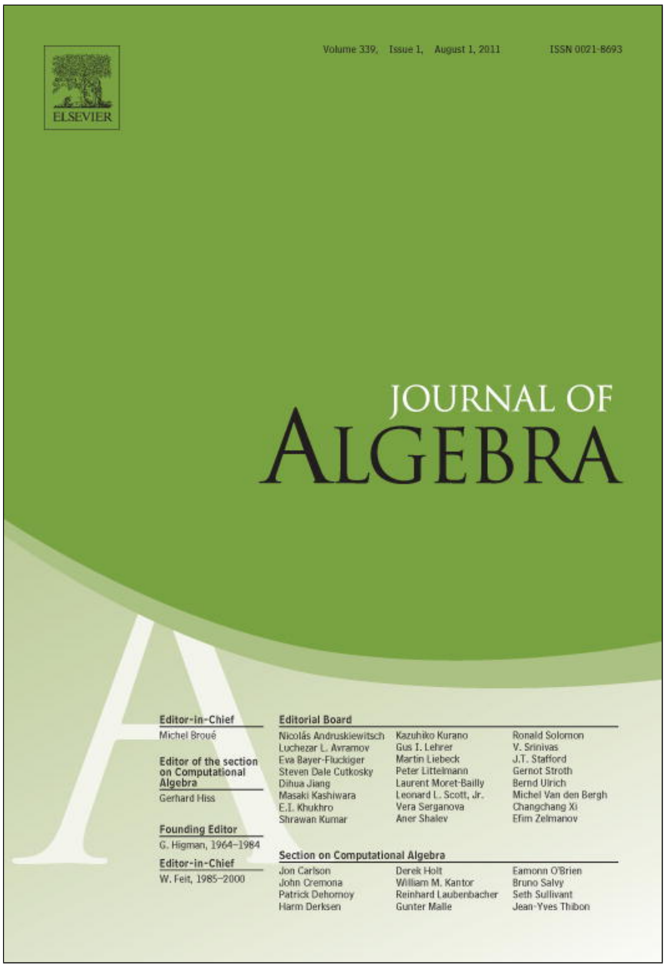
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Cohomology ring of differential operator rings

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ABSTRACT

We compute the multiplicative structure in the Hochschild cohomology ring of a differential operators ring and the cap product of Hochschild cohomology on the Hochschild homology.

Introduction

Let k be a field and A an associative k-algebra with 1. An extension E/A of A is a *differential operator ring* on A if there exist a Lie k-algebra $\mathfrak{g}$ and a k-vector space embedding $x \mapsto \bar{x}$, of $\mathfrak{g}$ into E, such that for all $x, y \in \mathfrak{g}$ and $a \in A$, the following conditions hold:

(1) $\bar{x}a - a\bar{x} = a^x$, where $a \mapsto a^x$ is a derivation,
(2) $\bar{x}\,\bar{y} - \bar{y}\,\bar{x} = \overline{[x, y]_{\mathfrak{g}}} + f(x, y)$, where $[-,-]_{\mathfrak{g}}$ is the bracket of $\mathfrak{g}$ and $f: \mathfrak{g} \times \mathfrak{g} \to A$ is a k-bilinear map,
(3) for a given basis $(x_i)_{i \in I}$ of $\mathfrak{g}$, the algebra E is a free left A-module with the standard monomials in the x_i's as a basis.

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This general construction was introduced in [Ch] and [Mc-R]. Several particular cases of this type of extensions have been considered previously in the literature. For instance:

- when $\mathfrak{g}$ is a one-dimensional vector space and f is the trivial cocycle, E is the Ore extension $A[x,\delta]$, where $\delta(a)=a^x$,
- when $A=k$, we obtain the algebras studied by Sridharan in [S], which are the quasi-commutative algebras E, whose associated graded algebra is a symmetric algebra,
- McConnell [Mc, §2] studies this type of extensions under the hypothesis that A is commutative and $(x,a)\mapsto a^x$ is an action, and Borho et al. [B-G-R, Theorem 4.2] consider the case in which the cocycle is trivial.

Blattner et al. [B-C-M] and Doi and Takeuchi [D-T] independently begun the study of the crossed products $A\#_f H$ of a k-algebra A by a Hopf k-algebra H, and in [M] it was proved that the differential operator rings on A are the crossed products of A by enveloping algebras of Lie algebras.

In [G-G1] the authors obtained complexes, simpler than the canonical ones, which compute the Hochschild homology and cohomology of a differential operator ring E with coefficients in an E-bimodule M. In this paper we continue this investigation by studying the Hochschild cohomology ring of E and the cap product

$$\mathrm{H}_p(E,M)\times \mathrm{HH}^q(E)\to \mathrm{H}_{p-q}(E,M)\quad (q\leqslant p),$$

in terms of the above mentioned complexes. Moreover we generalize the results of [G-G1] by considering the (co)homology of E relative to a subalgebra K of A which is stable under the action of $\mathfrak{g}$ (which we also call the Hochschild (co)homology of the K-algebra E). We also seize the opportunity to fix some minor mistakes and to simplify some proofs in [G-G1].

The paper is organized as follows: In Section 1 we obtain a projective resolution (X_*,d_*) of the E-bimodule E, relative to the family of all epimorphisms of E-bimodules which split as (E,K)-bimodule maps. In Section 2 we determine and study comparison maps between (X_*,d_*) and the normalized Hochschild resolution $(E\otimes_K \overline{E}^{\otimes_K^*}\otimes_K E, b'_*)$ of E, relative to K. In Sections 3 and 4 we apply the above results in order to obtain complexes $(\overline{X}_*^K(M),\bar{d}_*)$ and $(\overline{X}_K^*(M),\bar{d}^*)$, simpler than the canonical ones, giving the Hochschild homology and cohomology of the K-algebra E with coefficients in an E-bimodule M, respectively. The main results are Theorems 3.4 and 4.4, in which we obtain morphisms

$$\overline{X}_K^*(E)\otimes\overline{X}_K^*(E)\to\overline{X}_K^*(E)\quad\text{and}\quad \overline{X}_*^K(M)\otimes\overline{X}_K^*(E)\to\overline{X}_*^K(M),$$

inducing the cup and cap product, respectively. Finally in Section 5 we obtain further simplifications, assuming that A is a symmetric algebra.

1. Preliminaries

Let k be a field. In this paper all the algebras are over k. Let A be an algebra and H a Hopf algebra. We are going to use the Sweedler notation $\Delta(h)=\sum_{(h)} h^{(1)}\otimes_k h^{(2)}$ for the comultiplication Δ of H. A *weak action* of H on A is a k-bilinear map $(h,a)\mapsto a^h$, from $H\times A$ to A, such that

(1) $(ab)^h=\sum_{(h)} a^{h^{(1)}}b^{h^{(2)}}$,
(2) $1^h=\epsilon(h)1$,
(3) $a^1=a$,

for $h\in H$, $a,b\in A$. By an *action* of H on A we mean a weak action such that

$$\left(a^l\right)^h=a^{hl}\quad\text{for all } h,l\in H,\ a\in A.$$

Let A be an algebra and let H be a Hopf algebra acting weakly on A. Given a k-linear map $f : H \otimes_k H \to A$ we let $A\#_f H$ denote the algebra (which is not necessarily associative nor with multiplicative unit) whose underlying vector space is $A \otimes_k H$ and whose multiplication is given by

$$(a \otimes_k h)(b \otimes_k l) = \sum_{(h)(l)} ab^{h^{(1)}} f\left(h^{(2)}, l^{(1)}\right) \otimes_k h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes_k h$ of $A\#_f H$ will usually be written $a\#h$. The algebra $A\#_f H$ is called a *crossed product* if it is associative with $1\#1$ as identity element. In [B-C-M] it was proved that this happens if and only if the map f and the weak action of H on A satisfy the following conditions:

(1) (Normality of f) for all $h \in H$ we have $f(h, 1) = f(1, h) = \epsilon(h) 1_A$,
(2) (Cocycle condition) for all $h, l, m \in H$ we have

$$\sum_{(h)(l)(m)} f\left(l^{(1)}, m^{(1)}\right)^{h^{(1)}} f\left(h^{(2)}, l^{(2)} m^{(2)}\right) = \sum_{(h)(l)} f\left(h^{(1)}, l^{(1)}\right) f\left(h^{(2)} l^{(2)}, m\right),$$

(3) (Twisted module condition) for all $h, l \in H$ and $a \in A$ we have

$$\sum_{(h)(l)} \left(a^{l^{(1)}}\right)^{h^{(1)}} f\left(h^{(2)}, l^{(2)}\right) = \sum_{(h)(l)} f\left(h^{(1)}, l^{(1)}\right) a^{h^{(2)} l^{(2)}}.$$

We assume from now on that H is the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In this case, item (1) of the definition of weak action implies that

$$(ab)^x = a^x b + ab^x$$

for each $x \in \mathfrak{g}$ and $a, b \in A$. So, a weak action determines a k-linear map

$$\delta : \mathfrak{g} \to \mathrm{Der}_k(A)$$

by $\delta(x)(a) = a^x$. Moreover if $(h, a) \mapsto a^h$ is an action, then δ is a homomorphism of Lie algebras. Conversely, given a k-linear map $\delta : \mathfrak{g} \to \mathrm{Der}_k(A)$, there exists a (generally non-unique) weak action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. When δ is a homomorphism of Lie algebras, there is a unique action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. For a proof of the previous results we refer to [B-C-M]. It is immediate to prove that each normal cocycle

$$f : U(\mathfrak{g}) \otimes_k U(\mathfrak{g}) \to A$$

is convolution invertible. For a proof see [G-G1, Remark 1.1].

Next we recall some results and notations from [G-G1] that we will need later. Let K be a subalgebra of A which is stable under the weak action of $\mathfrak{g}$ (that is $\lambda^x \in K$ for all $\lambda \in K$ and $x \in \mathfrak{g}$) and let $E = A\#_f U(\mathfrak{g})$ be a crossed product. We are going to modify the sign of some boundary maps in order to obtain simpler expressions for the comparison maps.

To begin, we fix some notations:

(1) The unadorned tensor product $\otimes$ means the tensor product $\otimes_K$ over K.

(2) For $B = A$ or $B = E$ and each $r \in \mathbb{N}$, we write $\overline{B} = B/K$,

$$B^r = B \otimes \cdots \otimes B \quad (r \text{ times}) \quad \text{and} \quad \overline{B}^r = \overline{B} \otimes \cdots \otimes \overline{B} \quad (r \text{ times}).$$

Moreover, for $b \in B$ we also let b denote the class of b in $\overline{B}$.
(3) For each Lie algebra $\mathfrak{g}$ and $s \in \mathbb{N}$, we write $\mathfrak{g}^{\wedge s} = \mathfrak{g} \wedge \cdots \wedge \mathfrak{g}$ (s times).
(4) Throughout this paper we will write $\mathbf{a}_{1r}$ for $a_1 \otimes \cdots \otimes a_r \in A^r$ and $\mathbf{x}_{1s}$ for $x_1 \wedge \cdots \wedge x_s \in \mathfrak{g}^{\wedge s}$.
(5) For $\mathbf{a}_{1r}$ and $0 \leqslant i < j \leqslant r$, we write $\mathbf{a}_{ij} = a_i \otimes \cdots \otimes a_j$.
(6) For $\mathbf{x}_{1s}$ and $1 \leqslant i \leqslant s$, we write $\mathbf{x}_{1\hat{\imath}s} = x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_s$.
(7) For $\mathbf{x}_{1s}$ and $1 \leqslant i < j \leqslant s$, we write $\mathbf{x}_{1\hat{\imath}\hat{\jmath}s} = x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_s$.

