_2 := -\tfrac{1}{2}x_1x_2D_1D_2, \quad q_3 := -x_2^2D_1^2.$$

STEP 2. $\text{lm}(h) = D_1^2$, $\text{lc}(h) = \frac{1}{2}x_1x_2^3$. In this case, $\text{lm}(f_j) \mid \text{lm}(f)$ only for $j = 2$ and we find that $\alpha_2 = (\alpha_{21}, \alpha_{22}) \in \mathbb{N}^3$ is such that $\alpha_j + \exp(\text{lm}(f_j)) = \exp(\text{lm}(h))$ is $\alpha = (1, 0)$; moreover, $\sigma^{\alpha}(\text{lc}(f_2))c_{\alpha,\beta} = x_2$ and $r = \frac{1}{2}x_1x_2^2$ is such that $\text{lc}(h) = rx_2$. Thus

$$h = h - rx^{\alpha_2}f_2 = -\tfrac{1}{2}x_1^2x_2^2D_1D_2 + x_1^2x_2D_2$$

and

$$q_1 := 3x_2D_1, \quad q_2 := -\tfrac{1}{2}x_1x_2D_1D_2 + \tfrac{1}{2}x_1x_2^2D_1, \quad q_3 := -x_2^2D_1^2.$$

STEP 3. Note that $\text{lm}(h) = D_1D_2$ and $\text{lm}(f_j) \mid \text{lm}(h)$ for $j = 1, 2, 3$. In this case we have:

$$\begin{aligned}
&(\alpha_{11}, \alpha_{12}) + (1, 1) = (1, 1) \;\Rightarrow\; \alpha_{11} = 0,\, \alpha_{12} = 0,\\
&\sigma^{\alpha_1}(\text{lc}(f_1))c_{\alpha_1,\beta_1} = x_1x_2,\\
&(\alpha_{21}, \alpha_{22}) + (1, 0) = (1, 1) \;\Rightarrow\; \alpha_{21} = 0,\, \alpha_{22} = 1,\\
&\sigma^{\alpha_2}(\text{lc}(f_2))c_{\alpha_2,\beta_2} = 2x_2,\\
&(\alpha_{31}, \alpha_{32}) + (0, 1) = (1, 1, 1) \;\Rightarrow\; \alpha_{31} = 1,\, \alpha_{32} = 0,\\
&\sigma^{\alpha_3}(\text{lc}(f_3))c_{\alpha_3,\beta_3} = x_1.
\end{aligned}$$

We solve

$$-\tfrac{1}{2}x_1^2x_2^2 = r_1x_1x_2 + r_2(2x_2) + r_3x_1 \;\Rightarrow\; r_1 = 3x_1x_2, \quad r_2 = -x_1^2x_2, \quad r_3 = -\tfrac{3}{2}x_1x_2^2;$$

thus

$$\begin{aligned}
h &= h - (r_1x^{\alpha_1}f_1 + r_2x^{\alpha_2}f_2 + r_3x^{\alpha_3}f_3)\\
&= h - \big(3x_1^2x_2^2D_1D_2 - x_1^2x_2^2(2D_1D_2 + x_2D_1 - x_1D_2) - \tfrac{3}{2}x_1^2D_1D_2\big)\\
&= x_1^2x_2^3D_1 + (x_1^2x_2 - x_1^3x_2^2)D_2
\end{aligned}$$

and also

$$q_1 := 3x_2D_1 - 3x_1x_2,\ q_2 := -\tfrac{1}{2}x_1x_2D_1D_2 + \tfrac{1}{2}x_1x_2^2D_1 - x_1^2x_2D_2,\ q_3 := -x_2^2D_1^2 - \tfrac{3}{2}x_1x_2^2D_1.$$

STEP 4. Finally, note that $h = x_1^2x_2^3D_1 + (x_1^2x_2 - x_1^3x_2^2)D_2 = x_1^2x_2^2f_1 + (x_1x_2 - x_1^2x_2^2)f_3$, thus

$$f = q_1f_1 + q_2f_2 + q_3f_3$$

where

$$\begin{aligned}
q_1 &:= 3x_2D_1 - 3x_1x_2, \quad q_2 := -\tfrac{1}{2}x_1x_2D_1D_2 + \tfrac{1}{2}x_1x_2^2D_1 - x_1^2x_2D_2 + x_1^2x_2^2,\\
q_3 &:= -x_2^2D_1^2 - \tfrac{3}{2}x_1x_2^2D_1 + x_1x_2 - x_1^2x_2^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\max\{\text{lm}(\text{lm}(q_1)\,\text{lm}(f_1)), \text{lm}(\text{lm}(q_2)\,\text{lm}(f_2)), \text{lm}(\text{lm}(q_3)\,\text{lm}(f_3))\}\\
&\qquad\qquad = \max\{D_1^2D_2, D_1^2D_2, D_1^2D_2\} = \text{lm}(f).
\end{aligned}$$

6.3. Gröbner bases of left ideals. Our next purpose is to recall the definition of a Gröbner basis for the left ideals of the skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$.

DEFINITION 95. Let $I \neq 0$ be a left ideal of A and let G be a non-empty finite subset of non-zero polynomials of I. We say that G is a *Gröbner basis* for I if each element $0 \neq f \in I$ is reducible with respect to G.

We will say that $\{0\}$ is a Gröbner basis for $I = 0$.

THEOREM 96 ([19, Theorem 24]). *Let $I \neq 0$ be a left ideal of A and let G be a finite subset of non-zero polynomials of I. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for I.*
(ii) *For any polynomial $f \in A$,*

$$f \in I \quad \textit{if and only if} \quad f \xrightarrow{G}_{+} 0.$$

(iii) *For any $0 \neq f \in I$ there exist $g_1, \ldots, g_t \in G$ such that $\mathrm{lm}(g_j) \,|\, \mathrm{lm}(f)$, $1 \leq j \leq t$ (i.e., there exist $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(\mathrm{lm}(g_j)) = \exp(\mathrm{lm}(f))$) and*

$$\mathrm{lc}(f) \in \langle \sigma^{\alpha_1}(\mathrm{lc}(g_1))c_{\alpha_1,g_1}, \ldots, \sigma^{\alpha_t}(\mathrm{lc}(g_t))c_{\alpha_t,g_t}\}.$$

(iv) *For $\alpha \in \mathbb{N}^n$, let $\langle \alpha, I\}$ be the left ideal of R defined by*

$$\langle \alpha, I\} := \langle \mathrm{lc}(f) \mid f \in I, \exp(\mathrm{lm}(f)) = \alpha\}.$$

Then $\langle \alpha, I\} = J$, with

$$J := \langle \sigma^{\beta}(\mathrm{lc}(g))c_{\beta,g} \mid g \in G, \ \textit{with}\ \beta + \exp(\mathrm{lm}(g)) = \alpha\}.$$

From this theorem we get the following consequences.

COROLLARY 97. *Let $I \neq 0$ be a left ideal of A. Then*

(i) *If G is a Gröbner basis for I, then $I = \langle G\}$.*
(ii) *Let G be a Gröbner basis for I. If $f \in I$ and $f \xrightarrow{G}_{+} h$, with h reduced, then $h = 0$.*
(iii) *Let $G = \{g_1, \ldots, g_t\}$ be a set of non-zero polynomials of I with $\mathrm{lc}(g_i) \in R^*$ for each $1 \leq i \leq t$. Then G is a Gröbner basis of I if and only if given $0 \neq r \in I$ there exists i such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(r)$.*

Proof. (i) This is a direct consequence of Theorem 96.

(ii) Let $f \in I$ and $f \xrightarrow{G}_{+} h$, with h reduced. Since $f - h \in \langle G\} = I$, we have $h \in I$; if $h \neq 0$, then h can be reduced by G, but this is not possible since h is reduced.

(iii) If G is a Gröbner basis of I, then given $0 \neq r \in I$, r is reducible with respect to G, hence there exists i such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(r)$. Conversely, if this condition holds for some i, then r is reducible with respect to G since the equation $\mathrm{lc}(r) = r_1\sigma^{\alpha_i}(\mathrm{lc}(g_i)c_{\alpha_i,g_i}$, with $\alpha_i + \exp(\mathrm{lm}(g_i)) = \exp(\mathrm{lm}(r))$, is soluble with solution $r_1 = \mathrm{lc}(r)c'_{\alpha_i,g_i}(\sigma^{\alpha_i}(\mathrm{lc}(g_i)))^{-1}$, where c'_{α_i,g_i} is a left inverse of c_{α_i,g_i}. ∎

6.4. Buchberger's algorithm for left ideals. In [19] Buchberger's algorithm was constructed for computing Gröbner bases of left ideals for the particular case of quasi-commutative bijective skew PBW extensions. In this subsection we extend Buchberger's procedure to the general case of bijective skew PBW extensions, i.e., without assuming that they are quasi-commutative. Complementing Remark 89, *from now on we will assume that* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *is bijective.*

We start by fixing some notation and proving a key lemma.

DEFINITION 98. Let $F := \{g_1, \dots, g_s\} \subseteq A$, let X_F be the least common multiple of $\mathrm{lm}(g_1), \dots, \mathrm{lm}(g_s)$, let $\theta \in \mathbb{N}^n$, $\beta_i := \exp(\mathrm{lm}(g_i))$ and $\gamma_i \in \mathbb{N}^n$ be such that $\gamma_i + \beta_i = \exp(X_F)$, $1 \leq i \leq s$. Then $B_{F,\theta}$ will denote a finite set of generators of

$$S_{F,\theta} := \mathrm{Syz}_R[\sigma^{\gamma_1+\theta}(\mathrm{lc}(g_1))c_{\gamma_1+\theta,\beta_1} \ \dots \ \sigma^{\gamma_s+\theta}(\mathrm{lc}(g_s))c_{\gamma_s+\theta,\beta_s}].$$

For $\theta = \mathbf{0} := (0, \dots, 0)$, $S_{F,\theta}$ will be denoted by S_F and $B_{F,\theta}$ by B_F.

REMARK 99. Let $(b_1, \dots, b_s) \in S_{F,\theta}$. Since A is bijective, there exists a unique $(b'_1, \dots, b'_s) \in S_F$ such that $b_i = \sigma^\theta(b'_i)c_{\theta,\gamma_i}$ for $1 \leq i \leq s$: in fact, the existence and uniqueness of $(b'_1, \dots, b'_s)$ follow from the bijectivity of A. Now, since $(b_1, \dots, b_s) \in S_{F,\theta}$, we have $\sum_{i=1}^s b_i\sigma^{\theta+\gamma_i}(\mathrm{lc}(g_i))c_{\theta+\gamma_i,\beta_i} = 0$. Replacing b_i by $\sigma^\theta(b'_i)c_{\theta,\gamma_i}$ in the last equation, we obtain

$$\sum_{i=1}^s \sigma^\theta(b'_i)c_{\theta,\gamma_i}\sigma^{\theta+\gamma_i}(\mathrm{lc}(g_i))c^{-1}_{\theta,\gamma_i}c_{\theta,\gamma_i}c_{\theta+\gamma_i,\beta_i} = 0;$$

multiplying by $c^{-1}_{\theta,\gamma_i+\beta_i}$ we get

$$\sum_{i=1}^s \sigma^\theta(b'_i)c_{\theta,\gamma_i}\sigma^{\theta+\gamma_i}(\mathrm{lc}(g_i))c^{-1}_{\theta,\gamma_i}c_{\theta,\gamma_i}c_{\theta+\gamma_i,\beta_i}c^{-1}_{\theta,\gamma_i+\beta_i} = 0.$$

Now we can use the identities of Remark 9, so

$$\sum_{i=1}^s \sigma^\theta(b'_i)\sigma^\theta\big(\sigma^{\gamma_i}(\mathrm{lc}(g_i))\big)\sigma^\theta(c_{\gamma_i,\beta_i}) = 0,$$

and since σ^θ is injective, we have $\sum_{i=1}^s b'_i\sigma^{\gamma_i}(\mathrm{lc}(g_i))c_{\gamma_i,\beta_i} = 0$, i.e., $(b'_1, \dots, b'_s) \in S_F$.

LEMMA 100. *Let* $g_1, \dots, g_s \in A$, $c_1, \dots, c_s \in R - \{0\}$ *and* $\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$ *be such that* $\alpha_1 + \exp(g_1) = \cdots = \alpha_s + \exp(g_s) := \delta$. *If* $\mathrm{lm}(\sum_{i=1}^s c_i x^{\alpha_i} g_i) \prec x^\delta$, *then there exist* $r_1, \dots, r_k \in R$ *and* $l_1, \dots, l_s \in A$ *such that*

$$\sum_{i=1}^s c_i x^{\alpha_i} g_i = \sum_{j=1}^k r_j x^{\delta-\exp(X_F)}\Big(\sum_{i=1}^s b_{ji}x^{\gamma_i}g_i\Big) + \sum_{i=1}^s l_i g_i,$$

where X_F *is the least common multiple of* $\mathrm{lm}(g_1), \dots, \mathrm{lm}(g_s)$, $\gamma_i \in \mathbb{N}^n$ *is such that* $\gamma_i + \exp(g_i) = \exp(X_F)$, $1 \leq i \leq s$, *and*

$$B_F := \{\mathbf{b}_1, \dots, \mathbf{b}_k\} := \{(b_{11}, \dots, b_{1s}), \dots, (b_{k1}, \dots, b_{ks})\}.$$

Moreover, $\mathrm{lm}(x^{\delta-\exp(X_F)}\sum_{i=1}^s b_{ji}x^{\gamma_i}g_i) \prec x^\delta$ *for every* $1 \leq j \leq k$, *and* $\mathrm{lm}(l_i g_i) \prec x^\delta$ *for every* $1 \leq i \leq s$.