Let $\Lambda(\mathfrak{g})$ be the exterior algebra generated by the k-vector space $\mathfrak{g}$ and let $\Lambda(\mathfrak{g})\#U(\mathfrak{g})$ be the smash product obtained by using the action of $U(\mathfrak{g})$ over $\Lambda(\mathfrak{g})$, determined by $x^{x'} := [x', x]_{\mathfrak{g}}$. We define Y_* as the algebra

$$E \otimes \big(\Lambda(\mathfrak{g})\#U(\mathfrak{g})\big) = \big(A\#_f U(\mathfrak{g})\big) \otimes \big(\Lambda(\mathfrak{g})\#U(\mathfrak{g})\big),$$

endowed with the gradation, obtained giving degree 0 to the elements

$$(a\#1) \otimes (1\#1), \qquad y_x := (1\#x) \otimes (1\#1) \quad \text{and} \quad \rho_x := (1\#1) \otimes (1\#x),$$

and degree 1 to the elements $e_x := (1\#1) \otimes (x\#1)$. If we identify each $a \in A$ with $(a\#1) \otimes (1\#1)$, then Y_* is the extension of A, generated by the elements y_x and ρ_x of degree 0, and e_x, of the degree 1, subject to the relations

$$\begin{aligned}
&y_{\lambda x+x'} = \lambda y_x + y_{x'}, && y_{x'} y_x = y_x y_{x'} + y_{[y',y]_{\mathfrak{g}}} + f\big(y', y\big) - f\big(y, y'\big),\\
&\rho_{\lambda x+x'} = \lambda \rho_x + \rho_{x'}, && \rho_{x'} z_y = y_x \rho_{x'},\\
&e_{\lambda x+x'} = \lambda e_x + e_{x'}, && e_{x'} y_x = y_x e_{x'},\\
&y_x a = a^x + a y_x, && \rho_{x'} \rho_x = \rho_x \rho_{x'} + \rho_{[x',x]_{\mathfrak{g}}},\\
&\rho_x a = a \rho_x, && e_{x'} \rho_x = \rho_x e_{x'} + e_{[x',x]_{\mathfrak{g}}},\\
&e_x a = a e_x, && e_x^2 = 0,
\end{aligned}$$

where $\lambda \in k$, x' and x in $\mathfrak{g}$ and $[-,-]_{\mathfrak{g}}$ denotes the Lie bracket in $\mathfrak{g}$. Note that E is a subalgebra of Y_* via the embedding that takes $a \in A$ to a and $1\#x$ to y_x for all $x \in \mathfrak{g}$. This gives rise to a structure of left E-module on Y_*. For all $x \in \mathfrak{g}$, let $z_x = y_x + \rho_x$. Since

$$\begin{aligned}
&z_{\lambda x+x'} = \lambda z_x + z_{x'},\\
&z_x a = a^x + a z_x,\\
&z_{x'} z_x = z_x z_{x'} + z_{[x',x]_{\mathfrak{g}}} + f\big(x', x\big) - f\big(x, x'\big),
\end{aligned}$$

there is also an algebra map from E to Y_* that takes $a \in A$ to a and $1\#x$ to z_x for all $x \in \mathfrak{g}$. This map is also an embedding, since it is a section, with a left inverse given by the algebra map from Y_* to E, that takes a to a, y_x to $1\#x$, ρ_x to 0 and e_x to 0.

Remark 1.1. The complex Y_* is slightly different from the similar complex introduced in [G-G1]. However we will obtain in Theorem 1.8 the same projective resolution of E as the one obtained in [G-G1]. We have two reasons to justify the present definition of Y_*. On one hand, it allows us to give a very simple proof of the following theorem (corresponding to [G-G1, Theorem 3.1.1]) and, on the other hand, it allows us to obtain a better contracting homotopy of the resolution that appears in Theorem 1.7. For instance the new contracting homotopy will be left E-linear.

Remark 1.2. In a first version of this paper we fixed in the following theorem a mistake at the beginning of Section 3.1 of [G-G1]. The error was that the weak action of $\mathfrak{g}$ on $A \otimes \Lambda(g)$ was poorly defined. Using the notation of that paper it was

$$(a \otimes e)^u = a^{\pi(u)} \otimes e + a \otimes e^u,$$

but should have been

$$(a \otimes e)^u = \sum_{(u)} a^{\pi(u^{(1)})} \otimes e^{\pi(u^{(2)})}.$$

In the current version this weak action does not appear.

Let $(g_i)_{i \in I}$ be a basis of $\mathfrak{g}$ with indexes running on an ordered set I. For each $i \in I$ let us write $y_i := y_{g_i}$, $z_i := z_{g_i}$, $e_i := e_{g_i}$ and $\rho_i := \rho_{g_i}$.

Theorem 1.3. *Each Y_s is a free left E-module with basis*

$$\rho_{i_1}^{m_1} e_{i_1}^{\delta_1} \cdots \rho_{i_l}^{m_l} e_{i_l}^{\delta_l} \quad \begin{pmatrix} l \geqslant 0,\ i_1 < \cdots < i_l \in I,\ m_j \geqslant 0,\ \delta_j \in \{0,1\} \\ m_j + \delta_j > 0,\ \delta_1 + \cdots + \delta_l = s \end{pmatrix}.$$

Proof. It is sufficient to see that

$$\overline{\rho}_{i_1}^{m_1} \bar{e}_{i_1}^{\delta_1} \cdots \overline{\rho}_{i_l}^{m_l} \bar{e}_{i_l}^{\delta_l} \quad \begin{pmatrix} l \geqslant 0,\ i_1 < \cdots < i_l \in I,\ m_j \geqslant 0,\ \delta_j \in \{0,1\} \\ m_j + \delta_j > 0,\ \delta_1 + \cdots + \delta_l = s \end{pmatrix},$$

where $\overline{\rho}_i := 1\#x_i$ and $\bar{e}_i := x_i\#1$, is a basis of $\Lambda(\mathfrak{g})\#U(\mathfrak{g})$ as a k-vector space, which follows easily from the fact that

$$x_{j_1} \wedge \cdots \wedge x_{j_s} \quad (j_1 < \cdots < j_l \in I)$$

is a basis of $\mathfrak{g}^{\wedge s}$ and, by the Poincaré–Birkhoff–Witt theorem,

$$x_{i_1}^{m_1} \cdots x_{i_l}^{m_l} \quad (l \geqslant 0,\ i_1 < \cdots < i_l \in I,\ m_j \geqslant 0)$$

is a basis of $U(\mathfrak{g})$. $\square$

Remark 1.4. A similar, but more involved argument, shows that each Y_s is a free right E-module with the same basis. We will not use this result.

Remark 1.5. The following result improves [G-G1, Theorem 3.1.3] in the sense that in the current version we obtain that the complex introduced there is contractible as a complex of (A, E)-bimodules and not only as a complex of k-modules.

Theorem 1.6. *Let $\tilde{\mu}: Y_0 \to E$ be the algebra map defined by $\tilde{\mu}(a) = a$ for $a \in A$ and $\tilde{\mu}(y_i) = \tilde{\mu}(z_i) = 1\# g_i$ for $i \in I$. There is a unique derivation $\partial_*: Y_* \to Y_{*-1}$ such that $\partial(e_i) = \rho_i$ for $i \in I$. Moreover, the chain complex of E-bimodules*

$$E \xleftarrow{\tilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} \cdots$$

is contractible as a complex of (E, A)-bimodules. A chain contracting homotopy

$$\sigma_0^{-1}: E \to Y_0, \qquad \sigma_{s+1}^{-1}: Y_s \to Y_{s+1} \quad (s \geqslant 0)$$

is given by

$$\sigma^{-1}(1) = 1,$$
$$\sigma^{-1}\bigl(\rho_{i_1}^{m_1} e_{i_1}^{\delta_1} \cdots \rho_{i_l}^{m_l} e_{i_l}^{\delta_l}\bigr) = \begin{cases} (-1)^s \rho_{i_1}^{m_1} e_{i_1}^{\delta_1} \cdots \rho_{i_{l-1}}^{m_{l-1}} e_{i_{l-1}}^{\delta_{l-1}} \rho_{i_l}^{m_l - 1} e_{i_l} & \text{if } \delta_l = 0, \\ 0 & \text{if } \delta_l = 1, \end{cases}$$

where we assume that $i_1 < \cdots < i_l$, $\delta_1 + \cdots + \delta_l = s$ and $m_l + \delta_l > 0$.

Proof. A direct computation shows that

- $\tilde{\mu} \circ \sigma^{-1}(1) = \tilde{\mu}(1) = 1$.
- $\sigma^{-1} \circ \tilde{\mu}(1) = \sigma^{-1}(1) = 1$ and $\partial \circ \sigma^{-1}(1) = \partial(0) = 0$.
- If $\mathbf{x} = \mathbf{x}' \rho_{i_l}^{m_l}$, where $m_l > 0$ and $\mathbf{x}' = \rho_{i_1}^{m_1} \cdots \rho_{i_{l-1}}^{m_{l-1}}$ with $i_1 < \cdots < i_l$, then

$$\sigma^{-1} \circ \tilde{\mu}(\mathbf{x}) = \sigma^{-1}(0) = 0 \quad \text{and} \quad \partial \circ \sigma^{-1}(\mathbf{x}) = \partial\bigl(\mathbf{x}' \rho_{i_l}^{m_l - 1} e_{i_l}\bigr) = \mathbf{x}.$$

- Let $\mathbf{x} = \mathbf{x}' \rho_{i_l}^{m_l} e_{i_l}^{\delta_l}$, where $m_l + \delta_l > 0$ and $\mathbf{x}' = \rho_{i_1}^{m_1} e_{i_1}^{\delta_1} \cdots \rho_{i_{l-1}}^{m_{l-1}} e_{i_{l-1}}^{\delta_{l-1}}$ with $i_1 < \cdots < i_l$ and $\delta_1 + \cdots + \delta_l = s > 0$. If $\delta_l = 0$, then

$$\sigma^{-1} \circ \partial(\mathbf{x}) = \sigma^{-1}\bigl(\partial\bigl(\mathbf{x}'\bigr)\rho_{i_l}^{m_l}\bigr) = (-1)^{s-1} \partial\bigl(\mathbf{x}'\bigr) \rho_{i_l}^{m_l - 1} e_{i_l},$$
$$\partial \circ \sigma^{-1}(\mathbf{x}) = \partial\bigl((-1)^s \mathbf{x}' \rho_{i_l}^{m_l - 1} e_{i_l}\bigr) = (-1)^s \partial\bigl(\mathbf{x}'\bigr) \rho_{i_l}^{m_l - 1} e_{i_l} + \mathbf{x},$$

  and if $\delta_l = 1$, then

$$\sigma^{-1} \circ \partial(\mathbf{x}) = \sigma^{-1}\bigl(\partial\bigl(\mathbf{x}'\bigr)\rho_{i_l}^{m_l} e_{i_l} + (-1)^{s-1} \mathbf{x}' \rho_{i_l}^{m_l + 1}\bigr) = \mathbf{x},$$
$$\partial \circ \sigma^{-1}(\mathbf{x}) = \partial(0) = 0.$$

The result follows immediately. □

For each $s \geqslant 0$ we consider $E \otimes_k \mathfrak{g}^{\wedge s}$ as a right K-module via $(\mathbf{c} \otimes_k \mathbf{x})\lambda = \mathbf{c}\lambda \otimes_k \mathbf{x}$. For $r, s \geqslant 0$, let $X_{rs} = (E \otimes_k \mathfrak{g}^{\wedge s}) \otimes \overline{A}^r \otimes E$. The groups X_{rs} are E-bimodules in an obvious way. Let us consider the diagram of E-bimodules and E-bimodule maps

$$
\begin{tikzcd}
\vdots \arrow[d, "\partial_3"] & & & \\
Y_2 \arrow[d, "\partial_2"] & X_{02} \arrow[l, "\mu_2"'] & X_{12} \arrow[l, "d^0_{12}"'] & \cdots \arrow[l, "d^0_{22}"'] \\
Y_1 \arrow[d, "\partial_1"] & X_{01} \arrow[l, "\mu_1"'] & X_{11} \arrow[l, "d^0_{11}"'] & \cdots \arrow[l, "d^0_{21}"'] \\
Y_0 & X_{00} \arrow[l, "\mu_0"'] & X_{10} \arrow[l, "d^0_{10}"'] & \cdots, \arrow[l, "d^0_{20}"']
\end{tikzcd}
$$

where $\mu_*: X_{0*} \to Y_*$ and $d^0_{**}: X_{**} \to X_{*-1,*}$, are defined by:

$$
\begin{aligned}
\mu(1 \otimes_k \mathbf{x}_{1s} \otimes 1) &= e_{x_1} \cdots e_{x_s}, \\
d^0(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) &= (-1)^s a_1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{2r} \otimes 1 \\
&\quad + \sum_{i=1}^{r-1} (-1)^{i+s} \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+1,r} \otimes 1 \\
&\quad + (-1)^{r+s} \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1,r-1} \otimes a_r.
\end{aligned}
$$

Each horizontal complex in this diagram is contractible as a complex of (E,K)-bimodules. A chain contracting homotopy is the family

$$
\sigma^0_{0s}: Y_s \to X_{0s}, \qquad \sigma^0_{r+1,s}: X_{rs} \to X_{r+1,s} \quad (r \geqslant 0),
$$

of (E,K)-bimodule maps, defined by

$$
\sigma^0(e_{x_1} \cdots e_{x_s} z_{x_{s+1}} \cdots z_{x_n}) = \sum_j a_j \otimes_k \mathbf{x}_{1s} \otimes 1 \# w_j,
$$

where $\sum_j a_j \# w_j = (1\#x_{s+1}) \cdots (1\#x_n)$, and

$$
\sigma^0(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes a_{r+1} \# w) = (-1)^{r+s+1} \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1,r+1} \otimes 1 \# w \quad (r \geqslant 0).
$$

(In order to prove that the σ^0's are right K-linear it is necessary to use that K is stable under the action of $\mathfrak{g}$.) Moreover, each X_{rs} is a projective E-bimodule relative to the family of all epimorphisms of E-bimodules which split as (E,K)-bimodule maps. We define E-bimodule maps

$$
d^l_{rs}: X_{rs} \to X_{r+l-1,s-l} \quad (r \geqslant 0 \text{ and } 1 \leqslant l \leqslant s)
$$

recursively by:

$$
d^l(\mathbf{y}) = \begin{cases}
-\sigma^0 \circ \partial \circ \mu(\mathbf{y}) & \text{if } l = 1 \text{ and } r = 0, \\
-\sigma^0 \circ d^1 \circ d^0(\mathbf{y}) & \text{if } l = 1 \text{ and } r > 0, \\
-\sum_{j=1}^{l-1} \sigma^0 \circ d^{l-j} \circ d^j(\mathbf{y}) & \text{if } l > 1 \text{ and } r = 0, \\
-\sum_{j=0}^{l-1} \sigma^0 \circ d^{l-j} \circ d^j(\mathbf{y}) & \text{if } l > 1 \text{ and } r > 0,
\end{cases}
$$

where $\mathbf{y} = 1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1$.

Theorem 1.7. *The complex*

$$E \xleftarrow{\overline{\mu}} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} \cdots, \tag{1}$$

where

$$\overline{\mu}(1\otimes 1)=1, \qquad X_n=\bigoplus_{r+s=n} X_{rs} \quad \text{and} \quad d_n=\sum_{\substack{r+s=n\\ r+l>0}}\sum_{l=0}^{s} d_{rs}^{l},$$

is a projective resolution of the E-bimodule E, relative to the family of all epimorphisms of E-bimodules which split as (E,K)-bimodule maps. Moreover an explicit contracting homotopy

$$\overline{\sigma}_0 : E \to X_0, \qquad \overline{\sigma}_{n+1} : X_n \to X_{n+1} \quad (n \geqslant 0)$$

of (1)*, as a complex of (E,K)-bimodules, is given by*

$$\overline{\sigma}_0=\sigma^0\circ\sigma_0^{-1} \quad \text{and} \quad \overline{\sigma}_{n+1}=-\sum_{l=0}^{n+1}\sigma_{l,n-l+1}^{l}\circ\sigma_{n+1}^{-1}\circ\mu_n+\sum_{r=0}^{n}\sum_{l=0}^{n-r}\sigma_{r+l+1,n-l-r}^{l},$$

where

$$\sigma_{l,s-l}^{l} : Y_s \to X_{l,s-l} \quad \text{and} \quad \sigma_{r+l+1,s-l}^{l} : X_{rs} \to X_{r+l+1,s-l} \quad (0<l\leqslant s,\ r\geqslant 0)$$

are recursively defined by

$$\sigma^l=-\sum_{j=0}^{l-1}\sigma^0\circ d^{l-j}\circ\sigma^j.$$

Proof. It follows from [G-G2, Corollary A.2]. □

The boundary maps of the projective resolution of E that we just found are defined recursively. Next we give closed formulas for them.

Theorem 1.8. *For $x_i, x_j \in \mathfrak{g}$, we put $\hat{f}_{ij}=f(x_i,x_j)-f(x_j,x_i)$. We have:*

$$\begin{aligned}
d^1(1\otimes_k \mathbf{x}_{1s}\otimes\mathbf{a}_{1r}\otimes 1) &= \sum_{i=1}^{s}(-1)^{i+1}\#x_i\otimes_k \mathbf{x}_{1\hat{i}s}\otimes\mathbf{a}_{1r}\otimes 1\\
&\quad+\sum_{i=1}^{s}(-1)^{i}\otimes_k \mathbf{x}_{1\hat{i}s}\otimes\mathbf{a}_{1r}\otimes 1\#x_i\\
&\quad+\sum_{\substack{i=1\\ 1\leqslant h\leqslant r}}^{s}(-1)^{i}\otimes_k \mathbf{x}_{1\hat{i}s}\otimes\mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes\mathbf{a}_{h+1,r}\otimes 1\\
&\quad+\sum_{1\leqslant i<j\leqslant s}(-1)^{i+j}\otimes_k [x_i,x_j]_{\mathfrak{g}}\wedge\mathbf{x}_{1\hat{i}\hat{j}s}\otimes\mathbf{a}_{1r}\otimes 1,
\end{aligned}$$

$$d^2(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) = \sum_{\substack{1 \leqslant i < j \leqslant s \\ 0 \leqslant h \leqslant r}} (-1)^{i+j+h+s} \otimes_k \mathbf{x}_{1\hat{i}\hat{j}s} \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,r} \otimes 1$$

and $d^l = 0$ for all $l \geqslant 3$.

Proof. The proof of [G-G1, Theorem 3.3] works in our more general context. □

2. The comparison maps

In this section we introduce and study comparison maps between (X_*, d_*) and the canonical normalized Hochschild resolution $(E \otimes \overline{E}^* \otimes E, b'_*)$ of the K-algebra E. It is well known that there are morphisms of E-bimodule complexes

$$\theta_* : (X_*, d_*) \to \big(E \otimes \overline{E}^* \otimes E, b'_*\big) \quad \text{and} \quad \vartheta_* : \big(E \otimes \overline{E}^* \otimes E, b'_*\big) \to (X_*, d_*),$$

such that $\theta_0 = \vartheta_0 = \mathrm{id}_{E \otimes E}$ and that these morphisms are inverse of each other up to homotopy. They can be recursively defined by $\theta_0 = \vartheta_0 = \mathrm{id}_{E \otimes E}$ and

$$\theta(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) = (-1)^n \theta \circ d(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) \otimes 1$$

and

$$\vartheta(1 \otimes \mathbf{c}_{1n} \otimes 1) = \overline{\sigma} \circ \vartheta \circ b'(1 \otimes \mathbf{c}_{1n} \otimes 1),$$

for $n \geqslant 1$, where $r + s = n$ and $\mathbf{c}_{1n} = c_1 \otimes \cdots \otimes c_n \in \overline{E}^n$. The following result was established without proof in [G-G1].

Proposition 2.1. *We have*:

$$\theta(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) = \sum_{\tau \in \mathfrak{S}_s} \mathrm{sg}(\tau) \otimes (1\#x_{\tau(1)} \otimes \cdots \otimes 1\#x_{\tau(s)}) * \mathbf{a}_{1r} \otimes 1,$$

where $\mathfrak{S}_s$ is the symmetric group in s elements and $*$ denotes the shuffle product, which is defined by

$$(\beta_1 \otimes \cdots \otimes \beta_s) * (\beta_{s+1} \otimes \cdots \otimes \beta_n) = \sum_{\sigma \in \{(s,n-s)-\text{shuffles}\}} \mathrm{sg}(\sigma) \beta_{\sigma(1)} \otimes \cdots \otimes \beta_{\sigma(n)}.$$

Proof. We proceed by induction on $n = r + s$. The case $n = 0$ is obvious. Suppose that $r + s = n$ and the result is valid for θ_{n-1}. By the recursive definition of θ, Theorem 1.8, and the inductive hypothesis we obtain that:

$$\begin{aligned} \theta(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) &= (-1)^n \theta \circ d(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) \otimes 1 \\ &= (-1)^n \theta \circ \big(d^0 + d^1 + d^2\big)(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1) \otimes 1 \\ &= \theta(1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1,r-1} \otimes a_r) \otimes 1 \\ &\quad + \theta\left(\sum_{i=1}^{s} (-1)^{i+n} \otimes_k \mathbf{x}_{1\hat{i}s} \otimes \mathbf{a}_{1r} \otimes 1\#x_i\right) \otimes 1. \end{aligned}$$

The desired result follows now using again the inductive hypothesis. □

Lemma 2.2. *Let $(g_i)_{i\in I}$ be the basis of $\mathfrak{g}$ considered in Theorem* 1.3. *As in that theorem, let us write $e_i = e_{g_i}$ for each $i \in I$. The following facts hold*:

(1) $\overline{\sigma}_{n+1} \circ \overline{\sigma}_n = 0$ *for all* $n \geqslant 0$.
(2) $\sigma^l((E \otimes_k \mathfrak{g}^{\wedge s}) \otimes \overline{A}^r \otimes K\#U(\mathfrak{g})) = 0$ *for all* $0 \leqslant l \leqslant s$.
(3) $\sigma^l(e_{i_1} \cdots e_{i_n}) = 0$ *for all* $0 < l \leqslant n$.
(4) $\sigma^l((E \otimes_k \mathfrak{g}^{\wedge s}) \otimes \overline{A}^r \otimes A) = 0$ *for all* $0 < l \leqslant s$.
(5) $\sigma^{-1} \circ \mu(A \otimes_k \mathfrak{g}^{\wedge n} \otimes A) = 0$.
(6) *Assume that* $i_1 < \cdots < i_n$. *Then,*

$$\sigma^{-1} \circ \mu(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1\#g_{i_{n+1}}) = \begin{cases} (-1)^n e_{i_1} \cdots e_{i_{n+1}} & \text{if } i_n < i_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) An inductive argument shows that there are maps (which are left E-linear and right K-linear)

$$\gamma_{rs}^l : X_{r+1,s} \to X_{r+l,s-l},$$

such that $\sigma_{r+l+1,s-l}^l = \sigma_{r+l+1,s-l}^0 \circ \gamma_{rs}^l \circ \sigma_{rs}^0$. Because of $\sigma^0 \circ \sigma^0 = 0$, this implies that $\sigma^{l'} \circ \sigma^l = 0$, for all $l, l' \geqslant 0$. Thus,

$$\overline{\sigma}_{n+1} \circ \overline{\sigma}_n = \sum_{l=0}^{n+1} \sigma^l \circ \sigma^{-1} \circ \mu \circ \sigma^0 \circ \sigma^{-1} \circ \mu = 0,$$

where the last equality holds because $\mu \circ \sigma^0 = \mathrm{id}$ and $\sigma^{-1} \circ \sigma^{-1} = 0$.