Proof. Let $x^{\beta_i} := \mathrm{lm}(g_i)$ for $1 \leq i \leq s$. Since $x^\delta = \mathrm{lm}(x_i^\alpha \mathrm{lm}(g_i))$, we have $\mathrm{lm}(g_i) \,|\, x^\delta$ and hence $X_F \,|\, x^\delta$, so there exists $\theta \in \mathbb{N}^n$ such that $\exp(X_F) + \theta = \delta$. On the other hand, $\gamma_i + \beta_i = \exp(X_F)$ and $\alpha_i + \beta_i = \delta$, so $\alpha_i = \gamma_i + \theta$ for every $1 \leq i \leq s$. Now, $\mathrm{lm}(\sum_{i=1}^s c_i x^{\alpha_i} g_i) \prec x^\delta$ implies that $\sum_{i=1}^s c_i\sigma^{\alpha_i}(\mathrm{lc}(g_i))c_{\alpha_i,\beta_i} = 0$. So we have

$\sum_{i=1}^{s} c_i\sigma^{\theta+\gamma_i}(\mathrm{lc}(g_i))c_{\theta+\gamma_i,\beta_i} = 0$. This implies that $(c_1,\dots,c_s)\in S_{F,\theta}$; from Remark 99 we know that there exists a unique $(c'_1,\dots,c'_s)\in S_F$ such that $c_i=\sigma^\theta(c'_i)c_{\theta,\gamma_i}$. Then

$$\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{i=1}^{s}\sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\alpha_i}g_i.$$

Now,

$$\begin{aligned} x^\theta c'_i x^{\gamma_i} &= (\sigma^\theta(c'_i)x^\theta + p_{c'_i,\theta})x^{\gamma_i} = \sigma^\theta(c'_i)x^\theta x^{\gamma_i} + p_{c'_i,\theta}x^{\gamma_i}\\ &= \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} + \sigma^\theta(c'_i)p_{\theta,\gamma_i} + p_{c'_i,\theta}x^{\gamma_i} = \sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} + p'_i \end{aligned}$$

where $p'_i := \sigma^\theta(c'_i)p_{\theta,\gamma_i} + p_{c'_i,\theta}x^{\gamma_i}$; note that $p'_i=0$ or $\mathrm{lm}(p'_i)\prec x^{\theta+\gamma_i}$ for each i. Thus, $\sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\theta+\gamma_i} = x^\theta c'_i x^{\gamma_i} + p_i$, with $p_i=0$ or $\mathrm{lm}(p_i)\prec x^{\theta+\gamma_i}$. Hence,

$$\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{i=1}^{s}\sigma^\theta(c'_i)c_{\theta,\gamma_i}x^{\alpha_i}g_i = \sum_{i=1}^{s}(x^\theta c'_i x^{\gamma_i}+p_i)g_i = \sum_{i=1}^{s}x^\theta c'_i x^{\gamma_i}g_i + \sum_{i=1}^{s}p_i g_i,$$

with $p_ig_i=0$ or $\mathrm{lm}(p_ig_i)\prec x^{\theta+\gamma_i+\beta_i}=x^\delta$. On the other hand, since $(c'_1,\dots,c'_s)\in S_F$, there exist $r'_1,\dots,r'_k\in R$ such that $(c'_1,\dots,c'_s)=r'_1\mathbf{b}_1+\cdots+r'_k\mathbf{b}_k = r'_1(b_{11},\dots,b_{1s})+\cdots+r'_k(b_{k1},\dots,b_{ks})$, thus $c'_i=\sum_{j=1}^{k} r'_j b_{ji}$. Using this, we have

$$\begin{aligned} \sum_{i=1}^{s} x^\theta c'_i x^{\gamma_i} g_i &= \sum_{i=1}^{s} x^\theta\Big(\sum_{j=1}^{k} r'_j b_{ji}\Big)x^{\gamma_i}g_i = \sum_{i=1}^{s}\Big(\sum_{j=1}^{k} x^\theta r'_j b_{ji}\Big)x^{\gamma_i}g_i\\ &= \sum_{i=1}^{s}\Big(\sum_{j=1}^{k}(\sigma^\theta(r'_j)x^\theta + p_{r'_j,\theta})b_{ji}\Big)x^{\gamma_i}g_i\\ &= \sum_{i=1}^{s}\Big(\sum_{j=1}^{k}\sigma^\theta(r'_j)x^\theta b_{ji}x^{\gamma_i}g_i + \sum_{j=1}^{k}p_{r'_j,\theta}b_{ji}x^{\gamma_i}g_i\Big)\\ &= \sum_{j=1}^{k}\sum_{i=1}^{s}\sigma^\theta(r'_j)x^\theta b_{ji}x^{\gamma_i}g_i + \sum_{i=1}^{s}\sum_{j=1}^{k}p_{r'_j,\theta}b_{ji}x^{\gamma_i}g_i\\ &= \sum_{j=1}^{k}\sigma^\theta(r'_j)x^\theta\sum_{i=1}^{s}b_{ji}x^{\gamma_i}g_i + \sum_{i=1}^{s}q_ig_i, \end{aligned}$$

where $q_i := \sum_{j=1}^{k} p_{r'_j,\theta}b_{ji}x^{\gamma_i}=0$ or $\mathrm{lm}(q_i)\prec x^{\theta+\gamma_i}$. Therefore,

$$\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{j=1}^{k} r_j x^\theta\sum_{i=1}^{s} b_{ji}x^{\gamma_i}g_i + \sum_{i=1}^{s} l_i g_i,$$

with $l_i := p_i+q_i$ for $1\le i\le s$ and $r_j:=\sigma^\theta(r'_j)$ for $1\le j\le k$. Finally, it is easy to see that $\mathrm{lm}(x^\theta(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}g_i))\prec x^\delta$ since $\mathrm{lm}(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}g_i)\prec x^{\gamma_i+\beta_i}$, and $\mathrm{lm}(l_ig_i)=\mathrm{lm}(p_ig_i+q_ig_i)\prec x^\delta$. ∎

With the notation of Definition 98 and Lemma 100, we can prove the main result of the present section.