(2) Since $\sigma^l = \sigma^0 \circ \gamma^l \circ \sigma^0$ for $l > 0$, we can assume that $l = 0$. In this case the assertion follows immediately from the definition of σ^0.

(3) By the definition of σ^0 and Theorem 1.8,

$$\sigma^0 \circ d^1 \circ \sigma^0(e_{i_1} \cdots e_{i_n}) = \sigma^0 \circ d^1(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1) = 0$$

and

$$\sigma^0 \circ d^2 \circ \sigma^0(e_{i_1} \cdots e_{i_n}) = \sigma^0 \circ d^2(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1) = 0.$$

Item (3) follows now easily by induction on l, since, by the recursive definition of σ^l and Theorem 1.8,

$$\sigma^1 = -\sigma^0 \circ d^1 \circ \sigma^0 \quad \text{and} \quad \sigma^l = -\sigma^0 \circ d^1 \circ \sigma^{l-1} - \sigma^0 \circ d^2 \circ \sigma^{l-2} \quad \text{for } l \geqslant 2.$$

(4) It is similar to the proof of item (3).

(5) Since $e_i a = a e_i$ for all $i \in I$ and $a \in A$,

$$\sigma^{-1} \circ \mu\big(a \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes a'\big) = \sigma^{-1}\big(a e_{i_1} \cdots e_{i_n} a'\big) = \sigma^{-1}\big(a a' e_{i_1} \cdots e_{i_n}\big) = 0,$$

where the last equality follows from the definition of σ^{-1}.

(6) We have

$$\begin{aligned}\sigma^{-1}\circ\mu(1\otimes_k g_{i_1}\wedge\cdots\wedge g_{i_n}\otimes 1\#g_{i_{n+1}}) &= \sigma^{-1}(e_{i_1}\cdots e_{i_n}z_{i_{n+1}})\\ &= \sigma^{-1}\big(e_{i_1}\cdots e_{i_n}(y_{i_{n+1}}+\rho_{i_{n+1}})\big)\\ &= \sigma^{-1}(y_{i_{n+1}}e_{i_1}\cdots e_{i_n})+\sigma^{-1}(e_{i_1}\cdots e_{i_n}\rho_{i_{n+1}}),\end{aligned}$$

where $z_{i_{n+1}}$, $y_{i_{n+1}}$ and $\rho_{i_{n+1}}$ are as in Theorem 1.3. So, in order to finish the proof it suffices to note that $\sigma^{-1}(y_{i_{n+1}}e_{i_1}\cdots e_{i_n})=0$ and

$$\sigma^{-1}(e_{i_1}\cdots e_{i_n}\rho_{i_{n+1}})=\begin{cases}(-1)^n e_{i_1}\cdots e_{i_{n+1}} & \text{if } i_n<i_{n+1},\\ 0 & \text{otherwise,}\end{cases}$$

which follows immediately from

$$e_{i_j}\rho_{i_{n+1}}=\rho_{i_{n+1}}e_{i_j}+e_{[x_{i_j},x_{i_{n+1}}]_{\mathfrak g}}\quad\text{for all } j \text{ such that } i_j>i_{n+1},$$

and the definition of σ^{-1}. $\square$

Theorem 2.3. *Let $(g_i)_{i\in I}$ be the basis of $\mathfrak g$ considered in Theorem 1.3. Assume that $\mathbf{c}_{1n}=c_1\otimes\cdots\otimes c_n\in\overline{E}^n$ is a simple tensor with $c_j\in A\cup\{1\#g_i\colon i\in I\}$ for all $j\in\{1,\dots,n\}$. If there exist $0\leqslant s\leqslant n$ and $i_1<\cdots<i_s$ in I, such that $c_j=1\#g_{i_j}$ for $1\leqslant j\leqslant s$ and $c_j\in A$ for $s<j\leqslant n$, then*

$$\vartheta(1\otimes\mathbf{c}_{1n}\otimes 1)=1\otimes_k g_{i_1}\wedge\cdots\wedge g_{i_s}\otimes\mathbf{c}_{s+1,n}\otimes 1.$$

Otherwise, $\vartheta(1\otimes\mathbf{c}_{1n}\otimes 1)=0$.

Proof. For all $n\geqslant 0$ we define P_n by $\mathbf{c}_{1n}\in P_n$ if there are $i_1<\cdots<i_s$ in I such that $c_j=1\#g_{i_j}$ for $j\leqslant s$ and $c_j\in A$ for $j>s$. We now proceed by induction on n. The case $n=0$ is immediate. Assume that the result is valid for ϑ_n. By item (1) of Lemma 2.2 and the recursive definition of ϑ_n, we have

$$\overline{\sigma}\circ\vartheta\big(\mathbf{c}'_{0n}\otimes 1\big)=\overline{\sigma}\circ\overline{\sigma}\circ\vartheta\circ b'\big(\mathbf{c}'_{0n}\otimes 1\big)=0,$$

and so

$$\vartheta(1\otimes\mathbf{c}_{1,n+1}\otimes 1)=(-1)^{n+1}\overline{\sigma}\circ\vartheta(1\otimes\mathbf{c}_{1,n+1}).$$

Assume that $c_j\in A\cup\{1\#g_i\colon i\in I\}$ for all $j\in\{1,\dots,n+1\}$. In order to finish the proof it suffices to show that:

- If $c_{1,n+1}\notin P_{n+1}$, then $\overline{\sigma}\circ\vartheta(1\otimes\mathbf{c}_{1,n+1})=0$.
- If $\mathbf{c}_{1,n+1}=1\#g_{i_1}\otimes\cdots\otimes 1\#g_{i_s}\otimes\mathbf{a}_{s+1,n+1}\in P_{n+1}$, then

$$\overline{\sigma}\circ\vartheta(1\otimes\mathbf{c}_{1,n+1})=(-1)^{n+1}\otimes_k g_{i_1}\wedge\cdots\wedge g_{i_s}\otimes\mathbf{a}_{s+1,n+1}\otimes 1.$$

If $\mathbf{c}_{1n}\notin P_n$, then $\vartheta(1\otimes\mathbf{c}_{1,n+1})=0$ by the inductive hypothesis. It remains to consider the case $\mathbf{c}_{1n}\in P_n$. We divide this into three subcases.

(1) If $\mathbf{c}_{1n} = 1\#g_{i_1} \otimes \cdots \otimes 1\#g_{i_s} \otimes \mathbf{a}_{s+1,n}$ and $c_{n+1} = a_{n+1} \in A$, then

$$\begin{aligned}\overline{\sigma} \circ \vartheta(1 \otimes \mathbf{c}_{1,n+1}) &= \overline{\sigma}(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s} \otimes \mathbf{a}_{s+1,n+1})\\ &= \sigma^0(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s} \otimes \mathbf{a}_{s+1,n+1})\\ &= (-1)^{n+1} \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s} \otimes \mathbf{a}_{s+1,n+1} \otimes 1,\end{aligned}$$

by the inductive hypothesis, items (4) and (5) of Lemma 2.2, and the definitions of $\overline{\sigma}$ and σ^0.

(2) If $\mathbf{c}_{1n} = 1\#g_{i_1} \otimes \cdots \otimes 1\#g_{i_s} \otimes \mathbf{a}_{s+1,n}$ with $s < n$ and $c_{n+1} = 1\#g_{i_{n+1}}$, then

$$\overline{\sigma} \circ \vartheta(1 \otimes \mathbf{c}_{1,n+1}) = \overline{\sigma}(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s} \otimes \mathbf{a}_{s+1,n} \otimes 1\#g_{i_{n+1}}) = 0,$$

by the inductive hypothesis, the definition of $\overline{\sigma}$ and item (2) of Lemma 2.2.

(3) If $\mathbf{c}_{1n} = 1\#g_{i_1} \otimes \cdots \otimes 1\#g_{i_n}$ and $c_{n+1} = 1\#g_{i_{n+1}}$, then

$$\begin{aligned}\overline{\sigma} \circ \vartheta(1 \otimes \mathbf{c}_{1,n+1}) &= \overline{\sigma}(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1\#g_{i_{n+1}})\\ &= -\sigma^0 \circ \sigma^{-1} \circ \mu(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1\#g_{i_{n+1}})\\ &= \begin{cases} (-1)^{n+1} \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_{n+1}} \otimes 1 & \text{if } \mathbf{c}_{1,n+1} \in P_{n+1},\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

by the inductive hypothesis, items (2), (3) and (6) of Lemma 2.2, and the definitions of $\overline{\sigma}$ and σ^0. $\square$

3. The Hochschild cohomology

Let $E = A\#_f U(\mathfrak{g})$ and let M be an E-bimodule. In this section we obtain a cochain complex $(\overline{X}^*_K(M), \bar{d}^*)$, simpler than the canonical one, giving the Hochschild cohomology of the K-algebra E with coefficients in M. When $K = k$ our result reduces to the one obtained in [G-G1, Section 5]. Then, we obtain an expression that gives the cup product of the Hochschild cohomology of E in terms of $(\overline{X}^*_K(E), \bar{d}^*)$. As usual, given $c \in E$ and $m \in M$, we let $[m, c]$ denote the commutator $mc - cm$.

3.1. The complex $(\overline{X}^*_K(M), \bar{d}^*)$

For $r, s \geqslant 0$, let

$$\overline{X}^{rs}_K(M) = \operatorname{Hom}_{K^e}\big(\overline{A}^r \otimes_k \mathfrak{g}^{\wedge s}, M\big),$$

where $\overline{A}^r \otimes_k \mathfrak{g}^{\wedge s}$ is considered as a K-bimodule via the canonical actions on $\overline{A}^r$. We define the morphism

$$\bar{d}^{rs}_l : \overline{X}^{r+l-1,s-l}_K(M) \to \overline{X}^{rs}_K(M) \quad \big(\text{with } 0 \leqslant l \leqslant \min(2, s) \text{ and } r + l > 0\big)$$

by:

$$\begin{aligned}\bar{d}_0(\varphi)(\mathbf{a}_{1r} \otimes_k \mathbf{x}_{1s}) &= a_1\varphi(\mathbf{a}_{2r} \otimes_k \mathbf{x}_{1s})\\ &\quad + \sum_{i=1}^{r-1}(-1)^i\varphi(\mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+2,r} \otimes_k \mathbf{x}_{1s})\\ &\quad + (-1)^r\varphi(\mathbf{a}_{1,r-1} \otimes_k \mathbf{x}_{1s})a_r,\end{aligned}$$

$$\begin{aligned}\bar{d}_1(\varphi)(\mathbf{a}_{1r}\otimes_k \mathbf{x}_{1s}) &= \sum_{i=1}^{s}(-1)^{i+r}\big[\varphi(\mathbf{a}_{1r}\otimes_k \mathbf{x}_{1\hat{\imath}s}), 1\#x_i\big]\\ &\quad+\sum_{\substack{i=1\\1\leqslant h\leqslant r}}^{s}(-1)^{i+r}\varphi\big(\mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes \mathbf{a}_{h+1,r}\otimes_k \mathbf{x}_{1\hat{\imath}s}\big)\\ &\quad+\sum_{1\leqslant i<j\leqslant s}(-1)^{i+j+r}\varphi\big(\mathbf{a}_{1r}\otimes_k [x_i,x_j]_{\mathfrak{g}}\wedge \mathbf{x}_{1\hat{\imath}\hat{\jmath}s}\big)\end{aligned}$$

and

$$\bar{d}_2(\varphi)(\mathbf{a}_{1r}\otimes_k \mathbf{x}_{1s}) = \sum_{\substack{1\leqslant i<j\leqslant s\\0\leqslant h\leqslant r}}(-1)^{i+j+h}\varphi(\mathbf{a}_{1h}\otimes \hat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes_k \mathbf{x}_{1\hat{\imath}\hat{\jmath}s}),$$

where $\hat{f}_{ij}=f(x_i,x_j)-f(x_j,x_i)$. Recall that $X_{rs}=(E\otimes_k \mathfrak{g}^{\wedge s})\otimes \overline{A}^r\otimes E$. Applying the functor $\mathrm{Hom}_{E^e}(-,M)$ to the complex (X_*,d_*) of Theorem 1.7, and using Theorem 1.8 and the identifications $\gamma^{rs}:\overline{X}_K^{rs}(M)\to \mathrm{Hom}_{E^e}(X_{rs},M)$, given by

$$\gamma(\varphi)(1\otimes_k \mathbf{x}_{1s}\otimes \mathbf{a}_{1r}\otimes 1)=(-1)^{rs}\varphi(\mathbf{a}_{1r}\otimes_k \mathbf{x}_{1s}),$$

we obtain the complex

$$\overline{X}_K^0(M)\xrightarrow{\bar{d}^1}\overline{X}_K^1(M)\xrightarrow{\bar{d}^2}\overline{X}_K^2(M)\xrightarrow{\bar{d}^3}\overline{X}_K^3(M)\xrightarrow{\bar{d}^4}\overline{X}_K^4(M)\xrightarrow{\bar{d}^5}\cdots,$$

where

$$\overline{X}_K^n(M)=\bigoplus_{r+s=n}\overline{X}_K^{rs}(M)\quad\text{and}\quad \bar{d}^n=\sum_{\substack{r+s=n\\r+l>0}}\sum_{l=0}^{\min(s,2)}\bar{d}_l^{rs}.$$