THEOREM 101. *Let $I\neq 0$ be a left ideal of A and let G be a finite subset of non-zero generators of I. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of I.*
(ii) *For all $F := \{g_1, \dots, g_s\} \subseteq G$, and for any $(b_1, \dots, b_s) \in B_F$,*

$$\sum_{i=1}^{s} b_i x^{\gamma_i} g_i \xrightarrow{G}_{+} 0.$$

Proof. (i)⇒(ii): We observe that $f := \sum_{i=1}^{s} b_i x^{\gamma_i} g_i \in I$, so by Theorem 96, $f \xrightarrow{G}_{+} 0$.

(ii)⇒(i): Let $0 \neq f \in I$. We will prove that condition (iii) of Theorem 96 holds. Let $G := \{g_1, \dots, g_t\}$. Then there exist $h_1, \dots, h_t \in A$ such that $f = h_1 g_1 + \cdots + h_t g_t$, and we can choose $\{h_i\}_{i=1}^{t}$ such that

$$x^{\delta} := \max\{\operatorname{lm}(\operatorname{lm}(h_i)\operatorname{lm}(g_i))\}_{i=1}^{t}$$

is minimal. Let $\operatorname{lm}(h_i) := x^{\alpha_i}$, $c_i := \operatorname{lc}(h_i)$, $\operatorname{lm}(g_i) = x^{\beta_i}$ for $1 \leq i \leq t$ and $F := \{g_i \in G \mid \operatorname{lm}(\operatorname{lm}(h_i)\operatorname{lm}(g_i)) = x^{\delta}\}$. Renumbering the elements of G we can assume that $F = \{g_1, \dots, g_s\}$. We will consider two possible cases:

Case 1: $\operatorname{lm}(f) = x^{\delta}$. Then $\operatorname{lm}(g_i) \mid \operatorname{lm}(f)$ for $1 \leq i \leq s$ and

$$\operatorname{lc}(f) = c_1 \sigma^{\alpha_1}(\operatorname{lc}(g_1)) c_{\alpha_1,\beta_1} + \cdots + c_s \sigma^{\alpha_s}(\operatorname{lc}(g_s)) c_{\alpha_s,\beta_s},$$

as required.

Case 2: $\operatorname{lm}(f) \prec x^{\delta}$. We will prove that this produces a contradiction. To begin, note that f can be written as

$$f = \sum_{i=1}^{s} c_i x^{\alpha_i} g_i + \sum_{i=1}^{s} (h_i - c_i x^{\alpha_i}) g_i + \sum_{i=s+1}^{t} h_i g_i; \tag{6.2}$$

we have $\operatorname{lm}((h_i - c_i x^{\alpha_i}) g_i) \prec x^{\delta}$ for every $1 \leq i \leq s$ and $\operatorname{lm}(h_i g_i) \prec x^{\delta}$ for every $s+1 \leq i \leq t$, so

$$\operatorname{lm}\Big(\sum_{i=1}^{s} (h_i - c_i x^{\alpha_i}) g_i\Big) \prec x^{\delta} \quad \text{and} \quad \operatorname{lm}\Big(\sum_{i=s+1}^{t} h_i g_i\Big) \prec x^{\delta},$$

and hence $\operatorname{lm}(\sum_{i=1}^{s} c_i x^{\alpha_i} g_i) \prec x^{\delta}$. By Lemma 100 (and its notation), we have

$$\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{j=1}^{k} r_j x^{\delta - \exp(X_F)} \Big(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i\Big) + \sum_{i=1}^{s} l_i g_i, \tag{6.3}$$

where $\operatorname{lm}(x^{\delta - \exp(X_F)} \sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i) \prec x^{\delta}$ for every $1 \leq j \leq k$ and $\operatorname{lm}(l_i g_i) \prec x^{\delta}$ for $1 \leq i \leq s$. By the hypothesis, $\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i \xrightarrow{G}_{+} 0$, whence, by Theorem 93, there exist $q_1, \dots, q_t \in A$ such that $\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i = \sum_{i=1}^{t} q_i g_i$, with

$$\operatorname{lm}\Big(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i\Big) = \max\{\operatorname{lm}(\operatorname{lm}(q_i)\operatorname{lm}(g_i))\}_{i=1}^{t}.$$

But $(b_{j1}, \dots, b_{js}) \in B_F$, so $\operatorname{lm}(\sum_{i=1}^{s} b_{ji} x^{\gamma_i} g_i) \prec X_F$ and hence $\operatorname{lm}(\operatorname{lm}(q_i)\operatorname{lm}(g_i)) \prec X_F$

for every $1 \leq i \leq t$. Thus,

$$\begin{aligned}\sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)}\Big(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}g_i\Big) &= \sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)}\Big(\sum_{i=1}^{t} q_i g_i\Big)\\ &= \sum_{i=1}^{t}\sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)} q_i g_i = \sum_{i=1}^{t} \widetilde{q}_i g_i,\end{aligned}$$

with $\widetilde{q}_i := \sum_{j=1}^{k} r_j x^{\delta-\exp(X_F)} q_i$ and $\operatorname{lm}(\widetilde{q}_i g_i) \prec x^{\delta}$ for every $1 \leq i \leq t$. Substituting $\sum_{i=1}^{s} c_i x^{\alpha_i} g_i = \sum_{i=1}^{t} \widetilde{q}_i g_i + \sum_{i=1}^{s} l_i g_i$ into equation (6.2), we obtain

$$f = \sum_{i=1}^{t} \widetilde{q}_i g_i + \sum_{i=1}^{s}(h_i - c_i x^{\alpha_i})g_i + \sum_{i=1}^{s} l_i g_i + \sum_{i=s+1}^{t} h_i g_i,$$

and so we have expressed f as a combination of polynomials $g_1, \ldots, g_t$, where every term has leading monomial $\prec x^{\delta}$. This contradicts the minimality of x^{δ} and finishes the proof. ∎

Corollary 102. *Let $F = \{f_1, \ldots, f_s\}$ be a set of non-zero polynomials of A. The algorithm below produces a Gröbner basis for the left ideal $\langle F\}$ of A (here $P(X)$ denotes the set of subsets of a set X):*

Buchberger's algorithm for bijective skew PBW extensions

INPUT: $F := \{f_1, \ldots, f_s\} \subseteq A$, $f_i \neq 0$, $1 \leq i \leq s$
OUTPUT: $G = \{g_1, \ldots, g_t\}$ *a Gröbner basis for* $\langle F\}$
INITIALIZATION: $G := \emptyset$, $G' := F$
WHILE $G' \neq G$ **DO**
  $D := P(G') - P(G)$
  $G := G'$
  FOR *each* $S := \{g_{i_1}, \ldots, g_{i_k}\} \in D$ **DO**
    Compute B_S
    FOR *each* $\mathbf{b} = (b_1, \ldots, b_k) \in B_S$ **DO**
      Reduce $\sum_{j=1}^{k} b_j x^{\gamma_j} g_{i_j} \xrightarrow{G'}_{+} r$, *with* r *reduced with respect to* G' *and* γ_j *defined as in Definition* 98
        IF $r \neq 0$ **THEN**
        $G' := G' \cup \{r\}$

From Theorem 14 and the previous corollary we get the following direct conclusion.

Corollary 103. *Each left ideal of A has a Gröbner basis.*

6.5. Gröbner bases of modules. In this subsection we present the general theory of Gröbner bases for submodules of A^m, $m \geq 1$, where $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ is a bijective skew PBW extension of R, with R an LGS ring (see Definition 88) and $\operatorname{Mon}(A)$ endowed with some monomial order (see Definition 83). A^m is the left free A-module of column vectors of length $m \geq 1$; since A is a left Noetherian ring (Theorem 14), A is an IBN ring (Invariant Basis Number, see [32]), and hence all bases of the free module A^m have m elements. Note moreover that A^m is left Noetherian, and hence any submodule of A^m

is finitely generated. This theory was studied in [25] and [26], but now we will extend Buchberger's algorithm to the general bijective case without assuming that A is quasi-commutative. The results presented in this section are an easy generalization of those of the previous sections, i.e., taking $m = 1$ we get the theory of Gröbner bases for the left ideals of A developed before. We will omit the proofs since most of them can be found in [25] and [26] or they are an easy adaptation of those of the previous sections. The theory presented in this section has also been studied by Gómez-Torrecillas et al. (see [6], [7]) for left PBW algebras over division rings and assuming some special commutativity conditions.

6.5.1. Monomial orders on $\mathrm{Mon}(A^m)$**.** In this subsection we will represent the elements of A^m as row vectors. We recall that the canonical basis of A^m is

$$\mathbf{e}_1 = (1, 0, \ldots, 0),\ \mathbf{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \mathbf{e}_m = (0, 0, \ldots, 1).$$

DEFINITION 104. A *monomial* in A^m is a vector $\mathbf{X} = X\mathbf{e}_i$, where $X = x^\alpha \in \mathrm{Mon}(A)$ and $1 \leq i \leq m$, i.e.,

$$\mathbf{X} = X\mathbf{e}_i = (0, \ldots, X, \ldots, 0),$$

where X is in the ith position, named the index of $\mathbf{X}$, $\mathrm{ind}(\mathbf{X}) := i$. A *term* is a vector $c\mathbf{X}$, where $c \in R$. The set of monomials of A^m will be denoted by $\mathrm{Mon}(A^m)$. Let $\mathbf{Y} = Y\mathbf{e}_j \in \mathrm{Mon}(A^m)$. We say that $\mathbf{X}$ *divides* $\mathbf{Y}$ if $i = j$ and X divides Y. We will say that any monomial $\mathbf{X} \in \mathrm{Mon}(A^m)$ divides the null vector $\mathbf{0}$. The *least common multiple* of $\mathbf{X}$ and $\mathbf{Y}$, denoted by $\mathrm{lcm}(\mathbf{X}, \mathbf{Y})$, is $\mathbf{0}$ if $i \neq j$, and $U\mathbf{e}_i$, where $U = \mathrm{lcm}(X, Y)$, if $i = j$. Finally, we let $\exp(\mathbf{X}) := \exp(X) = \alpha$ and $\deg(\mathbf{X}) := \deg(X) = |\alpha|$.

We now define monomials orders on $\mathrm{Mon}(A^m)$.

DEFINITION 105. A *monomial order* on $\mathrm{Mon}(A^m)$ is a total order $\succeq$ satisfying:

(i) $\mathrm{lm}(x^\beta x^\alpha)\mathbf{e}_i \succeq x^\alpha\mathbf{e}_i$ for every monomial $\mathbf{X} = x^\alpha\mathbf{e}_i \in \mathrm{Mon}(A^m)$ and any monomial x^β in $\mathrm{Mon}(A)$.
(ii) If $\mathbf{Y} = x^\beta\mathbf{e}_j \succeq \mathbf{X} = x^\alpha\mathbf{e}_i$, then $\mathrm{lm}(x^\gamma x^\beta)\mathbf{e}_j \succeq \mathrm{lm}(x^\gamma x^\alpha)\mathbf{e}_i$ for every monomial $x^\gamma \in \mathrm{Mon}(A)$.
(iii) $\succeq$ is degree compatible, i.e., $\deg(\mathbf{X}) \geq \deg(\mathbf{Y}) \Rightarrow \mathbf{X} \succeq \mathbf{Y}$.

If $\mathbf{X} \succeq \mathbf{Y}$ but $\mathbf{X} \neq \mathbf{Y}$ we will write $\mathbf{X} \succ \mathbf{Y}$, and $\mathbf{Y} \preceq \mathbf{X}$ means that $\mathbf{X} \succeq \mathbf{Y}$.

PROPOSITION 106. *Every monomial order on* $\mathrm{Mon}(A^m)$ *is a well-order.*

Given a monomial order $\succeq$ on $\mathrm{Mon}(A)$, we can define two natural orders on $\mathrm{Mon}(A^m)$.

DEFINITION 107. Let $\mathbf{X} = X\mathbf{e}_i$ and $\mathbf{Y} = Y\mathbf{e}_j \in \mathrm{Mon}(A^m)$.

(i) The TOP (*term over position*) order is defined by

$$\mathbf{X} \succeq \mathbf{Y} \Leftrightarrow \begin{cases} X \succeq Y \\ \text{or} \\ X = Y \text{ and } i > j. \end{cases}$$

(ii) The TOPREV order is defined by

$$\mathbf{X} \succeq \mathbf{Y} \Leftrightarrow \begin{cases} X \succeq Y \\ \text{or} \\ X = Y \text{ and } i < j. \end{cases}$$

Remark 108. (i) Note that

$$\mathbf{e}_m \succ \mathbf{e}_{m-1} \succ \cdots \succ \mathbf{e}_1 \quad \text{for TOP},$$
$$\mathbf{e}_1 \succ \mathbf{e}_2 \succ \cdots \succ \mathbf{e}_m \quad \text{for TOPREV}.$$

(ii) The POT (*position over term*) and POTREV orders defined in [1] and [30] for modules over classical polynomial commutative rings are not degree compatible.