Note that if $f(\mathfrak{g}\otimes_k\mathfrak{g})\subseteq K$, then the cochain complex $(\overline{X}_K^*(M),\bar{d}^*)$ is the total complex of the double complex $(\overline{X}_K^{**}(M),\bar{d}_0^{**},\bar{d}_1^{**})$.

Theorem 3.1. *The Hochschild cohomology $\mathrm{H}_K^*(E,M)$, of the K-algebra E with coefficients in M, is the cohomology of $(\overline{X}_K^*(M),\bar{d}^*)$.*

Proof. It is an immediate consequence of the above discussion. □

3.2. The comparison maps

The maps θ_* and ϑ_*, introduced in Section 2, induce quasi-isomorphisms

$$\bar{\theta}^*:\big(\mathrm{Hom}_{K^e}\big(\overline{E}^*,M\big),b^*\big)\to\big(\overline{X}_K^*(M),\bar{d}^*\big)$$

and

$$\bar{\vartheta}^*:\big(\overline{X}_K^*(M),\bar{d}^*\big)\to\big(\mathrm{Hom}_{K^e}\big(\overline{E}^*,M\big),b^*\big)$$

which are inverse of each other up to homotopy.

Proposition 3.2. *We have*

$$\overline{\theta}(\psi)(\mathbf{a}_{1r}\otimes_k \mathbf{x}_{1s}) = \sum_{\tau\in\mathfrak{S}_s}(-1)^{rs}\operatorname{sg}(\tau)\psi\big((1\#x_{\tau(1)}\otimes\cdots\otimes 1\#x_{\tau(s)})*\mathbf{a}_{1r}\big).$$

Proof. This follows immediately from Proposition 2.1. □

In the sequel we consider that $\overline{X}_K^{rs}\subseteq\overline{X}_K^{r+s}$ in the canonical way.

Theorem 3.3. *Let $(g_i)_{i\in I}$ be the basis of $\mathfrak{g}$ considered in Theorem 1.3 and let $\varphi\in\overline{X}_K^{rs}$. Assume that $\mathbf{c}_{1,r+s}=c_1\otimes\cdots\otimes c_{r+s}\in\overline{E}^{r+s}$ is a simple tensor with $c_j\in A\cup\{1\#g_i\colon i\in I\}$ for all $j\in\{1,\dots,r+s\}$. If $c_j=1\#g_{i_j}$ with $i_1<\cdots<i_s$ in I for $1\leqslant j\leqslant s$ and $c_j\in A$ for $s<j\leqslant r+s$, then*

$$\overline{\vartheta}(\varphi)(\mathbf{c}_{1,r+s}) = (-1)^{rs}\varphi(\mathbf{c}_{s+1,r+s}\otimes_k g_{i_1}\wedge\cdots\wedge g_{i_s}).$$

Otherwise, $\overline{\vartheta}(\varphi)(\mathbf{c}_{1,r+s})=0$.

Proof. This follows immediately from Theorem 2.3. □

As usual, in the following subsection we will write $\mathrm{HH}_K^*(E)$ instead of $\mathrm{H}_K^*(E,E)$.

3.3. *The cup product*

Recall that the cup product of $\mathrm{HH}_K^*(E)$ is given in terms of $(\mathrm{Hom}_{K^e}(\overline{E}^*,E),b^*)$, by

$$\big(\psi\smile\psi'\big)(\mathbf{c}_{1,m+n}) = \psi(\mathbf{c}_{1m})\psi'(\mathbf{c}_{m+1,m+n}),$$

where $\psi\in\mathrm{Hom}_{K^e}(\overline{E}^m,E)$ and $\psi'\in\mathrm{Hom}_{K^e}(\overline{E}^n,E)$. In this subsection we compute the cup product of $\mathrm{HH}_K^*(E)$ in terms of the small complex $(\overline{X}_K^*(E),\bar{d}^*)$. Given

$$\varphi\in\overline{X}_K^{rs}(E)\quad\text{and}\quad\varphi'\in\overline{X}_K^{r's'}(E)$$

we define $\varphi\bullet\varphi'\in\overline{X}_K^{r+r',s+s'}(E)$ by

$$\big(\varphi\bullet\varphi'\big)(\mathbf{a}_{1r''}\otimes_k\mathbf{x}_{1s''}) = \sum_{1\leqslant j_1<\cdots<j_s\leqslant s''}\operatorname{sg}(j_{1s})\varphi(\mathbf{a}_{1r}\otimes_k\mathbf{x}_{j_{1s}})\varphi'(\mathbf{a}_{r+1,r''}\otimes_k\mathbf{x}_{l_{1s'}}),$$

where

- $\operatorname{sg}(j_{1s})=(-1)^{r's+\sum_{u=1}^{s}(j_u-u)}$,
- $r''=r+r'$ and $s''=s+s'$,
- $1\leqslant l_1<\cdots<l_{s'}\leqslant s''$ denote the set defined by

$$\{j_1,\dots,j_s\}\cup\{l_1,\dots,l_{s'}\}=\big\{1,\dots,s''\big\},$$

- $\mathbf{x}_{j_{1s}}=x_{j_1}\wedge\cdots\wedge x_{j_s}$ and $\mathbf{x}_{l_{1s'}}=x_{l_1}\wedge\cdots\wedge x_{l_{s'}}$.

Theorem 3.4. *The cup product of $\mathrm{HH}_K^*(E)$ is induced by the operation $\bullet$ in the complex $(\overline{X}_K^*(E),\bar{d}^*)$.*

Proof. Let $\varphi \in \overline{X}_K^{rs}(E)$ and $\varphi' \in \overline{X}_K^{r's'}(E)$. Let r'' and s'' be natural numbers satisfying $r''+s''=r+r'+s+s'$ and let $\mathbf{a}_{1r''} \otimes_k \mathbf{x}_{1s''} \in X^K_{r''s''}$. Let $(g_i)_{i\in I}$ be the basis of $\mathfrak{g}$ considered in Theorem 1.3. Clearly we can assume that there exist $i_1 < \cdots < i_{s''}$ in I such that $x_j = g_{i_j}$ for all $1 \leqslant j \leqslant s''$. By Proposition 3.2,

$$\bar{\theta}\big(\bar{\vartheta}(\varphi) \smile \bar{\vartheta}(\varphi')\big)(\mathbf{a}_{1r''} \otimes_k \mathbf{x}_{1s''}) = \big(\bar{\vartheta}(\varphi) \smile \bar{\vartheta}(\varphi')\big)(T)$$

where

$$T = \sum_{\tau \in \mathfrak{S}_{s''}} (-1)^{r''s''} \operatorname{sg}(\tau)\big((1\#x_{\tau(1)}) \otimes \cdots \otimes (1\#x_{\tau(s'')})\big) * \mathbf{a}_{1r''}.$$

In order to finish the proof it suffices to note that by Theorem 3.3, this is zero if $r'' \neq r+r'$ and this is $(\varphi \bullet \varphi')(\mathbf{a}_{1r''} \otimes_k \mathbf{x}_{1s''})$ if $r''=r+r'$. $\square$

4. The Hochschild homology

Let $E = A\#_f U(\mathfrak{g})$ and let M be an E-bimodule. In this section we obtain a chain complex $(\overline{X}^K_*(M), \bar{d}_*)$, simpler than the canonical one, giving the Hochschild homology of the K-algebra E with coefficients in M. When $K=k$ our result reduces to the one obtained in [G-G1, Section 4]. Then, we obtain an expression that gives the cap product of $\mathrm{H}^K_*(E,M)$ in terms of $(\overline{X}^*_K(E), \bar{d}^*)$ and $(\overline{X}^K_*(E,M), \bar{d}_*)$. As in the previous section $[m,c]$ denotes the commutator $mc-cm$ of $m \in M$ and $c \in E$.

4.1. The complex $(\overline{X}^K_*(M), \bar{d}_*)$

For $r,s \geqslant 0$, let

$$\overline{X}^K_{rs}(M) = \frac{M \otimes \overline{A}^r}{[M \otimes \overline{A}^r, K]} \otimes \mathfrak{g}^{\wedge s},$$

where $[M \otimes \overline{A}^r, K]$ is the k-vector space generated by the commutators $[m \otimes \mathbf{a}_{1r}, \lambda]$, with $\lambda \in K$ and $m \otimes \mathbf{a}_{1r} \in M \otimes \overline{A}^r$. We let $\overline{m \otimes \mathbf{a}_{1r}}$ denote the class of $m \otimes \mathbf{a}_{1r}$ in $M \otimes \overline{A}^r/[M \otimes \overline{A}^r, K]$. We define the morphism

$$\bar{d}^l_{rs} : \overline{X}^K_{rs}(M) \to \overline{X}^K_{r+l-1,s-l}(M) \quad \big(\text{with } 0 \leqslant l \leqslant \min(2,s) \text{ and } r+l>0\big)$$

by:

$$\begin{aligned}
\bar{d}^0(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) &= \overline{ma_1 \otimes \mathbf{a}_{2r}} \otimes_k \mathbf{x}_{1s} \\
&\quad + \sum_{i=1}^{r-1} (-1)^i \overline{m \otimes \mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+2,r}} \otimes_k \mathbf{x}_{1s} + (-1)^r \overline{a_r m \otimes \mathbf{a}_{1,r-1}} \otimes_k \mathbf{x}_{1s}, \\
\bar{d}^1(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) &= \sum_{i=1}^{s} (-1)^{i+r} \overline{\big[(1\#x_i), m\big] \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1\hat{\imath}s} \\
&\quad + \sum_{\substack{i=1 \\ 1 \leqslant h \leqslant r}}^{s} (-1)^{i+r} \overline{m \otimes \mathbf{a}_{1,h-1} \otimes a_h^{x_i} \otimes \mathbf{a}_{h+1,r}} \otimes_k \mathbf{x}_{1\hat{\imath}s} \\
&\quad + \sum_{1 \leqslant i < j \leqslant s} (-1)^{i+j+r} \overline{m \otimes \mathbf{a}_{1r}} \otimes_k [x_i, x_j]_{\mathfrak{g}} \wedge \mathbf{x}_{1\hat{\imath}\hat{\jmath}s}
\end{aligned}$$

and

$$\bar{d}^2(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) = \sum_{\substack{1 \leqslant i < j \leqslant s \\ 0 \leqslant h \leqslant r}} (-1)^{i+j+h} \overline{m \otimes \mathbf{a}_{1h} \otimes \hat{f}_{ij} \otimes \mathbf{a}_{h+1,r}} \otimes_k \mathbf{x}_{1\hat{i}\hat{j}s},$$

where $\hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i)$. Recall that $X_{rs} = (E \otimes_k \mathfrak{g}^{\wedge s}) \otimes \bar{A}^r \otimes E$ and let E^e be enveloping algebra of E. By tensoring on the left X_{rs} over E^e with M, and using Theorem 1.8 and the identifications $\gamma_{rs} : \bar{X}^K_{rs}(M) \to M \otimes_{E^e} X_{rs}$, given by

$$\gamma(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) = (-1)^{rs} m \otimes_{E^e} (1 \otimes_k \mathbf{x}_{1s} \otimes \mathbf{a}_{1r} \otimes 1),$$

we obtain the complex

$$\bar{X}^K_0(M) \xleftarrow{\bar{d}_1} \bar{X}^K_1(M) \xleftarrow{\bar{d}_2} \bar{X}^K_2(M) \xleftarrow{\bar{d}_3} \bar{X}^K_3(M) \xleftarrow{\bar{d}_4} \bar{X}^K_4(M) \xleftarrow{\bar{d}_5} \cdots,$$

where

$$\bar{X}^K_n(M) = \bigoplus_{r+s=n} \bar{X}^K_{rs}(M) \quad \text{and} \quad \bar{d}_n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=0}^{\min(s,2)} \bar{d}^l_{rs}.$$