(iii) Other examples of monomial orders on $\mathrm{Mon}(A^m)$ are considered in [7], e.g., orders with weight.

Fix a monomial order on $\mathrm{Mon}(A)$. Let $\mathbf{f} \neq \mathbf{0}$ be a vector of A^m. Then we may write $\mathbf{f}$ as a sum of terms in the following way:

$$\mathbf{f} = c_1\mathbf{X}_1 + \cdots + c_t\mathbf{X}_t,$$

where $c_1, \ldots, c_t \in R - 0$ and $\mathbf{X}_1 \succ \cdots \succ \mathbf{X}_t$ are monomials in $\mathrm{Mon}(A^m)$.

Definition 109. With the above notation, we say that:

(i) $\mathrm{lt}(\mathbf{f}) := c_1\mathbf{X}_1$ is the *leading term* of $\mathbf{f}$.
(ii) $\mathrm{lc}(\mathbf{f}) := c_1$ is the *leading coefficient* of $\mathbf{f}$.
(iii) $\mathrm{lm}(\mathbf{f}) := \mathbf{X}_1$ is the *leading monomial* of $\mathbf{f}$.

For $\mathbf{f} = \mathbf{0}$ we define $\mathrm{lm}(\mathbf{0}) = \mathbf{0}$, $\mathrm{lc}(\mathbf{0}) = 0$, $\mathrm{lt}(\mathbf{0}) = \mathbf{0}$, and if $\succeq$ is a monomial order on $\mathrm{Mon}(A^m)$, then we define $\mathbf{X} \succ \mathbf{0}$ for any $\mathbf{X} \in \mathrm{Mon}(A^m)$. So, we extend $\succeq$ to $\mathrm{Mon}(A^m) \cup \{\mathbf{0}\}$.

6.5.2. Division algorithm and Gröbner bases for submodules of A^m. The reduction process, Theorem 93 and the division algorithm for left ideals can be easily adapted for submodules of A^m.

Definition 110. Let $M \neq 0$ be a submodule of A^m and let G be a non-empty finite subset of non-zero vectors of M. We say that G is a *Gröbner basis* for M if each element $\mathbf{0} \neq \mathbf{f} \in M$ is reducible with respect to G.

We will say that $\{\mathbf{0}\}$ is a Gröbner basis for $M = 0$.

Theorem 111 ([26]). *Let $M \neq 0$ be a submodule of A^m and let G be a finite subset of non-zero vectors of M. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis for M.*
(ii) *For any vector $\mathbf{f} \in A^m$,*

$$\mathbf{f} \in M \quad \textit{if and only if} \quad \mathbf{f} \xrightarrow{G}_{+} \mathbf{0}.$$

(iii) *For any $\mathbf{0} \neq \mathbf{f} \in M$ there exist $\mathbf{g}_1, \ldots, \mathbf{g}_t \in G$ such that $\mathrm{lm}(\mathbf{g}_j) \mid \mathrm{lm}(\mathbf{f})$, $1 \leq j \leq t$, (i.e., $\mathrm{ind}(\mathrm{lm}(\mathbf{g}_j)) = \mathrm{ind}(\mathrm{lm}(\mathbf{f}))$ and there exist $\alpha_j \in \mathbb{N}^n$ such that $\alpha_j + \exp(\mathrm{lm}(\mathbf{g}_j)) =$*

$\exp(\operatorname{lm}(\mathbf{f})))$ *and*

$$\operatorname{lc}(\mathbf{f}) \in \langle \sigma^{\alpha_1}(\operatorname{lc}(\mathbf{g}_1))c_{\alpha_1,\mathbf{g}_1}, \dots, \sigma^{\alpha_t}(\operatorname{lc}(\mathbf{g}_t))c_{\alpha_t,\mathbf{g}_t}\}.$$

(iv) *For* $\alpha \in \mathbb{N}^n$ *and* $1 \leq u \leq m$, *let* $\langle \alpha, M\}_u$ *be the left ideal of* R *defined by*

$$\langle \alpha, M\}_u := \langle \operatorname{lc}(\mathbf{f}) \mid \mathbf{f} \in M,\ \operatorname{ind}(\operatorname{lm}(\mathbf{f})) = u,\ \exp(\operatorname{lm}(\mathbf{f})) = \alpha\}.$$

Then $\langle \alpha, M\}_u = J_u$, *with*

$$J_u := \langle \sigma^{\beta}(\operatorname{lc}(\mathbf{g}))c_{\beta,\mathbf{g}} \mid \mathbf{g} \in G,\ \operatorname{ind}(\operatorname{lm}(\mathbf{g})) = u \text{ and } \beta + \exp(\operatorname{lm}(\mathbf{g})) = \alpha\}.$$

From this theorem we get the following consequences.

COROLLARY 112. *Let* $M \neq 0$ *be a submodule of* A^m. *Then*

(i) *If* G *is a Gröbner basis for* M, *then* $M = \langle G \rangle$.
(ii) *Let* G *be a Gröbner basis for* M. *If* $\mathbf{f} \in M$ *and* $\mathbf{f} \xrightarrow{G}_{+} \mathbf{h}$, *with* $\mathbf{h}$ *reduced, then* $\mathbf{h} = \mathbf{0}$.
(iii) *Let* $G = \{\mathbf{g}_1, \dots, \mathbf{g}_t\}$ *be a set of non-zero vectors of* M *with* $\operatorname{lc}(\mathbf{g}_i) \in R^*$ *for each* $1 \leq i \leq t$. *Then* G *is a Gröbner basis of* M *if and only if given* $0 \neq \mathbf{r} \in M$ *there exists* i *such that* $\operatorname{lm}(\mathbf{g}_i)$ *divides* $\operatorname{lm}(\mathbf{r})$.

Proof. The proof is an easy adaptation of the proof of Corollary 97. ■

Note that the remainder of $\mathbf{f} \in A^m$ with respect to a Gröbner basis is not unique. Moreover, if the term order is changed, a Gröbner basis may not be a Gröbner basis any further. In fact, a counterexample was given in [30] for the trivial case when $A = R[x_1, \dots, x_n]$ is the commutative polynomial ring.

6.5.3. Buchberger's algorithm for modules. Recall that we are assuming that A is a bijective skew PBW extension. We will observe that every submodule M of A^m has a Gröbner basis, and we will construct the Buchberger algorithm for computing such bases. The results obtained here improve those of [26] and [25] and generalize the results obtained in Section 6.4 for left ideals.

We start by fixing some notation and proving a preliminary general result.

DEFINITION 113. Let $F := \{\mathbf{g}_1, \dots, \mathbf{g}_s\} \subseteq A^m$ be such that the least common multiple of $\operatorname{lm}(\mathbf{g}_1), \dots, \operatorname{lm}(\mathbf{g}_s)$, denoted by $\mathbf{X}_F$, is non-zero. Let $\theta \in \mathbb{N}^n$, $\beta_i := \exp(\operatorname{lm}(\mathbf{g}_i))$ and $\gamma_i \in \mathbb{N}^n$ be such that $\gamma_i + \beta_i = \exp(\mathbf{X}_F)$, $1 \leq i \leq s$. Then $B_{F,\theta}$ will denote a finite set of generators of

$$S_{F,\theta} := \operatorname{Syz}_R[\sigma^{\gamma_1+\theta}(\operatorname{lc}(\mathbf{g}_1))c_{\gamma_1+\theta,\beta_1} \ \cdots \ \sigma^{\gamma_s+\theta}(\operatorname{lc}(\mathbf{g}_s))c_{\gamma_s+\theta,\beta_s}].$$

For $\theta = \mathbf{0} := (0, \dots, 0)$, $S_{F,\theta}$ will be denoted by S_F and $B_{F,\theta}$ by B_F.

LEMMA 114. *Let* $\mathbf{g}_1, \dots, \mathbf{g}_s \in A^m$, $c_1, \dots, c_s \in R - \{0\}$ *and* $\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$ *be such that* $\operatorname{lm}(x^{\alpha_1}\operatorname{lm}(\mathbf{g}_1)) = \cdots = \operatorname{lm}(x^{\alpha_s}\operatorname{lm}(\mathbf{g}_s)) =: \mathbf{X}_\delta$. *If* $\operatorname{lm}(\sum_{i=1}^{s} c_i x^{\alpha_i}\mathbf{g}_i) \prec \mathbf{X}_\delta$, *then there exist* $r_1, \dots, r_k \in R$ *and* $l_1, \dots, l_s \in A$ *such that*

$$\sum_{i=1}^{s} c_i x^{\alpha_i}\mathbf{g}_i = \sum_{j=1}^{k} r_j x^{\delta - \exp(\mathbf{X}_F)}\Big(\sum_{i=1}^{s} b_{ji}x^{\gamma_i}\mathbf{g}_i\Big) + \sum_{i=1}^{s} l_i\mathbf{g}_i,$$

where $\gamma_i \in \mathbb{N}^n$ *is such that* $\gamma_i + \exp(\mathbf{g}_i) = \exp(\mathbf{X}_F)$, $1 \leq i \leq s$, *and*

$$B_F := \{\mathbf{b}_1, \dots, \mathbf{b}_k\} := \{(b_{11}, \dots, b_{1s}), \dots, (b_{k1}, \dots, b_{ks})\}.$$

Moreover, $\mathrm{lm}(x^{\delta-\exp(\mathbf{X}_F)}\sum_{i=1}^{s} b_{ji}x^{\gamma_i}\mathbf{g}_i) \prec \mathbf{X}_\delta$ *for every* $1 \leq j \leq k$*, and* $\mathrm{lm}(l_i\mathbf{g}_i) \prec \mathbf{X}_\delta$ *for every* $1 \leq i \leq s$.

Proof. It is easy to adapt the proof of Lemma 100. ■

THEOREM 115. *Let* $M \neq 0$ *be a submodule of* A^m *and let* G *be a finite subset of non-zero generators of* M*. Then the following conditions are equivalent:*

(i) *G is a Gröbner basis of M.*
(ii) *For all* $F := \{\mathbf{g}_1, \ldots, \mathbf{g}_s\} \subseteq G$*, with* $\mathbf{X}_F \neq \mathbf{0}$*, and for any* $(b_1, \ldots, b_s) \in B_F$*,*

$$\sum_{i=1}^{s} b_i x^{\gamma_i}\mathbf{g}_i \xrightarrow{G}_{+} 0.$$

Proof. See the proof of Theorem 101. ■

COROLLARY 116. *Let* $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_s\}$ *be a set of non-zero vectors in* A^m*. The algorithm below produces a Gröbner basis for the submodule* $\langle \mathbf{f}_1, \ldots, \mathbf{f}_s\rangle$ *(again* $P(X)$ *denotes the set of subsets of a set* X*):*

Buchberger's algorithm for modules over bijective skew PBW extensions

INPUT: $F := \{\mathbf{f}_1, \ldots, \mathbf{f}_s\} \subseteq A^m$, $\mathbf{f}_i \neq \mathbf{0}$, $1 \leq i \leq s$
OUTPUT: $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_t\}$ *a Gröbner basis for* $\langle F\rangle$
INITIALIZATION: $G := \emptyset$, $G' := F$
WHILE $G' \neq G$ **DO**
  $D := P(G') - P(G)$
  $G := G'$
  FOR *each* $S := \{\mathbf{g}_{i_1}, \ldots, \mathbf{g}_{i_k}\} \in D$*, with* $\mathbf{X}_S \neq \mathbf{0}$, **DO**
    Compute B_S
    FOR *each* $\mathbf{b} = (b_1, \ldots, b_k) \in B_S$ **DO**
      Reduce $\sum_{j=1}^{k} b_j x^{\gamma_j}\mathbf{g}_{i_j} \xrightarrow{G'}_{+} \mathbf{r}$*, with* $\mathbf{r}$ *reduced with respect to* G' *and* γ_j *defined as in Definition* 113
        IF $\mathbf{r} \neq \mathbf{0}$ **THEN**
        $G' := G' \cup \{\mathbf{r}\}$

From Theorem 14 and the previous corollary we get the following direct conclusion.