Note that if $f(\mathfrak{g} \otimes_k \mathfrak{g}) \subseteq K$, then the chain complex $(\bar{X}^K_*(M), \bar{d}_*)$ is the total complex of the double complex $(\bar{X}^K_{**}(M), \bar{d}^0_{**}, \bar{d}^1_{**})$.

Theorem 4.1. *The Hochschild homology $\mathrm{H}^K_*(E, M)$, of the K-algebra E with coefficients in M, is the homology of $(\bar{X}^K_*(M), \bar{d}_*)$.*

Proof. It is an immediate consequence of the above discussion. □

4.2. The comparison maps

The maps θ_* and ϑ_*, introduced in Section 2, induce quasi-isomorphisms

$$\bar{\theta}_* : (\bar{X}^K_*(M), \bar{d}_*) \to \left(\frac{M \otimes \bar{E}^*}{[M \otimes \bar{E}^*, K]}, b_*\right)$$

and

$$\bar{\vartheta}_* : \left(\frac{M \otimes \bar{E}^*}{[M \otimes \bar{E}^*, K]}, b_*\right) \to (\bar{X}^K_*(M), \bar{d}_*)$$

which are inverse one of each other up to homotopy.

Proposition 4.2. *We have*

$$\bar{\theta}(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) = \sum_{\tau \in \mathfrak{S}_s} (-1)^{rs} \operatorname{sg}(\tau) \overline{m \otimes (1\#x_{\tau(1)} \otimes \cdots \otimes 1\#x_{\tau(s)}) * \mathbf{a}_{1r}}.$$

Proof. This follows immediately from Proposition 2.1. □

Theorem 4.3. *Let $(g_i)_{i\in I}$ be the basis of $\mathfrak{g}$ considered in Theorem 1.3. Assume that $\mathbf{c}_{1n} = c_1 \otimes \cdots \otimes c_n \in \overline{E}^n$ is a simple tensor with $c_j \in A \cup \{1\#g_i\colon\ i \in I\}$ for all $j \in \{1,\dots,n\}$. If there exist $0 \leqslant s \leqslant n$ and $i_1 < \cdots < i_s$ in I, such that $c_j = 1\#g_{i_j}$ for $1 \leqslant j \leqslant s$ and $c_j \in A$ for $s < j \leqslant n$, then*

$$\bar{\vartheta}(\overline{m \otimes \mathbf{c}_{1n}}) = (-1)^{s(n-s)}\overline{m \otimes \mathbf{c}_{s+1,n}} \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s}.$$

Otherwise, $\vartheta(\overline{m \otimes \mathbf{c}_{1n}}) = 0$.

Proof. This follows immediately from Theorem 2.3. $\square$

4.3. The cap product

Recall that the cap product

$$\mathrm{H}_p^K(E,M) \times \mathrm{HH}_K^q(E) \to \mathrm{H}_{p-q}^K(E,M) \quad (q \leqslant p)$$

is defined in terms of $(\frac{M\otimes\overline{E}^*}{[M\otimes\overline{E}^*,K]}, b_*)$ and $(\mathrm{Hom}_{K^e}(\overline{E}^*,E), b^*)$, by

$$\overline{m \otimes \mathbf{c}_{1p}} \frown \psi = \overline{m\psi(\mathbf{c}_{1q}) \otimes \mathbf{c}_{q+1,p}},$$

where $\psi \in \mathrm{Hom}_{K^e}(\overline{E}^q, E)$. In this subsection we compute the cap product in terms of the small complexes $(\overline{X}_*^K(M), \bar{d}_*)$ and $(\overline{X}_K^*(E), \bar{d}^*)$. Given

$$\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s} \in \overline{X}_{rs}^K(M) \quad \text{and} \quad \varphi' \in \overline{X}_K^{r's'}(E) \quad \text{with } r \geqslant r' \text{ and } s \geqslant s',$$

we define $(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) \bullet \varphi' \in \overline{X}_{r-r',s-s'}^K(M)$ by

$$(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) \bullet \varphi' = \sum_{1\leqslant j_1<\cdots<j_{s'}\leqslant s} \mathrm{sg}(j_{1s'})\overline{m\varphi'(\mathbf{a}_{1r'} \otimes_k \mathbf{x}_{j_{1s'}}) \otimes \mathbf{a}_{r'+1,r}} \otimes_k \mathbf{x}_{l_{1,s-s'}},$$

where

- $\mathrm{sg}(j_{1s'}) = (-1)^{rs'+r's'+\sum_{u=1}^{s'}(j_u-u)}$,
- $1 \leqslant l_1 < \cdots < l_{s-s'} \leqslant s$ denote the set defined by

$$\{j_1,\dots,j_{s'}\} \cup \{l_1,\dots,l_{s-s'}\} = \{1,\dots,s\},$$

- $\mathbf{x}_{j_{1s'}} = x_{j_1} \wedge \cdots \wedge x_{j_{s'}}$ and $\mathbf{x}_{l_{1,s-s'}} = x_{l_1} \wedge \cdots \wedge x_{l_{s-s'}}$.

Theorem 4.4. *In terms of the complexes $(\overline{X}_*^K(M), \bar{d}_*)$ and $(\overline{X}_K^*(E), \bar{d}^*)$, the cap product*

$$\mathrm{H}_p^K(E,M) \times \mathrm{HH}_K^q(E) \to \mathrm{H}_{p-q}^K(E,M)$$

is induced by $\bullet$.

Proof. Let $\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s} \in \overline{X}^K_{rs}(M)$ and $\varphi' \in \overline{X}_K^{r's'}(E)$. Let $(g_i)_{i \in I}$ be the basis of $\mathfrak{g}$ considered in Theorem 1.3. Clearly we can assume that there exist $i_1 < \cdots < i_s$ in I such that $x_j = g_{i_j}$ for all $1 \leqslant j \leqslant s$. By Proposition 4.2,

$$\overline{\vartheta}\big(\overline{\theta}(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) \frown \overline{\vartheta}\big(\varphi'\big)\big) = \overline{\vartheta}\big(T \frown \overline{\vartheta}\big(\varphi'\big)\big),$$

where

$$T = \sum_{\sigma \in \mathfrak{S}_s} (-1)^{rs} \operatorname{sg}(\sigma)\big((1 \# x_{\sigma(1)}) \otimes \cdots \otimes (1 \# x_{\sigma(s)})\big) * \mathbf{a}_{1r}.$$

Hence, by Theorem 3.3, if $r' > r$ or $s' > s$, then

$$\overline{\vartheta}\big(\overline{\theta}(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) \frown \overline{\vartheta}\big(\varphi'\big)\big) = 0,$$

and, if $r' \leqslant r$ and $s' \leqslant s$, then

$$\overline{\vartheta}\big(\overline{\theta}(\overline{m \otimes \mathbf{a}_{1r}} \otimes_k \mathbf{x}_{1s}) \frown \overline{\vartheta}\big(\varphi'\big)\big) = \sum_{1 \leqslant j_1 < \cdots < j_{s'} \leqslant s} \overline{\vartheta}\big(m\varphi'(\mathbf{a}_{1r'} \otimes_k \mathbf{x}_{j_{1s'}}) \otimes T'_{l_{l,s-s'}}\big),$$

where

$$T'_{l_{l,s-s'}} = \sum_{\tau \in \mathfrak{S}_{s-s'}} (-1)^{rs+r's} \operatorname{sg}(\tau)\big((1 \# x_{l_{\tau(1)}}) \otimes \cdots \otimes (1 \# x_{l_{\tau(s-s')}})\big) * \mathbf{a}_{r'+1,r}.$$

In order to finish the proof it suffices to apply Theorem 4.3. □

5. The (co)homology of $S(V)\#_f U(\mathfrak{g})$

In this section we obtain complexes $(\overline{Z}_*(M), \overline{\delta}_*)$ and $(\overline{Z}^*(M), \overline{\delta}^*)$, simpler than $(\overline{X}^*_K(M), \overline{d}^*)$ and $(\overline{X}^K_*(M), \overline{d}_*)$ respectively, giving the Hochschild homology of the K-algebra $E := A\#_f U(\mathfrak{g})$ with coefficients in an E-bimodule M, when

- $K = k$ and A is a symmetric algebra $S(V)$,
- $v^x \in k \oplus V$ for all $v \in V$ and $x \in \mathfrak{g}$,
- $f(x_1, x_2) \in k \oplus V$ for all $x_1, x_2 \in \mathfrak{g}$.

Then, we obtain an expression that gives the cup product of $\mathrm{HH}^*_K(E)$ in terms of $(\overline{Z}^*(E), \overline{\delta}^*)$, and we obtain an expression that gives the cap product of $\mathrm{H}^K_*(E, M)$ in terms of $(\overline{Z}_*(M), \overline{\delta}_*)$ and $(\overline{Z}^*(E), \overline{\delta}^*)$.