COROLLARY 117. *Every submodule of* A^m *has a Gröbner basis.*

EXAMPLE 118. We will illustrate the above algorithm with the bijective skew PBW extension $\mathcal{R}$ of Example 21. For computational reasons, we rewrite the generators and relations for this algebra in the following way:

$$x := b, \quad y := a, \quad z := c, \quad w := d$$

and

$$\begin{aligned} &yx = q^{-1}xy, \quad wx = qxw, \quad zy = qyz, \quad wz = qzw,\\ &zx = \mu^{-1}xz, \quad wy = yw + (q - q^{-1})xz, \end{aligned}$$

and therefore $\mathcal{R} \cong \sigma(k[x])\langle y, z, w\rangle$. On $\mathrm{Mon}(\mathcal{R})$ we consider the deglex order with $y \succ z \succ w$, and on $\mathrm{Mon}(A^2)$ the TOPREV order, whence $\mathbf{e}_1 > \mathbf{e}_2$. Moreover, we will take

$K=\mathbb{Q}$, $\mu=\frac{1}{2}$ and $q=\frac{2}{3}$. From the above relations, we obtain $\sigma_1(x)=\frac{3}{2}x$, $\sigma_2(x)=2x$ and $\sigma_3(x)=\frac{2}{3}x$. Let $\mathbf{f}_1=xyw\mathbf{e}_1+w\mathbf{e}_2$ and $\mathbf{f}_2=zw\mathbf{e}_1+xy\mathbf{e}_2$. We will construct a Gröbner basis for $M:=\langle\mathbf{f}_1,\mathbf{f}_2\rangle$:

Step 1. We start with $G:=\emptyset$, $G':=\{\mathbf{f}_1,\mathbf{f}_2\}$. Since $G'\neq G$, we set $D:=P(G')-P(G)$, i.e., $D:=\{S_1,S_2,S_{1,2}\}$, where $S_1:=\{\mathbf{f}_1\}$, $S_2:=\{\mathbf{f}_2\}$, $S_{1,2}:=\{\mathbf{f}_1,\mathbf{f}_2\}$. We also set $G:=G'$, and for every $S\in D$ such that $\mathbf{X}_S\neq\mathbf{0}$ we compute B_S:

- For S_1 we have $\mathrm{Syz}_{\mathbb{Q}[x]}[\sigma^{\gamma_1}(\mathrm{lc}(\mathbf{f}_1))c_{\gamma_1,\beta_1}]$, where $\beta_1=\exp(\mathrm{lm}(\mathbf{f}_1))=(1,0,1)$, $\gamma_1=(0,0,0)$ and $c_{\gamma_1,\beta_1}=1$; thus $B_{S_1}=\{0\}$ and we do not add any vector to G'.
- For S_2 we have an identical situation.
- For $S_{1,2}$ we have $X_{1,2}=\mathrm{lcm}(\mathrm{lm}(f_1),\mathrm{lm}(f_2))=yzw\mathbf{e}_1$, thus $\gamma_1=(0,1,0)$ and $\gamma_2=(1,0,0)$. Since $zyw=\frac{2}{3}yzw$, we find $c_{\gamma_1,\beta_1}=\frac{2}{3}$ and $\sigma^{\gamma_1}(\mathrm{lc}(f_1))=\sigma_2(x)=2x$. Analogously, $c_{\gamma_2,\beta_2}=1$ and $\sigma^{\gamma_2}(\mathrm{lc}(f_2))=\sigma_1(x^2)=\frac{9}{4}x^2$. Hence, we must compute a system of generators for $\mathrm{Syz}_{\mathbb{Q}[x]}\left[\frac{4}{3}x,\frac{9}{4}x^2\right]$. Such generator set can be $B_{S_{1,2}}=\left\{\left(\frac{3}{4}x,-\frac{4}{9}\right)\right\}$. From this we get

$$\begin{aligned}\tfrac{3}{4}xz\mathbf{f}_1-\tfrac{4}{9}y\mathbf{f}_2&=\tfrac{3}{4}xz(xyw\mathbf{e}_1+w\mathbf{e}_2)-\tfrac{4}{9}y(x^2zw\mathbf{e}_1+xy\mathbf{e}_2)\\&=x^2zywe_1+\tfrac{3}{4}xzw\mathbf{e}_2-x^2yzw\mathbf{e}_1-\tfrac{2}{3}xy^2\mathbf{e}_2=-\tfrac{2}{3}xy^2\mathbf{e}_2+\tfrac{3}{4}xzw\mathbf{e}_2:=\mathbf{f}_3,\end{aligned}$$

Observe that $\mathbf{f}_3$ is reduced with respect to G'. We set $G':=\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$.

Step 2. Since $G=\{\mathbf{f}_1,\mathbf{f}_2\}\neq G'=\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$, we set $D:=\mathcal{P}(G')-\mathcal{P}(G)$, i.e., $D:=\{S_3,S_{1,3},S_{2,3},S_{1,2,3}\}$, where $S_1:=\{\mathbf{f}_1\}$, $S_{1,3}:=\{\mathbf{f}_1,\ \mathbf{f}_3\}$, $S_{2,3}:=\{\mathbf{f}_2,\mathbf{f}_3\}$, $S_{1,2,3}:=\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$. We further set $G:=G'$, and for every $S\in D$ such that $\mathbf{X}_S\neq\mathbf{0}$ we must compute B_S. Since $\mathbf{X}_{S_{1,3}}=\mathbf{X}_{S_{2,3}}=\mathbf{X}_{S_{1,2,3}}=\mathbf{0}$, we only need to consider S_3.

- We have to compute

$$\mathrm{Syz}_{\mathbb{Q}[x]}[\sigma^{\gamma_3}(\mathrm{lc}(\mathbf{f}_3))c_{\gamma_3,\beta_3}],$$

where $\beta_3=\exp(\mathrm{lm}(\mathbf{f}_3))=(2,0,0)$, $\mathbf{X}_{S_3}=\mathrm{lcm}\{\mathrm{lm}(\mathbf{f}_3)\}=\mathrm{lm}(\mathbf{f}_3)=y^2\mathbf{e}_2$, $\exp(\mathbf{X}_{S_3})=(0,2,0)$, $\gamma_3=\exp(\mathbf{X}_{S_3})-\beta_3=(0{,}0,\ 0)$, $x^{\gamma_3}x^{\beta_3}=y^2$, so $c_{\gamma_3,\beta_3}=1$. Hence

$$\sigma^{\gamma_3}(\mathrm{lc}(\mathbf{f}_3))c_{\gamma_3,\beta_3}=\sigma^{\gamma_3}(-\tfrac{2}{3}x)1=\sigma_2^0\sigma_3^0(-\tfrac{2}{3}x)=-\tfrac{2}{3}x$$

and $\mathrm{Syz}_{\mathbb{Q}[x]}\left[-\frac{2}{3}x\right]=\{0\}$, i.e., $B_{S_3}=\{0\}$. This means that we do not add any vector to G', and hence $G=\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$ is a Gröbner basis for M.

Remark 119. There are some classical and elementary applications of Gröbner theory that we will study in a forthcoming paper; for example, we can solve the membership problem, we can compute the syzygy module, the intersection and the quotient of ideals and submodules, the matrix presentation of a finitely presented module, the kernel and the image of homomorphism between modules, the one-sided inverse of a matrix, etc. With this, we can make constructive the theory of projective modules, stably free modules and Hermite rings studied in the present work.

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