For $r, s \geqslant 0$, let $Z_{rs} = E \otimes \mathfrak{g}^{\wedge s} \otimes V^{\wedge r} \otimes E$. The groups Z_{rs} are E-bimodules in an obvious way. Let

$$\delta^l_{rs} : Z_{rs} \to Z_{r+l-1,s-l} \quad \big(0 \leqslant l \leqslant \min(2, s) \text{ and } r + l > 0\big)$$

be the E-bimodule morphisms defined by

$$\begin{aligned}
\delta^0(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) &= \sum_{i=1}^{r} (-1)^{i+s}(v_i \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1\hat{i}r} \otimes 1 - 1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1\hat{i}r} \otimes v_i),\\
\delta^1(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) &= \sum_{i=1}^{s} (-1)^{i+1} \# x_i \otimes \mathbf{x}_{1\hat{i}s} \otimes \mathbf{v}_{1r} \otimes 1\\
&\quad + \sum_{i=1}^{s} (-1)^{i} \otimes \mathbf{x}_{1\hat{i}s} \otimes \mathbf{v}_{1r} \otimes 1 \# x_i
\end{aligned}$$

$$+ \sum_{\substack{i=1 \\ 1\leqslant h\leqslant r}}^{s} (-1)^i \otimes \mathbf{x}_{1\hat{\imath}s} \otimes \mathbf{v}_{1,h-1} \wedge v_h^{\bar{x}_i} \wedge \mathbf{v}_{h+1,r} \otimes 1$$

$$+ \sum_{1\leqslant i<j\leqslant s} (-1)^{i+j} \otimes [x_i, x_j]_{\mathfrak{g}} \wedge \mathbf{x}_{1\hat{\imath}\hat{\jmath}s} \otimes \mathbf{v}_{1r} \otimes 1$$

and

$$\delta^2(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) = \sum_{1\leqslant i<j\leqslant s} (-1)^{i+j+s} \otimes \mathbf{x}_{1\hat{\imath}\hat{\jmath}s} \otimes \hat{f}_{ij} \wedge \mathbf{v}_{1r} \otimes 1,$$

where

- $\mathbf{v}_{hl} = v_h \wedge \cdots \wedge v_l$,
- $v_h^{\bar{x}_i}$ is the V-component of $v_h^{x_i}$ (that is $v_h^{\bar{x}_i} \in V$ and $v_h^{x_i} - v_h^{\bar{x}_i} \in k$),
- $\hat{f}_{ij} = f_V(x_i, x_j) - f_V(x_j, x_i)$ in which $f_V(x_i, x_j)$ and $f_V(x_j, x_i)$ are the V-components of $f(x_i, x_j)$ and $f(x_j, x_i)$, respectively.

Theorem 5.1. *The complex*

$$E \xleftarrow{\overline{\mu}} Z_0 \xleftarrow{\delta_1} Z_1 \xleftarrow{\delta_2} Z_2 \xleftarrow{\delta_3} Z_3 \xleftarrow{\delta_4} Z_4 \xleftarrow{\delta_5} Z_5 \xleftarrow{\delta_6} \cdots,$$

where

$$\overline{\mu}(1 \otimes 1) = 1, \qquad Z_n = \bigoplus_{r+s=n} Z_{rs} \quad \textit{and} \quad \delta_n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=0}^{\min(s,2)} \delta_{rs}^l,$$

is a projective resolution of the E-bimodule E. Moreover, the family of maps

$$\Gamma_* : Z_* \to X_*,$$

given by

$$\Gamma(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) = \sum_{\sigma\in\mathfrak{S}_r} \operatorname{sg}(\sigma) \otimes \mathbf{x}_{1s} \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes 1,$$

defines a morphism of E-bimodule complexes from (Z_*, δ_*) to (X_*, d_*).

Proof. It is clear that each Z_n is a projective E-bimodule and a direct computation shows that Γ_* is a morphism of complexes. Let

$$G_*^0 \subseteq G_*^1 \subseteq G_*^2 \subseteq G_*^3 \subseteq \cdots \quad \text{and} \quad F_*^0 \subseteq F_*^1 \subseteq F_*^2 \subseteq F_*^3 \subseteq \cdots$$

be the filtrations of (Z_*, δ_*) and (X_*, d_*), defined by

$$G_n^i = \bigoplus_{\substack{r+s=n \\ s\leqslant i}} Z_{rs} \quad \text{and} \quad F_n^i = \bigoplus_{\substack{r+s=n \\ s\leqslant i}} X_{rs},$$

respectively. In order to see that Γ_* is a quasi-isomorphism it is sufficient to show that it induces a quasi-isomorphism between the graded complexes associated with the filtrations introduced above. In other words, the maps

$$\Gamma_{*s} : \big(Z_{*s}, \delta^0_{*s}\big) \to \big(X_{*s}, d^0_{*s}\big) \quad (s \geqslant 0),$$

defined by

$$\Gamma(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sg}(\sigma) \otimes \mathbf{x}_{1s} \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes 1,$$

are quasi-isomorphisms, which follows easily from Proposition 2.1. □

5.1. Hochschild cohomology

Let M be an E-bimodule. For $r, s \geqslant 0$, let

$$\overline{Z}^{rs}(M) = \operatorname{Hom}_k\big(V^r \otimes \mathfrak{g}^{\wedge s}, M\big).$$

We define the morphism

$$\bar{\delta}^{rs}_l : \overline{Z}^{r+l-1,s-l}(M) \to \overline{Z}^{rs}(M) \quad \big(\text{with } 0 \leqslant l \leqslant \min(2,s) \text{ and } r+l > 0\big)$$

by:

$$\begin{aligned}
\bar{\delta}_0(\varphi)(\mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) &= \sum_{i=1}^{r} (-1)^i \big[v_i, \varphi(\mathbf{v}_{1\hat{\imath}r} \otimes \mathbf{x}_{1s})\big],\\
\bar{\delta}_1(\varphi)(\mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) &= \sum_{i=1}^{s} (-1)^{i+r} \big[\varphi(\mathbf{v}_{1r} \otimes \mathbf{x}_{1\hat{\imath}s}), 1\#x_i\big]\\
&\quad + \sum_{\substack{i=1\\ 1 \leqslant h \leqslant r}}^{s} (-1)^{i+r} \varphi\big(\mathbf{v}_{1,h-1} \wedge v_h^{\bar{x}_i} \wedge \mathbf{v}_{h+1,r} \otimes \mathbf{x}_{1\hat{\imath}s}\big)\\
&\quad + \sum_{1 \leqslant i < j \leqslant s} (-1)^{i+j+r} \varphi\big(\mathbf{v}_{1r} \otimes [x_i, x_j]_{\mathfrak{g}} \wedge \mathbf{x}_{1\hat{\imath}\hat{\jmath}s}\big)
\end{aligned}$$

and

$$\bar{\delta}_2(\varphi)(\mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) = \sum_{1 \leqslant i < j \leqslant s} (-1)^{i+j} \varphi(\hat{f}_{ij} \wedge \mathbf{v}_{1r} \otimes \mathbf{x}_{1\hat{\imath}\hat{\jmath}s}).$$

Applying the functor $\operatorname{Hom}_{E^e}(-, M)$ to the complex (Z_*, δ_*), and using Theorem 5.1 and the identifications $\xi^{rs} : \overline{Z}^{rs}(M) \to \operatorname{Hom}_{E^e}(Z_{rs}, M)$, given by

$$\xi(\varphi)(1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1) = (-1)^{rs} \varphi(\mathbf{v}_{1r} \otimes \mathbf{x}_{1s}),$$

we obtain the complex

$$\overline{Z}^0(M) \xrightarrow{\bar{\delta}^1} \overline{Z}^1(M) \xrightarrow{\bar{\delta}^2} \overline{Z}^2(M) \xrightarrow{\bar{\delta}^3} \overline{Z}^3(M) \xrightarrow{\bar{\delta}^4} \overline{Z}^4(M) \xrightarrow{\bar{\delta}^5} \cdots,$$

where

$$\overline{Z}^n(M) = \bigoplus_{r+s=n} \overline{Z}^{rs}(M) \quad \text{and} \quad \bar{\delta}^n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=0}^{\min(s,2)} \bar{\delta}_l^{rs}.$$

Note that if $f(\mathfrak{g}\otimes\mathfrak{g}) \subseteq k$, then the cochain complex $(\overline{Z}^*(M), \bar{\delta}^*)$ is the total complex of the double complex $(\overline{Z}^{**}(M), \bar{\delta}_0^{**}, \bar{\delta}_1^{**})$.

Theorem 5.2. *The Hochschild cohomology* $\mathrm{H}^*(E, M)$, *of* E *with coefficients in* M, *is the cohomology of* $(\overline{Z}^*(M), \bar{\delta}^*)$.

The map $\Gamma_*: (Z_*, \delta_*) \to (X_*, d_*)$ induces a quasi-isomorphism

$$\overline{\Gamma}^*: \left(\overline{X}_k^*(M), \bar{d}_*\right) \to \left(\overline{Z}^*(M), \bar{\delta}^*\right).$$

Proposition 5.3. *We have*

$$\overline{\Gamma}(\varphi)(\mathbf{v}_{1r}\otimes\mathbf{x}_{1s}) = \sum_{\sigma\in\mathfrak{S}_r} \operatorname{sg}(\sigma)\varphi(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r)}\otimes\mathbf{x}_{1s}).$$

Proof. This follows immediately from Theorem 5.1. □

5.2. The cup product

In this subsection we compute the cup product of $\mathrm{HH}^*(E)$ in terms of the complex $(\overline{Z}^*(E), \bar{\delta}^*)$. Given $\phi \in \overline{Z}^{rs}(E)$ and $\phi' \in \overline{Z}^{r's'}(E)$, we define $\phi \star \phi' \in \overline{Z}^{r+r',s+s'}(E)$ by

$$\left(\phi\star\phi'\right)(\mathbf{v}_{1r''}\otimes\mathbf{x}_{1s''}) = \sum_{\substack{1\leqslant i_1<\cdots<i_r\leqslant r''\\ 1\leqslant j_1<\cdots<j_s\leqslant s''}} \operatorname{sg}(i_{1r}, j_{1s})\phi(\mathbf{v}_{i_{1r}}\otimes\mathbf{x}_{j_{1s}})\phi'(\mathbf{v}_{h_{1r'}}\otimes\mathbf{x}_{l_{1s'}}),$$

where

- $\operatorname{sg}(i_{1r}, j_{1s}) = (-1)^{r's+\sum_{u=1}^{r}(i_u-u)+\sum_{u=1}^{s}(j_u-u)}$,
- $r'' = r + r'$ and $s'' = s + s'$,
- $1 \leqslant h_1 < \cdots < h_{r'} \leqslant r''$ denote the set defined by

$$\{i_1, \ldots, i_r\} \cup \{h_1, \ldots, h_{r'}\} = \left\{1, \ldots, r''\right\},$$

- $1 \leqslant l_1 < \cdots < l_{s'} \leqslant s''$ denote the set defined by

$$\{j_1, \ldots, j_s\} \cup \{l_1, \ldots, l_{s'}\} = \left\{1, \ldots, s''\right\},$$

- $\mathbf{v}_{i_{1r}} = v_{i_1}\wedge\cdots\wedge v_{i_r}$ and $\mathbf{v}_{h_{1r'}} = v_{h_1}\wedge\cdots\wedge v_{h_{r'}}$,
- $\mathbf{x}_{j_{1s}} = x_{j_1}\wedge\cdots\wedge x_{j_s}$ and $\mathbf{x}_{l_{1s'}} = x_{l_1}\wedge\cdots\wedge x_{l_{s'}}$.

Theorem 5.4. *The cup product of* $\mathrm{HH}^*(E)$ *is induced by the operation* $\star$ *in the complex* $(\overline{Z}^*(E), \bar{\delta}^*)$.

Proof. By Theorem 3.4 it suffices to prove that

$$\overline{\Gamma}(\varphi \bullet \varphi') = \overline{\Gamma}(\varphi) \star \overline{\Gamma}(\varphi') \tag{2}$$

for all $\varphi \in \overline{X}_k^{rs}(E)$ and $\varphi' \in \overline{X}_k^{r's'}(E)$. Let $\phi = \overline{\Gamma}(\varphi)$ and $\phi' = \overline{\Gamma}(\varphi')$. On one hand

$$\begin{aligned}(\phi \star \phi')(\mathbf{v}_{1r''} \otimes \mathbf{x}_{1s''}) &= \sum_{\substack{1 \leqslant i_1 < \cdots < i_r \leqslant r'' \\ 1 \leqslant j_1 < \cdots < j_s \leqslant s''}} \operatorname{sg}(i_{1r}, j_{1s}) \phi(\mathbf{v}_{i_{1r}} \otimes \mathbf{x}_{j_{1s}}) \phi'(\mathbf{v}_{h_{1r'}} \otimes \mathbf{x}_{l_{1s'}}) \\ &= \sum_{\substack{1 \leqslant i_1 < \cdots < i_r \leqslant r'' \\ 1 \leqslant j_1 < \cdots < j_s \leqslant s'' \\ \tau \in \mathfrak{S}_r,\, \nu \in \mathfrak{S}_{r'}}} \operatorname{sg}(i_{1r}, j_{1s}) \operatorname{sg}(\tau) \operatorname{sg}(\nu) \varphi(\mathbf{v}_{i_{\tau(1r)}} \otimes \mathbf{x}_{j_{1s}}) \varphi'(\mathbf{v}_{h_{\nu(1r')}} \otimes \mathbf{x}_{l_{1s'}}),\end{aligned}$$

where

$$\mathbf{v}_{i_{\tau(1r)}} = v_{i_{\tau(1)}} \otimes \cdots \otimes v_{i_{\tau(r)}} \quad \text{and} \quad \mathbf{v}_{h_{\nu(1r')}} = v_{h_{\nu(1)}} \otimes \cdots \otimes v_{h_{\nu(r')}}.$$

On the other hand

$$\begin{aligned}\overline{\Gamma}(\varphi \bullet \varphi')(\mathbf{v}_{1r''} \otimes \mathbf{x}_{1s''}) &= \sum_{\sigma \in \mathfrak{S}_{r''}} \operatorname{sg}(\sigma)(\varphi \bullet \varphi')(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r'')} \otimes \mathbf{x}_{1s''}) \\ &= \sum_{\substack{1 \leqslant j_1 < \cdots < j_s \leqslant s'' \\ \sigma \in \mathfrak{S}_{r''}}} \operatorname{sg}(\sigma) \operatorname{sg}(j_{is}) \varphi(\mathbf{v}_{\sigma(1r)} \otimes \mathbf{x}_{j_{1s}}) \varphi'(\mathbf{v}_{\sigma(r+1,r'')} \otimes \mathbf{x}_{l_{1s'}}),\end{aligned}$$

where

$$\mathbf{v}_{\sigma(1r)} = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \quad \text{and} \quad \mathbf{v}_{\sigma(r+1,r'')} = v_{\sigma(r+1)} \otimes \cdots \otimes v_{\sigma(r'')}.$$

Now, formula (2) follows immediately from these facts. □

5.3. *Hochschild homology*

Let M be an E-bimodule. For $r, s \geqslant 0$, let

$$\overline{Z}_{rs}(M) = M \otimes V^{\wedge r} \otimes \mathfrak{g}^{\wedge s}.$$

We define the morphisms

$$\bar{\delta}^l_{rs} : \overline{Z}_{rs}(M) \to \overline{Z}_{r+l-1,s-l}(M) \quad \big(0 \leqslant l \leqslant \min(2, s) \text{ and } r + l > 0\big)$$

by:

$$\bar{\delta}^0(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) = \sum_{i=1}^{r} (-1)^i [m, v_i] \otimes \mathbf{v}_{1\hat{i}r} \otimes \mathbf{x}_{1s},$$

$$\begin{aligned}\bar{\delta}^1(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) &= \sum_{i=1}^{s} (-1)^{i+r} [1\#x_i, m] \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1\hat{i}s} \\ &\quad + \sum_{\substack{i=1 \\ 1\leqslant h\leqslant r}}^{s} (-1)^{i+r} m \otimes \mathbf{v}_{1,h-1} \wedge v_h^{\bar{x}_i} \wedge \mathbf{v}_{h+1,r} \otimes \mathbf{x}_{1\hat{i}s} \\ &\quad + \sum_{1\leqslant i<j\leqslant s} (-1)^{i+j+r} m \otimes \mathbf{v}_{1r} \otimes [x_i, x_j]_{\mathfrak{g}} \wedge \mathbf{x}_{1\hat{i}\hat{j}s}\end{aligned}$$

and

$$\bar{\delta}^2(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) = \sum_{1\leqslant i<j\leqslant s} (-1)^{i+j} m \otimes \hat{f}_{ij} \wedge \mathbf{v}_{1r} \otimes \mathbf{x}_{1\hat{i}\hat{j}s}.$$

By tensoring on the left the complex (Z_*, δ_*) over E^e with M, and using Theorem 5.1 and the identifications $\xi_{rs} : \overline{Z}_{rs}(M) \to M \otimes_{E^e} Z_{rs}$, given by

$$\xi(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) = (-1)^{rs} m \otimes_{E^e} (1 \otimes \mathbf{x}_{1s} \otimes \mathbf{v}_{1r} \otimes 1),$$

we obtain the complex

$$\overline{Z}_0(M) \xleftarrow{\bar{\delta}_1} \overline{Z}_1(M) \xleftarrow{\bar{\delta}_2} \overline{Z}_2(M) \xleftarrow{\bar{\delta}_3} \overline{Z}_3(M) \xleftarrow{\bar{\delta}_4} \overline{Z}_4(M) \xleftarrow{\bar{\delta}_5} \cdots,$$

where

$$\overline{Z}_n(M) = \bigoplus_{r+s=n} \overline{Z}_{rs}(M) \quad \text{and} \quad \bar{\delta}_n = \sum_{\substack{r+s=n \\ r+l>0}} \sum_{l=0}^{\min(s,2)} \bar{\delta}^l_{rs}.$$

Note that if $f(\mathfrak{g} \otimes \mathfrak{g}) \subseteq k$, then the chain complex $(\overline{Z}_*(M), \bar{\delta}_*)$ is the total complex of the double complex $(\overline{Z}_{**}(M), \bar{\delta}^0_{**}, \bar{\delta}^1_{**})$.

Theorem 5.5. *The Hochschild homology* $\mathrm{H}_*(E, M)$*, of* E *with coefficients in* M*, is the homology of* $(\overline{Z}_*(M), \bar{\delta}_*)$*.*

Proof. It is an immediate consequence of the above discussion. □

The map $\Gamma_* : (Z_*, \delta_*) \to (X_*, d_*)$ induces a quasi-isomorphism

$$\overline{\Gamma}_* : \big(\overline{Z}_*(M), \bar{\delta}_*\big) \to \big(\overline{X}^k_*(M), \bar{d}_*\big).$$

Proposition 5.6. *We have*

$$\overline{\Gamma}(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sg}(\sigma) m \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes \mathbf{x}_{1s}.$$

Proof. This follows immediately from Theorem 5.1. □

5.4. The cap product

In this subsection we compute the cap product

$$\mathrm{H}_p(E,M)\times \mathrm{HH}^q(E)\to \mathrm{H}_{p-q}(E,M)\quad (q\leqslant p),$$

in terms of the complexes $(\overline{Z}_*(M),\overline{\delta}_*)$ and $(\overline{Z}^*(E),\overline{\delta}^*)$. Given

$$m\otimes \mathbf{v}_{1s}\otimes \mathbf{x}_{1s}\in \overline{Z}_{rs}(M)\quad\text{and}\quad \phi'\in \overline{Z}^{r's'}(E)\quad \text{with } r\geqslant r' \text{ and } s\geqslant s',$$

we define $(m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s})\star\phi'\in \overline{Z}_{r-r',s-s'}(M)$ by

$$(m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s})\star\phi' = \sum_{\substack{1\leqslant i_1<\cdots<i_{r'}\leqslant r\\ 1\leqslant j_1<\cdots<j_{s'}\leqslant s}} \mathrm{sg}(i_{1r'},j_{1s'})m\phi'(\mathbf{v}_{i_{1r'}}\otimes \mathbf{x}_{j_{1s'}})\otimes \mathbf{v}_{h_{1,r'-r}}\otimes \mathbf{x}_{l_{1,s'-s}},$$

where

- $\mathrm{sg}(i_{1r'},j_{1s'}) = (-1)^{rs'+r's'+\sum_{u=1}^{r'}(i_u-u)+\sum_{u=1}^{s'}(j_u-u)}$,
- $1\leqslant h_1<\cdots<h_{r-r'}\leqslant r$ denote the set defined by

$$\{i_1,\dots,i_{r'}\}\cup\{h_1,\dots,h_{r-r'}\}=\{1,\dots,r\},$$

- $1\leqslant l_1<\cdots<l_{s-s'}\leqslant s$ denote the set defined by

$$\{j_1,\dots,j_{s'}\}\cup\{l_1,\dots,l_{s-s'}\}=\{1,\dots,s\},$$

- $\mathbf{v}_{i_{1r'}}=v_{i_1}\wedge\cdots\wedge v_{i_{r'}}$ and $\mathbf{v}_{h_{1,r-r'}}=v_{h_1}\wedge\cdots\wedge v_{h_{r-r'}}$,
- $\mathbf{x}_{j_{1s'}}=x_{j_1}\wedge\cdots\wedge x_{j_{s'}}$ and $\mathbf{x}_{l_{1,s-s'}}=x_{l_1}\wedge\cdots\wedge x_{l_{s-s'}}$.

Theorem 5.7. *The cap product*

$$\mathrm{H}_p(E,M)\times \mathrm{HH}^q(E)\to \mathrm{H}_{p-q}(E,M)\quad (q\leqslant p)$$

is induced by $\star$*, in terms of the complexes* $(\overline{Z}_*(M),\overline{\delta}_*)$ *and* $(\overline{Z}^*(E),\overline{\delta}^*)$.

Proof. By Theorem 4.4 it suffices to prove that

$$\overline{\Gamma}(m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s})\bullet\varphi' = \overline{\Gamma}\big((m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s})\star\overline{\Gamma}(\varphi')\big) \tag{3}$$

for all $m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s}\in \overline{Z}_{rs}(M)$ and $\varphi'\in \overline{X}_k^{r's'}(E)$. Let $\phi'=\overline{\Gamma}(\varphi')$. On one hand

$$\overline{\Gamma}(m\otimes \mathbf{v}_{1r}\otimes \mathbf{x}_{1s})\bullet\varphi' = \sum_{\substack{1\leqslant j_1<\cdots<j_{s'}\leqslant s\\ \sigma\in\mathfrak{S}_r}} \mathrm{sg}(\sigma)\,\mathrm{sg}(j_{1,s'})m\varphi'(\mathbf{v}_{\sigma(1r')}\otimes \mathbf{x}_{j_{1s'}})\otimes \mathbf{v}_{\sigma(r'+1,r)}\otimes \mathbf{x}_{l_{1,s-s'}},$$

where

$$\mathbf{v}_{\sigma(1r')}=v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(r')}\quad\text{and}\quad \mathbf{v}_{\sigma(r'+1,r)}=v_{\sigma(r'+1)}\otimes\cdots\otimes v_{\sigma(r)}.$$

On the other hand

$$(m \otimes \mathbf{v}_{1r} \otimes \mathbf{x}_{1s}) \star \phi' = \sum_{\substack{1 \leqslant i_1 < \cdots < i_{r'} \leqslant r \\ 1 \leqslant j_1 < \cdots < j_{s'} \leqslant s}} \operatorname{sg}(i_{1r'}, j_{1s'}) m\phi'(\mathbf{v}_{i_{1r'}} \otimes \mathbf{x}_{j_{1s'}}) \otimes \mathbf{v}_{h_{1,r'-r}} \otimes \mathbf{x}_{l_{1,s'-s}}$$

$$= \sum_{\substack{1 \leqslant i_1 < \cdots < i_{r'} \leqslant r \\ 1 \leqslant j_1 < \cdots < j_{s'} \leqslant s \\ \tau \in \mathfrak{S}_{r'}}} \operatorname{sg}(\tau) \operatorname{sg}(i_{1r'}, j_{1s'}) m\varphi'(\mathbf{v}_{i_{\tau(1r')}} \otimes \mathbf{x}_{j_{1s'}}) \otimes \mathbf{v}_{h_{1,r'-r}} \otimes \mathbf{x}_{l_{1,s'-s}},$$

where $\mathbf{v}_{i_{\tau(1r')}} = v_{i_{\tau(1)}} \otimes \cdots \otimes v_{i_{\tau(r')}}$. Formula (3) follows immediately. $\square$

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